TORUS KNOTS AND CONTACT SURGERIES

TORUS DÜĞÜMLERİ VE KONTAKT AMELİYATLAR

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ABSTRACT

TORUS KNOTS AND CONTACT SURGERIES

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A closed curve homeomorphic to the unit circle S^1 in a 3-manifold is called a knot. In particularly, a knot that can be drawn on a torus without intersecting itself is called a torus knot. In this thesis, we study torus knots and their topological properties, invariants and polynomials. We study Dehn surgery on torus knots in topological 3-manifolds. Then, we study contact 3-manifolds. We study a special class of Legendrian knots which have topological knot type as torus knots. The aim of this thesis is to study lens spaces by using contact surgery techniques. For this purpose, obtaining lens spaces $L(4m+3, 4)$ by Legendrian surgery along the negative torus knots $\mathtt{T}(2, -(2m+1))$ where $m \geq 1$ are studied in detail.

Keywords: Torus knot, contact structure, Legendrian knot, contact surgeries

ÖZET

TORUS DÜĞÜMLERİ VE KONTAKT AMELİYATLAR

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3-manifoldlar içerisinde kendi kendisini kesmeyen kapalı eğrilere düğüm denir. Özel olarak, 2-boyutlu torus üzerinde kendi kendisini kesmeyecek şekilde çizilebilen düğüm tiplerine torus düğümleri denir. Bu tezde topolojik 3-manifoldlar içerisindeki torus düğümlerinin yanı sıra kontakt 3-manifoldlar içerisindeki Legendre torus düğümleri çalışılacaktır. Torus düğümlerinin topolojik özellikleri, değişmezleri ve polinomları çalışılıp hesaplanacaktır. Torus düğümlerine yapılan topolojik Dehn ameliyatları ve Legendre düğümlerine yapılan kontakt ameliyatlar çalışılacaktır. Bu tezde Legendre torus düğümleri de çalışılmıştır. Bu tezin amacı lens uzaylarını kontakt ameliyat tekniklerini kullanarak çalışmaktır. Bu amaçla, $m \geq 1$ olmak üzere $L(4m+3, 4)$ lens uzaylarının $\mathtt{T}(2, -(2m+1))$ negatif torus düğümlerine Legendre kontakt ameliyat yapılarak elde edilme tekniği detaylı olarak incelenmiştir.

Anahtar Kelimeler: Torus düğümü, kontakt yapı, Legendre düğümü, kontakt ameliyatlar

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Contents

List of Figures

1 INTRODUCTION

Knot theory is one of the main fields of topology in mathematics. A knot in a 3-manifold can be thought as a simple knotted circle which is a main example of 1-manifolds. Classification of knots is an important problem in knot theory. W. Thurston divides all knots into three groups which are torus knots, satellite knots and hyperbolic knots [1]. Torus knots are a member of special class of knots where one can draw on a torus without any self-intersection. In this thesis, torus knots are studied in detail.

Distinguishing two knots is another important problem in knot theory. Invariants are used to show that two knots are different. Polynomials are one of the most important invariants. The first polynomial invariant was invented by J. Alexander in 1928 [2]. It is called Alexander polynomial which is an invariant such that each knot corresponds to a polynomial. He showed that if the Alexander polynomials of two knots are different, then they are different. Another powerful polynomial invariant was introduced by V. Jones [3]. He also gave a formula for Jones polynomials of torus knots [4].

Several techniques are used to obtain 3-manifolds. However, the most important technique is Dehn surgery which was introduced by Max Dehn. Dehn surgery applied to a knot is an operation such that a tubular neighborhood of the knot is taken out in a given manifold, and by using homeomorphism of its neighborhood it is glued back. So, a new 3-manifold is formed, and this provides to understand 3-manifolds better. Lickorish and Wallace demonstrated that each closed, orientable 3-manifold is attained by surgery applied to some links in S^3 [5] [6].

Dehn surgery on a 3-manifold can give a different 3-manifold, but Kirby moves on a 3-manifold do not change the manifold. These moves were defined by Robion Kirby. [7]. The first move is taking a connected sum with S^3 or canceling. The second move is done by sliding one component of the framed link on another.

Lens spaces can be obtained by an operation called Dehn surgery applied to knots in many ways. Moser classified the lens spaces that are result of Dehn surgery applied to torus knots [8]. Unlike Moser, Bailey & Rolfsen presented the first example for obtaining the lens space $L(23, 7)$ by an integral surgery along an iterated cable knot which is not a torus knot [9]. The fact that surgeries on which knots give lens spaces is an important problem. Culler, Gordon, Luecke and Shalen showed that there are at most two surgery coefficients which give a lens space in case of different knots from torus knots [10]. In the simplest sense, the lens space $L(m, n)$ is constructed by $(-m/n)$-surgery along an unknot in 3-sphere. Rasmussen reached an important conclusion about this problem. He showed that there is only one way to obtain the lens space $L(4m+3, 4)$ by a single integral surgery on 3-sphere. He showed that $-(4m+3)$-surgery along $\mathtt{T}(2, -(2m+1))$ is the only integral surgery which gives $L(4m+3, 4)$ in [11].

Contact structures take place on smooth and odd dimensional manifolds, and these are maximally non-integrable two plane fields distributed all over the manifold. Marinet gave a proof that there is a contact structure on every 3-manifold [12]. There are two categories for contact 3-manifolds which are overtwisted and tight [13], [14]. Similar to topological 3-manifolds, contact 3-manifolds include knots which are Legendrian knots and transverse knots.

A knot in a contact 3-manifold that it is tangent to contact planes everywhere is a special kind of knots that are called Legendrian knots. Similar to topological knots, Legendrian knot classification is an important problem. Many mathematicians studied on Legendrian knot classification in tight contact 3-manifolds. The first Legendrian knot classification result was given by Eliashberg & Fraser. They classified Legendrian unknots in standard tight S^3 [15], see also [16]. After Eliashberg and Fraser, Legendrian torus knots were classified based on Legendrian isotopy by Etnyre and Honda. They also classified the figure eight knot in standard tight contact S^3 [17]. Their classification theorems of Legendrian torus knots are given in this thesis. Legendrian torus knots which exist in other contact 3-manifolds different than S^3 are also studied. Onaran classified Legendrian positive torus knots in universally tight contact lens spaces [18]. Moreover, Legendrian knot classification problem in overtwisted contact 3-manifolds is another important problem. Classification of exceptional unknots in overtwisted S^3 was given by Eliashberg and Fraser [16]. The first nontrivial knot type classification was done by Geiges and Onaran [19]. Exceptional torus knots were classified in overtwisted S^3 in [19].

Contact surgery is a surgery operation which can be applied to Legendrian knots in contact 3-manifolds. Similar to Dehn surgery which is a topological operation, a contact surgery applied to Legendrian knots constructs new contact 3-manifolds. The fact that every closed contact 3-manifolds can be formed by (± 1)-contact surgery in standard tight contact S^3 was shown by Ding & Geiges [20].

In this thesis, lens spaces obtained by a single contact surgery with contact framing (-1) applied to a single Legendrian negative torus knot are studied. Also, contact surgery techniques given by Geiges and Onaran in [21] are studied.

In Section 2 Background section, the main definitions and examples are well noted. After defining knots which are the fundemental examples of 1-manifolds, definitions and examples of torus knots are given. Some polynomial invariants of torus knots are introduced, and examples are given. Also, definition of surfaces is given, and then Seifert surfaces for torus knots are constructed. Dehn surgery and Kirby moves are presented with main examples. Contact 3-manifolds are introduced. Legendrian knots in contact 3-manifolds and their classical invariants are given with examples.

In Section 3 Legendrian Torus Knots section, definition of Legendrian torus knots and classification theorems of Legendrian torus knots in [17] are given. Also, contact surgery techniques and classical invariants of Legendrian knots from surgery diagrams are introduced. In the same section, lens spaces are studied by using contact surgery techniques. The proof of obtaining lens spaces $L(7,4)$ by contact (-1)-surgery along Legendrian left handed trefoil in some contact structure on 3-sphere [21] is given. Also, the general case, the proof of obtaining lens spaces $L(4m+3,4)$ by contact surgery with contact framing (-1) applied to Legendrian $\mathtt{T}(2,-(2m+1))$ in some contact structure on 3-sphere [21] is studied in detail.

Finally in Section 4 Conclusion section, contact surgery techniques are analyzed to obtain lens spaces from Legendrian negative torus knots. Some open problems about Legendrian torus knots are listed.

2 BACKGROUND

2.1 Knots in the 3-Sphere

Definition 2.1.1. [22] A simple closed curve in 3-sphere S^3 which does not have a self-intersection is called a **topological knot** in S^3 up to isotopy.

Example 2.1.1. Some examples of knots are given in the Figure 1. The unknot or the trivial knot which has no crossing is the simplest knot. The next one is a right handed trefoil knot which has 3 crossings. Also, figure eight is another example of a knot which has 4 crossings.

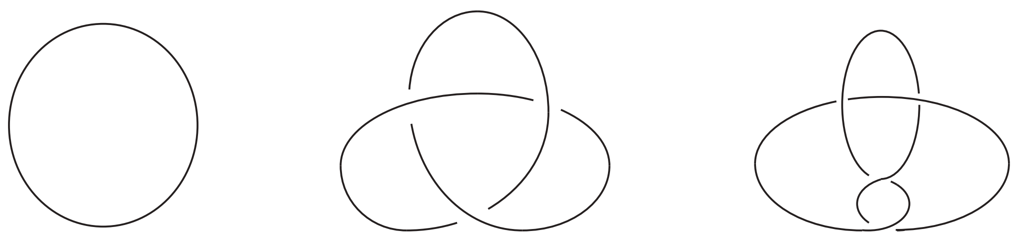


Figure 1: Unknot, right handed trefoil knot and figure 8 knot in S^3.

2.2 Knot Types in the 3-Sphere

Before 1974, it had been known to exist only two classes of knots which were torus and satellite knot. Robert Riley proved that figure 8 knot was hyperbolic [23]. After Robert Rilley, William Thurston showed that all knots are either torus knots, sattelite knots or hyperbolic knots [1].

Definition 2.2.1. [24] A knot which one can draw on the torus so that it has no self-intersecting is called a *torus knot*.

Torus knots will be studied comprehensively in Chapter 2.3.

Definition 2.2.2. [22] Assume that K_1 is a knot inside a solid torus and K_2 is any knot different from the unknot. The solid torus is cut from a meridian of it. A knot which is obtained from gluing the solid torus which has the form of the knot K_2 is named as a *satellite knot*. The second knot K_2 is called the *companion knot*.

Example 2.2.1. [22] Let K_1 be a trefoil in a solid torus and K_2 be a figure 8. The composite knot $K_1 \# K_2$ in the Figure 2 is an example of satellite knots.

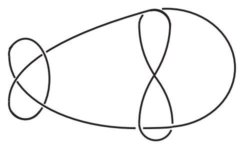


Figure 2: The composite knot $K_1\#K_2$.

Definition 2.2.3. [22] Consider a knot K in the 3-sphere S^3. It is called a *hyperbolic knot* on condition that 3-manifold $S^3 \setminus K$ is a hyperbolic 3-manifold. (A 3-manifold is called a hyperbolic if there is a metric which has a constant curvature -1 on it.)

Example 2.2.2. [23] Figure 8 is an example of hyperbolic knots.

Theorem 2.2.1. *[1] "(Classification of Knots) All knots fall into three categories of knots that are satellite knots, hyperbolic knots, and the last category that is the torus knots."*

2.3 Torus Knots in the 3-Sphere

Example 2.3.1. Right handed trefoil is the main example of torus knots.

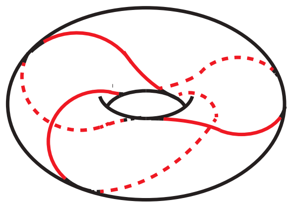


Figure 3: Right handed trefoil on a torus.

Every torus knot is represented by (l, m)-torus knot for relatively prime integers l and m.

Definition 2.3.1. [22] A (l, m)-torus knot means that the knot on the torus that wraps l times in the meridional direction and m times in the longitudinal direction for relatively prime integers l and m. It is denoted by $\mathtt{T}(l, m)$. If both l and m are positive, $\mathtt{T}(l, m)$ is called a *positive torus knot*. If one of these two integers is negative, it is called a *negative torus knot*.

In this thesis, negative torus knots are denoted by $\mathtt{T}(l,-m)$ for positive integers l, m. Also, the positive torus knots $\mathtt{T}(l,m)$ where $m > l > 0$, and the negative torus knots $\mathtt{T}(l,-m)$ where $m > l > 0$ will be studied.

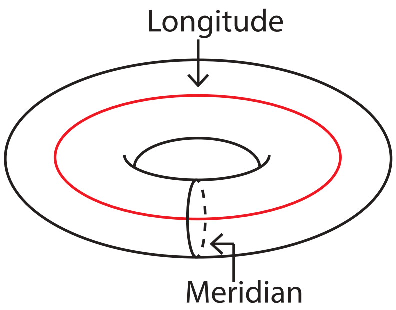


Figure 4: A longitude and meridian on a torus.

Example 2.3.2. Right handed trefoil is a positive torus knot denoted by $\mathtt{T}(2,3)$. Left handed trefoil is a negative torus knot denoted by $\mathtt{T}(2,-3)$. Both of them intersect the longitude 2 times and the meridian 3 times.

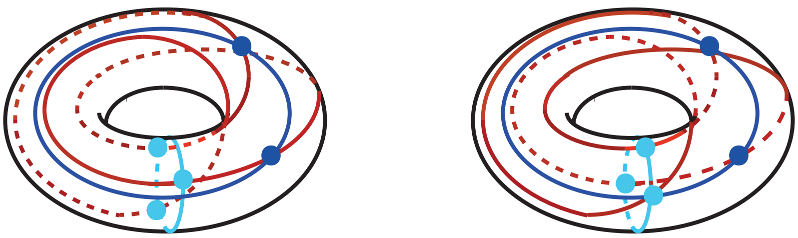


Figure 5: $\mathtt{T}(2,3)$ and $\mathtt{T}(2,-3)$.

Theorem 2.3.1. *[22] "A torus knot $\mathtt{T}(l,m)$ and a torus knot $\mathtt{T}(m,l)$ are equal for relatively prime integers l, m."*

The process of proving this theorem is given in [22]. Assume that a (m,n)-torus knot is taken. In this process, a disk which does not intersect the knot is removed from the torus. The torus with one boundary is converted to two interconnected bands. The shorter band represents a meridian of the torus, and the longer one represents a longitude of the torus. The two bands is turned inside out one by one, and this corresponds to a new torus. Then, the longer band represents a meridian of the new torus and the shorter one represents a longitude of the new torus. So, the torus knot $\mathtt{T}(l,m)$ equals the torus knot $\mathtt{T}(m,l)$ on the new torus [22].

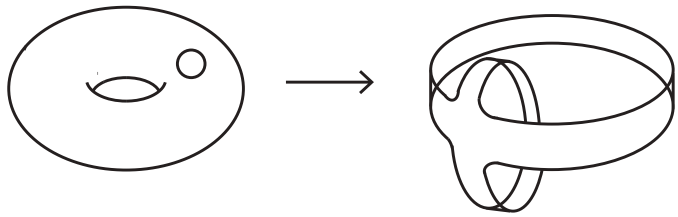


Figure 6: Torus with one boundary and two interconnected bands.

In knot theory, distinguishing knots is the most important problem. To show that two knots are not same, some invariants of knots are needed. Invariants of knots do not change under isotopy, and they are independent of knot diagrams. Crossing number, unknotting number and knot polynomial are some examples of invariants of knots.

Definition 2.3.2. [22] The minimum crossing number considered in all topological diagrams of a given knot K is named as *the crossing number* of the knot K. It is denoted by c_K.

Example 2.3.3. Since knots with crossing number 1 and knots with crossing number 2 are isotopic to the simplest knot which is the unknot, the first nontrivial knot that is the right handed trefoil knot has crossing number equals to 3, and given in Figure 7.

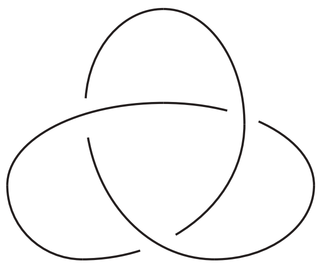


Figure 7: Right handed trefoil $\mathtt{T}(2,3)$ with 3 crossings.

Definition 2.3.3. [22] Consider a knot K. If one can find a topological diagram of K so that changing c crossings converts K into the unknot, and there is no diagram such that changing crossings less than c converts the knot K into the unknot, then the number c is named as *the unknotting number* for K . It is denoted by u_K.

Theorem 2.3.2. *[25]"The crossing number of a torus knot* $\mathtt{T}(l,m)$ *is* $c = min\{l(m-1), m(l-1)\}$ *where* l, $m \geq 2$*."*

Example 2.3.4. The crossing number of a right handed trefoil $\mathtt{T}(2,3)$ which is given in Figure 7 can be found as $3 = min\{2(3-1), 3(2-1)\}$ by Theorem 2.3.2.

Theorem 2.3.3. *[26] "The unknotting number of a torus knot* $\mathtt{T}(l,m)$ *is* $u = \frac{1}{2}(l-1)(m-1)$ *for positive integer* l *and* m*."*

Example 2.3.5. The unknotting numbers of $\mathtt{T}(2,3)$ knot is 1 by Theorem 2.3.3.

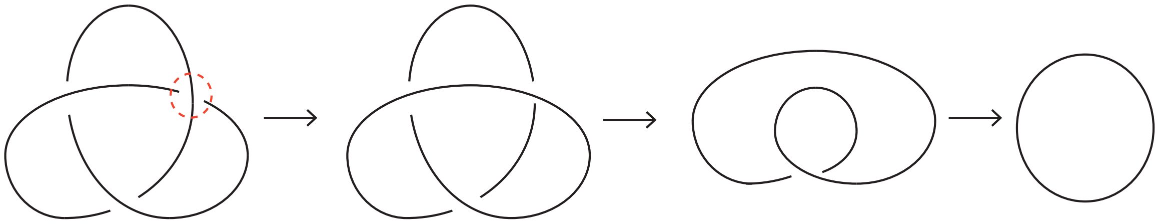


Figure 8: Unknotting number is 1.

2.3.1 Seifert Surfaces of Torus Knots

Definition 2.3.4. [27] Topological connected 2-manifold is called a *surface.*

Example 2.3.6. 2-dimensional sphere S^2 and torus T^2 are examples of surfaces.

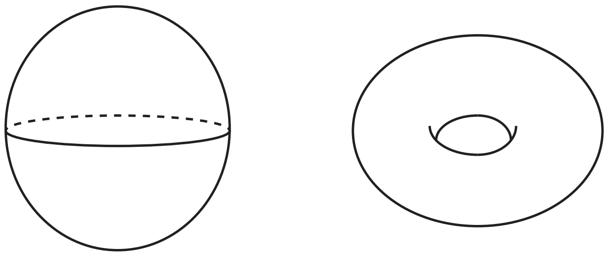


Figure 9: $2-$dimensional sphere S^2 and torus T^2.

Definition 2.3.5. [22] If a consistent normal vector can be chosen at each point on a surface, then the surface is called an *orientable surface.* Otherwise, it is called a *non-orientable surface.*

Definition 2.3.6. [22][28] A *surface with n-boundary* is a surface such that one can obtain by taking out the interiors of n disjoint disks from a surface without boundary. Also, a compact surface without boundary is called *closed surface.*

Example 2.3.7. 2-dimensional sphere S^2 is an orientable surface without boundary since it has an outward normal vector at each point on it. However, Möbius band is a non-orientable surface with 1-boundary since it does not have a consistent normal vector at each point on it.

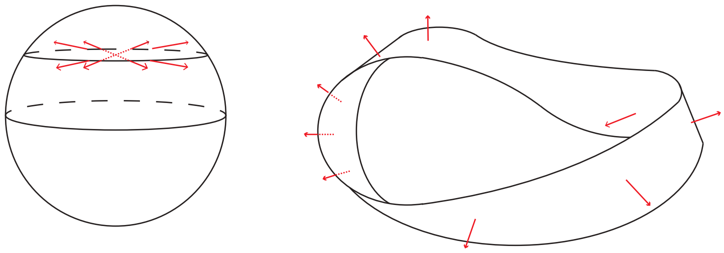


Figure 10: Orientable surface S^2 and non-orientable surface Möbius band.

Definition 2.3.7. [22] The number of holes in a surface is called the *genus*. A surface with genus g is denoted by Σ_g. Use Σ_g^n to denote a genus g surface with n-boundary components.

Definition 2.3.8. Take two surfaces S_1 and S_2. Take a disc D_1 on S_1 and a disc D_2 on S_2. The interior of these discs are removed from the surfaces. Then, two surfaces with boundary $S_1 \setminus int(D_1)$ and $S_2 \setminus int(D_2)$ are glued via a homeomorphism along their boundaries. This operation is called *connected sum of two surfaces* and the resulting new surface is denoted by $S_1 \# S_2$.

Definition 2.3.9. [29] Consider a genus g closed surface Σ_g. One can define the *Euler characteristic* of Σ_g is by $\chi(\Sigma_g) = V - E + F$, where V is the total number of vertices, E is the total number of edges and F is the total number of faces in a triangulation of Σ_g.

Lemma 2.3.1. [22] "The Euler characteristic of a closed orientable surface of genus g is $\chi(\Sigma_g) = 2 - 2g$."

Proof. Apply induction on the genus g.

i) Closed surfaces with genus $g = 0$ homeomorphic to the 2-dimensional surface S^2. Closed surfaces with genus $g = 1$ homeomorphic to the torus T^2. Triangulations of the surfaces S^2 and T^2 are given in Figure 11, respectively.

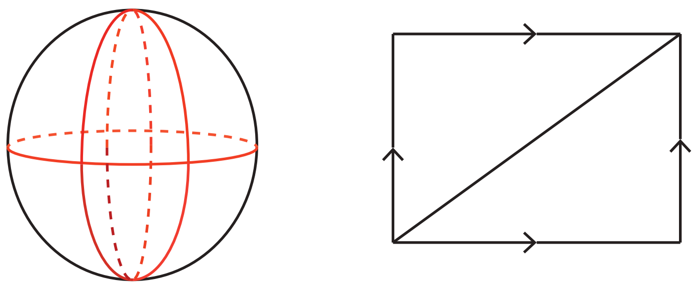


Figure 11: Triangulations of the surfaces S^2 and T^2, respectively.

The Euler characteristics of S^2 and T^2 are $\chi(S^2) = V - E + F = 6 - 12 + 8 = 2$ and $\chi(T^2) = V - E + F = 1 - 3 + 2 = 0$. So, the formula $\chi(\Sigma_g) = 2 - 2g$ is true for genus $g = 0$ and $g = 1$.

ii) Assume that the given formula $\chi(\Sigma_g) = 2 - 2g$ is true for genus $g = n - 1$. By the assumption,

$$\chi(\Sigma_{(n-1)}) = 2 - 2(n-1).$$

A surface with genus $g = n$ is obtained from the surface Σ_{n-1} by taking a connecting sum with torus T^2. So,

$$\chi(\Sigma_n) = \chi(\Sigma_{(n-1)} \# T^2) = 2 - 2(n-1) - 1 + 0 - 1 = 2 - 2n.$$

□

In 1934 the following theorem was proved by Herbert Seifert with an algorithm which is called Seifert Algorithm. Seifert Algorithm is an algorithm that constructs an orientable surface with boundary for a given knot such that the surface has a boundary as the given knot. Seifert algorithm has several steps:

1) Determine an orientation for a projection of the knot K.

2) Resolve all crossings of K to find Seifert circles which are obtained from eliminating the crossings as shown in Figure 12.

3) All Seifert circles are boundaries of disks at different heights in a plane.

4) Connect disks via twisted bands corresponding to the crossing of K to obtain a surface which has a boundary component K [30].

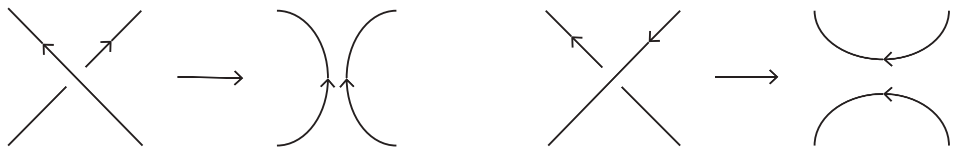


Figure 12: Eliminating crossings.

Theorem 2.3.4. *[30] "Every oriented knot in S^3 is a boundary of an orientable surface."*

In other words, this theorem says that for every knot K in S^3 there exists an orientable surface Σ such that $\partial\Sigma = K$.

Definition 2.3.10. [24] Consider a knot in S^3. An orientable surface with one boundary component that is the given knot is called a *Seifert surface*.

If a disk is removed from a surface without boundary, the Euler characteristic decreases by 1 since removing a disk means removing a face in a triangulation of the given surface. Therefore, the Seifert surface has Euler characteristic that is $\chi(\Sigma_g^1) = \chi(\Sigma_g) - 1 = 1 - 2g$.

Example 2.3.8. A Seifert surface of $\mathtt{T}(2,3)$ is obtained by Seifert algorithm in Figure 13.

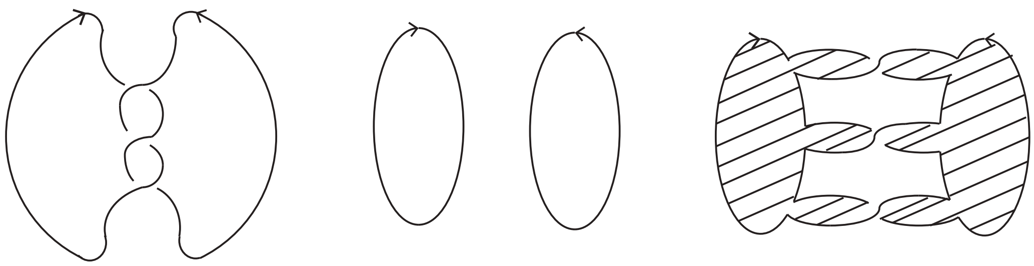


Figure 13: Seifert algorithm.

A Seifert surface is homotopic to the one in Figure 14:

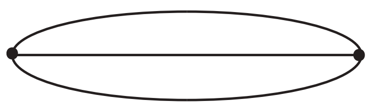


Figure 14: Homotopy.

So, the Euler characteristic of a Seifert surface given in Figure 13 is

$$\chi(\mathtt{T}(2,3)) = V - E + F = 2 - 3 + 0 = -1.$$

Then, $\chi(\mathtt{T}(2,3)) = 1 - 2g = -1$ and the genus of this Seifert surface of $\mathtt{T}(2,3)$ is 1.

Example 2.3.9. The Seifert circles of $\mathtt{T}(2,5)$ are obtained by resolving crossings of $\mathtt{T}(2,5)$ in Figure 15.

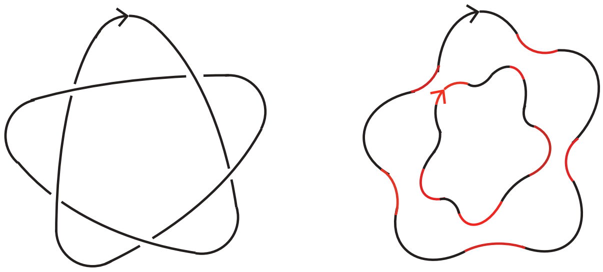


Figure 15: $\mathtt{T}(2,5)$ and Seifert circles of it.

A Seifert surface of $\mathtt{T}(2,5)$ is obtained by connecting these two Seifert circles via 5 twisted bands. A Seifert surface of $\mathtt{T}(2,5)$ which is constructed using Figure 15 is homotopic to Figure 16:

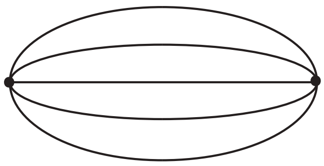


Figure 16: Homotopy type of a Seifert surface of $\mathtt{T}(2,5)$.

So, the Euler characteristic is in Figure 16

$$\chi(\mathtt{T}(2,5)) = V - E + F = 2 - 5 + 0 = -3.$$

Then, $\chi(\mathtt{T}(2,5)) = 1 - 2g = -3$ and the genus of this Seifert surface of $\mathtt{T}(2,5)$ is 2.

Example 2.3.10. A Seifert circles of $(2,m)$-torus knot are obtained by resolving crossings of $\mathtt{T}(2,m)$ in Figure 17 where $m \geq 2$.

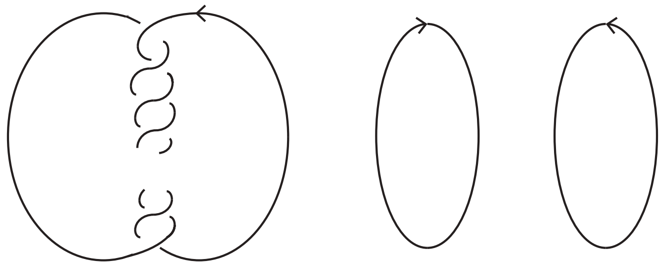


Figure 17: $\mathtt{T}(2,m)$ and Seifert circles of $\mathtt{T}(2,m)$.

A Seifert surface of $\mathtt{T}(2,m)$ is obtained by connecting these two Seifert circles via m twisted bands. The Euler characteristic of $\mathtt{T}(2,m)$ is

$$\chi(\mathtt{T}(2,m)) = V - E + F = 2 - m + 0 = 2 - m.$$

Then, $\chi(\mathtt{T}(2,m)) = 1 - 2g = 2 - m$ and the genus of this Seifert surface of $\mathtt{T}(2,m)$ is $(m-1)/2$.

Example 2.3.11. The Seifert circles of $(3,4)$-torus knot are obtained by resolving crossings of $\mathtt{T}(3,4)$ in Figure 18.

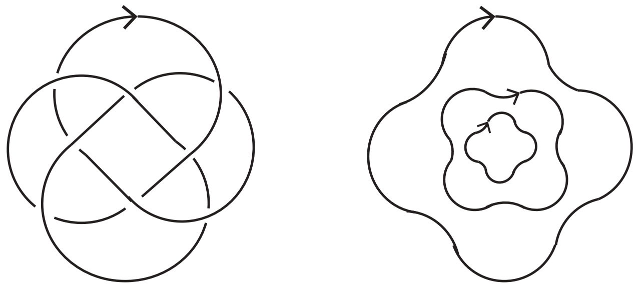


Figure 18: $\mathtt{T}(3,4)$ and Seifert circles of it.

A Seifert surface of $\mathtt{T}(3,4)$ is obtained by connecting these 3 Seifert circles via 5 twisted bands. A Seifert surface of $\mathtt{T}(3,4)$ obtained from Figure 18 is homotopic to the following graph given in Figure 19:

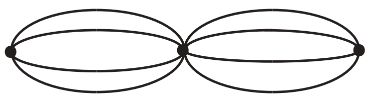


Figure 19: Homotopy type.

So, the Euler characteristic of the Seifert surface in Figure 18 is

$$\chi(\mathtt{T}(3,4)) = V - E + F = 3 - 8 + 0 = -5.$$

Then, $\chi(\mathtt{T}(3,4)) = 1 - 2g = -5$ and the genus of this Seifert surface of $\mathtt{T}(3,4)$ is 3.

Definition 2.3.11. Consider all Seifert surfaces of a given knot. Then, the genus for the given knot is the genus of the Seifert surfaces among all which has the minimal genus.

Theorem 2.3.5. *[24] "The genus of a torus knot* $\mathtt{T}(l,m)$ *is* $\dfrac{(l-1)(m-1)}{2}$ *for coprime positive integers* l, m*."*

2.3.2 Seifert Framing of Knots

Definition 2.3.12. [22] [28] A set which consists of disjoint knots is called as a *a link.*

Example 2.3.12. Unlink and Hopf link in Figure 20 are examples of links, respectively. Both have two link components which are unknots.

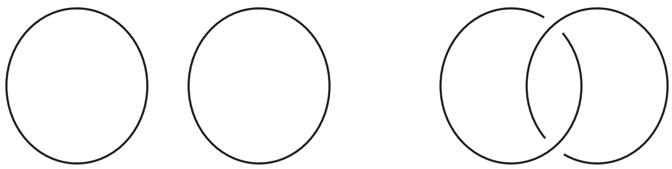


Figure 20: Unlink and Hopf link.

Definition 2.3.13. [22] [28] A sign with ± 1 value for each crossing of a given knot or link can be given in Figure 21. The *linking number between* L_1 *and* L_2, denoted by $lk(L_1, L_2)$, is defined as the half of the total sign of each crossings between L_1 and L_2.

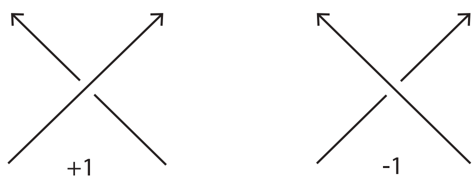


Figure 21: $+1$ crossing and -1 crossing.

Definition 2.3.14. [28] Consider a null-homologous knot in a 3-manifold, and consider a Seifert surface of the given knot. Take a parallel push of the knot which stays on the Seifert surface, and the linking number of the knot and its parallel push off is zero. Then, the parallel push off is the Seifert framing of the knot.

Example 2.3.13. A Seifert surface of an unknot U is a disc. The Seifert framing of U is given in Figure 22.

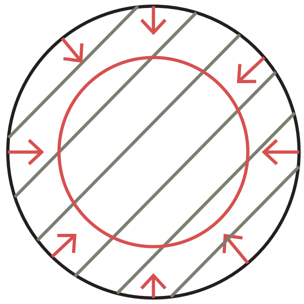


Figure 22: Seifert framing of the unknot.

2.3.3 Some Polynomials of Torus Knots

In knot theory, classifying knots is the fundamental aim. There are some invariants to show whether any two knots are different or not. Polynomials are one of the most important invariants. Each knot corresponds to a polynomial which is an invariant for the knot since it is the same for any diagram of the knot. In this section, some polynomials of $\mathtt{T}(l, m)$-torus knots will be studied.

2.3.3.1 Alexander Polynomial of Torus Knots

In 1928, J. Alexander discovered the first polynomial invariant which corresponds to knots [2]. This polynomial is called *Alexander polynomial*. Alexander defined it by labeling the regions which are bounded by the arcs of a knot diagram. After Alexander, K. Reidemeister gave a first description of the Alexander polynomial based on the arcs. Now, there are many definitions of it in different ways [31].

Consider an oriented knot K which has n crossings that are labelled $1, 2, \ldots, n$. Let each arc of it be labelled $y_1, y_2, \ldots, y_n$. Form an $n \times n$ matrix M where r^{th} row represents r^{th} crossing, and i^{th} column represents arc y_i of K. If the linking number of r^{th} crossing is positive, as appearing in Figure 23 (a), the entries of M are

$$m_{ri} = 1 - \tau$$

$$m_{rj} = -1$$

$$m_{rk} = \tau$$

and $m_{rs} = 0$ for $s \neq i, j, k$. Otherwise,

$$m_{ri} = 1 - \tau$$

$$m_{rj} = \tau$$

$$m_{rk} = -1$$

and $m_{rs} = 0$ for $s \neq i, j, k$. See Figure 23 [31].

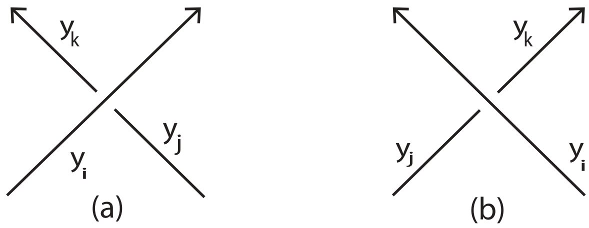


Figure 23: Positive and negative crossing.

Definition 2.3.15. [31] Form the $(n-1)\times(n-1)$ matrix from the associated matrix M described above by deleting n^{th} row and n^{th} column of M. It is named as the Alexander matrix of the knot. It is denoted by A_K. The Alexander matrix has determined denoted by $\Delta_K(t)$. This determinant is called *the Alexander polynomial of the knot.*

Example 2.3.14. The associated matrix M of trefoil $\mathtt{T}(2,3)$ obtained by Figure 24 is:

$$\mathbf{M} = \begin{pmatrix} 1-\tau & -1 & \tau \\ \tau & 1-\tau & -1 \\ -1 & \tau & 1-\tau \end{pmatrix}_{3\times 3}$$

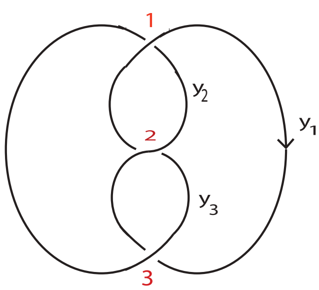


Figure 24: A labelling of the arcs and crossings of trefoil.

Then, the Alexander matrix of $\mathtt{T}(2,3)$ is:

$$A_{\mathtt{T}(2,3)} = \begin{pmatrix} 1-\tau & -1 \\ \tau & 1-\tau \end{pmatrix}$$

The Alexander polynomial of $\mathtt{T}(2,3)$ is $\Delta_{\mathtt{T}(2,3)}(t) = |A_{\mathtt{T}(2,3)}| = \tau^2 - \tau + 1$.

Example 2.3.15. The associated matrix M of $\mathtt{T}(2,5)$ obtained by Figure 25 is:

$$\mathbf{M} = \begin{pmatrix} 1-\tau & \tau & 0 & 0 & -1 \\ -1 & 1-\tau & \tau & 0 & 0 \\ 0 & -1 & 1-\tau & \tau & 0 \\ 0 & 0 & -1 & 1-\tau & \tau \\ \tau & 0 & 0 & -1 & 1-\tau \end{pmatrix}_{3\times 3}$$

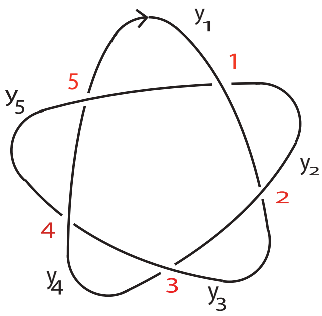


Figure 25: A labeling of the arcs and crossings of $\mathtt{T}(2,5)$.

Then, the Alexander matrix of $\mathtt{T}(2,5)$ is

$$A_{\mathtt{T}(2,5)} = \begin{pmatrix} 1-\tau & \tau & 0 & 0 \\ -1 & 1-\tau & \tau & 0 \\ 0 & -1 & 1-\tau & \tau \\ 0 & 0 & -1 & 1-\tau \end{pmatrix}_{3\times 3}.$$

Thus, the Alexander polynomial of $\mathtt{T}(2,5)$ is $\Delta_{\mathtt{T}(2,5)}(\tau) = |A_{\mathtt{T}(2,5)}| = \tau^4 - \tau^3 + \tau^2 - \tau + 1$.

Torus knots $\mathtt{T}(2,3)$ and $\mathtt{T}(2,5)$ are different torus knots because of their different Alexander polynomials.

Example 2.3.16. [31] The Alexander matrix of $\mathtt{T}(2,m)$ by Figure 26 where $gcd(2,m) =$

1 is:

$$A_{\mathtt{T}(2,m)} = \begin{pmatrix} 1-\tau & -1 & 0 & 0 & \cdots & \tau \\ \tau & 1-\tau & -1 & 0 & \cdots & 0 \\ 0 & \tau & 1-\tau & -1 & \cdots & 0 \\ & & & \vdots & & \\ & & & & 1-t & -1 \\ 0 & \cdots & \cdots & 0 & \tau & 1-\tau \end{pmatrix}_{(m-1)\times(m-1)}$$

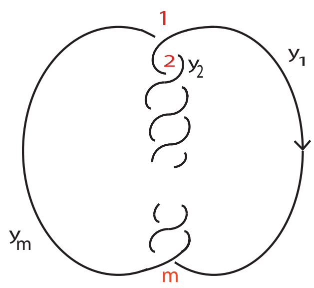


Figure 26: A labeling of the arcs and crossings of $\mathtt{T}(2,m)$.

By induction on $m > 0$, the Alexander polynomial of $\mathtt{T}(2,m)$ is

$$\Delta_{\mathtt{T}(2,m)}(\tau) = |A_{\mathtt{T}(2,m)}| = \tau^{m-1} - \tau^{m-2} + \ldots + \tau^2 - \tau + 1.$$

Then, the Alexander polynomial of $\mathtt{T}(2,m)$ is

$$\Delta_{\mathtt{T}(2,m)}(\tau) = \frac{(\tau^m + 1)}{(\tau + 1)}.$$

Theorem 2.3.6. *"[24] Consider a torus knot $\mathtt{T}(l,m)$. The formula of Alexander polynomial which is*

$$\Delta_{\mathtt{T}(l,m)}(\tau) = \frac{(\tau^{|lm|} - 1)(\tau - 1)}{(\tau^{|l|} - 1)(\tau^{|m|} - 1)}."$$

2.3.3.2 Jones Polynomial of Torus Knots

V. Jones discovered another knot polynomial which is called *Jones polynomial* in 1984 [3]. He found the new polynomial for knots when he was studying on operator algebra which is not related to knot theory.

Definition 2.3.16. [22] Consider an oriented knot or link diagram. One of crossings on the knot diagram is resolved according to K_+, K_- and K_0 given in Figure 27.

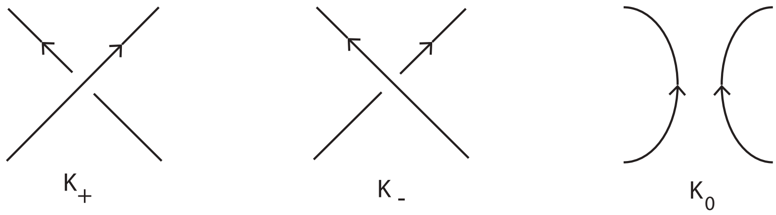


Figure 27: Resolving link or knot projections.

The Jones polynomial of K is a Laurent polynomial such that it satisfies the following conditions:

1) $V_u(\tau) = 1$, where u is a trivial knot

2) $\tau^{-1}V_{K_+}(\tau) - \tau V_{K_-}(\tau) = (\tau^{1/2} - \tau^{-1/2})V_{K_0}(\tau)$ (Skein relation).

Example 2.3.17. The red crossing of positive Hopf link H in Figure 28 is a positive crossing. So, let say $H = K_+$. Then, K_- and K_0 is obtained by resolving the red crossing of H according to Figure 28. Here K_- is the unlink with two components u_2 and K_0 is the unknot u.

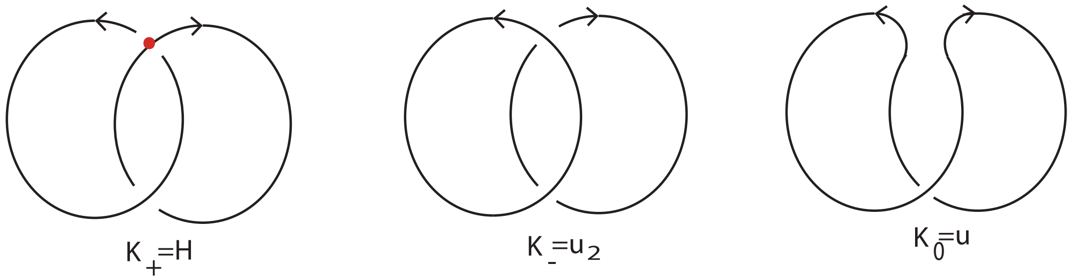


Figure 28: K_+, K_- and K_0.

Then, Hopf link satisfies the Skein relation:

$$\tau^{-1}V_H(\tau) - \tau V_{u_2} = (\tau^{1/2} - \tau^{-1/2})V_u(\tau)$$

$$\tau^{-1}V_H(\tau) + \tau(-\tau^{-1/2} - \tau^{1/2}) = (\tau^{1/2} - \tau^{-1/2}).$$

Therefore, the Jones polynomial of Hopf link is $V_H(\tau) = -\tau^{1/2} - \tau^{5/2}$.

Example 2.3.18. Consider the right handed trefoil $T(2,3)$ in Figure 29. Since the red crossing in Figure 29 is a positive crossing, assume that K_+ is $\mathtt{T}(2,3)$. To find Jones polynomial of $\mathtt{T}(2,3)$, the red crossing is resolved according to K_- and K_0. As seen in Figure 29, K_- and K_0 are the unknot u and the positive Hopf link H, respectively.

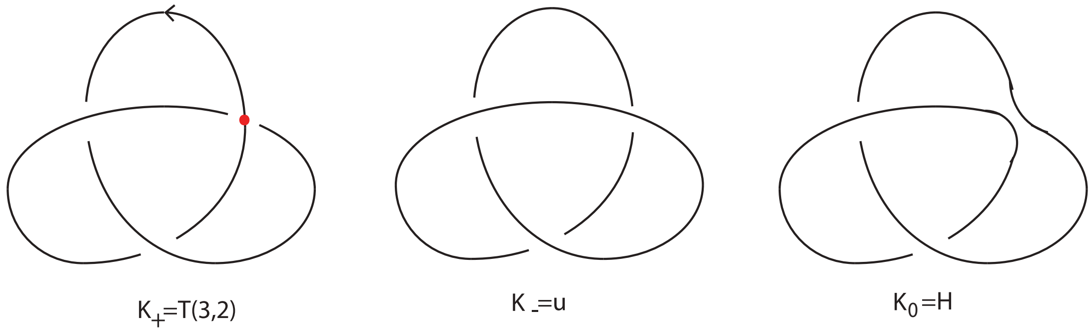


Figure 29: The right handed trefoil (K_+), the unknot (K_-) and the positive Hopf link (K_0) .

By the Skein relation,

$$\tau^{-1}V_{\mathtt{T}(2,3)} - \tau V_u = (\tau^{1/2} - \tau^{-1/2})V_H$$

$V_H(t) = -\tau^{1/2} - \tau^{5/2}$ computed in Example 2.3.17.

$$\tau^{-1}V_{\mathtt{T}(2,3)} - \tau = (\tau^{1/2} - \tau^{-1/2})(-\tau^{1/2} - \tau^{5/2})$$

The Jones polynomial of $\mathtt{T}(2,3)$ is $V_{\mathtt{T}(2,3)}(\tau) = -\tau^4 + \tau^3 + \tau$.

Lemma 2.3.2. "[4] Consider a torus knot $\mathtt{T}(l,m)$. The Jones polynomial is

$$V_{(l,m)}(\tau) = \frac{t^{(l-1)(m-1)/2}}{1-\tau^2}(1 - \tau^{m+1} - \tau^{m+1} + \tau^{l+m})."$$

2.4 Dehn surgery along a knot

Dehn surgery was introduced by Max Dehn in 1910. When he started to study for constructing Poincaré spaces, he found a new method that is called Dehn surgery, and today his method is very useful to study 3-manifolds. Roughly, Dehn surgery applied to a knot in S^3 is an operation such that a tubular neighbourhood of the knot is removed, and then glued back it to the knot exterior along their boundaries via a homoeomorphism. Generally, Dehn surgery along a link in a 3-manifold is an operation such that a tubular neighbourhood of each component of the link is removed, and then these neighbourhoods are glued back via homeomorphisms of their neighbourhoods. After

Dehn, Lickorish and Wallace demonstrated that each closed, orientable 3-manifold is attained by surgery applied to some links in S^3 [5] [6], for this reason Dehn surgery is important to understand topological 3-manifolds.

Definition 2.4.1. [32] Consider a closed and orientable topological $3-$manifold M. Consider a knot $K \subset M$. $N(K)$ is a tubular neighbourhood of K. i.e. $N(K) \cong S^1 \times D^2$. The boundary is a torus $\partial(N(K)) \cong S^1 \times S^1$. The tubular neighbourhood $N(K)$ is removed from M. A new solid torus is glued to the knot exterior $M \setminus N(K)$ via a diffeomorphism $h : \partial(S^1 \times D^2) \to \partial(M \setminus N(K))$. Then, a new closed orientable 3-manifold M' is obtained where $M' = M \cup_h (S^1 \times D^2)$. This operation is named as *Dehn surgery applied to the given knot* K. The new manifold M' changes depending on the diffeomorphism h.

If $M = S^3$, a surgery applied to the knot K is identified by coprime integers (p, q). A solid torus $S^1 \times D^2$ can be considered as the union of a 3-dimensional 2-handle and a 3-handle. Since a 3-handle is glued by an unique way, the gluing map is determined by the gluing curve $\{a\} \times \partial D^2$ that is an embedded simple closed curve in $\partial(S^3 \setminus N(K))$, and let $\beta = h(\{a\} \times \partial D^2) \in \partial(S^3 \setminus N(K))$. Homology class of the curve $\beta \in H_1(\partial(S^3 \setminus N(K), \mathbb{Z})) \cong H_1(S^1 \times S^1, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and there are two homological generator curves on $\partial(S^3 \setminus N(K))$ which are meridian $\mathtt{m}$ and longitude $\mathtt{l}$. A meridian $\mathtt{m}$ is a curve on $\partial(S^3 \setminus N(K))$ which bounds a disc in $(S^3 \setminus N(K))$, and it is a basis element for $H_1(\partial(S^3 \setminus N(K)$. A longitude $\mathtt{l}$ is another curve on $(S^3 \setminus N(K))$ which is determined according to the Seifert framing of K, i.e. $lk(K, \mathtt{l}) = 0$. Then, curves on $\partial(S^3 \setminus N(K))$ are identified by two coprime integers p and q, and the gluing map is defined by the following homeomorphism.

$$h : \partial(S^1 \times D^2) \to \partial(S^3 \setminus N(K))$$

$$\{a\} \times \partial D^2 \mapsto \beta = q\mathtt{m} + p\mathtt{l}$$

This operation is named as (p/q)-surgery along K [33].

Example 2.4.1. [28] [34] Consider an unknot K in S^3 and the tubular neighbourhood of K; $N(K) = S^1 \times D^2$. $N(K)$ is removed from S^3, then a solid torus $S^3 \setminus N(K) \cong S^1 \times D^2$ is obtained. The gluing map for the 0-surgery on K is:

$$h : \partial(S^1 \times D^2) \to \partial(S^1 \times D^2); h(\alpha) = q\mathtt{m} + p\mathtt{l} = m$$

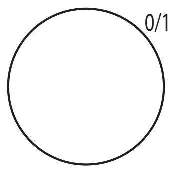


Figure 30: Obtaining $S^1 \times S^2$ by a Dehn Surgery .

Then, 0-surgery on K is $S^1 \times S^2$.

Example 2.4.2. [28] [34] The result of ∞-surgery on the unknot K is S^3. The gluing map for the $1/0 = \infty$-surgery is:

$$h : \partial(S^1 \times D^2) \to \partial(S^1 \times D^2); h(\alpha) = q\mathtt{m} + p\mathtt{l} = l.$$

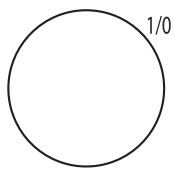


Figure 31: Obtaining S^3 by a Dehn Surgery .

Thus, S^3 is obtained by 1/0-Dehn Surgery.

Example 2.4.3. [28] The conclusion of (+1)-surgery applied to $\mathtt{T}(2,3)$ and (-1)-surgery applied to $\mathtt{T}(2,-3)$ in S^3 is the Poincaré manifold, respectively [28]. However, these Poincaré manifolds have opposite orientations.

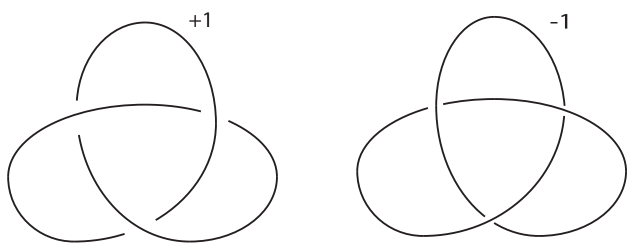


Figure 32: Obtaining Poincaré manifold by a Dehn surgery.

Example 2.4.4. [28] The lens space $L(m, n)$ is a result of a Dehn surgery with Seifert framing $(-m/n)$ applied to an unknot in S^3.

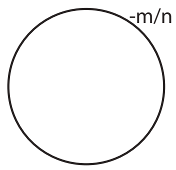


Figure 33: Obtaining $L(m, n)$.

Theorem 2.4.1. *"[32] Every lens space $L(m, n)$ is a result of a surgery diagram as in Figure 34 where $-m/n = [s_1, \ldots, s_k]$ is the continued fraction decomposition.*

$$[s_1, \ldots, s_k] =: s_1 - \cfrac{1}{s_2 - \cfrac{1}{s_3 - \ldots \cfrac{1}{s_k}}}. \text{"}$$

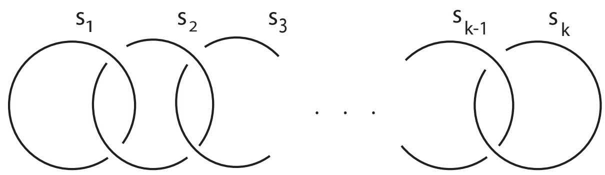


Figure 34: Surgery diagram for $L(m, n)$.

Example 2.4.5. In Figure 35, the result of each three surgery diagrams in S^3 is the lens space $L(7, 4)$.

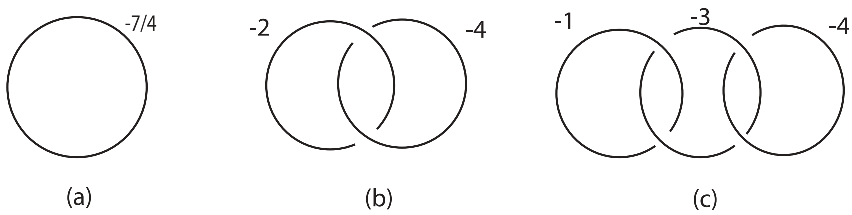


Figure 35: Obtaining $L(7, 4)$ from different Dehn Surgery diagrams.

As seen in Figure 35, there are many ways to obtain $L(7, 4)$. However, Rasmussen showed that the only way to obtain $L(7, 4)$ by applying an integral surgery on an only one knot in S^3 is a surgery with framing (-7) applied to the left handed trefoil given in Figure 36 [11].

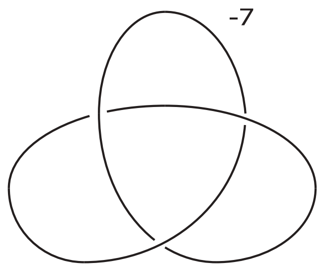


Figure 36: Obtaining $L(7, 4)$ by a Dehn Surgery .

2.5 Kirby Moves

Different surgeries applied to different knots may result in the same manifold. Kirby moves are used for showing the resulting manifolds that are the same. For more explanation see [7], [32] and [34].

Definition 2.5.1. [7][32] *Kirby moves* are the following two operations on a given link $\mathcal{L}$ such that the manifold stay the same.

1) Kirby move 1 (KM1): Attach or remove an unknot with Seifert framing ± 1 which have no intersection with the other components of $\mathcal{L}$. This movement equivalent to taking a connected sum of S^3 and a manifold or canceled, which results in the same manifold M; $M = M \# S^3$.

Figure 37: Kirby move 1.

2) Kirby move 2 (KM2): Slide one components of a link over another. Given L_1 and L_2 which are knots with Seifert framing f_1 and f_2, respectively. Sliding L_1 over L_2 is substituting $L_\# \cup L_2$ for $L_1 \cup L_2$ where $L_\# = L_1 \#_c L_2'$ and L_2' is a parallel push of L_2 such that it links with L_2' with f_2 times. The band c connects L_1 to L_2' as in Figure 38. After this slide, the framing of L_1 unchanges while the framing of L_2 changes and the framing of $L_\#$ is:

$$f_\# = f_1 + f_2 + 2lk(L_1, L_2)$$

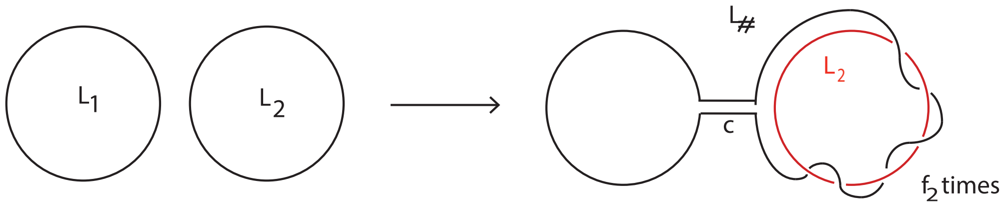


Figure 38: Kirby move 2.

Definition 2.5.2. [7] [32] The operation removing an unknot with the framing ± 1 from a link $\mathcal{L}$ is named as *blow-down*, otherwise adding an unknot with framing ± 1 to $\mathcal{L}$ is named as *blow-up*, see Figure 39.

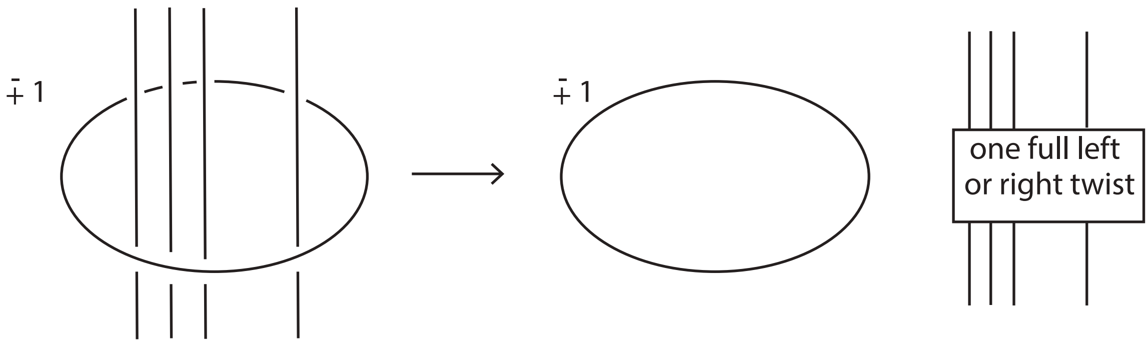


Figure 39: General ± 1 blowing up/blowing down.

Example 2.5.1. After blow-downs, the given framed link converts into an unlink with three components whose framings are 1. Since $(+1)$-surgery on the unknot is S^3, the framed link in Figure 40 represents S^3.

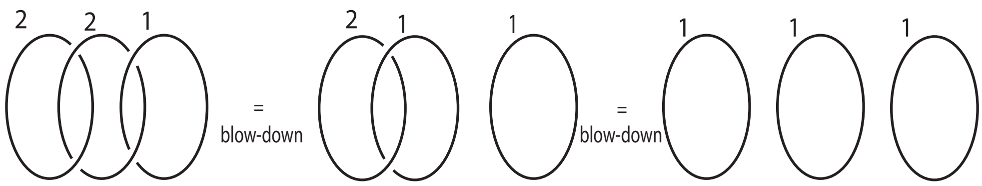


Figure 40: Converting a framed link into an unlink via blow-down.

2.6 Contact 3-manifolds

Assume that M is a 3-manifold. Consider its tangent space T_pM at $p \in M$. Moreover, consider its tangent bundle TM. $TM = \bigcup_{p \in M} T_pM$.

Definition 2.6.1. [35] A 2-plane field ξ on a 3-manifold M is named as a *contact structure* if there exists a 1-form $\alpha : TM \to \mathbb{R}$ such that locally $\xi = ker\alpha = \{v \in TM \mid \alpha(v) = 0\}$ and $\alpha \wedge d\alpha \neq 0$. Also, such 1-form α is named as a *contact form.*

Definition 2.6.2. [35] A 3-manifold M is named as a *contact* 3*-manifold* if there exists a contact structure ξ on M. It is denoted by the pair (M, ξ).

Example 2.6.1. [36] Let $\xi_{std} = ker\alpha$ in $\mathbb{R}^3$ where 1-form $\alpha = dz - ydx$. Since

$$\begin{aligned}
\alpha \wedge d\alpha &= (dz - ydx) \wedge d(dz - ydx) \\
&= (dz - ydx) \wedge d(dz) - d(ydx) \text{ (since } d \text{ is a linear map)} \\
&= (dz - ydx) \wedge (-dy \wedge dx) \text{ (since } d(dz) = 0) \\
&= (dz - ydx) \wedge (dx \wedge dy) \text{ (since } -dy \wedge dx = dx \wedge dy) \\
&= (dz \wedge dx \wedge dy) - \underbrace{(ydx \wedge dx \wedge dy)}_{=0} \text{ (since } dx \wedge dx = 0) \\
&= dz \wedge dx \wedge dy \neq 0,
\end{aligned}$$

the 1-form α is a contact form. So, ξ_{std} is a contact structure on $\mathbb{R}^3$. Also, ξ_{std_p} at any point p is generated by the following set:

$$ker\alpha_p = span\Big\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\Big\} \subset T\mathbb{R}^3$$

where $\alpha_p : T_p\mathbb{R}^3 \to \mathbb{R}$ and $T_p\mathbb{R}^3 = span\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big\}$ at each point $p \in \mathbb{R}^3$.

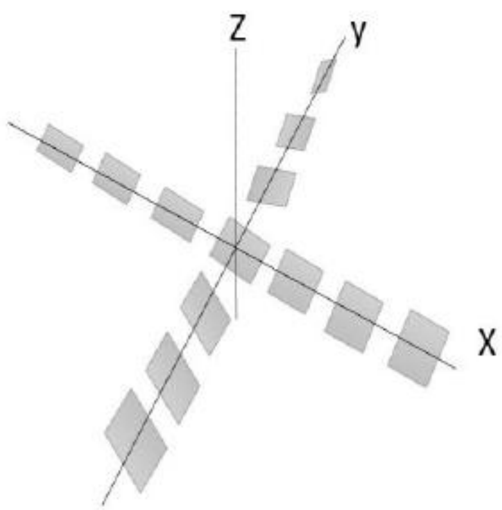


Figure 41: Standard contact structure on $\mathbb{R}^3$.

Example 2.6.2. [36] For a 1-form $\alpha_1 = dz+xdy$, $\xi_1 = ker\alpha_1$ is also a contact structure on $\mathbb{R}^3$. Indeed, 1-form α_1 is a contact form:

$$\begin{aligned}
\alpha_1 \wedge d\alpha_1 &= (dz + xdy) \wedge d(dz + xdy) \\
&= (dz + xdy) \wedge d(dz) + d(xdy) \text{ (since } d \text{ is a linear map)} \\
&= (dz + xdy) \wedge (dx \wedge dy) \text{ (since } d(dz) = 0) \\
&= (dz \wedge dx \wedge dy) + \underbrace{(xdy \wedge dx \wedge dy)}_{=0} \text{ (since } dy \wedge dy = 0) \\
&= dz \wedge dx \wedge dy \neq 0.
\end{aligned}$$

So, ξ_1 is a contact structure on $\mathbb{R}^3$. At any point p, ξ_{1_p} is generated by the following set:

$$ker\alpha_{1_p} = span\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\Big\} \subset T\mathbb{R}^3$$

where $\alpha_{1_p} : T_p\mathbb{R}^3 \to \mathbb{R}$ and $T_p\mathbb{R}^3 = span\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big\}$ at each point $p \in \mathbb{R}^3$.

Example 2.6.3. [36] An another sample of a contact structure on $\mathbb{R}^3$ is *the symmetric contact structure* $\xi_{sym} = ker\alpha_2$ where 1-from $\alpha_2 = dz - ydx + xdy$. Indeed, 1-from $\alpha_2 = dz - ydx + xdy$ is a contact form:

$$\begin{aligned}
\alpha_2 \wedge d\alpha_2 &= (dz - ydx + xdy) \wedge d(dz - ydx + xdy) \\
&= (dz - ydx + xdy) \wedge d(dz) - d(ydx) + d(xdy) \text{ (since } d \text{ is a linear map)} \\
&= (dz - ydx + xdy) \wedge (-dy \wedge dx + dx \wedge dy) \text{ (since } d(dz) = 0) \\
&= (dz - ydx + xdy) \wedge (2dx \wedge dy) \text{ (since } -dy \wedge dx = dx \wedge dy) \\
&= (dz \wedge 2dx \wedge dy) = 2dx \wedge dy \wedge dz \neq 0.
\end{aligned}$$

So, ξ_{sym} is a contact structure on $\mathbb{R}^3$. At any point p, ξ_{sym} is generated by the following sets:

$$\xi_{sym_p} = ker\alpha_{2_p} = span\Big\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x}\Big\} \text{ if } y \neq 0$$

$$\xi_{sym_p} = ker\alpha_{2_p} = span\Big\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\Big\} \text{ if } x \neq 0$$

$$\xi_{sym_p} = ker\alpha_{2_p} = span\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big\} \text{ if } x = y = 0.$$

where $\alpha_{2_p} : T_p\mathbb{R}^3 \to \mathbb{R}$ and $T_p\mathbb{R}^3 = span\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big\}$ at any point $p \in \mathbb{R}^3$. In cylindrical coordinates, we put $x = rcos\theta$ and $y = rsin\theta$ where r and θ are as in the Figure 42.

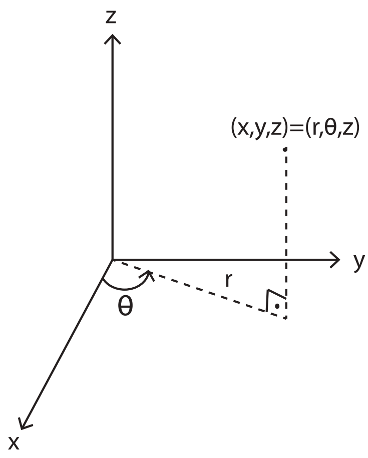


Figure 42: Cylindrical coordinates.

Then, we find $dx = cos\theta dr - rsin\theta d\theta$ and $dy = sin\theta dr + rcos\theta d\theta$. Now, substitute x, y, dx and dy in the $1-$form α_2.

$$\begin{aligned} \alpha_2 = dz - ydx + xdy &= dz - rsin\theta(cos\theta dr - rsin\theta d\theta) + rcos\theta(sin\theta dr + rcos\theta d\theta) \\ &= dz + r^2 d\theta. \end{aligned}$$

Therefore, the contact 1-form $\alpha_2 = dz + r^2 d\theta$ is in cylindrical coordinates.

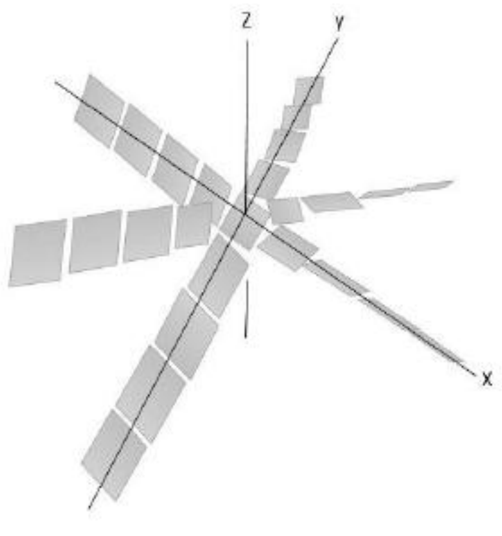


Figure 43: Symmetric contact structure on $\mathbb{R}^3$.

Example 2.6.4. [35][36] For a 1-form $\alpha_3 = cosrdz + rsinrd\theta$ in $\mathbb{R}^3$ with cylindrical coordinates, $\xi_{ot} = ker\alpha_3$ is another contact structure on $\mathbb{R}^3$. Indeed, $1-$form α_3 is a contact form:

$$\begin{aligned} d\alpha_3 &= d(cosrdz + rsinrd\theta) \\ &= -sinrdr \wedge dz + cosrd(dz) + (sinr + rcosr)dr \wedge d\theta + rsinrd(d\theta) \\ &= -sinrdr \wedge dz + (sinr + rcosr)dr \wedge d\theta \text{ (since } d(dz) = d(d\theta) = 0). \end{aligned}$$

Now, we will compute $\alpha_3 \wedge d\alpha_3$.

$$\begin{aligned} \alpha_3 \wedge d\alpha_3 &= (cosrdz + rsinrd\theta) \wedge (-sinrdr \wedge dz + (sinr + rcosr)dr \wedge d\theta) \\ &= cosrsinrdz \wedge dr \wedge d\theta + rcos^2rdz \wedge dr \wedge d\theta - rsin^2rd\theta \wedge dr \wedge dz \\ &= cosrsinrdz \wedge dr \wedge d\theta + rcos^2rdz \wedge dr \wedge d\theta + rsin^2rdz \wedge dr \wedge d\theta \\ &= (cosrsinr + r)dz \wedge dr \wedge d\theta \\ &= \left(\frac{cosrsinr}{r} + 1\right) dz \wedge dr \wedge d\theta \neq 0 \text{ (If } r > 0,\ \frac{cosrsinr}{r} + 1 > 0 \text{)} \end{aligned}$$

So, ξ_{ot} is a contact structure on $\mathbb{R}^3$. Also, ξ_{ot_p} at any point p is generated by the following set:

$$ker\alpha_{3_p} = span\Big\{\frac{\partial}{\partial r}, -rsinr\frac{\partial}{\partial z} + cosr\frac{\partial}{\partial \theta}\Big\} \subset T\mathbb{R}^3$$

where $\alpha_{3_p} : T_p\mathbb{R}^3 \to \mathbb{R}$ and $T_p\mathbb{R}^3 = span\Big\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\Big\}$ at each point $p = (r, \theta, z) \in \mathbb{R}^3$.

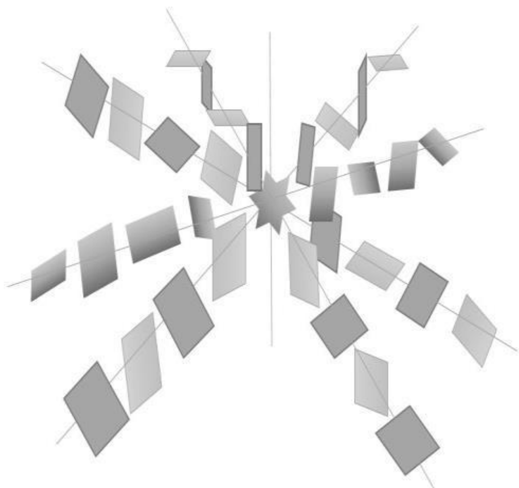


Figure 44: The overtwisted contact structure on $\mathbb{R}^3$.

Example 2.6.5. [36] [33] Consider 1-form $\alpha^{\sim} = xdy - ydx + wdz - zdw$ on the unit 3-sphere $S^3 \subset \mathbb{R}^4$.

$$\begin{aligned} \alpha^{\sim} \wedge d\alpha^{\sim} &= (xdy - ydx + wdz - zdw) \wedge d(xdy - ydx + wdz - zdw) \\ &= (xdy - ydx + wdz - zdw) \wedge (2dx \wedge dy + 2dw \wedge dz) \\ &= 2xdy \wedge dw \wedge dz - 2ydx \wedge dw \wedge dz + 2wdx \wedge dy \wedge dz - 2dx \wedge dy \wedge dw. \end{aligned}$$

The tangent space T_pS^3 can be generated by the following set

$$\left\{\frac{\partial}{\partial x_1} - \frac{x_1}{y_1}\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} - \frac{x_2}{y_2}\frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_1} - \frac{x_1}{y_2}\frac{\partial}{\partial y_2}\right\}.$$

On this basis for the tangent space T_pS^3, $\alpha^{\sim} \wedge d\alpha^{\sim} \neq 0$. So, $\alpha^{\sim}$ is a contact form and $\xi_{std} = ker(\alpha^{\sim})$ is a contact structure on S^3. This contact structure ξ_{std} is called *the standard contact structure* on S^3, and this contact 3-manifold is denoted by (S^3, ξ_{std}).

Definition 2.6.3. [35] Consider two contact 3−manifolds (M^1, ξ^1) and (M^2, ξ^2). Assume there exists a map $\Phi : M^1 \to M^2$ which is a diffeomorphism such that $d\Phi_p(\xi^1_p) = \xi^2_{\Phi(p)}$ for all $p \in M^1$ where $d\Phi_p : T_pM^1 \to T_{\Phi(p)}M^2$, then the two contact 3−manifolds are called **contactomorfic**.

[33] In the previous examples of contact structures on $\mathbb{R}^3$, $(\mathbb{R}^3, \xi_{std})$, $(\mathbb{R}^3, \xi_1)$ and $(\mathbb{R}^3, \xi_{sym})$ are all contactomorfic contact structures on $\mathbb{R}^3$.

Example 2.6.6. [33] $(\mathbb{R}^3, \xi_1)$ and $(\mathbb{R}^3, \xi_{sym})$ are contactomorphic via a diffeomorphism $\Phi : (\mathbb{R}^3, \xi_1) \to (\mathbb{R}^3, \xi_{sym})$ such that $\Phi(x, y, z) = (x, \frac{y}{2}, z + \frac{xy}{2}) = (x', y', z')$. The differential of Φ for a point $p \in \mathbb{R}^3$ is a linear map $d\Phi_p : T_p\mathbb{R}^3 \to T_{\Phi(p)}\mathbb{R}^3$ as the following:

$$d\Phi_p(w_1, w_2, w_3) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ y/2 & x/2 & 1 \end{pmatrix}}_{J\Phi_p} \underbrace{\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}}_{[w]_\beta}.$$

where $w = (w_1, w_2, w_3) \in T_p\mathbb{R}^3$ and $\beta = \left\{\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right\}$ is a basis for $T_p\mathbb{R}^3$. From Example 2.6.2 and Example 2.6.3, the contact plane ξ_{1_p} and ξ_{sym_p} at any point p are generated by the sets $\left\{\frac{\partial}{\partial x}, -x\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right\}$ and $\left\{x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'}, y'\frac{\partial}{\partial z'} + \frac{\partial}{\partial x'}\right\}$ where $y' \neq 0$, respectively.

$$\begin{aligned} d\Phi_p(\frac{\partial}{\partial x}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ y/2 & x/2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ y/2 \end{pmatrix} \\ &= \frac{\partial}{\partial x'} + \frac{y}{2}\frac{\partial}{\partial z'} \\ &= \frac{\partial}{\partial x'} + y'\frac{\partial}{\partial z'} \end{aligned}$$

and

$$\begin{aligned}
d\Phi_p(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ y/2 & x/2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -x \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ -x/2 \end{pmatrix} \\
&= \frac{1}{2}\frac{\partial}{\partial y'} - \frac{x}{2}\frac{\partial}{\partial z'} \\
&= \frac{1}{2}\frac{\partial}{\partial y'} - \frac{x'}{2}\frac{\partial}{\partial z'}
\end{aligned}$$

where $y' \neq 0$. So,

$$\begin{aligned}
d\Phi_p(\xi_{1_p}) &= span\Big\{\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial z'}, \frac{1}{2}\frac{\partial}{\partial y'} - \frac{x'}{2}\frac{\partial}{\partial z'}\Big\} \\
&= span\Big\{\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial z'}, \Big(\frac{1}{2}\frac{\partial}{\partial y'} - \frac{x'}{2}\frac{\partial}{\partial z'}\Big) + \frac{x'}{2y'}\Big(\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial z'}\Big)\Big\} \\
&= span\Big\{\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial z'}, \frac{1}{2y'}\big(x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'}\big)\Big\} \\
&= span\Big\{\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial z'}, x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'}\Big\} \\
&= \xi_{sym_p}.
\end{aligned}$$

Therefore, the diffeomorphism Φ is a contactomorphism from $(\mathbb{R}^3, \xi_1)$ to $(\mathbb{R}^3, \xi_{sym})$.

Theorem 2.6.1. *(Darboux's Theorem)[37] "Let M be a contact 3-manifold M and p be a point on M. Then, there exists a neighbourhood $U \subset M$ of p such that it is contactomorphic to a neighbourhood V of $p_1 = (0,0,0)$ in $(\mathbb{R}^3, \xi_{std})$."*

2.7 Tight and Overtwisted Contact Manifolds

2.7.1 Overtwisted Contact Manifolds

Definition 2.7.1. [35] Suppose that there is a disk $\mathcal{D}$ which is embedded in (M, ξ). Assume that its boundary which is denoted by $\partial\mathcal{D}$ is tangent to contact planes ξ everywhere such that contact framing and Seifert framing of $\partial\mathcal{D}$ are equal. Then $\mathcal{D}$ is called *an overtwisted disc.*

Example 2.7.1. In Example 2.6.4, the disc $\mathcal{D} = \{(r,\theta,z) \in \mathbb{R}^3 | z = 0, r \leq \pi\}$ is an overtwisted disc in $(\mathbb{R}^3, \xi_{ot})$. Its boundary $\mathcal{D}$ is tangent to the contact planes everywhere.

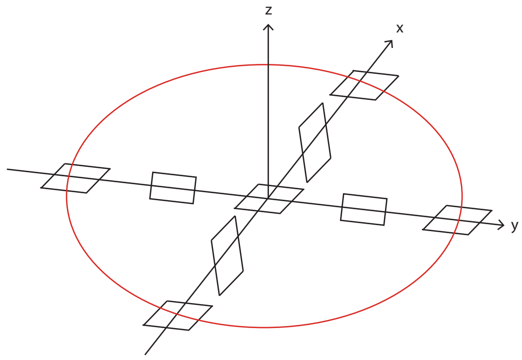


Figure 45: The overtwisted disc in $(\mathbb{R}^3, \xi_{0t})$.

Definition 2.7.2. [35] If there exists an overtwisted disc in a contact 3-manifold, then such a contact 3-manifold is called *an overtwisted contact manifold.*

Example 2.7.2. In Example 2.6.4, the contact manifold $(\mathbb{R}^3, \xi_{ot})$ is an example of an overtwisted contact 3-manifold since it has an embedded overtwisted disc in $(\mathbb{R}^3, \xi_{ot})$.

2.7.2 Tight Contact Manifolds

Definition 2.7.3. [35] If a contact 3-manifold (M, ξ) does not have any overtwisted disc in it, then it is called a *tight contact* 3*-manifold.*

Example 2.7.3. In Example 2.6.1 and Example 2.6.3, the contact 3-manifolds $(\mathbb{R}^3, \xi_{std})$ and $(\mathbb{R}^3, \xi_{sym})$ are examples of tight contact manifolds [14]. In these two contactomorphic contact manifolds, the contact planes twists slowly.

2.8 Legendrian knots in the $3-$sphere

In a contact 3-manifold, a knot which is tangent to contact 2-planes in each place is called a Legendrian knot [38]. We will study Legendrian knots in contact S^3 in this section.

Definition 2.8.1. [35][38] Let L be a knot in (M, ξ) which is an embedded S^1. The knot is called a **Legendrian knot** provided that the tangent space at p is in ξ_p; $T_pL \in \xi_p$ for all $p \in L$. In other words, an embedded curve $\varphi : S^1 \to M$ which is a parameterization of L satisfies this condition: $\varphi'(\theta) \in \xi_{\varphi(\theta)}$ where $\varphi' : T_\theta S^1 \to T_{\varphi(\theta)}M$ is a linear map for all $\theta \in S^1$.

Darboux proved that every contact structure is locally contactomorphic to $(\mathbb{R}^3, \xi_{std})$. We can think a Legendrian knot L in $\mathbb{R}^3$ as a Legendrian knot in S^3 since S^3 is obtained from $\mathbb{R}^3$ by adding a point at infinity to $\mathbb{R}^3$, $L \subset S^3 = \mathbb{R}^3 \cup \{p\}$.

Front projection is a projection map such that it projects curves in $\mathbb{R}^3$ to xz-plane. It is useful to picturize the knots.

Definition 2.8.2. [35] Let $\varphi(\theta) = (x(\theta), y(\theta), z(\theta))$ be a parameterized curve in $(\mathbb{R}^3, \xi_{std})$. **The front projection** of the curve φ is the curve $\varphi_f(\theta) = (x(\theta), z(\theta))$.

Take a Legendrian knot L in $(\mathbb{R}^3, \xi_{std})$. A parameterization φ of L is defined as:

$$\varphi : S^1 \to \mathbb{R}^3; \theta \mapsto (x(\theta), y(\theta), z(\theta)).$$

Suppose that φ is a C^1 immersion which means that φ is differentiable and its derivative $d_p\varphi : T_pS^1 \to T_{\varphi(p)}\mathbb{R}^3$ is an injective map for each point $p \in S^1$. By definition of a Legendrian knot, $\varphi'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) \in \xi_{std_{\varphi(\theta)}}$. Since $\xi_{std} = ker(dz - ydx)$, the following equation is obtained:

$$z'(\theta) - y(\theta)x'(\theta) = 0. \tag{1}$$

Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ can be pictured by their front projections. The image of $\varphi_f(\theta) : S^1 \to \mathbb{R}^2 : \theta \mapsto (x(\theta), z(\theta))$ where φ is a parameterization of L is named as the front projection of L. Though φ was an immersion, the front projection φ_f is not an immersion. If $x'(\theta)$ vanishes, $z'(\theta)$ also vanishes from Equation (1). So, if the front projection φ_f was an immersion, then $x'(\theta)$ must never 0. Thus, the front projection of L does not have vertical tangencies.

For a generic C^1 smooth L if $x'(\theta) = 0$, then the point θ is an isolated point where there exists a well-defined horizontal tangent line in the front projection of L. Such points are named as *generalized cusps.*

Moreover, y-coordinate of φ can be regenerated from Equation (1):

$$y(\theta) = \frac{z'(\theta)}{x'(\theta)} \tag{2}$$

Thus, a front projection of a Legendrian knot has three properties. The first one is that it does not have vertical tangencies. The second one is that its sole non-smooth points are generalized cusps, and the third one is that the slope of the overcrossing is smaller than the slope of the undercrossing in the front projection. [38].

Example 2.8.1. The front projections of a Legendrian unknot and a Legendrian right handed trefoil are given in Figure 46, respectively.

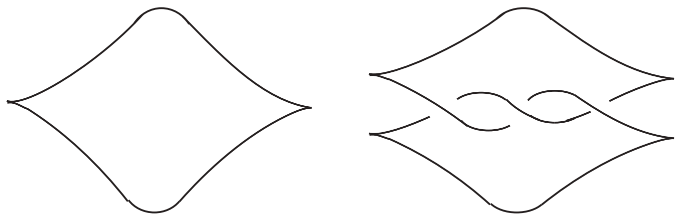


Figure 46: Legendrian unknot and Legendrian right handed trefoil knot.

Theorem 2.8.1. *[38] For every topological knot $K \subset S^3$, there exists a Legendrian knot representing the knot K.*

Proof. Any topological knot K can be converted into a Legendrian representative by the following movements.

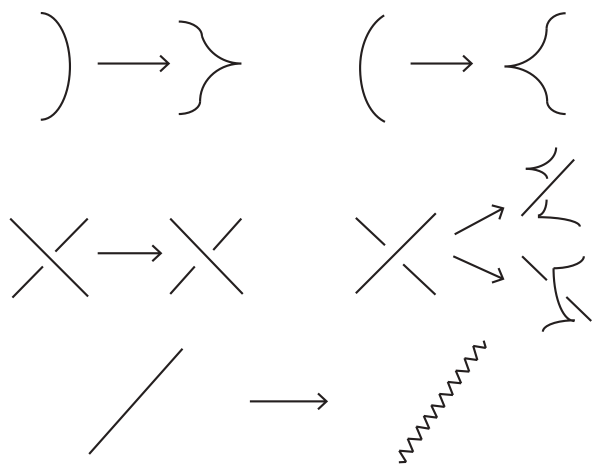


Figure 47: The movements of converting any topological knot into a Legendrian knot.

□

Example 2.8.2. Topological left handed trefoil is converted into Legendrian left handed trefoil by the movements in Figure 47.

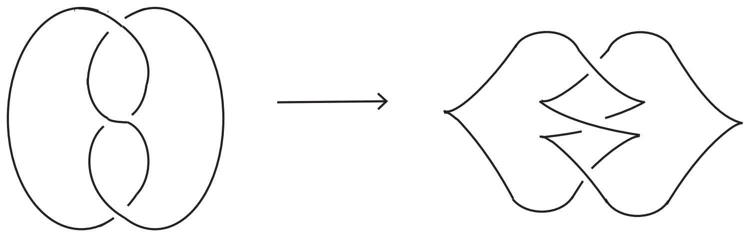


Figure 48: A Legendrian realization of a left handed trefoil knot.

2.8.1 Contact Framing of Legendrian Knots

Definition 2.8.3. [33] [35] Consider a Legendrian knot $L \subseteq (M, \xi)$. Consider a non-zero vector field v and perpendicular to L. Parallel push of L in the direction v which stays in ξ is called *a contact framing of L*.

Example 2.8.3. Let L be a Legendrian unknot in $(\mathbb{R}^3, \xi_{std})$. The vector field $v = \dfrac{\partial}{\partial z}$ is perpendicular to the standard contact planes ξ_{std}. The parallel push of L in direction v is the contact framing of L seeing in Figure 49.

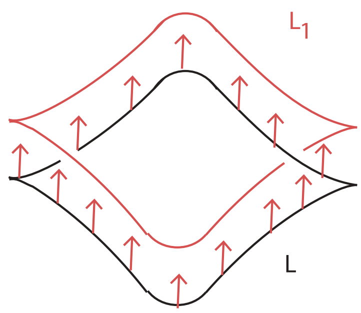


Figure 49: Contact framing of a Legendrian unknot.

2.9 Invariants of Legendrian Knots

2.9.1 Topological Knot Type

[38] Legendrian knot's topological knot type is an invariant for the knot.

Example 2.9.1. Legendrian left handed trefoil is given in Figure 50. It is different from the knot in Figure 51 since both have different knot types. The knot type of the Legendrian knot in Figure 51 is a right handed trefoil.

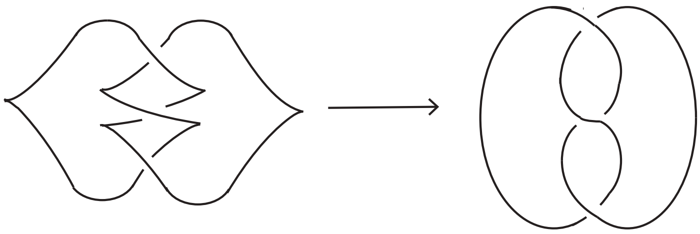


Figure 50: The knot type of Legendrian left handed trefoil.

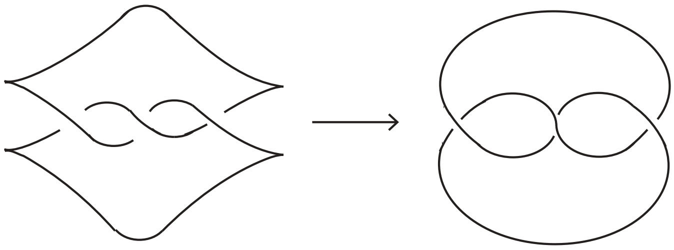


Figure 51: The knot type of Legendrian right handed trefoil.

2.9.2 Thurston-Bennequin Invariant

Definition 2.9.1. [38] Consider a Legendrian knot L in (M, ξ). Assume that L is null-homologous. Measurement of the twisting contact plane ξ around L is called *Thurston-Bennequin invariant* denoted by $\mathtt{tb}(L)$. In other words, Thurston-Bennequin invariant is the twisting contact framing as regards the Seifert framing of the knot.

Consider a Legendrian knot which is null-homologous in $(\mathbb{R}^3, \xi_{std})$. Take a non-zero vector field v transverse to ξ_{std}. Take a parallel push of the knot in the direction of v which stays in contact planes. Then, Thurston-Bennequin invariant $\mathtt{tb}(L)$ is defined as the linking between the knot and its parallel push off. Crossings between the knot and its parallel push off are obtained at the crossings and the cusps of the knot. In this situation, the following formula is obtained:

$$\mathtt{tb}(L) = lk(L, L_1) = writhe(\varphi_f(L)) - \frac{1}{2}(\text{total number of cusps in } \varphi_f(L)).$$

Here *writhe number* is the right number which is sum of signs of each crossing in the front projections.

Example 2.9.2. Consider a Legendrian unknot U in Figure 52(a). The vector field $v = \dfrac{\partial}{\partial z}$ is always perpendicular to ξ_{std}. A parallel push of U in the direction $\dfrac{\partial}{\partial z}$ is U'

in Figure 52(b). Then, the Thurston-Bennequin invariant of U is $lk(U, U') = -2$, or $tb(U) = writhe(\varphi_f(L)) - \frac{1}{2}(\# \text{ cusps}) = 0 - \frac{1}{2}4 = -2$.

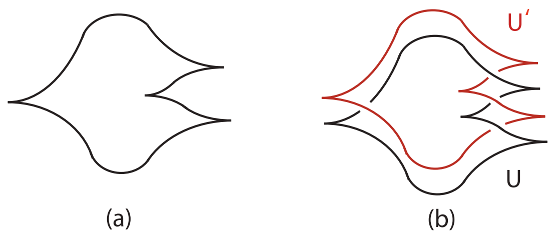


Figure 52: Legendrian unknot and parallel push off of it.

2.9.3 Maximal Thurston-Bennequin Invariant

Definition 2.9.2. The maximal Thurston-Bennequin invariant of the knot type K is the maximum of Thurston-Bennequin invariants over all Legendrian presentation of the knot type K. It is denoted by $\texttt{maximaltb}(K)$.

2.9.4 Rotation Number Invariant

Definition 2.9.3. [38] Consider an oriented Legendrian knot which is null-homologous. Consider its Seifert surface Σ_g^1. A trivial two dimensional tangent bundle is obtained by the restriction of contact planes ξ to Σ_g^1. A trivialization $\xi|_L = L \times \mathbb{R}^2$ is derived from the trivialization of $\xi|_{\Sigma_g^1}$. Take a vector field v which has the following properties:

1) It is non-zero,

2) It is tangent to the knot,

3) It points in the same direction as the knot.

Then, v can be considered as the set of non-zero vectors in $\mathbb{R}^2$. The winding number of v is defined as *the rotation number of L*. It is denoted by $\texttt{rot}(L)$.

Consider an oriented Legendrian knot in $(\mathbb{R}^3, \xi_{std})$. The vector field $w = \dfrac{\partial}{\partial y}$ is a non-zero part of ξ_{std}. The trivialization of the knot is formed by using w. Take a vector field v which satisfies three conditions above. Then, the rotation number $\texttt{rot}(L)$ is equivalent to the signed number of how many times w and v point in the same direction. If v passes w counter clockwise, it is called $(+1)$. Otherwise, it is (-1).

If the cusp is going down, the intersection will be positive. Otherwise, it is negative. In this process, the number of times v intersects both w and $-w$ is counted. For this reason, it must be divided by two to find $\texttt{rot}(L)$. Let C_u be the total number of up cusps in the front projection. Let C_d be the total number of down cusps in the front projection. Then, rotation number of the knot is

$$\texttt{rot}(L) = \frac{1}{2}(C_d - C_u).$$

Example 2.9.3. The Legendrian unknots with two different orientations U_1 and U_2 are given in Figure 53. The rotation number of U_1 in Figure 53(a) is $rot(U_1) = \frac{1}{2}(3-1) = 1$, and the rotation number of U_2 in Figure 53(b) is $rot(U_2) = \frac{1}{2}(1-3) = -1$. Therefore, the rotation number depends on the orientation.

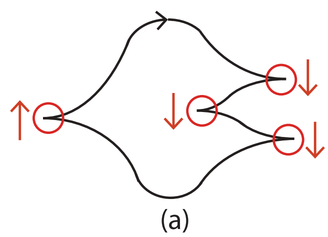


Figure 53: Oppositely oriented Legendrian unknot.

Example 2.9.4. Consider Legendrian right handed trefoil and Legendrian left handed trefoil in Figure 54 in $(\mathbb{R}^3, \xi_{std})$. They are different Legendrian knots because their classical invariants are different.

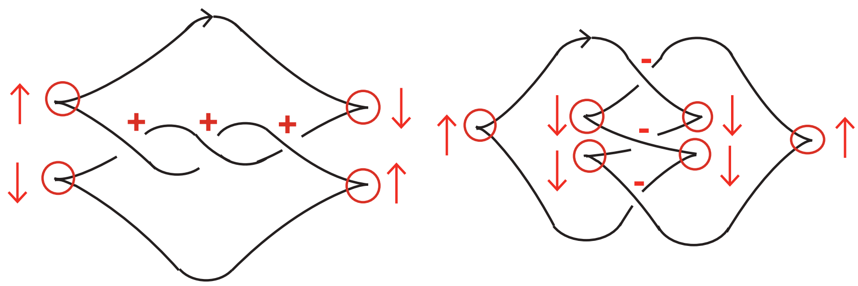


Figure 54: Legendrian trefoil knots.

Classical invariants for the Legendrian right trefoil L_r are the following:

$$tb(L_r) = writhe(\varphi_f(L_r)) - \frac{1}{2}(\#\text{cusps in } \varphi_f(L_r)) = 3 - \frac{1}{2}4 = 1$$

$$rot(L_r) = \frac{1}{2}(C_d - C_u) = \frac{1}{2}(2-2) = 0.$$

Classical invariants for the Legendrian left handed trefoil L_l are the following:

$$tb(L_l) = writhe(\varphi_f(L_l)) - \frac{1}{2}(\#\text{cusps in } \varphi_f(L_l)) = -3 - \frac{1}{2}6 = -6$$

$$rot(L_l) = \frac{1}{2}(C_d - C_u) = \frac{1}{2}(4-2) = 1.$$

2.10 Stabilizations

Stabilization for Legendrian knots is defined in $(\mathbb{R}^3, \xi_{std})$, and performed locally on the front projection of Legendrian knots. By Darboux's theorem, Legendrian knot stabilization operation can be done locally in any contact 3-manifold (M, ξ) [38].

Definition 2.10.1. [38] Consider an oriented Legendrian knot. The operation to convert the knot into another Legendrian knot having equal topological knot type with L by locally applying the movements in Figure 55 is called a *stabilization*. Adding down cusps is named as a positive stabilization, and adding up cusps is named as a negative stabilization. They are denoted by $P_+(L)$ and $N_-(L)$, respectively.

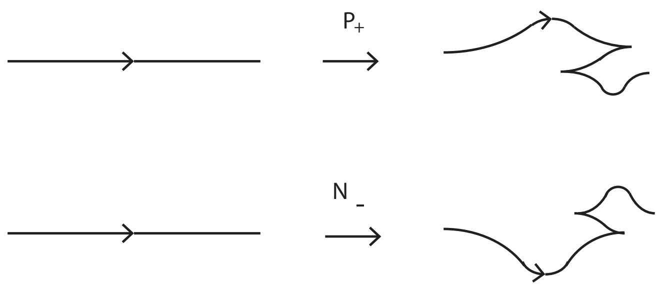


Figure 55: Positive and negative stabilization.

In the positive stabilization, the number of down cusps are increased by 2. In the negative stabilization, the number of up cusps are increased by 2. Although the knot type stays equal after the stabilization, the other two invariants alter as:

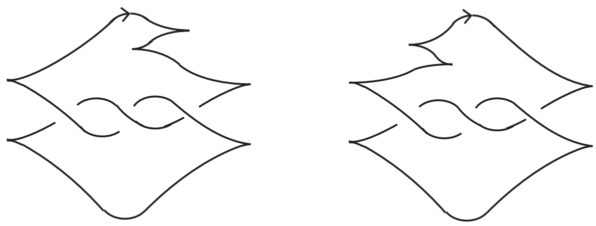
$$\texttt{tb}(P_+(L)) = \texttt{tb}(L) - 1, \quad \texttt{tb}(N_-(L)) = \texttt{tb}(L) - 1$$

and

$$\texttt{rot}(P_+(L)) = \texttt{rot}(L) + 1, \quad \texttt{rot}(N_-(L)) = \texttt{rot}(L) - 1.$$

Example 2.10.1. Consider the given Legendrian $\texttt{T}(2,3)$ in $(\mathbb{R}^3, \xi)$ in Figure 54. Thurston-Bennequin invariant of Legendrian $\texttt{T}(2,3)$ is $\texttt{tb} = 1$. Rotation number of Legendrian

$\mathtt{T}(2,3)$ is $\mathtt{rot} = 0$. After a positive stabilization P_+ in Figure 56(a), the classical invariants are $\mathtt{tb}(P_+) = 0$ and $\mathtt{rot}(P_+) = 1$. After a negative stabilization N_- in Figure 56(b), the classical invariants are $\mathtt{tb}(N_-) = 0$ and $\mathtt{rot}(N_-) = -1$.



(a) (b)

Figure 56: A positive stabilization P_+ of Legendrian $\mathtt{T}(2,3)$. A negative stabilization $N_-(L)$ of Legendrian $\mathtt{T}(2,3)$.

2.11 Classification Types of Legendrian Knots

Legendrian knots are classified by two ways which are by contactomorphism and by Legendrian isotopy.

Definition 2.11.1. [38] Consider two oriented Legendrian knots L_1 and L_2 in (M^3, ξ). If there exists a map φ from (M, ξ) to (M, ξ) so that $\varphi(L_1) = L_2$ and so that φ is a contactomorphism and φ is isotopic to the identity function, then two knots L_1, L_2 are called *Legendrian isotopic knots.*

Definition 2.11.2. [38] Consider two Legendrian knots L_1 & L_2 in (M, ξ). On the condition that there is a contactomorphism φ from (M, ξ) to (M, ξ) such that $\varphi(L_1) = L_2$, then L_1 and L_2 are *equivalent up to contactomorphism.*

2.12 Homotopy Invariants of Contact Structures

There can be one contact structure or more on a 3-manifold. Some invariants are necessary to distinguish contact structures on 3-manifolds. There are two homotopy invariants of contact structures on closed, orientable 3-manifolds. These are the d_2-invariant and the d_3-invariant and these invariants are defined in [39].

The Euler class d_2-invariant determines the homotopy over 2-skeleton of a closed, orientable 3-manifold. The other one is d_3-invariant of ξ which is a rational number such that it determines the homotopy obstruction over the 3-skeleton of the 3-manifold. In other words, any two homotopic 2-plane fields over the 2-skeleton of a closed, orientable 3-manifold are homotopic over the 3-manifold if and only if their d_3-invariants are the same [39].

These two homotopy invariants of ξ can be computed by using surgery diagrams. In this thesis, calculation of these two invariants will not be given. Calculation of these invariants are given in detail [39], [40].

Followings are two useful lemmas for determining types of contact 3-manifolds.

Lemma 2.12.1. [40] The d_3-invariant of ξ_{std} given in Example 2.6.5 on S^3 is $d_3(\xi_{std}) = -\frac{1}{2}$.

Lemma 2.12.2. [40] The d_3-invariant of ξ_{ot} on S^3 obtained by contact surgery with contact framing $(+1)$ applied to the unknot with $\texttt{tb}(L) = -2$ and $\texttt{rot}(L) = 1$ in (S^3, ξ_{std}) is $d_3(\xi_{ot}) = \frac{1}{2}$. Note that ξ_{ot} is overtwisted.

2.13 Knots in Overtwisted Contact 3-Manifolds

Definition 2.13.1. [16] Take a Legendrian knot K in an overtwisted (M, ξ). If the complement $(M \setminus K, \xi|_{M \setminus K})$ is tight, then it is called an *exceptional knot* . Otherwise, it is called a *loose knot.*

Example 2.13.1. Boundary of an overtwisted disc in an overtwisted manifold is an example of a loose knot. In fact, it is a loose unknot.

3 Legendrian Torus Knots

3.1 Classification of Legendrian torus knots in (S^3, ξ_{std})

Classification problem is an important problem for Legendrian knots. Classification problem is studied by many mathematicians. For example, in 1998, Eliashberg and Fraser classified Legendrian unknot in (S^3, ξ_{std}) [15]. Then, in 2001 Etnyre & Honda gave the classification of Legendrian torus knots. They also classified Legendrian figure eight knot in (S^3, ξ_{std}) in the same paper. Many mathematicians studied other Legendrian knots in other contact 3-manifolds, too. Linear Legendrian curves in contact 3-torus T^3 were classified by Ghiggini in [41]. Furthermore, Legendrian torus knots in other contact 3-manifolds are studied by Onaran. Legendrian positive torus knots in universally tight contact structures on lens spaces were classified in [18]. Recently, classification problem of Legendrian knots in overtwisted S^3's has also been studied. Eliashberg and Fraser classified exceptional unknots in overtwisted S^3's in [16]. Classification of Legendrian rational unknots in lens spaces $L(p, 1)$ for odd p was done by Etnyre and Baker in [42]. The same classification for any value of p was done by Geiges and Onaran in [43]. First nontrivial knot type classifications was also done by Geiges and Onaran. Exceptional Legendrian torus knots in overtwisted S^3's are classified in [19].

Definition 3.1.1. [17] Assume that S^3 is formed by two solid tori U and V, i.e. $S^3 = U \cup_T V$ where T is the common boundary of solid tori U and V. If μ and λ are the unique curves which bound a disk in U and V, respectively, and μ and λ are two homological generator curves on T, then any simple closed curve on T is in the form $l\mu+m\lambda$ for relatively prime integers l and m. The simple closed curve $l\mu+m\lambda$ is called a (l, m)-torus knot in S^3. A Legendrian $\mathtt{T}(l, m)$ torus knot in (S^3, ξ_{std}) is a Legendrian knot having knot type as $\mathtt{T}(l, m)$ torus knot.

Etnyre and Honda classified Legendrian torus knots in (S^3, ξ_{std}) in [17].

Theorem 3.1.1. *"[17] Two oriented Legendrian torus knots in (S^3, ξ_{std}) are Legendrian isotopic if and only if they have the same classical invariants."*

Example 3.1.1. By Theorem 3.1.1, the following two realizations of Legendrian left handed trefoil in (S^3, ξ_{std}) with $\mathtt{tb} = -6$ and $\mathtt{rot} = 1$ are Legendrian isotopic because their classical invariants are the same.

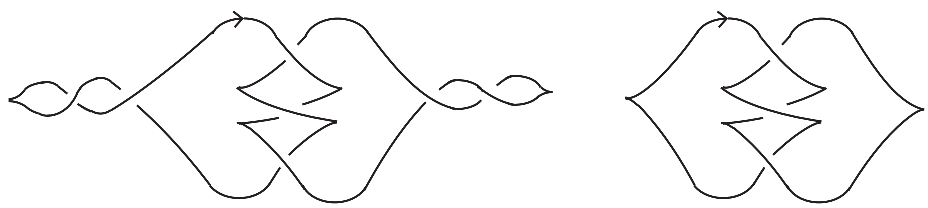


Figure 57: Two realizations of Legendrian left handed trefoil in (S^3, ξ_{std}).

There are two types of Legendrian torus knots: positive Legendrian torus knots and negative Legendrian torus knots.

3.1.1 Positive Legendrian Torus Knots

Definition 3.1.2. A Legendrian torus knot $\mathtt{T}(l,m)$ in (S^3,ξ_{std}) where $gcd(l,m)=1$ is called *a positive Legendrian torus knot* if $l,m>0$.

In this thesis, positive Legendrian torus knots $\mathtt{T}(l,m)$ where $m>l>0$ are studied. The following theorem of Etnyre and Honda is about Legendrian positive torus knots.

Theorem 3.1.2. *[17] " If L is an oriented Legendrian* $\mathtt{T}(l,m)$ *knot where* $m>l>0$, *then the rotation number of L with* $\mathtt{maximaltb}(L)=lm-l-m$ *is* $\mathtt{rot}(L)=0$. *If* $\mathtt{tb}(L)=ml-m-l-k$ *where k is a non-negative integer, then* $\mathtt{rot}(L)\in\{-k,-k+2,\ldots,k\}$*."*

Legendrian right handed trefoil in standard contact S^3 has the maximal Thurston-Bennequin invariant $\mathtt{maximaltb}(L_r)=1$. Theorem 3.1.2 gives a table of the invariants of Legendrian torus knots which have the same topological knot type. In this table, every dot represents a pair of invariants of Legendrian torus knots, and every arrow corresponds to a stabilization. For example, a table of the invariants of Legendrian $\mathtt{T}(2,3)$ where the knot is denoted by L_r is given Figure 58. Its maximal Thurston-Bennequin invariant $\mathtt{maximaltb(L_r)}=\mathtt{1}$ which is unique and it has the rotation number $\mathtt{rot(L_r)}=\mathtt{0}$.

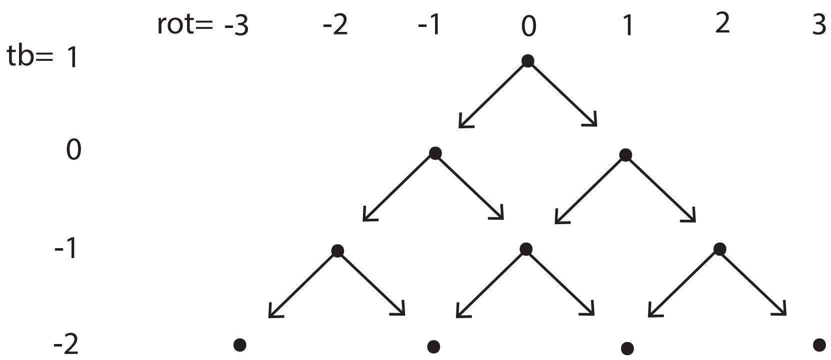


Figure 58: Table of some Legendrian right handed trefoils with pairs of invariants.

The unique positive Legendrian $\mathtt{T}(2,3)$ knot with $\mathtt{maximaltb(L_r)}=\mathtt{1}$ and $\mathtt{rot(L_r)}=\mathtt{0}$, the positive Legendrian $\mathtt{T}(2,3)$ knot with $\mathtt{maximaltb(L_r)}=\mathtt{0}$ and $\mathtt{rot(L_r)}=\mathtt{1}$, and the positive Legendrian $\mathtt{T}(2,3)$ knot with $\mathtt{maximaltb(L_r)}=\mathtt{0}$ and $\mathtt{rot(L_r)}=-\mathtt{1}$ are given in Figure 59(a), Figure 59(b) and Figure 59(c), respectively.

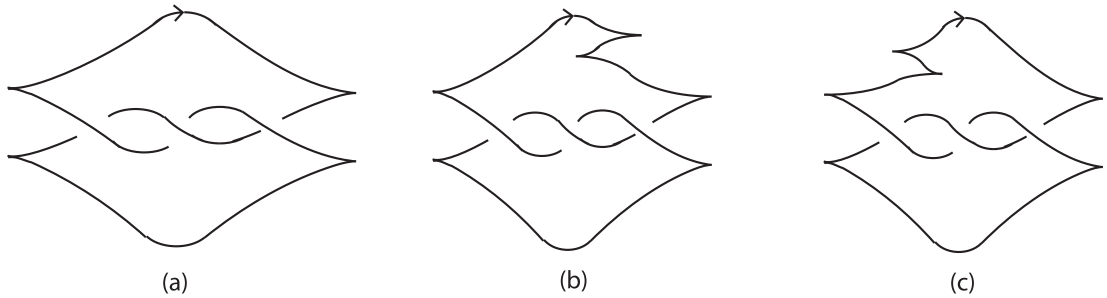


Figure 59: Three Legendrian realizations of positive right handed trefoil.

3.1.2 Negative Legendrian Torus Knots

Definition 3.1.3. A Legendrian torus knot $\mathtt{T}(l, m)$ in (S^3, ξ_{std}) where $gcd(l, m) = 1$ is called *a negative Legendrian torus knot* if $l < 0$ and $m > 0$, or $l > 0$ and $m < 0$.

In this thesis, negative Legendrian torus knots $\mathtt{T}(l, -m)$ where $m > l > 0$ are studied. The following theorem of Etnyre and Honda is about Legendrian negative torus knots.

Theorem 3.1.3. *[17] " If L is an oriented Legendrian* $\mathtt{T}(l, -m)$ *where* $m > l > 0$*, then the rotation number of* L *with* $\mathtt{maximaltb}(L) = -lm$ *is* $\mathtt{rot}(L) \in \{\pm(m - l - 2lk) : k \in \mathbb{Z}, 0 \leq k \leq \dfrac{(m-l)}{l}\}$*. "*

Legendrian left handed trefoil in standard contact S^3 has the maximal Thurston-Bennequin invariant $\mathtt{maximaltb}(L_l) = -6$. Theorem 3.1.3 gives a table of the invariants of Legendrian torus knots which have the same topological knot type. For example, a table of the invariants of Legendrian $\mathtt{T}(2, -3)$ where the knot is denoted by L_l is given Figure 60. There are two Legendrian $\mathtt{T}(2, -3)$ with $\mathtt{maximaltb(L_r)} = -6$. Their rotation numbers are $\mathtt{rot(L_r)} = \pm 1$.

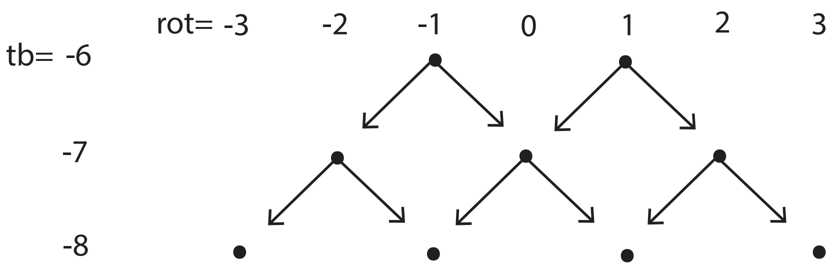


Figure 60: Table of some Legendrian left handed trefoils with pairs of invariants.

The negative Legendrian $\mathtt{T}(2,-3)$ with $\mathtt{maximaltb}(\mathtt{L_r}) = -6$, and $\mathtt{rot}(\mathtt{L_r}) = -1$ can be seen in Figure 61(a). The negative Legendrian $\mathtt{T}(2,-3)$ with $\mathtt{maximaltb}(\mathtt{L_r}) = -6$, and $\mathtt{rot}(\mathtt{L_r}) = 1$ can be seen in Figure 61(b).

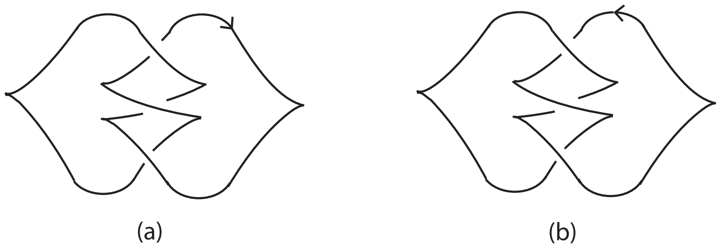


Figure 61: Table of some Legendrian left handed trefoils with pairs of invariants.

3.2 Contact Surgery in Contact 3-Manifolds Along Legendrian Knots

Contact surgery applies to Legendrian knots while Dehn surgery applies to topological knots. To apply a contact surgery to a Legendrian knot L in (M,ξ), first a tubular neighbourhood of the Legendrian knot is removed. Then, a new tight solid torus is glued to the exterior $M \setminus L$ along the boundaries via a homeomorphism of the tight solid torus such that one can extended the contact structure on M over the solid torus [20], [40].

When a contact 3-manifold (S^3, ξ_{std}) is considered, contact surgery applied to a Legendrian knot L is determined by coprime integers (p,q). Take a tubular neighbourhood $N(L)$. Homological generators of $(S^3 \setminus N(L), \xi_{std})$ are the meridian m and the longitude l which come from the contact framing. Contact surgery applied to L is gluing a tight solid torus $(S^1 \times D^2, \xi')$ to $(S^3 \setminus N(L), \xi_{std})$ along boundaries via the gluing map g:

$$g : \partial(S^1 \times D^2, \xi') \to \partial((S^3 \setminus N(L)), \xi_{std})$$

$$\{a\} \times \partial D^2 \mapsto qm + pl.$$

Therefore, after the contact surgery along L, a new contact 3-manifold $(M,\xi) = ((S^3 \setminus N(L)), \xi_{std})\ U_g\ (S^1 \times D^2, \xi')$ is formed.

Definition 3.2.1. A contact surgery with contact framing (-1) applied to a Legendrian knot is called *a Legendrian surgery.*

Theorem 3.2.1. *[20] "Every closed and contact 3-manifold can be formed by contact surgery with contact framing* (± 1) *applied to a Legendrian link in* (S^3, ξ_{std})*. "*

Theorem 3.2.2. *[40] "Contact surgery with contact framing* $(+1)$ *applied to a Legendrian unknot with* $\texttt{tb} = -1$ *and* $\texttt{rot} = 0$ *is the tight contact 3-manifold* $(S^1 \times S^2, \xi)$*."*

Theorem 3.2.3. *[44] "If a contact manifold* (M_1, ξ_1) *is formed by a Legendrian surgery in a tight* (M, ξ)*, then* (M_1, ξ_1) *is also tight."*

Observe that the result of a contact surgery can be overtwisted although the manifold on which the surgery is applied is tight.

Example 3.2.1. [40] In Figure 62, the Legendrian unknot L in the standard contact structures on S^3 produces an overtwisted S^3.

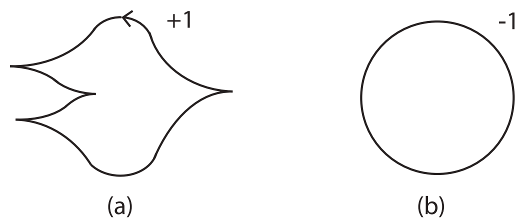


Figure 62: (a) Contact (+1)-surgery and (b) -1-Dehn surgery.

The classical invariants of L in Figure 62(a) is $\texttt{tb}(L) = -2$ and $\texttt{rot}(L) = 1$, and contact framing of L is $(+1)$. By definition of Thurston-Bennequin invariant, L has Seifert framing -1. So, the contact $(+1)$-surgery applied to the Legendrian knot L corresponds to topological Dehn surgery with surgery coefficient -1 applied to a topological unknot in Figure 62(b). Thus, the resulting manifold is S^3. In Figure 63(a), Legendrian unknot L' with contact framing c and $\texttt{tb}(L') = -1$ is the boundary of an overtwisted disc. Indeed, its Seifert framing is also c by Kirby move 1 as seeing Figure 63(b) after converting the contact surgery diagram into topological surgery diagram. Since $\texttt{tb}(L') = c - c = 0$, L' bounds an overtwisted disc by definition of an overtwisted disc. Therefore, S^3 in Figure 62(a) is an overtwisted manifold.

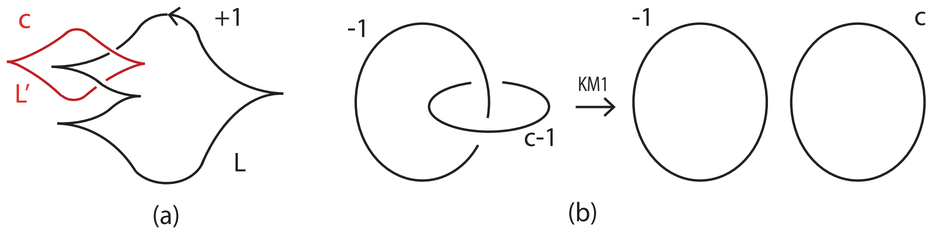


Figure 63: (a) Contact surgery diagram and (b) topological surgery diagram.

Lemma 3.2.1. "[45] (Cancellation Lemma) Take a Legendrian knot in (M, ξ) and take its parallel push. Then, contact surgery with contact framing (-1) applied to the knot and contact surgery with contact framing $(+1)$ applied to its parallel push off cancel each other. The resulting manifold (M_1, ξ_1) and (M, ξ) are contactomorphic."

Lemma 3.2.2. Consider a loose knot. Then, contact (p/q)-surgery applied to the loose knot always results in an overtwisted contact 3-manifold.

Proof. Since the given knot is a loose knot, the complement of the knot is an overtwisted contact 3-manifold. So, an overtwisted disc exists in the complement. Contact (p/q)-surgery along the loose knot can be done far away from this overtwisted disc in the complement. Then, the overtwisted disc in the complement of L remains after the contact (p/q)-surgery along the loose knot so that the resulting manifold is overtwisted. Therefore, any contact surgery along a loose knot is always overtwisted. □

3.3 Classical Invariants of Legendrian Knots from Surgery Diagrams

Definition 3.3.1. [43] [46] Let the surgery link $\mathcal{L} = K_1 \cup K_2 \cup \ldots \cup K_n$ in (S^3, ξ_{std}) be a contact (± 1)-surgery representation of closed and contact 3-manifold (M, ξ) such that $\mathcal{L}$ is an oriented link with the integral surgery coefficients $f_i = tb(K_i) \pm 1$ of each components K_i where $i = 1, 2, \ldots, n$. Take a Legendrian knot K_0 in (S^3, ξ_{std}) disjoint from $\mathcal{L}$. Then the matrix

$$A = (a_{i,j}) = \begin{cases} f_i & ;i = j \\ lk(K_i, K_j) & ;i \neq j \end{cases}$$

is called the *linking matrix* of $\mathcal{L}$, and the matrix

$$A_0 = (a_{i,j}) = \begin{cases} 0 & ;i = j = 0 \\ f_i & ;i = j \neq 0 \\ lk(K_i, K_j) & ;i \neq j \end{cases}$$

is called an *extended linking matrix.* In other words, A_0 is the linking matrix of $K_0 \cup K$.

Lemma 3.3.1. [46] [43] " Let the surgery link $\mathcal{L} = K_1 \cup K_2 \cup \ldots \cup K_n$ be a contact $(\pm)$-surgery representation of (M, ξ). Take a Legendrian knot K_0 in (S^3, ξ_{std}) disjoint from $\mathcal{L}$. The Thurston-Bennequin invariant of K_0 in (M, ξ) is $\mathtt{tb}(K_0) = tb_0(K_0) + \dfrac{detA_0}{detA}$ where $\mathtt{tb}_0(K_0)$ is the Thurston-Bennequin invariant of K_0 before the surgery. The rotation number of K_0 in (M, ξ) is $\mathtt{rot}(K_0) = \mathtt{rot}_0(K_0) - < \underline{\mathtt{rot}}, A^{-1} \cdot \underline{lk} >$ where $\mathtt{rot}_0(K_0)$ is the rotation number of K_0 before the surgery, $\underline{\mathtt{rot}}$ is the vector of rotation numbers, and $\underline{lk}$ is the vector of linking numbers between K_0 and K_i for $i = 1, 2, \ldots, n$."

Lemma 3.3.1 above is Lemma 6.6 of [46] and Lemma 2 of [43]. This lemma extended to more general contact 3-manifolds in [47], see Lemma 6.4 of [47]. See also Theorem 4.3 in [48] for computing rotation number in contact surgery diagrams.

3.4 Contact Surgery along Legendrian Torus Knots

Before studying contact surgery along Legendrian torus knots, Dehn surgery along torus knots is studied in this section. In 1971, Moser studied Dehn surgery applied to torus knots.

Theorem 3.4.1. *" [8] Take a torus knot $\mathtt{T}(l, m)$ in S^3 where $|m| > l > 0$ and M is the new manifold which is obtained by a topological Dehn surgery with framing $(-q/p)$ applied to the torus knot. Assume $\sigma = lmp + q$.*

1) *When $|\sigma| \neq 0$, M is a Seifert manifold such that it is singularly fibered by simple closed curves over S^2 with singularities of types $\alpha_1 = m$, $\alpha_2 = l$ and $\alpha_3 = |\sigma|$.*

2) *When $\sigma = \pm 1$, there are only two singular fibers of types $\alpha_1 = m$ and $\alpha_2 = l$, and M is the Lens space $L(|q|, pl^2)$.*

3) *When $|\sigma| = 0$, M is $L(l, m) \# L(m, l)$, and it is not singularly fibered."*

It is shown that there are different ways to obtain lens space $L(7,4)$ in Example 2.4.5. Similarly, there are different ways to obtain the lens space $L(7,4)$ by a surgery along a (l,m)-torus knot $\mathtt{T}(l,m)$ in S^3 for relatively prime integer l, m.

At this point, Rasmussen gave an important result. He proved that the only way to obtain the lens space $L(7,4)$ by an integral surgery on S^3 is (-7)-surgery along a left handed trefoil knot. Rasmussen gave the generalization of this result in [11]. Rasmussen states his result in [11] in terms of positive torus knots, here in this thesis, the same result in terms of negative torus knots is stated.

Corollary 3.4.1. [11] "The sole way to obtain the lens space $L(4m+3,4)$ by a topological integral surgery applied to a single knot in S^3 is an integral surgery with framing $-(4m+3)$ applied to $\mathtt{T}(2,-(2m+1))$."

In contact perspective, there are three tight contact structures on contact $L(7,4)$. Plamenevskaya claimed that only one tight contact structure of the total three tight contact structures on the contact lens space $L(7,4)$ can be realized by contact surgery with contact framing (-1) applied to a Legendrian $\mathtt{T}(2,-3)$ in some contact structure on S^3 using Rasmussen's result [49]. Geiges and Onaran showed that there is a mistake in Plamenevskaya's result in [21]. They showed that all three contact structures which are tight on the lens space $L(7,4)$ can be obtained by a single contact surgery with contact framing (-1) applied to a Legendrian left handed trefoil in S^3. Also, they generalized the result for the lens spaces $L(4m+3,4)$ in [21].

In this chapter, the results of Geiges and Onaran and their surgery techniques in [21] are studied in detail.

3.4.1 Contact Structures on Contact Lens Spaces

Honda, independently Giroux gave the exact number of tight contact structures that can exist on special class of 3-manifolds which are lens spaces [50] [51].

Theorem 3.4.2. *"[50] [51] Let $L(p,q)$ be a lens space where $p > q > 0$ and $gcd(p,q) = 1$. The exact number of contact structures which are tight on a given lens space is*

$$(r_0-1)(r_1-1)\dots(r_k-1)$$

where the $r_i \geq 2$ which are the terms in the following continued fraction expansion

$$\frac{p}{q} = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cdots \cfrac{1}{r_k}}} =: [r_0, \ldots, r_k].\text{''}$$

3.4.2 Contact Lens Space $L(7,4)$

In this section, contact 3-manifold $L(7,4)$ and the three tight contact structures on it analyzed by using contact surgery techniques in [21] for understanding lens spaces and contact surgery techniques better.

Theorem 3.4.3. *[21] All three tight contact structures on the lens space $L(7,4)$ can be formed via an only one (-1)-contact surgery applied to a Legendrian realization of $\mathtt{T}(2,-3)$ in some contact structure on S^3.*

Proof. There are three different tight contact structures ξ_1, ξ_2, ξ_3 on $L(7,4)$ by Theorem 3.4.2 where the continued fraction is $\frac{7}{4} = 2 - \frac{1}{4} =: [2,4]$. Their d_3-invariants are $d_3(\xi_1) = d_3(\xi_2) = -2/7$ and $d_3(\xi_3) = 0$ which are calculated in [21]. Two of them have the same d_3-invariants but their d_2-invariants are different. Using their d_2-invariants, they are distinguished. The contact structure ξ_3 is different from ξ_1 and ξ_2 since $d_3(\xi_3) = 0 \neq -2/7 = d_3(\xi_1) = d_3(\xi_2)$.

There are many ways to obtain the lens space $L(7,4)$. However, Rasmussen showed in Corollary 3.4.1 that there is an only one way to obtain $L(7,4)$ by an integral surgery on S^3 that is a (-7)-surgery along $\mathtt{T}(2,-3)$ in S^3 .

Legendrian $\mathtt{T}(2,-3)$ having Thurston-Bennequin invariant $\mathtt{tb} = sf - cf = -7 - (-1) = -6$ in (S^3, ξ_{std}) or an exceptional left handed trefoil knot in an overtwisted S^3 have to be considered if a single (-1)-contact surgery along Legendrian left handed trefoil $\mathtt{T}(2,-3)$ in some contact structure on S^3 is used for getting $L(7,4)$.

The maximal Thurston-Bennequin invariant of Legendrian $\mathtt{T}(2,-3)$ in (S^3, ξ_{std}) is $\mathtt{maximaltb} = -6$ by Theorem 3.1.3. There are two different realizations of $\mathtt{T}(2,-3)$ with $\mathtt{maximaltb} = -6$. Their rotation numbers are ± 1 which are shown in Figure 64.

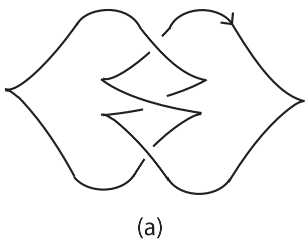


Figure 64: Two different realizations of Legendrian left handed trefoil with $tb = -6$ in (S^3, ξ_{std}).

The results of (-1)-contact surgeries along the two Legendrian left handed trefoils in (S^3, ξ_{std}) give two different contact structures ξ_1 and ξ_2 on the lens space $L(7,4)$ such that $d_3(\xi_1) = d_3(\xi_2) = -2/7$, but their d_2-invariants are different because their rotation numbers are different. So, ξ_1 and ξ_2 are different by Lisca and Matić [52]. In this thesis, calculation of d_3-invariant will not be done, but the detailed calculations of d_3-invariants can be found in [21]. Also, these two different contact manifolds $(L(7,4), \xi_1)$ and $(L(7,4), \xi_2)$ are tight by Theorem 3.2.3.

The other realization of $\mathtt{T}(2,-3)$ is an exceptional knot K which is given in Figure 65 in an overtwisted contact S^3. The given surgery diagram is overtwisted S^3 since its d_3-invariant is $3/2$ which is different from $-1/2$. This invariant is calculated in detail in [21].

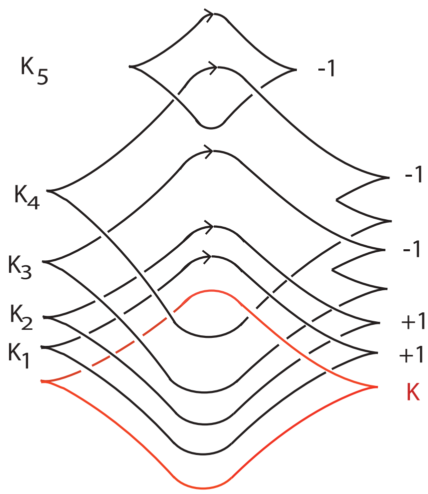


Figure 65: An exceptional left handed trefoil in an overtwisted S^3.

Contact surgery with contact framing (-1) is applied to the knot K in Figure 65 to show that it is an exceptional knot in an overtwisted contact S^3. By the cancellation lemma, contact surgery with contact framing (-1) cancels contact surgery with contact framing $(+1)$ applied to the parallel push off of the knot. Contact surgery with contact framing $(+1)$ applied to Legendrian unknot results in the tight $S^1 \times S^2$ by Theorem 3.2.2. By Wand's theorem, remaining other contact surgeries with contact framing (-1) result in a tight contact structure by Theorem 3.2.3. So, contact surgery with contact framing (-1) applied to the knot K is a tight contact manifold. So, K must be an exceptional knot in an overtwisted S^3 by Lemma 3.2.2.

Kirby moves is used to show that the exceptional realization of the left handed trefoil K in Figure 65 corresponds to a topological left handed trefoil as seeing in Figure 66 and 67. To find the Thurston-Bennequin invariant of the knot K in the surgered manifold, the linking matrix of the surgery is formed as follows according to Definition 3.3.1. Seifert framings of K_i for each $i = 1, 2, 3, 4, 5$ are obtained as $0, 0, -3, -3, -2$ from Figure 65, respectively. Also, the linking numbers $lk(K_i, K_j) = -1$ for $i, j = 1, 2, 3, 4$ and $lk(K_5, K_j) = 0$ for $j = 1, 2, 3$ and $lk(K_5, K_4) = -1$. Then, using Definition 3.3.1 the linking matrix and the extended linking matrix of the knot K can be found as the following.

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & -3 & -1 & 0 \\ -1 & -1 & -1 & -3 & -1 \\ 0 & 0 & 0 & -1 & -2 \end{pmatrix}$$

$$\mathbf{A_0} = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -3 & -1 & 0 \\ -1 & -1 & -1 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{pmatrix}$$

Thus, using Lemma 3.3.1 the Thurston-Bennequin invariant of the exceptional left handed trefoil in the surgery diagram in Figure 65 is $\mathtt{tb}(K) = \mathtt{tb}_0(K) + \dfrac{detA_0}{detA} = -1 + \dfrac{5}{-1} = -6$ where the Thurston-Bennequin invariant of K before the surgery is

$\mathtt{tb}_0(K) = -1$.

Contact surgery with contact framing (-1) applied to the exceptional left handed trefoil knot K in the overtwisted S^3 gives another tight contact structure ξ_3 on $L(7,4)$ such that $d_3(\xi_3) = 0$. The detailed calculations of d_3-invariants can be found in [21].

Therefore, the three different tight contact structures ξ_1, ξ_2, ξ_3 on $L(7,4)$ can be observed by contact surgery with contact framing (-1) applied to the Legendrian left handed trefoil knots in Figure 64, and the exceptional Legendrian $\mathtt{T(2,-3)}$ in Figure 65. The tight contact structures ξ_1 and ξ_2 on $L(7,4)$ come from contact surgery with contact framing (-1) applied to Legendrian left handed trefoil knots in the standard tight contact S^3 in Figure 64(a) and Figure 64(b), respectively. The other tight contact structure ξ_3 on the contact lens space $L(7,4)$ is obtained by contact (-1)-surgery along exceptional left handed trefoil in an overtwisted contact structure on S^3 in Figure 65.

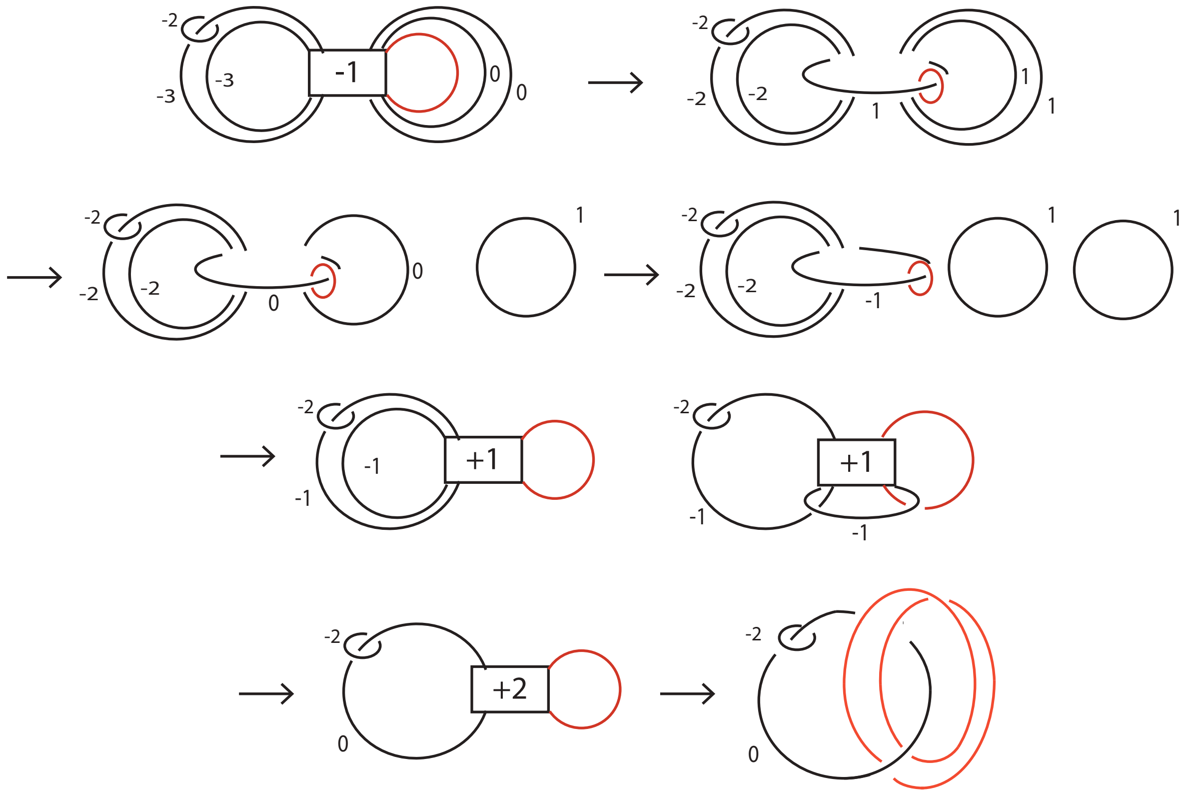


Figure 66: Kirby moves for the exceptional left handed trefoil in an overtwisted S^3.

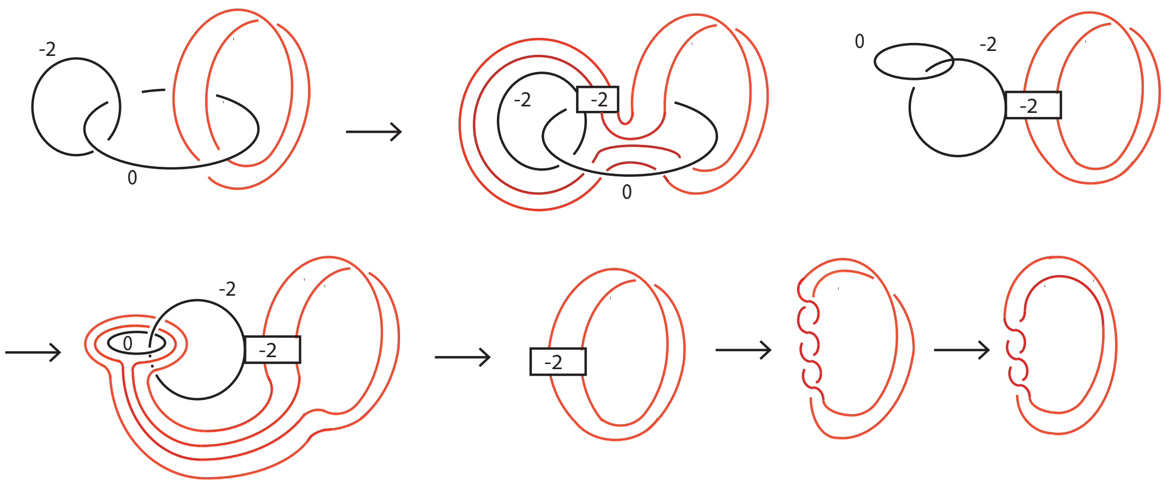


Figure 67: Handle slides for the exceptional left handed trefoil in an overtwisted S^3.

□

3.4.3 Contact Lens Spaces $L(4m+3,4)$

In this section, contact 3-manifold $L(4m+3,4)$ where the integer $m \geq 1$ and the number of tight contact structures on it analyzed by using contact surgery techniques in [21] for understanding lens spaces and contact surgery techniques better. Note that $m=1$ case, $L(7,4)$ case, is already studied in previous Section 3.4.2.

Theorem 3.4.4. *" [21] Each tight contact structures on the lens space $L(4m+3,4)$ can be formed via a single Legendrian surgery along a suitable Legendrian realization of $\mathtt{T}(2,-(2m+1))$ in some contact structure on S^3 where $m \geq 1$."*

Proof. There are $3m$ tight contact structures ξ_1, ξ_2, ... , ξ_{3m} on $L(4m+3,4)$ by Theorem 3.4.2 where the continued fraction is $\dfrac{4m+3}{4} = m+1-\dfrac{1}{4} =: [m+1,4]$.

Similar to the case where $m=1$, the lens space $L(7,4)$, there are many ways to obtain the lens space $L(4m+3,4)$ for $m \geq 1$. However, Rasmussen showed in Corollary 3.4.1 that there is an only one way to obtain $L(4m+3,4)$ by an integral surgery on S^3 that is a $-(4m+3)$-surgery along a negative torus knot $\mathtt{T}(2,-(2m+1))$ in S^3 .

Legendrian $\mathtt{T}(2,-(2m+1))$ with $\mathtt{tb} = sf - cf = -(4m+3)-(-1) = -(4m+2)$ in (S^3,ξ_{std}), or an exceptional $\mathtt{T}(2,-(2m+1))$ in some overtwisted contact structures on S^3 should be found if a (-1)-contact surgery along a Legendrian $\mathtt{T}(2,-(2m+1))$ in some contact structure on S^3 is used for getting $L(4m+3,4)$.

The maximal Thurston-Bennequin invariant for Legendrian $\mathtt{T}(2,-(2m+1))$ in (S^3, ξ_{std}) is $\mathtt{maximaltb} = -(4m+2)$ by Theorem 3.1.3. There are $2m$ Legendrian $\mathtt{T}(2,-(2m+1))$ with $\mathtt{maximaltb} = -(4m+2)$ such that their rotation numbers are in the following set

$$rot \in \{-2m+1, -2m+3, \ldots, 2m-3, 2m-1\}.$$

The results of contact (-1)-surgery along these $2m$ Legendrian negative torus knots $\mathtt{T}(2,-(2m+1))$ in (S^3, ξ_{std}) give $2m$ different contact structures ξ_1, ξ_2, $\ldots$, ξ_{2m} on $L(4m+3,4)$. These $2m$ contact structures are different since they come from $2m$ Legendrian $T(2,-(2m+1))$ knots having the same Thurston-Bennequin invariant but having different rotation numbers. Then, this will correspond to different contact structures on $L(4m+3,4)$ by [52]. In other words, these $2m$ contact structures are different by their different rotation numbers. Also, the $2m$ different contact structures are tight by Theorem 3.2.3.

The remaining m realizations of $\mathtt{T}(2,-(2m+1))$ have to be exceptional knots in some overtwisted contact structures on S^3 in Figure 68, and they are given in [46]. This surgery diagrams in Figure 68 give overtwisted contact structures on S^3 since $d_3 = 2a + 3/2 \neq -1/2$ for $a \in \mathbb{N}_0$. It is calculated in detail in [21].

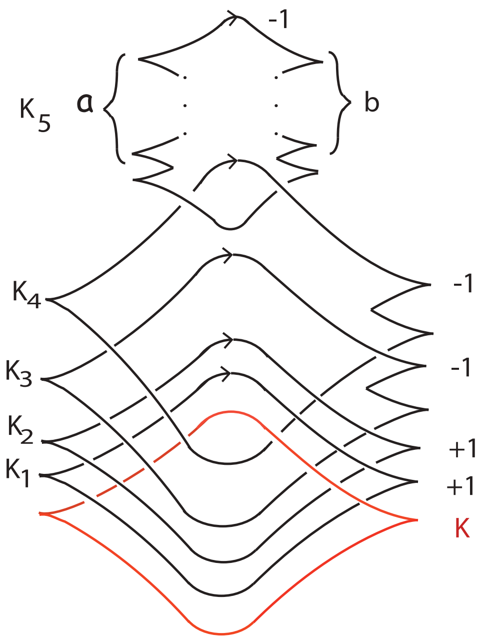


Figure 68: Exceptional realizations of $\mathtt{T}(2,-(2m+1))$ where $m = a+b+1$, $a, b \in \mathbb{N}_0$.

Similar to the case $L(7,4)$, contact surgery with contact framing (-1) applied to the exceptional torus knots K in Figure 68 gives a tight contact manifold by the cancellation lemma. So, the realisations of $\mathtt{T}(2,-(2m+1))$ must be exceptional in overtwisted S^3 by Lemma 3.2.2.

The exceptional torus knots K in Figure 68 correspond to the topological torus knots $\mathtt{T}(2,-(2m+1))$ by Kirby moves. The given surgery diagram in Figure 68 corresponds to the topological surgery diagram which is similar to the surgery diagram in Figure 66 where the $-(m+1) = -n$-framing occurs instead of the (-2)-framing. The exceptional torus knots K in Figure 68 separates from the surgery link by using $1+n$ handle slides instead of the $1+2$ slides in Figure 67. Then, the topological $\mathtt{T}(2,-(2m+1))$ in S^3 is obtained. The surgery linking matrix and the extended matrix of K are used for showing Thurston-Bennequin invariant of the knot K in Figure 68. By using Definition 3.3.1, the linking matrix of K is

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & -3 & -1 & 0 \\ -1 & -1 & -1 & -3 & -1 \\ 0 & 0 & 0 & -1 & -(m+1) \end{pmatrix}.$$

Also, by using Definition 3.3.1 again, the extended linking matrix of K is

$$\mathbf{A_0} = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -3 & -1 & 0 \\ -1 & -1 & -1 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 & -1 & -(m+1) \end{pmatrix}.$$

Thus, the classical invariants of K can be computed by Lemma 3.3.1.

$$\mathtt{tb}(K) = \mathtt{tb}_0(K) + \frac{detA_0}{detA} = -1 + \frac{-3+4(m+1)}{-1} = -(4m+2)$$

where the Thurston-Bennequin invariants of the knot K in unsurgered manifold is by $\mathtt{tb}_0(K) = -1$. The rotation numbers can be computed by

$$\mathtt{rot}(K) = \mathtt{rot}_0(K) - < \underline{\mathtt{rot}}, A^{-1} \cdot \underline{lk} >$$

$$\texttt{rot}(K) = 0- < (0,0,1,1,b-a)^t, A^{-1} \cdot (-1,-1,-1,-1,0)^t >$$

$$\texttt{rot}(K) = -(4m+3) + 2(b-a) = -(4m+3) + 2(m-1-2a)$$

for the integer $m \geq 1$ and for the numbers a, $b \in \mathbb{N} \cup \{0\}$, $a + b = m - 1$ by Lemma 3.3.1. So, the rotation numbers of K in $modulo(4m + 3)$ is:

$$\texttt{rot}(K) = -(4m+3) + 2(m-1-2a) \equiv 2(m-1-2a)\ mod(4m+3)$$

where $0 \leq a \leq m - 1$. Then, there are m distinct rotation numbers of K in the following set

$$\texttt{rot}(K) \in \{-2m+2, -2m+6, \ldots, 2m-6, 2m-2\}.$$

Contact surgery with contact framing (-1) applied to the exceptional negative torus knots $\texttt{T}(2, -(2m + 1))$ in some overtwisted contact structures on S^3 gives the other m tight contact structures ξ_{2m+1}, ... , ξ_{3m} on $L(4m + 3, 4)$. These m tight contact structures are different since they are obtained by Legendrian $T(2, -(2m+1))$ having the same Thurston-Bennequin invariant but having different rotation numbers. So, ξ_{2m+1}, ... , ξ_{3m} are different by Lisca and Matić [52]

Therefore, the $3m$ tight contact structures $\xi_1, \xi_2, \ldots, \xi_{3m}$ on $L(4m + 3, 4)$ can be observed by a single contact surgery with contact framing (-1) applied to Legendrian $\texttt{T}(2, -(2m + 1))$ in (S^3, ξ_{std}) and the exceptional negative torus knots in Figure 68 in some overtwisted contact structures on S^3. The tight contact structures $\xi_1, \ldots, \xi_{2m}$ on $L(4m + 3, 4)$ come from contact surgery with contact framing (-1) applied to Legendrian $\texttt{T}(2, -(2m + 1))$ with Thurston-Bennequin invariant $\texttt{tb} = -(4m + 2)$ in a standard contact structure on S^3 and their rotation numbers are in the set $\{-2m + 1, -2m + 3, \ldots, 2m - 3, 2m - 1\}$. The other m tight contact structure $\xi_{2m+1}, \ldots, \xi_{3m}$ on $L(4m + 3, 4)$ come from contact surgery with contact framing (-1) applied to exceptional negative torus knot $\texttt{T}(2, -(2m+1))$ in some overtwisted contact structures on S^3 in Figure 68. □

4 Conclusion

During the historical development of contact topology, lens spaces are obtained by surgery, but deciding which knots are used is the real problem. Many mathematicians studied on this problem. As noted by Moser in [8], the lens spaces obtained by Dehn surgery along torus knots were classified. In another respect, Rasmussen showed in [11] that there is only one way to produce a special class of lens spaces which are $L(4m+3,4)$ by an integral surgery.

This thesis helps to understand lens spaces by using contact surgeries. This study highlighted the importance of learning contact surgery techniques.

There are lots of ways to obtain lens spaces, but how to obtain a lens space by a single contact surgery along a single knot is a problem. Also, how to obtain all tight contact structures on the lens space is another problem. In this thesis, the lens spaces that are obtained by a contact surgery with contact framing (-1) applied to a single Legendrian negative torus knot are studied. Also, the techniques for obtaining all tight contact structures on the lens space are learned.

For this purpose, it is studied how to be formed lens spaces $L(4m+3,4)$ by a contact surgery with contact framing (-1) applied to a single Legendrian negative torus knot. Also, the fact that tight contact structures which are different on $L(4m+3,4)$ result from Legendrian surgery along Legendrian realisations of Legendrian negative torus knot is examined to understand 3-manifold $L(4m+3,4)$ better. The contact surgery techniques given by Geiges and Onaran are learned during the study [21].

For future research, the question of how to classify negative Legendrian torus knots in universally tight lens spaces will be studied. It would be also interesting to study on surgeries along positive Legendrian torus knots. In the future work, the following listed open problems are planned to be studied.

Open Problem 1: Classify negative Legendrian torus knots in universally tight lens spaces.

Open Problem 2: Study surgeries along positive Legendrian torus knots.

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