SOME FAMILIES OF COMBINATORIAL MATRICES AND THEIR ALGEBRAIC PROPERTIES

KOMBİNATORYAL MATRİSLERİN BAZI AİLELERİ VE ONLARIN CEBİRSEL ÖZELLİKLERİ

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Talha ARIKAN

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21/09/2018

TALHA ARIKAN

ABSTRACT

SOME FAMILIES OF COMBINATORIAL MATRICES AND THEIR ALGEBRAIC PROPERTIES

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Doctor of Philosophy, Department of Mathematics

Supervisor: Prof. Dr. Adnan TERCAN

Co-Supervisor: Prof. Dr. Emrah KILIÇ

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In this thesis, we will study some properties of certain families of combinatorial matrices. While some of the families will be examined throughout this thesis are new and firstly investigated, the others are the generalizations of some of the previously known matrices. We gather our studies into six different groups. They are non-symmetric band matrices with Gaussian q-binomial entries, generalization of the super Catalan matrix, families of Max and Min matrices, a non-symmetric variant of the Filbert matrix, a nonlinear generalization of the Filbert matrix and some certain Hessenberg matrices. For all matrices will be studied except the Hessenberg matrices, we present explicit formulæ for the LU-decompositions, determinants, inverse and LU-decompositions of the inverses of the matrices as well as the Cholesky decompositions when the matrix is symmetric. Additionally, we evaluate some certain Hessenberg determinants via generating function method. We use some new and existing methods to prove our claims. Particularly, we present a new method to evaluate determinants of some Hessenberg matrices whose entries consist of terms of higher order linear recursive sequences.

Keywords: LU and Cholesky decomposition, inverse matrix, determinant, Fibonacci numbers, Gaussian q-binomial and Fibonomial coefficients, Zeilberger's algorithm, Filbert matrix, Hessenberg matrices.

ÖZET

KOMBİNATORYAL MATRİSLERİN BAZI AİLELERİ VE ONLARIN CEBİRSEL ÖZELLİKLERİ

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Bu tez çalışmasında belirli kombinatoryal matris ailelerinin bazı özelliklerini çalışacağız. Tez boyunca incelenecek matris ailelerinin bazıları yeni ve ilk olarak araştırılacak olup diğerleri ise daha önceden bilinen bazı matrislerin genellemeleri olacaktır. Çalışmalarımızı, girdileri Gauss q-katsayıları olan simetrik olmayan bant matrisleri, süper Catalan matrisinin genellemesi, Max ve Min matris aileleri, Filbert matrisinin simetrik olmayan bir varyantı, Filbert matrisinin lineer olmayan bir genellemesi ve bazı Hessenberg matrisleri olmak üzere altı farklı grupta bir araya getirdik. Hessenberg matrisleri dışındaki çalışacağımız tüm matrislerin, LU-ayrışımları, determinantları, tersleri ve terslerinin LU-ayrışımları için net formüller vereceğiz. Bunun yanında matrislerin simetrik olmaları durumunda Cholesky ayrışımları için de ilgili formülleri sunacağız. Ayrıca bazı belirli Hessenberg determinantlarını üreteç fonksiyonları yardımıyla hesaplayacağız. Bu iddialarımızı ispatlamak için bazı yeni ve daha önceden kullanılan metotları kullanacağız. Özel olarak, girdileri yüksek mertebeden lineer indirgeme dizilerinin terimlerinden oluşan bazı Hessenberg matrislerinin determinantlarını hesaplamak için yeni bir yöntem vereceğiz.

Anahtar Kelimeler: LU ve Cholesky ayrışımı, ters matris, determinant, Fibonacci sayıları, Gauss q-binom ve Fibonomial katsayıları, Zeilberger'in algoritması, Filbert matrisi, Hessenberg matrisleri.

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CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

Symbols

F_n	nth Fibonacci number
L_n	nth Lucas number
P_n	nth Pell number
$U_n(p,q)$	nth generalized Fibonacci number
$V_n(p,q)$	nth generalized Lucas number
U_n	$U_n(p,1)$
V_n	$V_n(p,1)$
Δ	p^2+4
W_n	nth Horadam number
T_n	nth Tribonacci number
$\mathscr{H}_n$	nth harmonic number
$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{C}$	complex numbers
$\binom{n}{k}$	binomial coefficient
$\begin{bmatrix} n \\ k \end{bmatrix}_q$	Gaussian q-binomial coefficient
$\begin{Bmatrix} n \\ k \end{Bmatrix}_U$	generalized Fibonomial coefficient
$\begin{Bmatrix} n \\ k \end{Bmatrix}_F$	Fibonomial coefficient
$[n]_q$	q-integer
$(x;q)_n$	q-Pochhammer symbol
$\mathbf{i}$	$\sqrt{-1}$
$(a)_k$	usual Pochhammer symbol (rising factorial $= a^{\overline{k}}$)
$a^{\underline{k}}$	falling factorial
$[P]$	Iverson notation (1 if P is true, 0 otherwise.)
M_N	square matrix M of order N
$[M_{kj}]$	infinite matrix M
$[M_{kj}]_{0\le k,j\le N-1}$	square matrix M of order N whose indexes start at 0
$[M_{kj}]_{1\le k,j\le N}$	square matrix M of order N whose indexes start at 1

$D(a_n)$	diagonal matrix with a_k in kth row
$\|M_N\|_\infty$	infinity-norm of the matrix M
$\|M_N\|_1$	1-norm of the matrix M
$\lfloor \quad \rfloor$	floor function
$[x^n]f(x)$	coefficient of x^n in $f(x)$
$\det M$, $\|M\|$	determinant of the matrix M
per$M$	permanent of the matrix M
M^{-1}	inverse of the matrix M
M^T	transpose of the matrix M

Abbreviations

RHS	right hand side
LHS	left hand side

LIST OF TABLES

1 INTRODUCTION

A matrix is a rectangular array of any algebraic objects for which addition and multiplication are defined. Matrices are the main subject of the fundamental mathematical branch linear algebra. Historically, it was not the matrix but a certain number associated with a square array of numbers called the determinant that was first recognized. The term **matrix** was coined by James Joseph Sylvester in 1850. Then Arthur Cayley developed algebraic properties of matrices. He firstly applied them to the study of systems of linear equations. So matrices are mostly used as a way to describe systems of linear equations as well as to represent data in a tabular view.

Matrix arises in several branches of science, as well as different mathematical disciplines. For example, they are frequently used in physics, computer graphics, probability theory and statistics. Moreover, in some social sciences like economics, the scientists often use matrices. Thus, manipulating matrices has drawn interest. A major branch of numerical analysis is devoted to the development of efficient algorithms for the computations of some properties of matrices. So certain matrices with known properties are important to check accuracy of newly developed algorithms.

In literature, there are various special matrices. Band, Toeplitz, Pascal, Lehmer, Hilbert, Filbert and Hessenberg matrices are some examples of these special matrices and the main interest of this thesis. Combinatorial matrices are the matrices whose entries consist of some combinatorial numbers. Pascal and Filbert matrices are the common examples of the combinatorial matrices. Band, Toeplitz and Hesssenberg matrices are mainly used in numerical analysis. Thus several authors have been studied certain properties of them. Pascal, Lehmer, Hilbert and Filbert matrices have nice algebraic properties so they are useful to test accuracy of algorithms. Some authors studied various generalizations and variants of these kinds of matrices.

The aim of this thesis is to present some new combinatorial matrix families, which have not been studied before, and some generalizations of already known combinatorial matrices and their properties. We study some certain properties such as LU-decompositions, determinants, inverses etc. of these families of combinatorial matrices. We hope that the matrices we have studied in this thesis could be also used as test matrices. We will give some new methods and use existing methods to prove our claims.

Especially, we present a new computational method to evaluate certain Hessenberg determinants. We derive explicit formulæ related with the properties of these matrices and our main tool for finding them is to guess relevant quantities. This was done by experiments with a computer algebra system and spotting patterns. This becomes increasingly complicated when more new parameters are introduced. We have frequently used computer algebra systems Mathematica and Maple for our calculations.

In Section 2, we provide some essential information, which will be used throughout the thesis. In Section 3.1, we present some basic notions about matrices and introduce some special kinds of matrix families. Besides, in Section 3.2, we present previous studies related to these kinds of matrix families. This main section is divided into parts and each part includes a special family. Moreover, at the end of each this part, we indicate our motivation and brief introduction to our problem related to that matrix family.

Section 4 is devoted to the results which obtained in this thesis. In Section 4.1, we present some auxiliary results, which we use to prove some of our main results. On the other hand, these auxiliary results could be also applied to other matrices which are not considered in this thesis. In Sections 4.2-4.7, we provide our main results. Each section includes the results for a special family of combinatorial matrices.

In Section 4.2, we present a class of non-symmetric Toeplitz band matrices with upper bandwidth s and lower bandwidth r whose entries are defined via the Gaussian q-binomial coefficients. We provide explicit formulæ for the LU-decomposition, determinant and LU-decomposition of the inverse matrix. The case $r = s$ is the generalization of the results given in [1]. Furthermore, we derive some complementary results related to matrix which includes usual binomial coefficients. Our results are presented in [2].

In Section 4.3, we obtain the generalizations with two additional parameters of the results in [3]. We also present the idea how one can obtain similar generalizations by the help of already known results. In [4], we publish our studies.

In Section 4.4, we define two new families of the matrices, which are called Max and Min matrices, whose entries run in left-reversed and up-reversed L-shaped pattern, respectively. Our results also cover the results given in [5]. In [5] the authors used a method based on another auxiliary matrix family. But we use elementary linear algebra tools to derive our results which are simpler and more convenient. As an application,

we obtain a sequential generalization of the Lehmer matrix and its reciprocal analogue.

In Section 4.5, we define a new non-symmetric matrix via q-integers. Non-symmetric variants of the Filbert and Lilbert matrices come out as corollaries for the special choices of the parameters. We derive explicit formulæ for the LU-decompositions, inverse matrices L^{-1} and U^{-1} and inverses for whole matrices.

Some authors have studied many generalizations and variant of the Filbert matrix as we do in Section 4.5. But so far no one has studied a generalization or variant where the indexes of the recursive sequence are in nonlinear form. In Section 4.6, we introduce a new nonlinear generalization of the Filbert matrix with indexes in geometric progression for some parameters as well as a nonlinear generalization of the Lilbert matrix. As in Section 4.5, we present the LU-decompositions, inverse matrices L^{-1} and U^{-1} and inverses for the nonlinear generalizations of the Filbert and Lilbert matrices as well as we provide the Cholesky decompositions when the matrices are symmetric. We present our results in [6].

Finally, in Section 4.7, we present the generating function method to evaluate the determinant of new three classes of the Hessenberg matrices. This method was introduced in [7]. We extend it and obtain some further results. We also provide many special examples to see how the method works. By the help of our results, many determinantal formula which have been found in the previous studies, can be easily retrieved. As an application, we give a new and an elegant method to compute the determinants of the Hessenberg matrices whose entries consist of the terms of the higher order linear recursive sequences, which based on to find an adjacency-factor matrix. Our results are published in [8].

2 PRELIMINARIES

In this section, we will present some fundamental notions, which will be used throughout the thesis.

2.1 Linear Recursive Sequences

A recursive sequence is defined by a rule which gives the next term as a function of the previous terms. This rule is called the **recurrence relation** of the corresponding sequence. If we denote the nth term of a given sequence by u_n, such a recurrence relation is of the form

$$u_n = f(u_{n-1}, u_{n-2}, \ldots, u_{n-k}),$$

where f is a function with k inputs.

Definition 2.1. *For any reals p_i such that $i \in \{1, 2, ..., k\}$ and $p_k \neq 0$, the kth order linear recursive sequence $\{u_n\}$ with constant coefficients is defined by the rule for $n \geq k$,*

$$u_n = p_1 u_{n-1} + p_2 u_{n-2} + \cdots + p_k u_{n-k} \tag{2.1}$$

with arbitrary initial values u_t for $0 \leq t < k$ and assumed that at least one of them is different from zero.

It is obviously seen that the terms of the kth order linear recursive sequence $\{u_n\}$ defined by the rule (2.1) are uniquely determined by the coefficients p_i's and its initial values. We give the most common special cases of the sequence $\{u_n\}$ with Table 1. These number sequences have been studied by many authors. We refer to [9, 10, 11, 12] for more details about them.

Now we shall give two important definitions about linear recursive sequences.

Definition 2.2. *The* **characteristic polynomial** *of the sequence $\{u_n\}$ defined by (2.1) is the polynomial*

$$f(x) = x^k - p_1 x^{k-1} - p_2 x^{k-2} - \cdots - p_k$$

and the equation

$$x^k - p_1 x^{k-1} - p_2 x^{k-2} - \cdots - p_k = 0$$

is called **characteristic equation** *of the sequence $\{u_n\}$.*

Coefficients	Initials	Sequence Name
$p_1 = p_2 = 1$	$u_0 = 0,\ u_1 = 1$	Fibonacci Seq. $\{F_n\}$
$p_1 = \ p_2 = 1$	$u_0 = 2,\ u_1 = 1$	Lucas Seq. $\{L_n\}$
$p_1 = 2,\ p_2 = 1$	$u_0 = 0,\ u_1 = 1$	Pell Seq. $\{P_n\}$
$p_1 = 1,\ p_2 = 2$	$u_0 = 0,\ u_1 = 1$	Jacobsthal Seq. $\{J_n\}$
$p_1 = p,\ p_2 = q$	$u_0 = 0,\ u_1 = 1$	Gen. Fibonacci Seq. $\{U_n(p,q)\}$
$p_1 = p,\ p_2 = q$	$u_0 = 2,\ u_1 = p$	Gen. Lucas Seq. $\{V_n(p,q)\}$
$p_1 = p,\ p_2 = -q$	$u_0 = a,\ u_1 = b$	Horadam Seq. $\{W_n\}$
$p_1 = 2,\ p_2 = -1$	$u_0 = 0,\ u_1 = 1$	Natural Numbers $\mathbb{N}$
$p_1 = p_2 = p_3 = 1$	$u_0 = 0,\ u_1 = u_2 = 1$	Tribonacci Seq. $\{T_n\}$
$p_1 = 0, p_2 = p_3 = 1$	$u_0 = 3, u_1 = 0, u_2 = 2$	Perrin Seq. $\{\mathcal{P}_n\}$

Table 1: Some particular linear recursive sequence examples

It is possible to derive any term of the sequence $\{u_n\}$ by the help of its recurrence relation. Nevertheless, it is not useful to compute higher terms. So one needs a closed formula such that

$$u_n = g(n) \tag{2.2}$$

to compute any desired term of the sequence $\{u_n\}$ with less computation.

Definition 2.3. *Such an explicit formula given by (2.2) is called* **Binet formula** *of the linear recursive sequence* $\{u_n\}$.

The following theorem provides us how to find the Binet formula of the linear recursive sequence $\{u_n\}$.

Theorem 2.1. *Let the characteristic polynomial of the sequence* $\{u_n\}$ *factor over the complex number as*

$$f(x) = (x - r_1)^{m_1}(x - r_2)^{m_2} \dots (x - r_d)^{m_d},$$

where $r_1, r_2, \dots, r_d$ *are distinct nonzero complex numbers and* $m_1, m_2, \dots, m_d$ *are positive integers such that* $m_1 + m_2 + \cdots + m_d = k$. *Then there exist polynomials* $g_1(n), g_2(n), \dots, g_d(n)$ *with* $\deg g_i \leq m_i - 1$ *for all* $i \in \{1, 2, \dots, d\}$ *such that*

$$u_n = g_1(n)r_1^n + g_2(n)r_2^n + \cdots + g_d(n)r_d^n,$$

for $n \geq 0$.

One can easily find the proof of the above theorem in any textbook about difference equations. As a special case, when $m_1 = m_2 = \cdots = m_k = 1$, i.e. the characteristic polynomial has no multiple root, the Binet formula of the sequence $\{u_n\}$ takes the form

$$u_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n,$$

where for all $i \in \{1, 2, \ldots, k\}$, c_i's are constants determined according to the initial values of the sequence $\{u_n\}$.

Now we will apply the above theorem to the generalized Fibonacci sequence $\{U_n(p,q)\}$, which is defined by the recurrence relation for $n > 2$,

$$U_n(p,q) = pU_{n-1}(p,q) + qU_{n-2}(p,q),$$

in order to find its Binet formula for the case $p^2+4q \neq 0$. The characteristic polynomial of it is

$$f(x) = x^2 - px - q = (x-\alpha)(x-\beta),$$

where $\alpha, \beta = \left(p \mp \sqrt{p^2+4q}\right)/2$. So the Binet formula is of the form

$$U_n(p,q) = c_1\alpha^n + c_2\beta^n.$$

Since $U_0(p,q) = 0$ and $U_1(p,q) = 1$, finally the Binet formula of the sequence $\{U_n(p,q)\}$ is found as

$$U_n(p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Similarly, the Binet formula of the generalized Lucas sequence $\{V_n(p,q)\}$ is

$$V_n(p,q) = \alpha^n + \beta^n.$$

Throughout the thesis, we will frequently study the generalized Fibonacci sequence $\{U_n(p,1)\}$ and Lucas sequence $\{V_n(p,1)\}$ and briefly denote them by $\{U_n\}$ and $\{V_n\}$, respectively, unless otherwise specified. They satisfy the recurrence relations for $n > 2$,

$$U_n = pU_{n-1} + U_{n-2},$$
$$V_n = pV_{n-1} + V_{n-2},$$

with initial values $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively. Especially, their Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n, \tag{2.3}$$

where $\alpha, \beta = \left(p \mp \sqrt{\Delta}\right)/2$ and $\Delta = p^2 + 4$, respectively.

We need the following known identity including the generalized Fibonacci numbers for later use.

Proposition 2.1. *For integers n, m and k, the following equation holds.*

$$U_m U_n = (-1)^{n+k} U_{m-n+k} U_k + U_{m+k} U_{n-k}. \tag{2.4}$$

The proof can be immediately done by the Binet formula. For the numerous identities and properties, we refer to comprehensive books [11, 12].

2.2 Binomial Coefficients

The binomial coefficients occur in almost all areas of mathematics. The binomial coefficients get their name from the **binomial theorem**, which describes the expansion of the powers of a binomial $(x + y)$. Also, the binomial coefficients have a combinatorial interpretation, which counts the number of the ways of choosing k objects among n objects without replacement, and are denoted by the symbol $\binom{n}{k}$. From the combinatorial meaning of them, it is easily seen that they are defined by the ratio for $0 < k \leq n$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

with $\binom{n}{0} = 1$, which represents the empty choice. The integers n and k are called the **upper index** and the **lower index**, respectively. The indexes are restricted to be non-negative integers by the combinatorial interpretation. Since the binomial coefficients have many usages besides its combinatorial interpretation, it is useful to define them for any real upper index. So for any real r and integer k, they are formally defined by

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\dots(r-k+1)}{k!} & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

The two variable sequence $\left\{\binom{n}{k}\right\}_{n,k\geq 0}$ is also a recursive sequence which satisfies the recurrence relation for $1 \leq k \leq n-1$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

with $\binom{n}{0} = \binom{n}{n} = 1$. This relation is easily verified by the definition of the binomial coefficients. By the help of this relation, one can prove that each binomial coefficient

with integer indexes is an integer. More explanation and information can be found in the book "Concrete Mathematics" written by Graham, Knuth and Patashnik [13].

Recall the well-known Vandermonde's identity for later use.

Proposition 2.2. *For nonnegative integers k, m and n,*

$$\binom{m+n}{k} = \sum_{d=0}^{k} \binom{m}{d}\binom{n}{k-d}. \tag{2.5}$$

We will recall three different proofs for the Vandermonde's identity in the forthcoming subsections. For many variants of it, we refer to [13, p. 169].

By replacing each integer appearing in the numerator and denominator of $\binom{n}{k}$ with its respective generalized Fibonacci number, i.e. replace i by U_i, we can define a recursive analogue of the binomial coefficients, which is called generalized Fibonomial coefficients introduced by Jarden and Motzkin [14]. Formally, we have the following definition.

Definition 2.4. *For integers such that $0 < k < n$, the **generalized Fibonomial coefficients** are defined by*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U := \frac{U_n U_{n-1} \dots U_1}{(U_1 U_2 \dots U_k)\,(U_1 U_2 \dots U_{n-k})}$$

with $\begin{Bmatrix} n \\ n \end{Bmatrix}_U = \begin{Bmatrix} n \\ 0 \end{Bmatrix}_U = 1$ and 0 otherwise.

When the case $p = 1$, i.e. $U_n = F_n$, the generalized Fibonomial coefficients $\begin{Bmatrix} n \\ k \end{Bmatrix}_U$ are reduced to the usual Fibonomial coefficients $\begin{Bmatrix} n \\ k \end{Bmatrix}_F$. We refer to [15, 16, 17, 18] for more details about the generalized and usual Fibonomial coefficients.

The generalized Fibonomial coefficients satisfy the following recurrence relation for $1 \le k \le n-1$,

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U = U_{k+1}\begin{Bmatrix} n-1 \\ k \end{Bmatrix}_U + U_{n-k-1}\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_U$$

with $\begin{Bmatrix} n \\ n \end{Bmatrix}_U = \begin{Bmatrix} n \\ 0 \end{Bmatrix}_U = 1$. This relation follows by the equation $U_n = U_{k+1}U_{n-k} + U_k U_{n-k-1}$ (This can be shown by taking $m = 1$ in (2.4) and using the fact that $U_{-n} = (-1)^{n+1}U_n$). Surprisingly, as in the binomial coefficients, the generalized Fibonomial coefficients are always integers. From the recurrence relation, it is easy to see this fact by induction. On the other hand, this is not always true for the generalized Fibonomial coefficients defined by the sequence $\{U_n(p,q)\}$.

2.3 q-World

In classical q-analysis, the q-analogue of a nonnegative integer, ***q*-integer**, is defined by

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k.$$

From the above definition, it is easily seen that

$$\lim_{q\to 1}[n]_q = n.$$

For nonnegative integer n, the ***q*-Pochhammer symbol** is defined as

$$(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$$

with $(x;q)_0 = 1$. So one can derive the q-analogue of $n!$ as follows

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n}.$$

Now we can move on the q-analogue of the binomial coefficient.

Definition 2.5. *For nonnegative integers such that $n \geq k$, the* ***Gaussian q-binomial coefficients*** *are defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

and 0 *otherwise.*

Obviously, we have

$$\lim_{q\to 1}\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

The Gaussian q-binomial coefficients satisfy the following two equivalent recurrence relations for $1 \leq k \leq n$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$. Inductively, by the above relations, it is seen that every Gaussian q-binomial coefficient is a polynomial in q.

It is possible to extend Definition 2.5 for any real upper index by considering

$$\begin{bmatrix} r \\ k \end{bmatrix}_q = \frac{(q^{1+r-k};q)_k}{(q;q)_k},$$

where r is an arbitrary real and $k \geq 0$ is an integer.

The Binet formulæ given by (2.3) can be rewritten as

$$U_n = \alpha^{n-1}\frac{1-q^n}{1-q} \tag{2.6}$$

and

$$V_n = \alpha^n(1+q^n) \tag{2.7}$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}q^{-1/2}$, where $\mathbf{i} = \sqrt{-1}$. The RHS of the equations (2.6) and (2.7) are the q-forms of the generalized Fibonacci and Lucas numbers, respectively.

Thus, by the help of the q-forms, the link between the generalized Fibonomial and Gaussian q-binomial coefficients is found as

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U = \alpha^{k(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}_q = \mathbf{i}^{k(n-k)}q^{\frac{1}{2}k(k-n)}\begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{2.8}$$

As it is seen from the above relationship, if we have an identity including generalized Fibonacci numbers then we can convert it into the q-form or vice versa. Since there are many useful identities and tools in q-analysis, studying the q-form of an identity is more advantageous than studying the original form. In this thesis, we will often use this idea. In other words, we will prove some q-identities for general parameter q then the identities including the generalized Fibonacci numbers or Fibonomial coefficients would come out as corollaries for special value of q.

Now we shall give some known identities. The following theorems are the one version of the **Cauchy binomial theorem** and **Rothe's formula**.

Theorem 2.2. *For $n \geq 0$,*

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}}x^k = (-xq;q)_n = \prod_{k=1}^{n}(1+xq^k). \tag{2.9}$$

Theorem 2.3. *For $n \geq 0$,*

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}}x^k = (x;q)_n = \prod_{k=0}^{n-1}(1-xq^k). \tag{2.10}$$

For the proofs of the above theorems and more identities, we refer to [19]. Furthermore, one can look at [20, 21] for more information about the q-analysis.

Now we present the q-analogue of the Vandermonde's identity (2.5).

Theorem 2.4 (q-Vandermonde identity)**.** *For nonnegative integers k, m and n,*

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_q = \sum_{d=0}^{k} \begin{bmatrix} m \\ k-d \end{bmatrix}_q \begin{bmatrix} n \\ d \end{bmatrix}_q q^{d(m-k+d)}. \tag{2.11}$$

A proof of this identity can be found in [22]. We will provide a computer-based proof of it later. Now suppose that we verify this identity. Then if we let $q \to 1$ in the equation (2.11), we will get the Vandermonde's identity (2.5). This approach gives us another advantage of studying q-identities. In other words, if we have a q-identity and the limit $q \to 1$ is applicable to it, then we achieve another useful identity, as well. Thus studying in q-world enables us to obtain more general results.

2.4 Generating Functions

Generating functions are one of the most useful inventions in mathematics. The generating function is a way of encoding a sequence $\{a_n\}_{n\geq 0}$ by treating them as the coefficients of a power series. Briefly, generating functions transform the problems about the sequences into the problems about power series or functions. In this manner, manipulating infinite sequences gets easier. Wilf's book "Generatingfunctionology" [23] is totally devoted to the generating functions. We refer to it in order to get more information about the generating functions.

For a given sequence $\{a_n\}_{n\geq 0}$, the **generating function** of $\{a_n\}_{n\geq 0}$ is the power series

$$A(x) := \sum_{n\geq 0} a_n x^n. \tag{2.12}$$

It is sometimes called **ordinary generating function** to distinguish from other types of generating functions (see [13, 22, 23]).

In general, the power series (2.12) is considered as a formal power series, i.e. an algebraic object. Thus we are not worried about the convergence. The power series (2.12) may also be considered as an analytic function on the interval of convergence of

it. This treatment allows us to get some asymptotic information about the terms of the sequence (see [24]).

Now, we shall compute the generating function of the Fibonacci sequence as an illustration. Let $F(x) = \sum_{n\geq 0} F_n x^n$. So we have

$$\begin{aligned}
F(x) = \;& F_0 + F_1 x + F_2 x^2 + \cdots + F_n x^n + \cdots \\
-xF(x) = \;& - F_0 x - F_1 x^2 - \cdots - F_{n-1} x^n - \cdots \\
-x^2 F(x) = \;& - F_0 x^2 - \cdots - F_{n-2} x^n - \cdots .
\end{aligned}$$

After adding these three equations, we obtain

$$\begin{aligned}
F(x)(1 - x - x^2) = F_0 + (F_1 - F_0)x + (F_2 - F_1 - F_0)x^2 + \cdots \\
+ (F_n - F_{n-1} - F_{n-2})x^n + \cdots .
\end{aligned}$$

Since for $n \geq 2$, $F_n - F_{n-1} - F_{n-2} = 0$ by its recurrence relation, the generating function of the Fibonacci numbers is

$$F(x) = \frac{x}{1 - x - x^2}.$$

Similarly, the generating function of the Lucas numbers is

$$L(x) = \frac{2 - x}{1 - x - x^2}.$$

By the above approach, one can easily compute the generating function of the kth order linear recursive sequence $\{u_n\}$, defined by (2.1), as

$$\sum_{n\geq 0} u_n x^n = \frac{p(x)}{1 - p_1 x - p_2 x^2 - \cdots - p_k x^k},$$

where $p(x)$ is a polynomial, which will be determined according to the initial values of the sequence $\{u_n\}$ such that $\deg p(x) < k$.

Let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$. Then for any complex constants c_1 and c_2, the following properties hold.

$$c_1 A(x) + c_2 B(x) = \sum_{n\geq 0} (c_1 a_n + c_2 b_n) x^n,$$

$$A(x)B(x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n. \tag{2.13}$$

The proofs are straightforward. Both of the above equations can be inductively generalized by considering arbitrary finite number of generating functions. We can select just even powers or odd powers out of the power series $A(x)$ as follows

$$\sum_{n\geq 0} a_{2n}x^{2n} = \frac{A(x)+A(-x)}{2} \text{ or } \sum_{n\geq 0} a_{2n+1}x^{2n+1} = \frac{A(x)-A(-x)}{2}, \tag{2.14}$$

respectively. For more properties and some special generating functions, we refer to [23].

We would like to recall the following useful theorem.

Theorem 2.5. *Let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$ be analytic complex functions in a non-empty open neighborhood D of zero. If $A(x) = B(x)$ for all $x \in D$, then $a_n = b_n$ for all $n \in \mathbb{Z}$.*

The proof of this theorem can be found in [25]. In [23], the author gave a method called "Snake Oil", which is based on this theorem.

Now we will present another proof of the Vandermonde's identity (2.5) by the help of Theorem 2.5. By the binomial theorem, we know that

$$\sum_{k\geq 0} \binom{m+n}{k} x^k = (1+x)^{m+n}$$

and by the identity (2.13), we write

$$\sum_{k\geq 0} \left(\sum_{d=0}^{k} \binom{m}{d} \binom{n}{k-d} \right) x^k = (1+x)^m (1+x)^n.$$

Since the both sides of (2.5) have the same generating function $(1+x)^{m+n}$ for all $x \in \mathbb{C}$, they must be equal for all nonnegative integers k, m and n by Theorem 2.5. We will also use this argument in Section 4.7.

2.5 Hypergeometric Series

Hypergeometric series appear in many areas of mathematics such as combinatorics, analysis, applied mathematics etc. The history of hypergeometric series was launched many years ago by Euler, Gauss, and Riemann. Although the topic is very old, it is still the subject of a lot of ongoing research. There are many books devoted only to

the hypergeometric series. Since we encountered some hypergeometric sums, which are truncated hypergeometric series, in this thesis, we would like to give very brief introductory information about them.

Definition 2.6. *The series*

$$\sum_{k\geq 0} t_k$$

is called ***hypergeometric series*** *if* $t_0 = 1$ *and the ratio* t_{k+1}/t_k *is a rational function of* k*, i.e.*

$$\frac{t_{k+1}}{t_k} = \frac{p(k)}{h(k)}, \tag{2.15}$$

where $p(k)$ *and* $h(k)$ *are polynomials of* k*. In this case,* t_k *is called* ***hypergeometric term****. The functions generated by hypergeometric terms are called* ***hypergeometric functions*** *and truncated hypergeometric series are called* ***hypergeometric sums****.*

For example, since

$$\frac{\binom{m}{d+1}\binom{n}{k-d-1}}{\binom{m}{d}\binom{n}{k-d}} = \frac{(m-d)(k-d)}{(d+1)(n-k+d+1)},$$

the sum on the RHS of the Vandermonde's identity (2.5) is a hypergeometric sum.

Consider the hypergeometric function

$$\sum_{k\geq 0} t_k z^k. \tag{2.16}$$

If the polynomials in (2.15) are completely factored, then we write

$$\frac{t_{k+1}}{t_k} = \frac{p(k)}{h(k)} = \frac{(k+a_1)(k+a_2)\dots(k+a_n)}{(k+b_1)(k+b_2)\dots(k+b_m)(k+1)}, \tag{2.17}$$

where the factor $(k+1)$ in the denominator is presented for some historical reasons of notation. If -1 is not a root of the polynomial $h(k)$, then we can multiply both the numerator and denominator with the factor $(k+1)$ for the convenience. Then the hypergeometric function (2.16) is notationally shown as

$$\sum_{k\geq 0} t_k z^k = {}_nF_m\left[\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_m \end{matrix}; z\right] = \sum_{k\geq 0} \frac{(a_1)_k (a_2)_k \dots (a_n)_k}{(b_1)_k (b_2)_k \dots (b_m)_k} \frac{z^k}{k!},$$

where $(a)_k$ is the **usual Pochhammer symbol** defined by $(a)_k = (a)(a+1)\dots(a+k-1)$, also known as rising factorial.

In general, the first term t_0 doesn't have to be 1. If it is not equal to 1 but different from zero and t_k satisfies the equation (2.17), then the hypergeometric function (2.16) is represented as follows

$$\sum_{k\geq 0} t_k z^k = t_0 \times {}_nF_m \left[\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_m \end{matrix} ; z \right].$$

For more extensive knowledge about hypergeometric series, we refer to [26, 27]. Nowadays, hypergeometric series are also well understood from an algorithmic point of view. There are some algorithms [28] to deal with the hypergeometric series, sums or functions and also their implementations to computer algebra systems, such as Maple and Mathematica. The most efficient and modern of them is celebrated Zeilberger's algorithm. We will mention about it after a while.

Inherently, the most natural question is "What is the q-analogue of the hypergeometric series?".

Definition 2.7. *The series*

$$\sum_{k\geq 0} t_k(q)$$

is called ***q-hypergeometric series*** *if the ratio t_{k+1}/t_k is a rational function of q^k, i.e.*

$$\frac{t_{k+1}(q)}{t_k(q)} = \frac{p(x)}{h(x)},$$

where $p(x)$ and $h(x)$ are polynomials of x and $x = q^k$. In this case, $t_k(q)$ is called ***q-hypergeometric term****. The functions generated by q-hypergeometric terms are called* ***q-hypergeometric functions*** *and truncated q-hypergeometric series are called* ***q-hypergeometric sums****.*

As an example, since

$$\frac{\begin{bmatrix} m \\ d+1 \end{bmatrix}_q \begin{bmatrix} n \\ k-d-1 \end{bmatrix}_q}{\begin{bmatrix} m \\ d \end{bmatrix}_q \begin{bmatrix} n \\ k-d \end{bmatrix}_q} = q^{m-k+1} \frac{(q^d - q^k)(q^d - q^n)}{(1-q^d q^{m-k+1})(1-qq^d)},$$

the sum on the RHS of the q-Vandermonde's identity (2.11) is a q-hypergeometric sum.

In order not to get out of the subject of our thesis, we don't want to give more details. For more details, see [19, 27, 29].

Naturally, there is q-analogue of the Zeilberger's algorithm, which deals with the q-hypergeometric series, sums and functions.

2.5.1 Zeilberger's Algorithm

Since the people have encountered hypergeometric sums and series in many areas of mathematics, some mechanical methods to deal with them have been derived. The first mechanical method for hypergeometric sums was discovered by Sister Mary Celine Fasenmyer [30] in 1945. Such methods serve to compute them directly or prove equalities [28].

One of the most popular mechanical methods is Gosper's algorithm [31], which computes some indefinite hypergeometric sums in terms of another hypergeometric term. This method based on rewriting the hypergeometric term into telescoping form. Unfortunately, it can find closed forms for only a few classes of the hypergeometric sums we meet in practice. Namely, it is applicable to the limited numbers of the hypergeometric sums.

In 1991, Doron Zeilberger [32, 33, 34] showed how to extend Gosper's algorithm so that it becomes even more effective, making it succeed in vastly more cases. With Zeilberger's extension, we can also handle hypergeometric series, not just sums. For very brief and understandable introduction, we refer to [13]. In a few words, it produces a polynomial recurrence for the hypergeometric series or sums. In the following years, some authors also did some improvements on Zeilberger's algorithm.

Moreover, as with Gosper's original method or other algorithms, for Zeilberger's algortihm, the calculations can be done by the help of computers. Maple packages of the Zeilberger's algorithm have been written by Zeilberger [35] and Koornwinder [36]. Paule and Schorn [37] implemented Zeilberger's algorithm for the Mathematica system. Thus one can easily manage these types of sums or series by using the computer algebra systems without much time-consuming.

Now by using the Mathematica implementation of Zeilberger's algorithm, we will prove the Vandermonde's identity (2.5) (We refer to [37] for the guide of Mathematica package). Let's denote the RHS of (2.5) by S_k, then the algorithm produces the following recurrence relation

$$S_{k+1} = \frac{m+n-k}{k+1} S_k.$$

So after solving this relation by going backward, we obtain

$$S_{k+1} = \frac{(m+n-k)(m+n-k+1)\dots(m+n)}{(k+1)!} S_0.$$

Since $S_0 = 1$, the proof of (2.5) follows. For many other examples and the Mathematica package, see [38].

2.5.2 q-Zeilberger Algorithm

Since the q-hypergeometric series have lots of applications on many areas of mathematics, such as combinatorics, partition theory etc., people need similar mechanical methods for the q-hypergeometric series or sums as an extension of Zeilberger's algorithm.

Zeilberger also observed that his algorithm can be carried over to the q-hypergeometric cases. He and Wilf [39, 40] extended his algorithm for the q-hypergeometric series and sums and wrote Maple package for this algorithm. Furhermore, Koornwinder [36] wrote another Maple package for this algorithm. Afterwards, Paule and Riese [41] developed the Mathematica implementation of the q-Zeilberger algorithm. We refer to [41] for the user guide of this package and some applied examples.

The q-Zeilberger algorithm is the q-analogue of Zeilberger's algorithm. In other words, the q-Zeilberger algorithm performs some computations for q-hypergeometric series and sums as same as Zeilberger's algorithm does for the hypergeometric series and sums.

As we seen before, the sum on the RHS of the q-Vandermonde identity (2.11) is a q-hypergeometric sum. Now we shall apply this algorithm to this sum as an illustration by using Mathematica implementation. Denote the RHS of (2.11) by S_k. Then q-Zeilberger algorithm gives the recurrence relation

$$S_k = \frac{1-q^{m+n-k+1}}{1-q^k} S_{k-1}.$$

By solving this relation and the fact that $S_0 = 1$, we obtain

$$S_k = \frac{(q^{m+n-k+1};q)_k}{(q;q)_k} = \begin{bmatrix} m+n \\ k \end{bmatrix}_q,$$

which is the LHS of (2.11). Thus we provide a computer based proof for the q-Vandermonde identity. We will use this algorithm to prove some of our results throughout the thesis.

It is worthwhile to mention that we encountered some cases in which although the summand term is q-hypergeometric, the q-Zeilberger algorithm does not work in general. This is interesting weakness of the q-Zeilberger algorithm. For this reason, we used different approaches to prove those identities.

Lastly, for the interested readers, we refer to [38] for the Mathematica packages of various symbolic computation methods and their user friendly guides.

3 LITERATURE REVIEW

In this section, firstly, we will present some basic definitions and notions about matrix theory and special kinds of matrix families. Afterwards, we will provide some historical background about these kinds of special matrix families.

3.1 About Matrices

Not only in every area of mathematics but also in fundamental sciences and engineering, somehow matrices occur and are used. Thus they have an important duty and have been studied for years.

As mentioned in Introduction, we will study some special combinatorial matrices in this thesis. Before giving previous works in the literature related to our thesis, we will recall some fundamental notions and the definitions of some special matrices, which we will study.

In general, we will obtain some algebraic properties of some combinatorial matrices such as LU-decomposition, Cholesky decomposition, inverse, determinant etc. Firstly, we would like to mention about these basic concepts and explain why these are important.

If the entries of a matrix are some combinatorial numbers such as Fibonacci numbers, binomial coefficients etc., then we call **combinatorial matrix**. Also in the literature, there is combinatorial matrix theory which investigates the combinatorial meanings of the matrices [42]. So the both differ from each other.

The **inverse** of a square matrix A is a matrix B if the equation

$$A \cdot B = B \cdot A = I,$$

holds, where I is the identity matrix, whose (i,j)th entry is $[i=j]$. Here $[\cdot]$ denotes the Iverson notation which means:

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{otherwise,} \end{cases}$$

where P is a statement that can be true or false. The inverse of the matrix A is denoted by A^{-1}. The inverse matrices are frequently used especially in solving linear system of equation and obtaining inverse transformations. There are some methods to

compute the inverse of a given matrix. It is worth mentioning that it is getting harder to identify the inverse matrix for the higher order matrix. So the matrices, whose inverses are explicitly known, are important.

It is often useful to summarize a multivariate phenomenen with a singular number. For the matrices, **determinant** is an example of this and it is denoted by $\det A$ or $|A|$ for a given square matrix A. Another example of this is **permanent** and it is denoted by per$A$. For more details and explanations about determinant and permanent, we refer to [43].

There are various different methods to evaluate the determinant of a matrix. Also in the literature, there are many determinant formulæ of some special matrices. Krattenthaler's surveys [44, 45] are elegant sources to find some of determinant evaluation methods and get the idea where we need determinants. Also they are comprehensive databases for some known determinant formulæ.

For a square matrix A, ***LU*-decomposition** refers to the factorization of A into two factors, a lower unit triangular matrix L and an upper triangular matrix U such that

$$A = L \cdot U.$$

The LU-decomposition can be considered as the matrix form of Gaussian elimination. Computer algebra systems usually use it to solve square systems of linear equations. Also it helps to find the inverse matrix and compute the determinant of the matrix. For example, since $\det L = 1$ and U is a triangular matrix, we can easily compute the determinant of A by the formula $\det A = \det U$. On the other hand, if we know the inverse matrices L^{-1} and U^{-1}, then we may find a formula for the inverse matrix A^{-1} by the fact $A^{-1} = U^{-1}L^{-1}$. We will also use these advantages of the LU-decomposition to evaluate the determinant and inverse of a matrix.

There may not exist the LU-decomposition of any square matrix A. The matrix should satisfy some conditions to have LU-decomposition. However, there is an alternative decomposition by the help of the permutation matrix, which is a square matrix that has exactly one entry of 1 in each row and each column and 0's elsewhere. We refer to [46] for the necessary and sufficient conditions for the existence of the LU-decomposition and this alternative decomposition.

The **Cholesky decomposition** of a symmetric positive-definite matrix A, that is a matrix such that for every non-zero column vector x, $x^TAx > 0$, is a factorization of the form

$$A = C \bullet C^T, \tag{3.1}$$

where C is a lower triangular matrix and C^T is the transpose of the matrix C. The Cholesky decomposition is unique for a symmetric positive-definite matrix. It is possible to extend this definition for complex valued matrix by considering Hermitian matrix and conjugate transpose instead of symmetric matrix and transpose, respectively. In this thesis, we are always interested in real valued matrix. Note that when the matrix is not positive-definite but symmetric matrix, we still use the phrase "Cholesky decomposition" only to point out the relation (3.1). Cholesky decomposition has similar advantages with the LU-decomposition, but it is more efficient than the LU-decomposition. For example, the Cholesky decomposition is nearly twice as efficient as the LU-decomposition for solving systems of linear equations. Thus LU-decomposition and Cholesky decomposition help to simplify computations, both theoretically and practically.

For more details, explanations, examples and advantages about the notions given just above, we refer to [43, 47].

Nowadays, computer is one of the best friends of the scientists. Since the notions, mentioned just above, are many advantages in matrix theory, one desire to compute them easily and correctly. In the literature, there are many different methods and algorithms to evaluate them. Thus people need some special matrices, whose certain algebraic properties are explicitly known, to apply these methods and algorithms to see the accuracy and efficiency. These types of matrices are known as **test matrices**. Briefly, test matrices are key to test the accuracy of an algorithm or a method. In this thesis, we will provide many explicit formulæ for some algebraic properties of various special combinatorial matrices. We hope that some matrices, we studied, will be used as test matrices.

Before mentioning about some special matrix families, we would like to give some notations and remarks, we will regularly use from now on.

Remarks:

1) In general, the size of the matrices does not really matter, so that we may think about an infinite matrix M and restrict it whenever necessary to the first N rows, respectively, columns and use the notation M_N.

2) We denote the (k,j)th entry of a given matrix M and its inverse M^{-1} by M_{kj} and M^{-1}_{kj}, respectively. If the size of the matrix M is N, then we denote its (k,j)th entry by $(M_N)_{kj}$. Furthermore, $[M_{kj}]$, $[M_{kj}]_{0\leq k,j\leq N-1}$ and $[M_{kj}]_{1\leq k,j\leq N}$ mean an infinite matrix M, a matrix M of size N whose indexes start at 0 and a matrix M of size N whose indexes start at 1, respectively. Unless otherwise specified, we assumed that the indexes start at 1.

3) We use the letters L, U and A, B for the LU-decompositions of a given matrix and its inverse, respectively. Also the letter C is used for the Cholesky decomposition. For the related matrix to given matrix M, we may frequently use calligraphic letter $\mathcal{M}$. In that cases, we apply the same representation to the factor matrices coming from LU-decomposition and Cholesky decomposition.

4) Since we will study many matrices, distinguishing them from each other is difficult. For this reason, the letter, which identifies the matrix, is only valid in the related subsection. In other words, we may use same letter for different matrices in different subsections.

5) We denote a sequence whose first term starts at the index 1 by $\{a_n\}$. Moreover, $\{a_n\}_{n\geq 0}$ stands for the sequence whose first term is a_0.

Now, we will introduce some special matrix families, which we will encounter throughout the thesis.

3.1.1 Special Matrices

Diagonal Matrix

A **diagonal matrix** is a square matrix in which the entries outside of the main diagonal are all zero. The matrix $D(a_n) = [D_{kj}]$ stands for a diagonal matrix constructed via a given sequence $\{a_n\}$, and is defined by

$$D_{kj} = \begin{cases} a_k & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Toeplitz Matrix

A **Toeplitz matrix** is a square matrix in which the entries on each descending diagonal from left to right are constant. It satisfies that $M_{kj} = M_{k+1,j+1}$ for all $k, j \geq 1$. In general, by considering this relation, the definition could be extended for the nonsquare matrices. As an example, any square Toeplitz matrix of size N is of the form:

$$M_N = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(N-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{N-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix}.$$

Band Matrix

A **band matrix** is a matrix whose nonzero entries are confined between an upper and a lower diagonal bands, comprising the main diagonal and zero outside. Formally, if M is a band matrix than there are nonnegative integers r and s such that

$$M_{kj} = 0 \text{ if } j < k - r \text{ or } j > k + s.$$

The quantities r and s are called the **lower bandwidth** and **upper bandwidth**, respectively. Moreover, the **bandwidth** of this band matrix is equal to $r + s + 1$. As an example, when $r = 2, s = 3$ and $N = 6$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{bmatrix},$$

where a_{ij}'s are arbitrary nonzero reals.

Diagonal matrices, upper and lower triangular matrices are most known examples of band matrices. When $r = s = 1$, then the corresponding band matrices are called the **tridiagonal matrices**.

Furthermore, kth diagonal band which is above (rsp. below) the main diagonal is called ***k*th superdiagonal** (rsp. ***k*th subdiagonal)**.

Another important class of band matrices is the family of Toeplitz band matrices, which are both Toeplitz and band matrices.

Hankel Matrix

A **Hankel matrix** is a square matrix in which each ascending skew-diagonal from left to right is constant. This could be considered as an upside down Toeplitz matrix. For a given sequence $\{a_n\}_{n\geq 0}$, Hankel matrix is of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Formally, any Hankel matrix M is defined for a given sequence $\{a_n\}_{n\geq 0}$, as follows for $k, j \geq 0$,

$$M_{kj} = M_{jk} = a_{k+j-2}.$$

Hankel matrices have very important applications, especially in operator theory. For more details about them, see [48].

Considering some particular number sequences instead of $\{a_n\}_{n\geq 0}$, there are many special matrices with nice algebraic properties. Some authors also studied the Hankel matrix by considering the reciprocal sequence of $\{a_n\}_{n\geq 0}$ of the form

$$\begin{bmatrix} \frac{1}{a_0} & \frac{1}{a_1} & \frac{1}{a_2} & \cdots \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} & \cdots \\ \frac{1}{a_2} & \frac{1}{a_2} & \frac{1}{a_4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Some of the known examples of the Hankel matrices are Hilbert and Filbert matrices, which we will discuss in Section 3.2.4.

Hessenberg Matrix

An **upper Hessenberg matrix** has zero entries below the first subdiagonal, and a

lower Hessenberg matrix has zero entries above the first superdiagonal. The lower Hessenberg matrix H_N is of the form

$$H_N = \begin{bmatrix} h_{11} & h_{12} & & & & 0 \\ h_{21} & h_{22} & h_{23} & & & \\ h_{31} & h_{32} & h_{33} & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ h_{N-1,1} & h_{N-1,2} & h_{N-1,3} & \cdots & \ddots & h_{N-1,N} \\ h_{N1} & h_{N2} & h_{N3} & \cdots & \cdots & h_{NN} \end{bmatrix}. \tag{3.2}$$

Similarly, the upper Hessenberg matrix of size N could be considered as transpose of the matrix H_N. In this thesis, we will study the lower Hessenberg matrices. Nevertheless, one can easily adapt our results to the upper Hessenberg matrices, as well. A triangular matrix is both lower and upper Hessenberg matrix. Moreover, a Hessenberg matrix is a band matrix whose one of the upper or lower bandwidth is 1.

Lehmer Matrix

The **Lehmer matrix** $[M_{kj}]$ (see [49]) is the symmetric matrix defined by

$$M_{kj} = \begin{cases} \frac{k}{j} & \text{if } j \geq k, \\ \frac{j}{k} & \text{if } j < k. \end{cases}$$

Equivalently, this may be written as

$$M_{kj} = \frac{\min(k,j)}{\max(k,j)}.$$

3.2 Previous Studies

In this subsection, we present some previous studies related to our results obtained in this thesis. We divide this subsection into different parts and each part includes some previous results about different matrix families. At the end of each part, we provide our motivations and what we will do in the following Results section. In other words, we briefly indicate our problems.

3.2.1 Band Matrices

Band matrices and their special cases such as Toeplitz band matrices, symmetric Toeplitz band matrices, especially tridiagonal matrices have been extensively studied

by many authors [50, 1, 51, 52, 53, 54, 55]. These matrices arise in many areas of mathematics and its applications. Especially, the band matrices have many applications in numerical analysis. The matrices from finite element or finite difference problems are often banded. Tridiagonal matrices are used in telecommunication system analysis, for solving linear recurrence systems with non-constant coefficients, etc. For these reasons, the band matrices with known algebraic properties are important.

In 1972, for $r \geq 0$ and $1 \leq k, j \leq N$, Hoskins and Ponzo [1] defined the $N \times N$ symmetric Toeplitz band matrix $M_N = [M_{kj}]$ of bandwidth $2r+1$ via the binomial coefficients as

$$M_{kj} = (-1)^{r+k-j} \binom{2r}{r+k-j}.$$

For example, when $r = 3$ and $N = 7$, M_7 is of the form

$$M_7 = \begin{bmatrix} -20 & 15 & -6 & 1 & 0 & 0 & 0 \\ 15 & -20 & 15 & -6 & 1 & 0 & 0 \\ -6 & 15 & -20 & 15 & -6 & 1 & 0 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ 0 & 1 & -6 & 15 & -20 & 15 & -6 \\ 0 & 0 & 1 & -6 & 15 & -20 & 15 \\ 0 & 0 & 0 & 1 & -6 & 15 & -20 \end{bmatrix}.$$

The authors gave formulæ for the determinant, the inverse matrix and the LU-decomposition of the matrix M_N. For example, they gave

$$\det M_N = (-1)^{N+r-1} \prod_{d=1}^{N} \binom{2r+d-1}{r} \binom{d+r-1}{r}^{-1}$$

and

$$(M_N^{-1})_{kj} = (-1)^r \binom{k+r-1}{r} \binom{j+r-1}{r} \times \sum_{d=1}^{N} \binom{d+r-1-k}{r-1} \binom{d+r-1-j}{r} \binom{d+r-1}{r}^{-1} \binom{d+2r-1}{r}^{-1}.$$

The authors only considered the symmetric band matrix M_N with upper bandwidth r and lower bandwidth r.

It is worthwhile to note that a non-symmetric band matrix with upper bandwidth s and lower bandwidth r via the binomial coefficients has not been considered and studied

up to now. In Section 4.2, we will consider non-symmetric Toeplitz band matrix via the Gaussian q-binomial coefficients defined by for $k, j \geq 0$,

$$H_{kj} = (-1)^{r(k+j)+j}\mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)}q^{\frac{1}{2}(k-j)(k-j-r+s)-\frac{1}{2}rs}\begin{bmatrix} r+s \\ r+j-k \end{bmatrix}_q. \quad (3.3)$$

We will obtain some algebraic properties of the matrix H. Also we will provide some further results derived from this matrix. Our results do not only generalize the results of Hoskins and Ponzo but also include new families of the band matrices.

3.2.2 Pascal Matrices

The Pascal matrices are defined via the binomial coefficients [56, 57]. They are mainly three kinds: the first is the left adjusted Pascal matrix P_N, the second is the right adjusted Pascal matrix Q_N and the third is the symmetric Pascal matrix S_N. They are defined by for $0 \leq k, j < N$,

$$P_{kj} = \binom{k}{j}, \; Q_{kj} = \binom{k}{N-1-j} \text{ and } S_{kj} = \binom{k+j}{k},$$

respectively. In [58], the author studied the reciprocal of the symmetric Pascal matrix $\left[\binom{k+j}{k}^{-1}\right]_{k,j\geq 0}$ and its some parametric generalizations.

Recently, Prodinger [3] defined a matrix whose entries consist of the super Catalan numbers $\left\{\frac{(2i)!(2j)!}{i!j!(i+j)!}\right\}_{i,j\geq 0}$. He also defined its reciprocal analogue as well as their q-analogues whose (k, j)th entries are defined by

$$\binom{2k}{k}\binom{2j}{j}\binom{k+j}{k}^{-1} \text{ and } \binom{2k}{k}^{-1}\binom{2j}{j}^{-1}\binom{k+j}{k},$$

and

$$\begin{bmatrix} 2k \\ k \end{bmatrix}_q \begin{bmatrix} 2j \\ j \end{bmatrix}_q \begin{bmatrix} k+j \\ k \end{bmatrix}_q^{-1} \text{ and } \begin{bmatrix} 2k \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} k+j \\ k \end{bmatrix}_q,$$

respectively. Then he gave some algebraic properties of these matrices.

In Section 4.3, we will study parametric generalizations of the just above matrices, introducing two additional parameters. We also mention how one can obtain further generalizations of these types of matrices.

3.2.3 Max and Min Matrices

In the literature, for some special sequences $\{a_n\}$, some authors studied the matrices $[\max(a_k, a_j)]_{1\leq k,j\leq N}$ and $[\min(a_k, a_j)]_{1\leq k,j\leq N}$. We listed them below:

- The author of [59] studied the matrix

$$[\max(N+1-k, N+1-j)]_{1\leq k,j\leq N},$$

which is called the Franc matrix.

- The author of [60] gave the Cholesky decomposition of the matrix

$$\left[\frac{1}{\max(k+1, j+1)}\right]_{k,j\geq 0},$$

which is called the loyal companion of the Hilbert matrix.

- In [61], the author found eigenvalues and eigenvectors of the matrices

$$[\min(k,j)]_{1\leq k,j\leq N} \quad \text{and} \quad [\min(2k-1, 2j-1)]_{1\leq k,j\leq N}.$$

- Fonseca [62] studied the general cases of the matrices considered in [61] by defining the matrix $[\min(ak-b, aj-b)]_{1\leq k,j\leq N}$ for $a>0$ and $a\neq b$. Then he computed eigenvalues and eigenvectors of this general matrix by computing its inverse.

Recently, Mattila and Haukkanen [5] studied more general matrix families. Let $T = \{a_1, a_2, \ldots, a_N\}$ be a finite multiset of real numbers, such that $a_1 \leq a_2 \leq \cdots \leq a_N$. They considered the matrices $[\max(a_k, a_j)]_{1\leq k,j\leq N}$ and $[\min(a_k, a_j)]_{1\leq k,j\leq N}$ defined on the set T. Clearly, they may be written explicitly as

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_2 & a_2 & a_3 & \cdots & a_N \\ a_3 & a_3 & a_3 & \cdots & a_N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_N & a_N & a_N & \cdots & a_N \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_1 & a_2 & a_2 & \cdots & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_N \end{bmatrix},$$

respectively. They computed the determinants, inverses, Cholesky decompositions of these matrices and examined the positive definiteness of them. They used the meet and join matrices, see [63], as a tool to obtain their results. Moreover, they indicated that it is difficult to verify their results by using only basic linear algebra methods.

In Section 4.4, we will introduce more general families by considering the matrices $\left[a_{\max(k,j)}\right]$ and $\left[a_{\min(k,j)}\right]$, whose entries run in left-reversed and up-reversed L-shaped pattern, respectively, as well as their reciprocal analogues for an arbitrary sequence

$\{a_n\}$. These matrices cover the all previous studied matrices and also generalize the results in [5]. We use basic linear algebra methods to prove our results, which also yield new and alternative proofs for the results in [5]. We will also present some interesting and useful applications of our results.

3.2.4 Hilbert, Filbert Matrices and Their Variants

The Hilbert matrix $H = [H_{kj}]$ is defined with entries

$$H_{kj} = \frac{1}{k+j-1}.$$

As a recursive analogue of the Hilbert matrix, Richardson [64] defined and studied the Filbert matrix $R = [R_{kj}]$ with entries

$$R_{kj} = \frac{1}{F_{k+j-1}},$$

where F_n is the nth Fibonacci number. Clearly, both the Hilbert and Filbert matrices are the examples of the Hankel matrices.

After the Filbert matrix, several generalizations and variants of it have been investigated and studied by some authors. We briefly summarize some of these:

- In [65], Kılıç and Prodinger studied a generalization of the Filbert matrix by defining the matrix $\left[\frac{1}{F_{k+j+r}}\right]$, where $r \geq -1$ is an arbitrary integer parameter.
- After this, Prodinger [66] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters by taking its (k,j)th entry as $\frac{x^k y^j}{F_{\lambda(k+j)+r}}$, where $r \geq -1$ and $\lambda \geq 1$ are arbitrary integers and x, y are any reals.
- In another study [67], Kılıç and Prodinger obtained two variants of the generalized Filbert matrix by considering the matrices $\left[\frac{F_{\lambda(k+j)+r}}{F_{\lambda(k+j)+s}}\right]$ and $\left[\frac{L_{\lambda(k+j)+r}}{L_{\lambda(k+j)+s}}\right]$, where s, r and λ are integer parameters such that $s \neq r$, and $s \geq -1$ and $\lambda \geq 1$. The second matrix is the first instance, where the entries of the matrix include the Lucas numbers.
- Kılıç and Prodinger [68] gave a further generalization of the Filbert matrix by defining the matrix Q with entries

$$Q_{kj} = \frac{1}{F_{k+j+r}F_{k+j+r+1}\dots F_{k+j+r+d-1}},$$

where $r \geq -1$ and $d \geq 1$ are arbitrary integers. The generalized Filbert matrix is the particular case, when $d = 1$, of the matrix Q.

- In another paper [69], Kılıç and Prodinger introduced the matrix G as a parametric generalization of the matrix Q by

$$G_{kj} = \frac{1}{F_{\lambda(k+j)+r} F_{\lambda(k+j+1)+r} \dots F_{\lambda(k+j+d-1)+r}},$$

where $r \geq -1$, $d \geq 1$ and $\lambda \geq 1$ are integer parameters.

- Kılıç and Prodinger [70] gave new four variants of the Filbert matrix, by defining the matrices P, T, Y and Z with entries

$$P_{kj} = \frac{1}{F_{\lambda k+\mu j+r}},\ T_{kj} = \frac{F_{\lambda k+\mu j+r}}{F_{\lambda k+\mu j+s}},\ Y_{kj} = \frac{1}{L_{\lambda k+\mu j+r}} \text{ and } Z_{kj} = \frac{L_{\lambda k+\mu j+r}}{L_{\lambda k+\mu j+s}},$$

respectively, where s, r, λ and μ are integer parameters such that $s \neq r$, $r, s \geq -1$ and $\lambda, \mu \geq 1$. When $\lambda = \mu = 1$, the matrix Y is also known as generalized Lilbert matrix, which is the Lucas analogue of the generalized Filbert matrix.

- More recently, as the Lucas analogue of the matrix G, Kılıç and Prodinger [71] defined the matrix W by

$$W_{kj} = \frac{1}{L_{\lambda(k+j)+r} L_{\lambda(k+j+1)+r} \dots L_{\lambda(k+j+d-1)+r}},$$

where λ and r are arbitrary integers and d is a positive integer.

The authors of the all-above mentioned works have studied various properties of the given matrices such as LU and Cholesky decompositions, determinants, inverses, etc. All these results yield some further combinatorial identities, as well. In many of them, firstly the authors converted the entries of the matrices into q-forms and obtained related results for these q-forms. Afterwards, they proved all their claims in the q-forms by the means of the celebrated q-Zeilberger algorithm for the general parameter q. But only in [67, 70], q-Zeilberger algorithm did not work and because of that they used some traditional methods. We will encounter the same situation in Section 4.5.

In Section 4.5, we will introduce a new non-symmetric variant of the Filbert matrix defined by the entries for $k, j \geq 0$,

$$\frac{U_{\lambda k-\mu j+d}}{U_{\lambda k+\mu j+d}}$$

with positive integers λ, μ and d, where U_n stands for the nth generalized Fibonacci number. Note that the interesting feature of this matrix is that it includes some zero terms as entries. Specially, when $\lambda = \mu = 1$, then the entries on the dth superdiagonal are all zero. Furthermore, it would be never a symmetric matrix for any choice of the parameters. For this reason, we will also obtain related results for the transposed matrix.

If we look closely, the indexes of the Fibonacci or Lucas numbers in the Filbert or Lilbert matrix and all its generalizations or variants, studied before, are in the linear forms. Any nonlinear forms of the indexes have not been studied anywhere, yet. In Section 4.6, we will present a new generalization of the Filbert matrix whose indexes will be in the nonlinear form. This will be the first example in the literature. In brief, we will study the matrix as a nonlinear generalization of the Filbert matrix defined with the entries

$$\frac{1}{U_{\lambda(k+r)^n+\mu(j+s)^m+c}},$$

where λ, μ, n and m are positive integers, r, s and c are any integers such that $\lambda(k+r)^n+\mu(j+s)^m+c > 0$ for all positive integers k and j. Moreover, we will present its Lucas analogue.

3.2.5 Hessenberg Matrices

Hessenberg matrices were firstly investigated by Karl Hessenberg (1904-1959), a German engineer.

They are one of the most important matrices in numerical analysis [47, 72]. For example, the Hessenberg decomposition played an important role in computation of the matrix eigenvalues [47]. So in applied mathematics, they have important role.

In [73, 74], authors introduced a constructive way to compute the inverse of the finite and infinite Hessenberg matrices, respectively.

Note that we indicated that we would use N for the order of the matrices. However, in the sections about Hessenberg, matrices we prefer to use n rather than N. Because we will consider the value of the determinant of the matrices as the sequences indexed with their order.

Cahill et al. [75] gave a recurrence relation for the determinant of the matrix defined by (3.2) as follows for $n > 0$,

$$\det H_n = h_{nn} \det H_{n-1} + \sum_{r=1}^{n-1} \left((-1)^{n-r} h_{nr} \prod_{j=r}^{n-1} h_{j,j+1} \det H_{r-1} \right),$$

where $H_0 = 1$. Unfortunately, this result is not useful for the higher order Hessenberg matrices.

In [76, 77, 78, 79, 80], the authors gave the relationships between some certain recursive sequences and the determinants or permanents of some certain Hessenberg matrices. Meanwhile, some authors computed the determinants and permanents of various type of tridiagonal matrices which are indeed Hessenberg matrices [81, 82, 83, 84]. For example, in [83], Kılıç provided the formula

$$\begin{vmatrix} 2 & 1 & & & 0 \\ -1 & 2 & 1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & -1 & 2 \end{vmatrix} = P_{n+1},$$

where P_n is the nth Pell number given in Table 1. The authors of the all works mentioned above used the cofactor expansion of the determinant as their main tool and then evaluated the determinants recursively.

The authors of [85] gave an algorithm to compute determinant of the Hessenberg matrices.

Moreover, the authors of [86, 87] evaluated the determinants of some special families of the Hessenberg matrices by using the combinatorial approaches.

Recently, Macfarlane [88] considered the Hessenberg matrix whose entries consist of

the terms of the sequence $\{W_n\}$:

$$M_n = \begin{bmatrix} W_1 & W_2 & W_3 & \cdots & W_{n-2} & W_{n-1} & W_n \\ -x & W_1 & W_2 & \cdots & W_{n-3} & W_{n-2} & W_{n-1} \\ & -x & W_1 & \cdots & W_{n-4} & W_{n-3} & W_{n-2} \\ & & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & \ddots & W_1 & W_2 & W_3 \\ & & & & -x & W_1 & W_2 \\ 0 & & & & & -x & W_1 \end{bmatrix},$$

where $\{W_n\}$ is the Horadam sequence given in Table 1. Again by using the cofactor expansion of the determinant, he showed that the sequence $\{\det M_n\}$ satisfies the recurrence relation for $n > 2$,

$$\det M_n = (b + px) \det M_{n-1} - qx\,(a + x) \det M_{n-2}.$$

More recently, by using generating functions, Merca [89] showed that the determinant of an $n \times n$ Toeplitz-Hessenberg matrix is expressed as a sum over the integer partitions of n.

In the literature, Getu [7] firstly computed the determinants of a class of the Hessenberg matrices by using the generating functions. He considered the infinite matrix

$$R = \begin{bmatrix} b_0 & 1 & 0 & 0 & \dots \\ b_1 & c_1 & 1 & 0 & \dots \\ b_2 & c_2 & c_1 & 1 & \dots \\ b_3 & c_3 & c_2 & c_1 & \dots \\ b_4 & c_4 & c_3 & c_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then he showed that if the equation

$$A(x) = \frac{B(x)}{C(x) + 1}$$

holds then

$$a_n = (-1)^n \det R_n,$$

where $A(x)$, $B(x)$ and $C(x)$ are the generating functions of the sequences $\{a_{n+1}\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_{n+1}\}_{n\geq 0}$, respectively.

As it is seen, the determinants of the Hessenberg matrices have drawn attentions of many researchers. In Section 4.7, we will use the generating functions to determine the relationships between the determinants of the new three classes of the Hessenberg matrices whose entries are terms of the certain number sequences and the generating functions of these sequences. This method is more efficient and applicable than cofactor expansion and the determinants of many previously studied Hessenberg matrices are easily computed by this method. Furthermore, we will give an elegant method to evaluate the determinants of the Hessenberg matrices whose entries consist of the terms of the higher order linear recursive sequences.

4 RESULTS

In this section, we present the results which are obtained in our thesis. Firstly, in the following subsection we give some auxiliary results for further use. The next subsections are devoted to our main results. In each these subsections, we present some results for the different combinatorial matrix families.

4.1 Auxiliary Results

The following propositions are the general results about some matrix families. We use them as tools to prove some of our results. They may apply to other matrices which are not considered throughout the thesis, as well. All of them are new and useful results about some special matrix families.

Firstly, we shall start with a proposition about Toeplitz matrices.

Proposition 4.1. *If M is a Toeplitz matrix of order N, then there exist the following relationships between the factor matrices coming from the LU-decompositions of the matrices M and M^{-1}, for $0 \leq k, j \leq N-1$,*

(i) $A_{kj} = L^{-1}_{N-1-j,N-1-k}$,

(ii) $A^{-1}_{kj} = L_{N-1-j,N-1-k}$,

(iii) $B_{kj} = U^{-1}_{N-1-j,N-1-k}$,

(iv) $B^{-1}_{kj} = U_{N-1-j,N-1-k}$,

(v) $M^{-1}_{kj} = M^{-1}_{N-1-j,N-1-k}$.

Proof. For the claims (i) and (ii), consider

$$\sum_{d=j}^{k} A_{kd}A^{-1}_{dj} = \sum_{d=j}^{k} L^{-1}_{N-1-d,N-1-k}L_{N-1-j,N-1-d}$$
$$= \sum_{d=N-1-k}^{N-1-j} L^{-1}_{d,N-1-k}L_{N-1-j,d} = [N-1-j, N-1-k] = [k=j],$$

which gives us $AA^{-1} = I$, as claimed. For the claims (iii) and (iv), we have

$$\sum_{d=k}^{j} B_{kd}B^{-1}_{dj} = \sum_{d=k}^{j} U^{-1}_{N-1-d,N-1-k}U_{N-1-j,N-1-d}$$

$$= \sum_{d=N-1-j}^{N-1-k} U^{-1}_{d,N-1-k} U_{N-1-j,d} = [N-1-j, N-1-k] = [k=j],$$

as desired.

For the LU-decomposition of M^{-1}, we should show that $M^{-1} = A \bullet B$ or equivalently $M = B^{-1} \bullet A^{-1}$. So it is sufficient to show that

$$\sum_{\max(k,j) \le d \le N-1} B^{-1}_{kd} A^{-1}_{dj} = M_{kj}.$$

Thus consider

$$\begin{aligned}
\sum_{\max(k,j) \le d \le N-1} B^{-1}_{kd} A^{-1}_{dj} &= \sum_{\max(k,j) \le d \le N-1} U_{N-1-d,N-1-k} L_{N-1-j,N-1-d} \\
&= \sum_{0 \le d \le N-1-\max(k,j)} L_{N-1-j,d} U_{d,N-1-k} \\
&= \sum_{0 \le d \le \min(N-1-j,N-1-k)} L_{N-1-j,d} U_{d,N-1-k}.
\end{aligned}$$

Since $M = L \bullet U$ and M is a Toeplitz matrix, we have $\sum_{0 \le d \le \min(k,j)} L_{kd} U_{dj} = M_{kj}$ and $M_{kj} = M_{N-1-j,N-1-k}$. Finally, we obtain

$$\sum_{\max(k,j) \le d \le N-1} B^{-1}_{kd} A^{-1}_{dj} = M_{N-1-j,N-1-k} = M_{kj},$$

which completes the proof. By the fact $M^{-1} = A \bullet B = U^{-1} \bullet L^{-1}$ and the relationships (i) and (iii), we have

$$\begin{aligned}
M^{-1}_{kj} &= \sum_{0 \le d \le \min(k,j)} L^{-1}_{N-1-d,N-1-k} U^{-1}_{N-1-j,N-1-d} \\
&= \sum_{N-1-\min(k,j) \le d \le N-1} L^{-1}_{d,N-1-k} U^{-1}_{N-1-j,d} \\
&= \sum_{\max(N-1-j,N-1-k) \le d \le N-1} L^{-1}_{d,N-1-k} U^{-1}_{N-1-j,d} = M^{-1}_{N-1-j,N-1-k}.
\end{aligned}$$

So the claim (v) follows. □

By the above proposition, one can easily derive the LU-decomposition of the inverse of a Toeplitz matrix from its LU-decomposition.

The next proposition is about the matrices whose entries include separable factors with respect to the indexes k and j.

Let $H = [H_{kj}]$ be a square matrix, whose LU-decomposition, inverse, LU-decomposition of its inverse and Cholesky decomposition are known with the matrices $L = [L_{kj}]$, $U = [U_{kj}]$, $H^{-1} = [H^{-1}_{kj}]$, $A = [A_{kj}]$, $B = [B_{kj}]$ and $C = [C_{kj}]$, respectively.

Proposition 4.2. *Assume that $\mathcal{H} = [\mathcal{H}_{kj}]$ is a square matrix and there exist the sequences $\{s_n\}$ and $\{m_n\}$ with nonzero terms such that $\mathcal{H}_{kj} = H_{kj}s_km_j$. Then for the matrix $\mathcal{H}$, one can determine the LU-decomposition, inverse, LU-decomposition of its inverse and Cholesky decomposition as shown*

$$\begin{aligned}
&\mathcal{L}_{kj} = L_{kj}\frac{s_k}{s_j} && \text{and} && \mathcal{U}_{kj} = U_{kj}s_km_j,\\
&\mathcal{L}^{-1}_{kj} = L^{-1}_{kj}\frac{s_k}{s_j} && \text{and} && \mathcal{U}^{-1}_{kj} = U^{-1}_{kj}\frac{1}{s_j}\frac{1}{m_k},\\
&\mathcal{H}^{-1}_{kj} = H^{-1}_{kj}\frac{1}{s_j}\frac{1}{m_k},\\
&\mathcal{A}_{kj} = A_{kj}\frac{m_j}{m_k} && \text{and} && \mathcal{B}_{kj} = B_{kj}\frac{1}{s_j}\frac{1}{m_k},\\
&\mathcal{A}^{-1}_{kj} = A^{-1}_{kj}\frac{m_j}{m_k} && \text{and} && \mathcal{B}^{-1}_{kj} = B^{-1}_{kj}s_km_j
\end{aligned}$$

and when for all $k \geq 1$, $s_k = m_k$,

$$\mathcal{C}_{kj} = C_{kj}s_k.$$

Proof. By the assumption for the matrix $\mathcal{H}$, firstly we can write

$$\mathcal{H} = D(s_n) \bullet H \bullet D(m_n),$$

where $D(a_n)$ is the diagonal matrix defined as before. Since the LU-decomposition of the matrix H is known, namely $H = L \bullet U$, we can write

$$\mathcal{H} = D(s_n) \bullet L \cdot U \bullet D(m_n) = D(s_n) \bullet L \bullet D\left(\frac{1}{s_n}\right) \bullet D(s_n) \bullet U \bullet D(m_n).$$

Here we see that $D(s_n)\bullet L\bullet D\left(\frac{1}{s_n}\right)$ is a unite lower triangular matrix and $D(s_n)\bullet U\bullet D(m_n)$ is an upper triangular matrix. So we have

$$\mathcal{L} = D(s_n) \bullet L \bullet D\left(\frac{1}{s_n}\right) \text{ and } \mathcal{U} = D(s_n) \bullet U \bullet D(m_n),$$

which gives the LU-decomposition of the matrix $\mathcal{H}$. Moreover, by the rule of the inverse of the multiplication of the matrices, we may immediately derive

$$\mathcal{H}^{-1} = D\left(\frac{1}{m_n}\right) \bullet H^{-1} \bullet D\left(\frac{1}{s_n}\right). \tag{4.1}$$

The matrices $\mathcal{A}$ and $\mathcal{B}$ follow after applying the LU-decomposition to the matrix $\mathcal{H}^{-1}$. The relations of the inverse matrices $\mathcal{L}^{-1}, \mathcal{U}^{-1}, \mathcal{A}^{-1}$ and $\mathcal{B}^{-1}$ can be easily derived as in (4.1). For the Cholesky decomposition of $\mathcal{H}$, consider

$$\mathcal{H} = D(s_n) \cdot H \cdot D(s_n) = D(s_n) \cdot C \cdot C^T \cdot D(s_n)^T = (D(s_n) \cdot C)\,(D(s_n) \cdot C)^T,$$

then the claim follows. □

This proposition is very useful to obtain new matrix identities. We will frequently use it in the forthcoming subsections.

Finally, we have the following proposition to derive the LU-decomposition of the transposed matrix.

Proposition 4.3. *Let K be a nonsingular square matrix whose LU-decomposition is known with the matrices $L = [L_{kj}]$, $U = [U_{kj}]$, respectively. Then for the LU-decomposition of the transposed matrix K^T, we have*

$$K^T = L' \cdot U',$$

where

$$L'_{kj} = \frac{U_{jk}}{U_{jj}} \quad \textit{and} \quad U'_{kj} = L_{jk} U_{kk}.$$

Proof. Since K is nonsingular, for all $k \geq 1$, we have $U_{kk} \neq 0$. Then consider

$$K^T = U^T \cdot L^T = U^T \cdot D\Big(\frac{1}{U_{nn}}\Big) \cdot D(U_{nn}) \cdot L^T.$$

Then $L' = U^T \cdot D(\frac{1}{U_{nn}})$ and $U' = D(U_{nn}) \cdot L^T$, which completes the proof. □

We have the following useful corollary.

Corollary 4.1. *Let S be a nonsingular symmetric matrix. Then S can be written as*

$$S = U^T \cdot D\left(\frac{1}{U_{nn}}\right) \cdot U,$$

where U is the factor matrix coming from the LU-decomposition of the matrix S. Furthermore, the Cholesky decomposition of S is derived as

$$C_{kj} = U_{jk} \frac{1}{\sqrt{U_{jj}}}.$$

Proof. Since S is a nonsingular symmetric matrix, by Proposition 4.3, the factor matrix L of S is equal to $U^T \bullet D\left(\frac{1}{U_{nn}}\right)$. So the first claim follows. Consider

$$\begin{aligned} S &= U^T \bullet D\left(\frac{1}{U_{nn}}\right) \bullet U = U^T \bullet D\left(\frac{1}{\sqrt{U_{nn}}}\right) \bullet D\left(\frac{1}{\sqrt{U_{nn}}}\right) \bullet U \\ &= U^T \bullet D\left(\frac{1}{\sqrt{U_{nn}}}\right) \bullet \left(U^T \bullet D\left(\frac{1}{\sqrt{U_{nn}}}\right)\right)^T. \end{aligned}$$

Thus $C = U^T \bullet D\left(\frac{1}{\sqrt{U_{nn}}}\right)$, which completes the proof. □

The above corollary allows us to derive the Cholesky decomposition of a matrix from its LU-decomposition if it is symmetric. Remind that the Cholesky decomposition means that the matrix satisfies the relation (3.1). If the matrix S is a positive definite, i.e. for all $n > 0$, $U_{nn} > 0$, then our result is valid for the general theory and also provides an alternative proof for the fact that the matrices comes from the Cholesky decomposition of a positive definite matrix consist of real entries.

Now we are ready to move to our main results.

4.2 A Family of the Non-Symmetric Band Matrices

As mentioned in Section 3.2.1, in this section we introduce a class of non-symmetric Toeplitz band matrices with upper bandwidth s and lower bandwidth r whose entries are defined via the Gaussian q-binomial coefficients to obtain the generalizations of the results of [1]. The case $s = r$ gives us the q-analogue of the result of [1]. When $s = r + 1$, we have a Toeplitz band matrix with even number of bands, which has not been studied before.

Briefly, we define a matrix H with bandwidth $r + s + 1$ via the Gaussian q-binomial coefficients. We provide explicit formulæ for the LU-decomposition, determinant and LU-decomposition of the inverse matrix H^{-1}. Furthermore, we derive some complementary results for the work [1] related to the case of bandwidth $r+s+1$. We presented obtained results in [2].

Our main tool is usually to guess relevant quantities. Then we use the q-Zeilberger algorithm to prove our claims. All identities we obtain hold for the general quantity q, so that results about the Fibonomial coefficients come out as corollaries for the special

choice of q. Finally, by the help of the limit $q \to 1$, we derive further results including the usual binomial coefficients.

For nonnegative arbitrary integers r and s, we define the matrix $H = [H_{kj}]_{k,j\geq 0}$ with upper bandwidth s and lower bandwidth r by

$$H_{kj} = (-1)^{r(k+j)+j}\mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)}q^{\frac{1}{2}(k-j)(k-j-r+s)-\frac{1}{2}rs}\begin{bmatrix} r+s \\ r+j-k \end{bmatrix}_q. \quad (4.2)$$

For example, when $r = 2$ and $s = 4$, the matrix H is of the form

$$\begin{bmatrix}
-q^{-4}\begin{bmatrix}6\\2\end{bmatrix}_q & -\mathbf{i}q^{-\frac{9}{2}}\begin{bmatrix}6\\3\end{bmatrix}_q & q^{-4}\begin{bmatrix}6\\4\end{bmatrix}_q & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\5\end{bmatrix}_q & -1 & & 0 \\
\mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\1\end{bmatrix}_q & -q^{-4}\begin{bmatrix}6\\2\end{bmatrix}_q & -\mathbf{i}q^{-\frac{9}{2}}\begin{bmatrix}6\\3\end{bmatrix}_q & q^{-4}\begin{bmatrix}6\\4\end{bmatrix}_q & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\5\end{bmatrix}_q & \ddots & \\
1 & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\1\end{bmatrix}_q & -q^{-4}\begin{bmatrix}6\\2\end{bmatrix}_q & \ddots & \ddots & \ddots & -1 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\5\end{bmatrix}_q \\
& & \ddots & \ddots & \ddots & \ddots & q^{-4}\begin{bmatrix}6\\4\end{bmatrix}_q \\
& & & \ddots & \ddots & \ddots & -\mathbf{i}q^{-\frac{9}{2}}\begin{bmatrix}6\\3\end{bmatrix}_q \\
0 & & & & 1 & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\1\end{bmatrix}_q & -q^{-4}\begin{bmatrix}6\\2\end{bmatrix}_q
\end{bmatrix}.$$

When $q = \beta/\alpha$, where $\alpha, \beta = (p \mp \sqrt{p^2+4})/2$, we get the Fibonomial analogue of the matrix H and denote it by $\mathcal{H} = [\mathcal{H}_{kj}]_{k,j\geq 0}$, where

$$\mathcal{H}_{kj} = (-1)^{r(k+j)+j(k+1)}\mathbf{i}^{j(j+1)+k(k+1)+r(r-1)}\begin{Bmatrix} r+s \\ r+j-k \end{Bmatrix}_U.$$

For $r = 2$ and $s = 4$, we have

$$\mathcal{H} = \begin{bmatrix}
-\begin{Bmatrix}6\\2\end{Bmatrix}_U & -\begin{Bmatrix}6\\3\end{Bmatrix}_U & \begin{Bmatrix}6\\4\end{Bmatrix}_U & \begin{Bmatrix}6\\5\end{Bmatrix}_U & -1 & & 0 \\
\begin{Bmatrix}6\\1\end{Bmatrix}_U & -\begin{Bmatrix}6\\2\end{Bmatrix}_U & -\begin{Bmatrix}6\\3\end{Bmatrix}_U & \begin{Bmatrix}6\\4\end{Bmatrix}_U & \begin{Bmatrix}6\\5\end{Bmatrix}_U & \ddots & \\
1 & \begin{Bmatrix}6\\1\end{Bmatrix}_U & -\begin{Bmatrix}6\\2\end{Bmatrix}_U & \ddots & \ddots & \ddots & -1 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \begin{Bmatrix}6\\5\end{Bmatrix}_U \\
& & \ddots & \ddots & \ddots & \ddots & \begin{Bmatrix}6\\4\end{Bmatrix}_U \\
& & & \ddots & \ddots & \ddots & -\begin{Bmatrix}6\\3\end{Bmatrix}_U \\
0 & & & & 1 & \begin{Bmatrix}6\\1\end{Bmatrix}_U & -\begin{Bmatrix}6\\2\end{Bmatrix}_U
\end{bmatrix}. \quad (4.3)$$

Before giving main results, it is worthwhile to note that one may ignore some power terms which are separable with respect to the indexes by the help of Proposition 4.2. We prefer this way because as it is seen in (4.3), the sign pattern from the lowest subdiagonal "$+ + - - + + - - \cdots$" looks nicer.

4.2.1 Main Results

Now we shall start with the matrices L and U as well as their inverses.

Theorem 4.1. *For $k, j \geq 0$,*

$$L_{kj} = (-1)^{r(k+j)+j}\mathbf{i}^{k+j+(r-s)(k-j)}q^{\frac{1}{2}(k-j)(k-j+s-r)}\begin{bmatrix} r \\ k-j \end{bmatrix}_q \begin{bmatrix} k \\ k-j \end{bmatrix}_q \begin{bmatrix} s+k \\ k-j \end{bmatrix}_q^{-1},$$

$$L_{kj}^{-1} = (-1)^{r(k+j)}\mathbf{i}^{(k-j)(r-1-s)}q^{\frac{1}{2}(k-j)(s-r+1)}\begin{bmatrix} k-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} s+j \\ j \end{bmatrix}_q \begin{bmatrix} s+k \\ k \end{bmatrix}_q^{-1},$$

$$\begin{aligned} U_{kj} = (-1)^{r(k+j)+j}&\mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)}q^{\frac{1}{2}(k-j)(k-j-r+s)-\frac{1}{2}rs} \\ &\times \begin{bmatrix} s \\ j-k \end{bmatrix}_q \begin{bmatrix} r+s+k \\ r+j \end{bmatrix}_q \begin{bmatrix} s+k \\ j \end{bmatrix}_q^{-1} \end{aligned}$$

and

$$\begin{aligned} U_{kj}^{-1} = (-1)^{(k+j)(r+1)}&\mathbf{i}^{k-j-r+(k+r-j)(r-s)}q^{\frac{1}{2}(k-j)(s-r-1)+\frac{1}{2}rs} \\ &\times \begin{bmatrix} j-k+s-1 \\ s-1 \end{bmatrix}_q \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} r+s+j \\ r \end{bmatrix}_q^{-1}. \end{aligned}$$

As their Fibonomial analogues, for the matrix $\mathcal{H}$, we have the following corollary.

Corollary 4.2. *For $k, j \geq 0$,*

$$\mathcal{L}_{kj} = (-1)^{r(k+j)+j(k+1)}\mathbf{i}^{k(k+1)+j(j+1)}\begin{Bmatrix} r \\ k-j \end{Bmatrix}_U \begin{Bmatrix} k \\ k-j \end{Bmatrix}_U \begin{Bmatrix} s+k \\ k-j \end{Bmatrix}_U^{-1},$$

$$\mathcal{L}_{kj}^{-1} = (-1)^{r(k+j)}\begin{Bmatrix} k-j+r-1 \\ r-1 \end{Bmatrix}_U \begin{Bmatrix} s+j \\ j \end{Bmatrix}_U \begin{Bmatrix} s+k \\ k \end{Bmatrix}_U^{-1},$$

$$\mathcal{U}_{kj} = (-1)^{r(k+j)+j(k+1)}\mathbf{i}^{j(j+1)+k(k+1)+r(r-1)}\begin{Bmatrix} s \\ j-k \end{Bmatrix}_U \begin{Bmatrix} r+s+k \\ r+j \end{Bmatrix}_U \begin{Bmatrix} s+k \\ j \end{Bmatrix}_U^{-1}$$

and

$$\mathcal{U}_{kj}^{-1} = (-1)^{(j+k)(r+1)}\mathbf{i}^{r(r-1)}\begin{Bmatrix} j-k+s-1 \\ s-1 \end{Bmatrix}_U \begin{Bmatrix} r+k \\ k \end{Bmatrix}_U \begin{Bmatrix} r+s+j \\ r \end{Bmatrix}_U^{-1}.$$

Consequently, we could give the values of the determinants of the matrices H_N and $\mathcal{H}_N$. They are simply evaluated as the products of the main diagonal entries of the upper triangular matrices U and $\mathcal{U}$, respectively.

Theorem 4.2. *For $N \geq 1$, we have*

$$\det H_N = \mathbf{i}^{r(r+s-1)N}q^{-\frac{1}{2}Nrs}\prod_{d=0}^{N-1}\begin{bmatrix} r+s+d \\ d \end{bmatrix}_q \begin{bmatrix} s+d \\ d \end{bmatrix}_q^{-1}.$$

As the Fibonomial analogue, we have

Corollary 4.3. *For $N \geq 1$,*

$$\det \mathcal{H}_N = \mathbf{i}^{r(r-1)N} \prod_{d=0}^{N-1} \begin{Bmatrix} r+s+d \\ r+d \end{Bmatrix}_U \begin{Bmatrix} s+d \\ d \end{Bmatrix}_U^{-1}.$$

Specially, when $r = s = 3$, we have the following nice formula

$$\det \mathcal{H}_N = (-1)^N \frac{U_{N+1}}{U_1} \frac{U_{N+2}^2}{U_2^2} \frac{U_{N+3}^3}{U_3^3} \frac{U_{N+4}^2}{U_4^2} \frac{U_{N+5}}{U_5}.$$

Now recall a result from [1]. For $r = 3$, the determinant of the matrix they studied is equal to

$$\frac{N+1}{1} \frac{(N+2)^2}{2^2} \frac{(N+3)^3}{3^3} \frac{(N+4)^2}{4^2} \frac{N+5}{5}.$$

As it is seen, there is a remarkable similarity.

Moreover, when $r = 5$ and $s = 4$, we have

$$\det \mathcal{H}_N = \frac{U_{N+1}}{U_1} \frac{U_{N+2}^2}{U_2^2} \frac{U_{N+3}^3}{U_3^3} \frac{U_{N+4}^4}{U_4^4} \frac{U_{N+5}^4}{U_5^4} \frac{U_{N+6}^3}{U_6^3} \frac{U_{N+7}^2}{U_7^2} \frac{U_{N+8}}{U_8}.$$

For the inverse matrix H^{-1}, unfortunately there isn't any explicit formula. Nevertheless by the LU-decomposition, we get the following theorem.

Theorem 4.3. *For $0 \leq k, j \leq N-1$,*

$$(H_N^{-1})_{kj} = (-1)^{r(j+k)+k} \mathbf{i}^{k(1+r-s)+j(1-r+s)+r(r-s-1)} q^{\frac{1}{2}((k-j)(s-r)+rs-j-k)} \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} s+j \\ j \end{bmatrix}_q$$
$$\times \sum_{d=0}^{N-1} q^d \begin{bmatrix} d-k+s-1 \\ s-1 \end{bmatrix}_q \begin{bmatrix} d-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} r+s+d \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} s+d \\ d \end{bmatrix}_q^{-1}.$$

Although there is no closed formula for the inverse matrix, we may express it in another way. The following theorem helps us to express the matrix H_N^{-1} by its LU-decomposition and moreover we can explicitly find the inverses of these factor matrices.

Theorem 4.4. *For $0 \leq k, j \leq N-1$,*

$$A_{kj} = (-1)^{r(k+j)} \mathbf{i}^{(k-j)(r-1-s)} q^{\frac{1}{2}(k-j)(s-r+1)}$$
$$\times \begin{bmatrix} k-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} s+N-1-k \\ s \end{bmatrix}_q \begin{bmatrix} s+N-1-j \\ s \end{bmatrix}_q^{-1},$$

$$A_{kj}^{-1} = (-1)^{r(k+j)+j} \mathbf{i}^{k+j+(r-s)(k-j)} q^{\frac{1}{2}(k-j)(k-j+s-r)}$$

$$\times \begin{bmatrix} r \\ k-j \end{bmatrix}_q \begin{bmatrix} N-1-j \\ k-j \end{bmatrix}_q \begin{bmatrix} s+N-1-j \\ k-j \end{bmatrix}_q^{-1},$$

$$B_{kj} = (-1)^{(k+j)(r+1)} \mathbf{i}^{k-j-r+(k+r-j)(r-s)} q^{\frac{1}{2}(k-j)(s-r-1)+\frac{1}{2}rs} \times \begin{bmatrix} j-k+s-1 \\ s-1 \end{bmatrix}_q \begin{bmatrix} r+N-1-j \\ r \end{bmatrix}_q \begin{bmatrix} r+s+N-1-k \\ r \end{bmatrix}_q^{-1}$$

and

$$B_{kj}^{-1} = (-1)^{r(k+j)+j} \mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)} q^{\frac{1}{2}(k-j)(k-j-r+s)-\frac{1}{2}rs} \times \begin{bmatrix} s \\ j-k \end{bmatrix}_q \begin{bmatrix} r+s+N-1-j \\ s+k-j \end{bmatrix}_q \begin{bmatrix} s+N-1-j \\ s+k-j \end{bmatrix}_q^{-1}.$$

As the Fibonomial analogue, we have

Corollary 4.4. *For* $0 \leq k, j \leq N-1$,

$$\mathcal{A}_{kj} = (-1)^{(k+j)r} \begin{Bmatrix} k-j+r-1 \\ r-1 \end{Bmatrix}_U \begin{Bmatrix} s+N-1-k \\ s \end{Bmatrix}_U \begin{Bmatrix} s+N-1-j \\ s \end{Bmatrix}_U^{-1},$$

$$\mathcal{A}_{kj}^{-1} = (-1)^{r(k+j)+j(k+1)} \mathbf{i}^{k(k+1)+j(j+1)} \times \begin{Bmatrix} r \\ k-j \end{Bmatrix}_U \begin{Bmatrix} N-1-j \\ k-j \end{Bmatrix}_U \begin{Bmatrix} s+N-1-j \\ k-j \end{Bmatrix}_U^{-1},$$

$$\mathcal{B}_{kj} = (-1)^{(j+k)(r+1)} \mathbf{i}^{r(r-1)} \begin{Bmatrix} j-k+s-1 \\ s-1 \end{Bmatrix}_U \times \begin{Bmatrix} r+N-1-j \\ r \end{Bmatrix}_U \begin{Bmatrix} r+s+N-1-k \\ r \end{Bmatrix}_U^{-1}$$

and

$$\mathcal{B}_{kj}^{-1} = (-1)^{r(k+j)+j(k+1)} \mathbf{i}^{j(j+1)+k(k+1)+r(r-1)} \times \begin{Bmatrix} s \\ j-k \end{Bmatrix}_U \begin{Bmatrix} r+s+N-1-j \\ s+k-j \end{Bmatrix}_U \begin{Bmatrix} s+N-1-j \\ s+k-j \end{Bmatrix}_U^{-1}.$$

Specially, when the case $p = 1$, i.e. $U_n = F_n$, our results become valid for the usual Fibonomial coefficients.

4.2.2 Proofs

Now we will give the proofs of our main results.

To show $H = L \bullet U$, it is sufficient to prove that the following equation holds.

$$\sum_{0\le d\le \min(k,j)} L_{kd}U_{dj} = H_{kj}.$$

Thus we need to show

$$\begin{aligned}
&\sum_{0\le d\le \min(k,j)} (-1)^{r(k+d)+d}\mathbf{i}^{(d+k)+(r-s)(k-d)} q^{\frac{1}{2}(k-d)(k-d+s-r)} \begin{bmatrix} r \\ k-d \end{bmatrix}_q \begin{bmatrix} k \\ k-d \end{bmatrix}_q \\
&\times \begin{bmatrix} s+k \\ k-d \end{bmatrix}_q^{-1} (-1)^{r(d+j)+j}\mathbf{i}^{d(1+r-s)+j(1-r+s)-r(1-s-r)} \\
&\times q^{\frac{1}{2}(d-j)(d-j-r+s)-\frac{1}{2}rs} \begin{bmatrix} s \\ j-d \end{bmatrix}_q \begin{bmatrix} r+s+d \\ r+j \end{bmatrix}_q \begin{bmatrix} s+d \\ j \end{bmatrix}_q^{-1} \\
&= (-1)^{r(k+j)+j}\mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)} q^{\frac{1}{2}(k-j)(k-j-r+s)-\frac{1}{2}rs} \begin{bmatrix} r+s \\ r+j-k \end{bmatrix}_q.
\end{aligned}$$

After some simplifications, we have the following equation to prove

$$\begin{aligned}
&\sum_{j-s\le d\le k} q^{-d(k+j)+d^2} \begin{bmatrix} r \\ k-d \end{bmatrix}_q \begin{bmatrix} k \\ k-d \end{bmatrix}_q \begin{bmatrix} s+k \\ k-d \end{bmatrix}_q^{-1} \begin{bmatrix} s \\ j-d \end{bmatrix}_q \begin{bmatrix} r+s+d \\ r+j \end{bmatrix}_q \begin{bmatrix} s+d \\ j \end{bmatrix}_q^{-1} \\
&= q^{-jk} \begin{bmatrix} r+s \\ r+j-k \end{bmatrix}_q.
\end{aligned}$$

Let's denote the LHS of the above equation by $\mathtt{SUM}_k$. Then the Mathematica package of the q-Zeilberger algorithm produces the recursion

$$\mathtt{SUM}_k = \frac{q^{-j}(1-q^{1+j-k+r})}{(1-q^{-j+k+s})}\mathtt{SUM}_{k-1}.$$

By going backward, we obtain

$$\mathtt{SUM}_k = \frac{q^{-jk}(1-q^{1+j-k+r})\cdots(1-q^{r+j})}{(1-q^{s-j+k})\cdots(1-q^{s-j+1})}\mathtt{SUM}_0,$$

where $\mathtt{SUM}_0 = \begin{bmatrix} r+s \\ r+j \end{bmatrix}_q$. After multiplying both the denominator and numerator of the above equation with $(q;q)_{j-k+r}$, we get

$$\mathtt{SUM}_k = q^{-jk} \begin{bmatrix} r+s \\ r+j-k \end{bmatrix}_q,$$

as claimed. So the proof of the LU-decomposition of the matrix H is completed.

Now we turn the inverse matrix L^{-1}. Since L and L^{-1} are lower triangular matrices, we only need to look at the entries indexed by (k,j) with $k \geq j$. So we should show that

$$\sum_{j \leq d \leq k} L_{kd} L^{-1}_{dj} = [k = j].$$

Then we have

$$\begin{aligned}\sum_{j \leq d \leq k} L_{kd} L^{-1}_{dj} &= (-1)^{r(k+j)} \mathbf{i}^{k+j+(k-j)(r-s)} q^{\frac{1}{2}\left(k^2-j+(r-s)(j-k)\right)} \begin{bmatrix} s+j \\ j \end{bmatrix}_q \\ &\times \sum_{j \leq d \leq k} (-1)^d q^{\frac{1}{2}(d^2+d)-kd} \begin{bmatrix} r \\ k-d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q \begin{bmatrix} s+k \\ k-d \end{bmatrix}_q^{-1} \begin{bmatrix} d-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} s+d \\ d \end{bmatrix}_q^{-1}.\end{aligned}$$

The q-Zeilberger algorithm computes the sum on the RHS of the above equation as 0 when $k \neq j$ and $r \neq 0$. For the case $r = 0$, H is an upper triangular matrix so that the claim is clear. For the case $k = j$, it is easy to see that $L_{kk} L^{-1}_{kk} = 1$. So the proof is completed.

Since U and U^{-1} are upper triangular matrices, we just need to look at the entries indexed by (k,j) with $j \geq k$. Thus we have

$$\begin{aligned}\sum_{k \leq d \leq j} U_{kd} U^{-1}_{dj} &= (-1)^{r(j+k+r-1)+j} \mathbf{i}^{(k-j)(r-s+1)} q^{\frac{1}{2}\left((r-s)(j-k)+k^2+j\right)} \begin{bmatrix} r+s+j \\ r \end{bmatrix}_q^{-1} \\ &\times \sum_{k \leq d \leq j} (-1)^d q^{\frac{1}{2}(d^2-d)-kd} \begin{bmatrix} s \\ d-k \end{bmatrix}_q \begin{bmatrix} r+s+k \\ s-d+k \end{bmatrix}_q \\ &\times \begin{bmatrix} s+k \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} j-d+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} r+d \\ r \end{bmatrix}_q.\end{aligned}$$

Similarly, the q-Zeilberger algorithm computes the sum on the RHS of the above equation as 0 when $k \neq j$ and $s \neq 0$. When we choose the number of superdiagonals of the matrix H as zero, that is the case $s = 0$, it is easy to check because the matrix H is a lower triangular matrix. If $k = j$, it is obvious that $U_{kk} U^{-1}_{kk} = 1$. Finally

$$\sum_{k \leq d \leq j} U_{kd} U^{-1}_{dj} = [k = j],$$

so the proof of Theorem 4.1 is completed.

For the inverse matrix H_N^{-1}, by using the fact that $H_N^{-1} = U_N^{-1} \bullet L_N^{-1}$, we can write

$$(H_N^{-1})_{kj} = \sum_{d=0}^{N-1} U^{-1}_{kd} L^{-1}_{dj}$$

$$= \sum_{d=0}^{N-1} (-1)^{(k+d)(r+1)} \mathbf{i}^{(k-d-r)+(k+r-d)(r-s)} q^{\frac{1}{2}((k-d)(s-r-1)+rs)}$$
$$\times \begin{bmatrix} d-k+s-1 \\ s-1 \end{bmatrix}_q \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} r+s+d \\ r \end{bmatrix}_q^{-1} (-1)^{r(d+j)} \mathbf{i}^{(d-j)(r-1-s)}$$
$$\times q^{\frac{1}{2}(d-j)(s-r+1)} \begin{bmatrix} d-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} s+j \\ j \end{bmatrix}_q \begin{bmatrix} s+d \\ d \end{bmatrix}_q^{-1}.$$

After some straightforward simplifications, we obtain Theorem 4.3.

Since the matrix H is a Toeplitz matrix, Theorem 4.4 arises as a consequence of Proposition 4.1.

Thus the proofs of all theorems are completed for the general real parameter q. The proofs of all corollaries follow by choosing $q = \beta/\alpha$.

4.2.3 The Case Bandwidth $r + s + 1$ with the Binomial Coefficients

The results will be presented in this subsection are direct generalizations of the results of [1] with upper bandwidth s and lower bandwidth r. The results for the case $r = s$ cover the results in [1].

For $r, s \geq 0$, we define non-symmetric Toeplitz band matrix $G = [G_{kj}]_{k,j\geq 0}$ via the binomial coefficients as

$$G_{kj} = (-1)^{k+j+r} \binom{r+s}{r+j-k}.$$

For example, when $r = 2$, $s = 4$ and $N = 7$, we have:

$$G_7 = \begin{bmatrix} 15 & -20 & 15 & -6 & 1 & 0 & 0 \\ -6 & 15 & -20 & 15 & -6 & 1 & 0 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ 0 & 1 & -6 & 15 & -20 & 15 & -6 \\ 0 & 0 & 1 & -6 & 15 & -20 & 15 \\ 0 & 0 & 0 & 1 & -6 & 15 & -20 \\ 0 & 0 & 0 & 0 & 1 & -6 & 15 \end{bmatrix}.$$

We list the results related to the LU-decomposition, inverse matrices L^{-1} and U^{-1} and determinant of the matrix G, respectively.

Theorem 4.5. *For $k, j \geq 0$,*

$$L_{kj} = (-1)^{k+j} \binom{r}{k-j} \binom{k}{j} \binom{s+k}{k-j}^{-1},$$

$$U_{kj} = (-1)^{k+j+r}\binom{s}{j-k}\binom{r+s+k}{r+j}\binom{s+k}{j}^{-1},$$

$$L_{kj}^{-1} = \binom{k-j+r-1}{r-1}\binom{s+j}{j}\binom{s+k}{k}^{-1}$$

and

$$U_{kj}^{-1} = (-1)^{r}\binom{j-k+s-1}{s-1}\binom{r+k}{k}\binom{r+s+j}{r}^{-1}.$$

Theorem 4.6. *For $N \geq 1$,*

$$\det G_N = (-1)^{rn}\prod_{d=0}^{N-1}\binom{r+s+d}{r+d}\binom{s+d}{d}^{-1}.$$

Similarly, we have the following result for the LU-decomposition of the inverse matrix G_N^{-1}.

Theorem 4.7. *For $0 \leq k, j \leq N-1$,*

$$A_{kj} = \binom{k-j+r-1}{r-1}\binom{s+N-1-k}{s}\binom{s+N-1-j}{s}^{-1},$$

$$A_{kj}^{-1} = (-1)^{k+j}\binom{r}{k-j}\binom{N-1-j}{k-j}\binom{s+N-1-j}{k-j}^{-1},$$

$$B_{kj} = (-1)^{r}\binom{j-k+s-1}{s-1}\binom{r+N-1-j}{r}\binom{r+s+N-1-k}{r}^{-1}$$

and

$$B_{kj}^{-1} = (-1)^{k+j+r}\binom{s}{j-k}\binom{r+s+N-1-j}{s+k-j}\binom{s+N-1-j}{s+k-j}^{-1}.$$

The proofs of the all above theorems can be done by using Zeilberger's algorithm similar to the previous section. For example, for the LU-decomposition, we have

$$\sum_{d=0}^{min(k,j)} L_{kd}U_{dj} = (-1)^{k+j+r}\sum_{d=0}^{min(k,j)}\binom{r}{k-d}\binom{k}{d}\binom{s+k}{k-d}^{-1} \times \binom{s}{j-d}\binom{r+s+d}{r+j}\binom{s+d}{j}^{-1}.$$

Denote the sum on the RHS by $\mathtt{SUM}_k$. Then Zeilberger's algorithm produces

$$\mathtt{SUM}_{k+1} = \frac{j-k+r}{j-k-s-1}\mathtt{SUM}_k.$$

After solving this recursion, we obtain the $\mathtt{SUM}_k = \binom{r+s}{r+j-k}$, as claimed. Other proofs can be done similarly.

On the other hand, here we would like to present a different approach to prove them. If we perform the limit $q \to 1$ for the results in Section 4.2.1, then we get the results for the matrices including the usual binomial coefficients. When $q \to 1$, the matrix H takes the form

$$\hat{H}_{kj} = (-1)^{r(k+j)+j}\mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)}\binom{r+s}{r+j-k}.$$

So it is seen that

$$G_{kj} = (-1)^{r(k+j)+k+r}\mathbf{i}^{k(s-r-1)+j(r-s-1)+r(1-s-r)}\hat{H}_{kj}.$$

By performing the limit $q \to 1$ to the results in Section 4.2.1, we obtain the algebraic properties of the matrix $\hat{H}$. If we chose the sequences $\{s_n\}$ and $\{m_n\}$ as $\{(-1)^{n(r+1)}\mathbf{i}^{n(s-r-1)+r(1-s-r)}\}$ and $\{(-1)^{r(n+1)}\mathbf{i}^{n(r-s-1)}\}$, respectively and then apply Proposition 4.2 to the results for the matrix $\hat{H}$, then we obtain the results for the matrix G, as desired. This is a useful prototype to show the efficiency of Proposition 4.2.

Now we present some results about the infinity-norm of the matrix G_N^{-1}, which is the maximum value of the absolute row sum, that is,

$$\left\|G_N^{-1}\right\|_\infty = \max_k \left(\sum_{j=0}^{N-1} \left|G_{kj}^{-1}\right|, 0 \le k \le N-1 \right).$$

Firstly, we have the following lemma:

Lemma 4.1. *For $0 \le k \le N-1$, the kth row sum, denoted by S_k, of the matrix G_N^{-1} is*

$$S_k = (-1)^r \binom{k+r}{r}\binom{N-k-1+s}{s}\binom{r+s}{r}^{-1}.$$

Proof. Let e_k be the unit vector of order N, where 1 is in the kth position and e be the vector of order N, where all entries consist of 1's. Then we may write

$$S_k = e_k^T G_N^{-1} e.$$

Since there is no closed formula for G_N^{-1}, we will use the the fact $G^{-1} = U^{-1}L^{-1}$, where the matrices L^{-1} and U^{-1} were given in Theorem 4.5. Thus we should compute

$$S_k = (e_k^T U_N^{-1})(L_N^{-1} e).$$

Here the first parenthesis gives the kth row of the matrix of U_N^{-1} and the second parenthesis gives the row sum of the matrix L_N^{-1}. So the sum of kth row of the matrix L_N^{-1}, denoted by s_k, is

$$s_k = \binom{s+k}{k}^{-1} \sum_{j=0}^{k} \binom{k-j+r-1}{r-1} \binom{s+j}{j},$$

which, by a variant of the Vandermonde identity (for more details see the Eq. (5.26) in [13]), equals

$$\binom{s+k}{k}^{-1} \binom{k+r+s}{r+s} = \binom{k+r+s}{r} \binom{r+s}{r}^{-1}.$$

Consequently, we have that

$$(L_N^{-1} e) = [s_0, s_1, \ldots, s_{N-1}]^T$$

and

$$(e_k^T U_N^{-1}) = [0, 0, \ldots, U_{kk}^{-1}, U_{k,k+1}^{-1}, \ldots, U_{k,N-1}^{-1}].$$

Finally, we obtain

$$\begin{aligned} S_k = \sum_{j=k}^{N-1} U_{kj}^{-1} s_j &= (-1)^r \binom{r+s}{s}^{-1} \binom{r+k}{r} \sum_{j=k}^{N-1} \binom{j-k+s-1}{s-1} \\ &= (-1)^r \binom{r+s}{s}^{-1} \binom{r+k}{r} \sum_{j=0}^{N-k-1} \binom{j+s-1}{j}, \end{aligned}$$

which by the formula $\sum_{k \le n} \binom{r+k}{k} = \binom{r+n+1}{n}$, equals

$$= (-1)^r \binom{r+s}{s}^{-1} \binom{r+k}{r} \binom{s+N-k-1}{s},$$

as claimed. □

Before going further, we would like to recall the definition of the unimodal sequence.

Definition 4.1. *A **unimodal sequence** is a finite sequence which first increases and then decreases. That is, a sequence $\{a_1, a_2, \ldots, a_n\}$ is unimodal if there exists an integer $t \in \{2, 3, \ldots, n-1\}$ such that*

$$a_1 \le a_2 \le \cdots \le a_t \text{ and } a_t \ge a_{t+1} \ge \cdots \ge a_n.$$

In order to find the infinity-norm of the matrix G_N^{-1}, we need the maximum value of $|S_k|$. For this, we investigate the unimodality of $\{|S_k|\}_{n \ge 0}$.

Lemma 4.2. *The sequence* $\{|S_k|\}_{n\geq 0}$ *is unimodal.*

Proof. Since the factor $\binom{r+s}{s}^{-1}$ is independent from the index k, we may show that

$$\{a_k\}_{n\geq 0} = \left\{ \binom{r+k}{r} \binom{s+N-k-1}{s} \right\}$$

is unimodal instead of showing the unimodality of the sequence $\{|S_k|\}_{n\geq 0}$. Consider

$$\begin{aligned}
a_k^2 &= \binom{r+k}{r}^2 \binom{s+N-k-1}{s}^2 \\
&= \frac{(k+r)(k+1)(N-k-1+s)(N-k)}{(k+r+1)k(N-k+s)(N-k-1)} a_{k-1}a_{k+1} \\
&= \left(1 - \frac{1}{k+r+1}\right)\left(1+\frac{1}{k}\right)\left(1-\frac{1}{N-k+s}\right)\left(1+\frac{1}{N-k-1}\right) a_{k-1}a_{k+1} \\
&= \left(1+\frac{r}{k^2+kr+k}\right)\left(1+\frac{s}{(k-N+1)\,(k-N-s)}\right) a_{k-1}a_{k+1} \\
&> a_{k-1}a_{k+1},
\end{aligned}$$

which gives that the sequence $\{a_k\}_{n\geq 0}$ is strictly log-concave that means $\{a_k\}_{n\geq 0}$ is unimodal (For more detail see [90]). Finally, the sequence $\{|S_k|\}_{n\geq 0}$ is unimodal, as well. □

Since the sequence $\{|S_k|\}_{n\geq 0}$ is unimodal, it has a maximum value for some k, where $k \in \{1, 2, \ldots, N-2\}$. Thus we can compute the $\left\|G_N^{-1}\right\|_\infty$:

Theorem 4.8. *For* $N \geq 1$,

$$\left\|G_N^{-1}\right\|_\infty = \frac{(r+t+1)^{\underline{r}}(s+N-t)^{\underline{s}}}{(r+s)!},$$

where $t = \left\lfloor \dfrac{Nr}{r+s} \right\rfloor$ *and the falling factorial is defined as* $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$.

Proof. We know that there exist an integer $k \in \{1, 2, \ldots, N-2\}$ so that $|S_k|$ is maximum. We shall find this value of k. Similarly, we only consider the sequence $\{a_k\}_{n\geq 0} = \left\{\binom{r+k}{r}\binom{s+N-k-1}{s}\right\}$ instead of the sequence $\{S_k\}_{n\geq 0}$ because it is enough to consider the factors only depend on k. Consider

$$\frac{a_{k+1}}{a_k} = \frac{\binom{k+1+r}{r}\binom{N-k-2+s}{s}}{\binom{k+r}{r}\binom{N-k-1+s}{s}}$$

$$
\begin{aligned}
&= \frac{(k+1+r)(N-k-1)}{(k+1)(N-k-1+s)} \\
&= \left(1+\frac{r}{k+1}\right)\left(1-\frac{s}{N-k-1+s}\right) \\
&= 1+\frac{Nr-(k+1)(s+r)}{(k+1)(N-k-1+s)}.
\end{aligned}
$$

If $Nr-(k+1)(s+r)>0$, then $k<\frac{Nr}{r+s}-1$ and so $\{a_k\}_{n\geq 0}$ is increasing for such k's. When $k>\frac{Nr}{r+s}-1$, the sequence $\{a_k\}_{n\geq 0}$ is decreasing. Since k is an integer, the sequence $\{a_k\}_{n\geq 0}$ takes the maximum value at $k=\left\lfloor\frac{Nr}{r+s}\right\rfloor$, which completes the proof. □

Denote the sum of the jth column entries of the matrix G_N^{-1} by $\overline{S}_j$. By Proposition 4.1(v), we can see that $S_k=\overline{S}_{N-1-k}$. So we derive the result

$$\left\|G_N^{-1}\right\|_1=\overline{S}_{N-1-t}=S_t,$$

where $t=\left\lfloor\dfrac{Nr}{r+s}\right\rfloor$ and $\|\cdot\|_1$ is the maximum absolute column sum norm.

At the end of this subsection, we would like to mention the relationship between Toeplitz and Hankel matrices. Let T be a Toeplitz matrix. Then the matrix obtained by

$$M=T\bullet J,$$

where J is a square matrix such that the entries outside of the skew main diagonal are all zero otherwise 1, is a Hankel matrix. So if we have a Toeplitz matrix then we obtain a Hankel matrix by the help of the matrix J, or vice versa (Note that $J^{-1}=J$).

Thus the matrix $H\bullet J$ is a Hankel matrix and by using the fact that $\det J_N=(-1)^{\binom{N}{2}}$ and the results in Section 4.2.1, we may compute the determinants of a new family of Hankel matrices and also obtain its Fibonomial analogue as well as the results including binomial coefficients. For example, for $r=3$, $s=2$ and $N=8$ by the results of the

matrix G, we obtain

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 5 & -10 \\ 0 & 0 & 0 & 0 & -1 & 5 & -10 & 10 \\ 0 & 0 & 0 & -1 & 5 & -10 & 10 & -5 \\ 0 & 0 & -1 & 5 & -10 & 10 & -5 & 1 \\ 0 & -1 & 5 & -10 & 10 & -5 & 1 & 0 \\ -1 & 5 & -10 & 10 & -5 & 1 & 0 & 0 \\ 5 & -10 & 10 & -5 & 1 & 0 & 0 & 0 \\ -10 & 10 & -5 & 1 & 0 & 0 & 0 & 0 \end{vmatrix} = \frac{9}{1} \times \frac{10^2}{2^2} \times \frac{11^2}{3^2} \times \frac{12}{4}.$$

4.3 A Generalization of the Super Catalan Matrix

As mentioned in Section 3.2.2, in this section we will give the generalizations with two additional parameters of the results in [3]. Briefly, we study the matrices $M = [M_{kj}]$ and $T = [T_{kj}]$ defined by for nonnegative integers r and s and $k, j \geq 0$,

$$M_{kj} = \binom{k+j}{k} \binom{2k+r}{k}^{-1} \binom{2j+s}{j}^{-1}$$

and

$$T_{kj} = \binom{k+j}{k}^{-1} \binom{2k+r}{k} \binom{2j+s}{j},$$

respectively. These matrices are the reciprocals of each other. Clearly, the results in [3] are the case $r = s = 0$.

In order to obtain more general results, we introduce the matrices $\mathcal{M}$ and $\mathcal{T}$ which are the q-analogues of the matrices M and T, respectively. They are reasonably defined by for $k, j \geq 0$,

$$\mathcal{M}_{kj} = \begin{bmatrix} k+j \\ k \end{bmatrix}_q \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1}$$

and

$$\mathcal{T}_{kj} = \begin{bmatrix} k+j \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q,$$

respectively.

For the both matrices, we derive explicit formulæ for the LU-decomposition, inverse matrices L^{-1} and U^{-1}, determinant and Cholesky decomposition when the matrix is symmetric, that is the case $r = s$. Unfortunately, the inverses do not have closed

formulæ. For this reason, we give explicit expressions for the LU-decomposition of the inverse matrices. Afterwards, when $q \to 1$, we get the results for the matrices M and T. The readers can also find our results in [4].

Similar to the previous section, in this section, our main approach is to guess relevant quantities and then we will use the q-Zeilberger algorithm and formulæ (2.9) and (2.10) to justify relevant equalities. All identities we will obtain hold for general q. One may also obtain Fibonomial analogues of these results by choosing $q = \beta/\alpha$ and the help of Proposition 4.2. But we don't prefer to present that results not only to avoid repetition but also they are a bit cumbersome.

4.3.1 The Matrix $\mathcal{M}$

For the matrix $\mathcal{M}$, we have the following theorem.

Theorem 4.9. *For $k, j \geq 0$,*

$$\mathcal{L}_{kj} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q,$$

$$\mathcal{L}_{kj}^{-1} = (-1)^{k+j} q^{\binom{k-j}{2}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q,$$

$$\mathcal{U}_{kj} = q^{k^2} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} j \\ k \end{bmatrix}_q$$

and

$$\mathcal{U}_{kj}^{-1} = (-1)^{k+j} q^{k(k+1)/2 - j(j+1)/2 - kj} \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q.$$

For $0 \leq k, j \leq N-1$,

$$\mathcal{A}_{kj} = (-1)^{k+j} q^{k(k+3)/2 - j(j+3)/2 - N(k-j)} \frac{1-q^{2j+1}}{1-q^{k+j+1}} \begin{bmatrix} N-j-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q \times \begin{bmatrix} k+j \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q,$$

$$\mathcal{A}_{kj}^{-1} = q^{(k-j)(k-N+1)} \begin{bmatrix} k+j \\ k \end{bmatrix}_q \begin{bmatrix} N-j-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q \begin{bmatrix} k+s \\ s \end{bmatrix}_q^{-1},$$

$$\mathcal{B}_{kj} = (-1)^{k+j} q^{(j+1)(j+2)/2 - N(k+j+1) + 3k(k+1)/2} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} N+k \\ k+j+1 \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \times \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q \begin{bmatrix} k+s \\ s \end{bmatrix}_q^{-1}$$

and

$$\mathcal{B}_{kj}^{-1} = q^{(k+j+1)(N-j-1)} \frac{1-q^{2j+1}}{1-q^{N-k}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} N+j \\ k+j \end{bmatrix}_q^{-1} \begin{bmatrix} j \\ k \end{bmatrix}_q \times \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q.$$

For $N \geq 1$,

$$\det \mathcal{M}_N = q^{N(N-1)(2N-1)/6} \prod_{d=0}^{N-1} \begin{bmatrix} 2d+r \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} 2d+s \\ d \end{bmatrix}_q^{-1}.$$

Finally, when $r = s$, *for* $k, j \geq 0$,

$$\mathcal{C}_{kj} = q^{j^2/2} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} k \\ j \end{bmatrix}_q.$$

Now we shall give the proof of this theorem.

Proof. For $\mathcal{L}$ and $\mathcal{L}^{-1}$,

$$\begin{aligned} \sum_{j \leq d \leq k} \mathcal{L}_{kd} \mathcal{L}_{dj}^{-1} &= \sum_{j \leq d \leq k} (-1)^{d+j} q^{\binom{d-j}{2}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2d+r \\ d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} 2d+r \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} d \\ j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \\ &= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \sum_{d=0}^{k-j} \begin{bmatrix} k-j \\ d \end{bmatrix}_q (-1)^d q^{\binom{d}{2}}. \end{aligned}$$

By Rothe's formula (2.10) if $k > j$ then the last sum on the RHS of the above equation equals $(1;q)_{k-j} = 0$ and if $k = j$, then it equals 1. Thus we conclude

$$\sum_{j \leq d \leq k} \mathcal{L}_{kd} \mathcal{L}_{dj}^{-1} = [k = j],$$

as claimed.

For $\mathcal{U}$ and $\mathcal{U}^{-1}$,

$$\begin{aligned} \sum_{k \leq d \leq j} \mathcal{U}_{kd} \mathcal{U}_{dj}^{-1} &= q^{k^2 - \binom{j+1}{2}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \\ &\quad \times q^{k(2j+k+1)/2} (-1)^{k+j} \sum_{0 \leq d \leq j-k} \begin{bmatrix} j-k \\ d \end{bmatrix}_q (-1)^d q^{\binom{d+1}{2} + d(k-j)}. \end{aligned}$$

By the Cauchy binomial theorem (2.9), if $j > k$ then the last sum on the RHS of the above equation equals $\prod_{d=1}^{j-k} (1 - q^{(k-j)+d}) = 0$. The case $k = j$ can be easily computed as 1. So we have

$$\sum_{k \leq d \leq j} \mathcal{U}_{kd} \mathcal{U}_{dj}^{-1} = [k = j],$$

as desired.

For the LU-decomposition, we should show that

$$\sum_{0\le d\le \min(k,j)} \mathcal{L}_{kd}\mathcal{U}_{dj} = \mathcal{M}_{kj}.$$

Firstly, we can assume that $k \le j$. Consider,

$$\sum_{0\le d\le \min(k,j)} \mathcal{L}_{kd}\mathcal{U}_{dj} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} (q;q)_k (q;q)_j \times \sum_{0\le d\le k} q^{d^2} \frac{1}{(q;q)_d^2 (q;q)_{k-d} (q;q)_{j-d}}. \tag{4.4}$$

Denote the sum in (4.4) by $\mathtt{SUM}_k$. The Mathematica package of the q-Zeilberger algorithm produces the recursion

$$\mathtt{SUM}_k = \frac{1-q^{j+k}}{(1-q^k)^2}\mathtt{SUM}_{k-1}.$$

Since $\mathtt{SUM}_0 = (q;q)_k^{-1}(q;q)_j^{-1}$, we obtain

$$\mathtt{SUM}_k = (q;q)_k^{-1}(q;q)_j^{-1} \begin{bmatrix} k+j \\ k \end{bmatrix}_q.$$

Since the sum in (4.4) is symmetric with respect to k and j, the case $j < k$ follows likewise. Eventually, we get that

$$\sum_{0\le d\le \min(k,j)} \mathcal{L}_{kd}\mathcal{U}_{dj} = \mathcal{M}_{kj},$$

which completes the proof of the LU-decomposition of the matrix $\mathcal{M}$.

For $\mathcal{A}$ and $\mathcal{A}^{-1}$, consider

$$\sum_{j\le d\le k} \mathcal{A}_{kd}\mathcal{A}_{dj}^{-1} = (-1)^k q^{k(k+3)/2-j+N(j-k)} \frac{(q;q)_{N-j-1}}{(q;q)_{N-k-1}} \times \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} k \\ j \end{bmatrix}_q \times \sum_{j\le d\le k} \begin{bmatrix} k-j \\ d-j \end{bmatrix}_q (-1)^d q^{d(d-1)/2-jd} \frac{(q;q)_{d+j}}{(q;q)_{d-j}} \frac{1-q^{2d+1}}{1-q^{k+d+1}}.$$

For the sum on the last line of the above equation, we get that it is equal to 0 provided that $k \ne j$ by the q-Zeilberger algorithm. If $k = j$, it is obvious that $\mathcal{A}_{kk}\mathcal{A}_{kk}^{-1} = 1$. Thus

$$\sum_{j\le d\le k} \mathcal{A}_{kd}\mathcal{A}_{dj}^{-1} = [k = j],$$

as claimed. Similarly, by the q-Zeilberger algorithm, we compute

$$\sum_{k\le d\le j} \mathcal{B}_{kd}\mathcal{B}^{-1}_{dj} = [k=j].$$

When $r=s$, then we have a symmetric matrix. Thus by Corollary 4.1, the Cholesky decomposition can be computed as

$$\begin{aligned}\mathcal{C}_{kj} &= \mathcal{U}_{jk}\mathcal{U}_{jj}^{-\frac{1}{2}} \\ &= q^{j^2}\begin{bmatrix}2j+r\\ j\end{bmatrix}_q^{-1}\begin{bmatrix}2k+r\\ k\end{bmatrix}_q^{-1}\begin{bmatrix}k\\ j\end{bmatrix}_q q^{-\frac{1}{2}j^2}\begin{bmatrix}2j+r\\ j\end{bmatrix}_q = q^{j^2/2}\begin{bmatrix}2k+r\\ k\end{bmatrix}_q^{-1}\begin{bmatrix}k\\ j\end{bmatrix}_q,\end{aligned}$$

as claimed.

For the LU-decomposition of $\mathcal{M}^{-1}$, we should show that $\mathcal{M}^{-1} = \mathcal{A}\cdot\mathcal{B}$ which is the same as $\mathcal{M} = \mathcal{B}^{-1}\cdot\mathcal{A}^{-1}$. So it is sufficient to show that

$$\sum_{\max(k,j)\le d\le N-1} \mathcal{B}^{-1}_{kd}\mathcal{A}^{-1}_{dj} = \mathcal{M}_{kj}.$$

After some arrangements, we have

$$\begin{aligned}\sum_{\max(k,j)\le d\le N-1} \mathcal{B}^{-1}_{kd}\mathcal{A}^{-1}_{dj} = \begin{bmatrix}2k+r\\ k\end{bmatrix}_q^{-1}\begin{bmatrix}2j+s\\ j\end{bmatrix}_q^{-1} \sum_{j\le d\le N-1} q^{(j+k+1)(N-1-d)} \\ \times \frac{1-q^{2d+1}}{1-q^{N-k}}\begin{bmatrix}d\\ k\end{bmatrix}_q\begin{bmatrix}N+d\\ k+d\end{bmatrix}_q^{-1}\begin{bmatrix}d+j\\ d\end{bmatrix}_q\begin{bmatrix}N-j-1\\ d-j\end{bmatrix}_q.\end{aligned}$$

By replacing $(N-1)$ with N, the sum on the RHS of the above equation equals

$$\sum_{j\le d\le N} q^{(j+k+1)(N-d)}\frac{1-q^{2d+1}}{1-q^{N+1-k}}\begin{bmatrix}d\\ k\end{bmatrix}_q\begin{bmatrix}N+1+d\\ k+d\end{bmatrix}_q^{-1}\begin{bmatrix}d+j\\ d\end{bmatrix}_q\begin{bmatrix}N-j\\ d-j\end{bmatrix}_q.$$

Denote this sum by $\mathtt{SUM}_N$. The q-Zeilberger algorithm gives the following recursion provided that $k\neq N$ and $j\neq N$

$$\mathtt{SUM}_N = \mathtt{SUM}_{N-1}.$$

So $\mathtt{SUM}_N = \mathtt{SUM}_j = \begin{bmatrix}k+j\\ k\end{bmatrix}_q$, which completes the proof except the case $(k,j)=(N-1,N-1)$. This case could be easily checked by hand.

Thus the proof is completed. □

4.3.2 The Matrix $\mathcal{T}$

For the matrix $\mathcal{T}$, we have the following result.

Theorem 4.10. *For $k, j \geq 0$,*

$$\mathcal{L}_{kj} = \begin{bmatrix} 2k+r \\ k+j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} k+r \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q.$$

For $j \geq 1$ and $k \geq 0$,

$$\mathcal{L}_{kj}^{-1} = (-1)^{k+j} q^{\binom{k-j}{2}} \frac{1-q^{2k}}{1-q^{k+j}} \begin{bmatrix} k+j \\ k-j \end{bmatrix}_q \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q,$$

for $k \geq 1$,

$$\mathcal{L}_{k0}^{-1} = (-1)^k (1+q^k) q^{\binom{k}{2}} \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1}$$

and $\mathcal{L}_{00}^{-1} = 1$. For $k \geq 1$ and $j \geq 0$,

$$\begin{aligned} \mathcal{U}_{kj} = (-1)^k q^{k(3k-1)/2} (1+q^k) \begin{bmatrix} 2j+s \\ k+j \end{bmatrix}_q \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} j-k+s \\ s \end{bmatrix}_q \\ \times \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \end{aligned}$$

and for $j \geq 0$,

$$\mathcal{U}_{0j} = \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q.$$

For $k, j \geq 0$,

$$\begin{aligned} \mathcal{U}_{kj}^{-1} = (-1)^k q^{k(k+1)/2 - j(k+j)} \frac{1-q^j}{1-q^{k+j}} \begin{bmatrix} k+j \\ j-k \end{bmatrix}_q \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q \\ \times \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q. \end{aligned}$$

For $0 \leq k, j \leq N-1$,

$$\begin{aligned} \mathcal{A}_{kj} = (-1)^{k+j} q^{(k+1)(k+2)/2 - (j+1)(j+2)/2 + N(j-k)} \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} N+k-1 \\ 2k \end{bmatrix}_q \\ \times \begin{bmatrix} N+j-1 \\ 2j \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q, \end{aligned}$$

$$\begin{aligned} \mathcal{A}_{kj}^{-1} = q^{(k-j)(k-N+1)} \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} N+k-1 \\ 2k \end{bmatrix}_q \begin{bmatrix} N+j-1 \\ 2j \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q^{-1} \\ \times \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q, \end{aligned}$$

$$\mathcal{B}_{kj} = q^{(j+1)(j+2)/2-N(N-1)/2-jN+k^2-1} \begin{bmatrix} N+j-1 \\ 2j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q^{-1} \times \begin{bmatrix} j+r \\ r \end{bmatrix}_q \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1}$$

and

$$\mathcal{B}_{kj}^{-1} = (-1)^{N+j+1} q^{k-kj-j(j+1)/2+kN+N(N-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} N+k-1 \\ 2k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q \times \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1}.$$

For $N \geq 1$,

$$\det \mathcal{T}_N = (-1)^{N(N-1)/2} \prod_{d=1}^{N-1} q^{d(3d-1)/2} \begin{bmatrix} 2d+s \\ s \end{bmatrix}_q \begin{bmatrix} 2d+r \\ r \end{bmatrix}_q \begin{bmatrix} d+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} d+s \\ s \end{bmatrix}_q^{-1}.$$

Finally, when $r = s$, for $j \geq 1$ and $k \geq 0$,

$$\mathcal{C}_{kj} = \mathbf{i}^j (1+q)^{j/2} q^{j(3j-1)/4} \begin{bmatrix} 2k+r \\ k+j \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} k-j+r \\ r \end{bmatrix}_q$$

and for $k \geq 0$,

$$\mathcal{C}_{k0} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q.$$

Proof. By the definitions of the matrices $\mathcal{L}$ and $\mathcal{L}^{-1}$, for the case $j = 0$, we have

$$\sum_{0 \leq d \leq k} \mathcal{L}_{kd} \mathcal{L}_{d0}^{-1} = \mathcal{L}_{kd} \mathcal{L}_{00}^{-1} + \sum_{1 \leq d \leq k} \mathcal{L}_{kd} \mathcal{L}_{d0}^{-1}.$$

If $k = 0$, we get 1 as $(0,0)$th entry of the multiplication $\mathcal{L} \bullet \mathcal{L}^{-1}$. If $k > 0$, after some rearrangements, we have

$$\begin{aligned} \sum_{1 \leq d \leq k} \mathcal{L}_{kd} \mathcal{L}_{d0}^{-1} &= \sum_{0 \leq d \leq n} \mathcal{L}_{n+1,d+1} \mathcal{L}_{d+1,0}^{-1} \\ &= \sum_{0 \leq d \leq n} (-1)^{d+1} (1+q^{d+1}) q^{(d^2+d)/2} \begin{bmatrix} 2n+2+r \\ n+d+2 \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} n+1 \\ d+1 \end{bmatrix}_q \begin{bmatrix} n+1+r \\ d+1 \end{bmatrix}_q^{-1}. \end{aligned}$$

The q-Zeilberger algorithm compute the sum on the RHS of the above equation as $-\begin{bmatrix} 2n+2+r \\ n+1 \end{bmatrix}_q$. By changing $n+1$ with k again, we get $-\begin{bmatrix} 2k+r \\ k \end{bmatrix}_q$. Finally, if $k > 0$,

$$\sum_{0 \leq d \leq k} \mathcal{L}_{kd} \mathcal{L}_{d0}^{-1} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q + \sum_{1 \leq d \leq k} \mathcal{L}_{kd} \mathcal{L}_{d0}^{-1}$$

$$= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q - \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q = 0,$$

as desired. For the case $j > 0$, we have

$$\sum_{j \le d \le k} \mathcal{L}_{kd}\mathcal{L}_{dj}^{-1} = \sum_{j \le d \le k} (-1)^{d+j} q^{\binom{d-j}{2}} \frac{1-q^{2d}}{1-q^{d+j}} \begin{bmatrix} 2k+r \\ k+d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q \times \begin{bmatrix} k+r \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} d+j \\ d-j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q.$$

Again by the q-Zeilberger algorithm, we obtain that it is equal to 0 provided that $k \neq j$. The case $k = j$ could be easily computed as 1. Finally,

$$\sum_{j \le d \le k} \mathcal{L}_{kd}\mathcal{L}_{dj}^{-1} = [k = j],$$

as claimed. Verification of the inverse of $\mathcal{U}$ can be similarly done by the help of the q-Zeilberger algorithm. Inverses of the matrices $\mathcal{A}$ and $\mathcal{B}$ may be shown as in the proof of Theorem 4.9. We would prefer to omit them due to the similarities.

For the LU-decomposition, we have to prove that

$$\sum_{0 \le d \le \min(k,j)} \mathcal{L}_{kd}\mathcal{U}_{dj} = \mathcal{T}_{kj}.$$

The cases $k = 0$, $j \ge 0$ and $j = 0$, $k \ge 0$ could be easily seen. For the other cases, consider

$$\begin{aligned}
\sum_{0 \le d \le \min(k,j)} \mathcal{L}_{kd}\mathcal{U}_{dj} &= \mathcal{L}_{k0}\mathcal{U}_{0j} + \sum_{1 \le d \le \min(k,j)} \mathcal{L}_{kd}\mathcal{U}_{dj} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q \\
&+ \sum_{1 \le d \le \min(k,j)} (-1)^d (1+q^d) q^{(3d-1)d/2} \begin{bmatrix} 2k+r \\ k+d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q \\
&\times \begin{bmatrix} k+r \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j+d \end{bmatrix}_q \begin{bmatrix} j-d+s \\ s \end{bmatrix}_q \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \\
&= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q + \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q \\
&\times \begin{bmatrix} 2k \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j \\ j \end{bmatrix}_q^{-1} \sum_{1 \le d \le \min(k,j)} (-1)^d (1+q^d) q^{(3d-1)d/2} \\
&\times \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q \begin{bmatrix} 2j \\ j+d \end{bmatrix}_q.
\end{aligned}$$

Without loss of generality, we may assume that $k \le j$. So consider the sum

$$\mathtt{SUM}_k = \sum_{-k \le d \le k} (-1)^d (1+q^d) q^{(3d-1)d/2} \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q \begin{bmatrix} 2j \\ j+d \end{bmatrix}_q.$$

Then the q-Zeilberger algorithm gives the following recurrence relation for the sum $\mathtt{SUM}_k$.

$$\mathtt{SUM}_k = \frac{(1+q^k)(1-q^{2k-1})}{(1-q^{j+k})}\mathtt{SUM}_{k-1}.$$

Since $\mathtt{SUM}_0 = 2\begin{bmatrix}2j\\ j\end{bmatrix}_q$, we obtain

$$\mathtt{SUM}_k = 2\begin{bmatrix}2k\\ k\end{bmatrix}_q\begin{bmatrix}2j\\ j\end{bmatrix}_q\begin{bmatrix}k+j\\ k\end{bmatrix}_q^{-1}.$$

Since

$$\begin{aligned}\mathtt{SUM}_k &= \sum_{-k\le d\le -1}(-1)^d(1+q^d)q^{(3d-1)d/2}\begin{bmatrix}2k\\ k+d\end{bmatrix}_q\begin{bmatrix}2j\\ j+d\end{bmatrix}_q + 2\begin{bmatrix}2k\\ k\end{bmatrix}_q\begin{bmatrix}2j\\ j\end{bmatrix}_q\\ &\quad+\sum_{1\le d\le k}(-1)^d(1+q^d)q^{(3d-1)d/2}\begin{bmatrix}2k\\ k+d\end{bmatrix}_q\begin{bmatrix}2j\\ j+d\end{bmatrix}_q\\ &= 2\begin{bmatrix}2k\\ k\end{bmatrix}_q\begin{bmatrix}2j\\ j\end{bmatrix}_q + 2\sum_{1\le d\le k}(-1)^d(1+q^d)q^{(3d-1)d/2}\begin{bmatrix}2k\\ k+d\end{bmatrix}_q\begin{bmatrix}2j\\ j+d\end{bmatrix}_q,\end{aligned}$$

then we have

$$\begin{aligned}\sum_{0\le d\le k}\mathcal{L}_{kd}\mathcal{U}_{dj} &= \begin{bmatrix}2k+r\\ k\end{bmatrix}_q\begin{bmatrix}2j+s\\ j\end{bmatrix}_q + \begin{bmatrix}2k+r\\ k\end{bmatrix}_q\begin{bmatrix}2j+s\\ j\end{bmatrix}_q\\ &\quad\times\begin{bmatrix}2k\\ k\end{bmatrix}_q^{-1}\begin{bmatrix}2j\\ j\end{bmatrix}_q^{-1}\left(\frac{1}{2}\mathtt{SUM}_k - \begin{bmatrix}2k\\ k\end{bmatrix}_q\begin{bmatrix}2j\\ j\end{bmatrix}_q\right)\\ &= \begin{bmatrix}2k+r\\ k\end{bmatrix}_q\begin{bmatrix}2j+s\\ j\end{bmatrix}_q\begin{bmatrix}k+j\\ k\end{bmatrix}_q^{-1} = \mathcal{T}_{kj},\end{aligned}$$

as desired.

The proof of the LU-decomposition of the inverse matrix $\mathcal{T}^{-1}$ could be similarly done as in the proof of Theorem 4.9. Similarly, when $r = s$, the Cholesky decomposition follows by Corollary 4.1. □

4.3.3 The Matrices M and T

Obviously, we have

$$\lim_{q\to 1}\mathcal{M} = M \text{ and } \lim_{q\to 1}\mathcal{T} = T.$$

Thus the related results for the matrices M and T can be easily obtained by performing the limit $q \to 1$ to the results in Sections 4.3.1 and 4.3.2, respectively. Without any efforts, we obtain new results including binomial coefficients. This is one of the greatest advantages of studying q-analogues.

Now we list the results for the matrices M and T, respectively.

Theorem 4.11. *For the matrix M, we have for $k, j \geq 0$,*

$$L_{kj} = \binom{2k+r}{k}^{-1}\binom{2j+r}{j}\binom{k}{j},$$

$$L_{kj}^{-1} = (-1)^{k+j}\binom{2k+r}{k}^{-1}\binom{2j+r}{j}\binom{k}{j},$$

$$U_{kj} = \binom{2k+r}{k}^{-1}\binom{2j+s}{j}^{-1}\binom{j}{k}$$

and

$$U_{kj}^{-1} = (-1(^{k+j}\binom{2k+s}{k}\binom{2j+r}{j}\binom{j}{k}.$$

For $0 \leq k, j \leq N-1$,

$$A_{kj} = (-1)^{k+j}\frac{1+2j}{k+j+1}\binom{N-j-1}{k-j}\binom{2k+s}{k}\binom{k+j}{k}^{-1}\binom{2j+s}{s}^{-1}\binom{j+s}{s},$$

$$A_{kj}^{-1} = \binom{k+j}{k}\binom{N-j-1}{k-j}\binom{2j+s}{j}^{-1}\binom{2k+s}{s}\binom{k+s}{s}^{-1},$$

$$B_{kj} = (-1)^{k+j}\binom{2j+r}{j}\binom{N+k}{k+j+1}\binom{j}{k}\binom{2k+s}{s}\binom{k+s}{s}^{-1}$$

and

$$B_{kj}^{-1} = \frac{2j+1}{N-k}\binom{2k+r}{k}^{-1}\binom{N+j}{k+j}^{-1}\binom{j}{k}\binom{2j+s}{s}^{-1}\binom{j+s}{s}.$$

For $N \geq 1$,

$$\det \mathcal{M}_N = \prod_{k=0}^{N-1}\binom{2k+r}{k}^{-1}\binom{2k+s}{k}^{-1}$$

and finally, when $r = s$, for $k, j \geq 0$,

$$C_{kj} = \binom{2k+r}{k}^{-1}\binom{k}{j}.$$

Theorem 4.12. *For the matrix T, we have for $k, j \geq 0$,*

$$L_{kj} = \binom{2k+r}{k+j}\binom{k}{j}\binom{k+r}{j}^{-1}\binom{2j+r}{r}^{-1}\binom{j+r}{r},$$

for $j \geq 1$ and $k \geq 0$,

$$L_{kj}^{-1} = (-1)^{k+j}\frac{2k}{k+j}\binom{k+j}{k-j}\binom{2k+r}{r}\binom{k+r}{r}^{-1}\binom{2j+r}{r}^{-1}\binom{j+r}{r},$$

$$L_{k0}^{-1} = 2(-1)^{k}\binom{2k+r}{r}\binom{k+r}{r}^{-1}$$

and $L_{00}^{-1} = 1$. For $k \geq 1$ and $j \geq 0$,

$$U_{kj} = (-1)^{k}2\binom{2j+s}{k+j}\binom{2k+r}{r}\binom{j-k+s}{s}\binom{k+r}{r}^{-1}\binom{j+s}{s}^{-1}$$

and $U_{0j} = \binom{2j+s}{j}$. *For* $k, j \geq 0$,

$$U_{kj}^{-1} = (-1)^k \frac{j}{k+j} \binom{k+j}{j-k} \binom{2j+r}{r}^{-1} \binom{j+r}{r} \binom{2k+s}{s}^{-1} \binom{k+s}{s}.$$

For $0 \leq k, j \leq N-1$,

$$A_{kj} = (-1)^{k+j} \binom{k}{j} \binom{N+k-1}{2k} \binom{N+j-1}{2j}^{-1} \binom{2k+s}{s}^{-1} \binom{k+s}{s} \times \binom{2j+s}{s} \binom{j+s}{s}^{-1},$$

$$A_{kj}^{-1} = \binom{k}{j} \binom{N+k-1}{2k} \binom{N+j-1}{2j}^{-1} \binom{2k+s}{s}^{-1} \binom{k+s}{s} \binom{2j+s}{s} \binom{j+s}{s}^{-1},$$

$$B_{kj} = \binom{N+j-1}{2j} \binom{j}{k} \binom{2k+s}{k}^{-1} \binom{j+r}{r} \binom{2j+r}{r}^{-1}$$

and

$$B_{kj}^{-1} = (-1)^{N+j+1} \binom{j}{k} \binom{N+k-1}{2k}^{-1} \binom{2j+s}{s} \binom{2k+r}{r} \binom{k+r}{r}^{-1}.$$

For $N \geq 1$,

$$\det T_N = (-1)^{N(N-1)/2} \prod_{d=1}^{N-1} \binom{2d+s}{s} \binom{2d+r}{r} \binom{d+r}{r}^{-1} \binom{d+s}{s}^{-1}.$$

Finally, when $r = s$, *for* $j \geq 1$ *and* $k \geq 0$,

$$C_{kj} = (-2)^{j/2} \binom{2k+r}{k+j} \binom{k+r}{r}^{-1} \binom{k-j+r}{r}$$

and for $k \geq 0$,

$$C_{k0} = \binom{2k+r}{k}.$$

We want to finish this section by giving a conclusion remark. We will show how one can obtain results for the matrices M and T or similar kind of matrices without the help of the q-analogues.

For example, we shall take the matrix M. We can consider the entries of it as

$$M_{kj} = \binom{k+j}{k} s_k m_j,$$

where $s_k = \binom{2k+r}{k}^{-1}$ and $m_j = \binom{2j+s}{j}^{-1}$. Thus the entries of the matrix M is separable with respect to indexes. So we can apply Proposition 4.2 if we know the properties of the matrix $\left[\binom{k+j}{k}\right]_{k,j\geq 0}$. Fortunately, the algebraic properties of this symmetric Pascal

matrix (also its reciprocal analogue) are studied in [58]. For example, the (k,j)th entry of the L matrix coming from the LU-decomposition of this symmetric Pascal matrix is $\binom{k}{j}$. So if we apply Proposition 4.2 for the matrix M, we obtain its L matrix as

$$L_{kj} = \binom{k}{j}\binom{2k+r}{k}^{-1}\binom{2j+r}{j},$$

which is the exactly same with the quantity given in Theorem 4.11. Likewise, all algebraic properties can be obtained in this way. Furthermore, since $\binom{k+j}{k} = \frac{(k+j)!}{k!j!}$, by choosing $s_k = k!$ and $m_j = j!$ in Proposition 4.2, we get new results for the matrix $[(k+j)!]_{k,j\geq 0}$, which is a Hankel matrix, as well. More generally, by Proposition 4.2, one can easily extend our results by considering the matrix with entries for $k, j \geq 0$,

$$\binom{k+j}{k}\prod_{l=1}^{m_1}\binom{a_l k+p_l}{k}\prod_{l=1}^{m_2}\binom{b_l k+r_l}{k}^{-1}\prod_{l=1}^{m_3}\binom{c_l j+s_l}{j}\prod_{l=1}^{m_4}\binom{d_l j+t_l}{j}^{-1},$$

as well as its reciprocal analogue, where the parameters m_1, m_2, m_3, m_4, p_l's, r_l's, s_l's and t_l's are nonnegative integers and a_l's, b_l's, c_l's and d_l's are positive integers.

Thus we see that Proposition 4.2 is very useful to derive new results as well as to prove existing identities.

4.4 General Family of the Max and Min Matrices

As mentioned in Section 3.2.3, recently the authors [5] studied the matrices of order N defined by $[\max(a_k, a_j)]_{1\leq k,j\leq N}$ and $[\min(a_k, a_j)]_{1\leq k,j\leq N}$ over the set $\{a_1, a_2, \ldots, a_N\}$, such that $a_1 \leq a_2 \leq \cdots \leq a_N$. They obtained some algebraic properties of these matrices. Note that their approach only works for the increasing sequence. Their method based on another auxiliary family of the matrices which called "meet and join matrices". They also indicated that the elementary tools are difficult to derive such results.

In this section, we generalize their results for an arbitrary sequence $\{a_n\}$ by defining the matrices conveniently. In other words, we define two new families of the matrices, which called Max and Min matrices, whose entries run in left-reversed and up-reversed L-shaped pattern, respectively. By any given sequence $\{a_n\}$, we define the matrices M_1, M_2 as

$$(M_1)_{kj} = a_{\max(k,j)}, \qquad (M_2)_{kj} = a_{\min(k,j)}$$

and their reciprocal analogues $\mathcal{M}_1$ and $\mathcal{M}_2$ as

$$(\mathcal{M}_1)_{kj} = \frac{1}{a_{\max(k,j)}}, \qquad (\mathcal{M}_2)_{kj} = \frac{1}{a_{\min(k,j)}}.$$

Here note that the size of the matrices does not matter, as well. That means we can consider them as infinite matrices. Clearly, the matrices M_1 and M_2 are of the forms

$$M_1 = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n & \cdots \\ a_2 & a_2 & a_3 & \cdots & a_n & \cdots \\ a_3 & a_3 & a_3 & \cdots & a_n & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ a_n & a_n & a_n & \cdots & a_n & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} a_1 & a_1 & a_1 & \cdots & a_1 & \cdots \\ a_1 & a_2 & a_2 & \cdots & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ a_1 & a_2 & a_3 & \cdots & a_n & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

respectively.

It is worthwhile to note that if the sequence $\{a_n\}$ is increasing, then $a_{\max(k,j)} = \max(a_k, a_j)$ and $a_{\min(k,j)} = \min(a_k, a_j)$. Conversely, if the sequence $\{a_n\}$ is decreasing, then $a_{\max(k,j)} = \min(a_k, a_j)$ and $a_{\min(k,j)} = \max(a_k, a_j)$. Thus the results of [5] will be the special cases of our results.

We will study various properties of the matrices M_1, M_2, $\mathcal{M}_1$ and $\mathcal{M}_2$, such as LU-decomposition, inverse, Cholesky decomposition, etc. by using elementary tools which are simpler and more convenient way than the method used in [5]. In Section 4.4.1, we go around the matrices M_1 and $\mathcal{M}_1$. Afterwards, in Section 4.4.2, we examine the matrices M_2 and $\mathcal{M}_2$. Finally, we give some further applications of our main results. For example, as a consequence of our results, we will give an idea how we can obtain a generalization of the Lehmer matrix and its reciprocal analogue.

We have the following lemma for later use.

Lemma 4.3. *Let $\{a_n\}$ be any real sequence. Then for all $k, j > 0$, we have*

$$a_{\max(k,j)} a_{\min(k,j)} = a_k a_j.$$

Proof. For all cases $k = j$, $k > j$ and $j > k$, it is obviously seen. □

Throughout this section, we assume that $\{a_n\}$ is any sequence such that $a_n \neq 0$ and $a_n \neq a_{n+1}$ for all $n \geq 1$, otherwise the matrices M_1 and M_2 become singular.

4.4.1 Max Matrices and Their Reciprocal Analogues

We derive the LU-decompositions, inverses, Cholesky decompositions and LU-decompositions of the inverses of the matrices M_1 and $\mathcal{M}_1$, respectively.

For the matrix M_1, we have the following results.

We start with the LU-decomposition:

Theorem 4.13. *For $k, j \geq 1$,*

$$L_{kj} = \begin{cases} \dfrac{a_k}{a_j} & \text{if } k \geq j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$U_{kj} = \begin{cases} a_j & \text{if } k = 1, \\ \dfrac{a_j\,(a_{k-1} - a_k)}{a_{k-1}} & \text{if } j \geq k > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we shall give the inverse matrices L^{-1} and U^{-1} by the following theorem.

Theorem 4.14. *For $k, j \geq 1$,*

$$L_{kj}^{-1} = \begin{cases} (-1)^{k+j}\,\dfrac{a_k}{a_j} & \text{if } 0 \leq k - j \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$U_{kj}^{-1} = \begin{cases} (-1)^{k+j}\,\dfrac{a_{j-1}}{a_k\,(a_{j-1} - a_j)} & \text{if } 0 \leq j - k \leq 1 \text{ and } j \neq 1, \\ \dfrac{1}{a_1} & \text{if } k = j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we compute the inverse matrix $(M_1)_N^{-1}$ as follows.

Theorem 4.15. *For $1 \leq k, j \leq N$, $(M_1)_N^{-1}$ is the symmetric tridiagonal matrix defined*

by

$$(M_1^{-1})_{kj} = \begin{cases} \dfrac{1}{a_1 - a_2} & \text{if } k = j = 1, \\ \dfrac{a_{k-1} - a_{k+1}}{(a_{k+1} - a_k)(a_k - a_{k-1})} & \text{if } 1 \neq k = j \neq N, \\ \dfrac{a_{N-1}}{a_N (a_{N-1} - a_N)} & \text{if } k = j = N, \\ \dfrac{1}{a_k - a_{k-1}} & \text{if } k = j + 1. \end{cases}$$

For the Cholesky decomposition, we have the following result.

Theorem 4.16. *For $k, j \geq 1$, C is the lower triangular matrix defined by*

$$C_{kj} = \begin{cases} \dfrac{a_k}{\sqrt{a_1}} & \text{if } j = 1, \\ \dfrac{a_k}{a_j a_{j-1}} \sqrt{a_j a_{j-1} (a_{j-1} - a_j)} & \text{if } j > 1. \end{cases}$$

We will give the LU-decomposition of $(M_1)_N^{-1}$ and the inverses of these factor matrices by the following theorems.

Theorem 4.17. *For $1 \leq k, j \leq N$,*

$$A_{kj} = \begin{cases} (-1)^{k+j} & \text{if } 0 \leq k - j \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_{kj} = \begin{cases} \dfrac{1}{a_N} & \text{if } k = j = N, \\ (-1)^{k+j} \dfrac{1}{a_k - a_{k+1}} & \text{if } 0 \leq j - k \leq 1 \text{ and } k \neq N, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.18. *For $1 \leq k, j \leq N$,*

$$A_{kj}^{-1} = \begin{cases} 1 & \text{if } k \geq j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_{kj}^{-1} = \begin{cases} a_N & \text{if } j = N, \\ a_j - a_{j+1} & \text{if } k \leq j < N, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In order to prove $M_1 = L \bullet U$, it is sufficient to show that

$$\sum_{d=1}^{\min(k,j)} L_{kd} U_{dj} = a_{\max(k,j)}.$$

Consider

$$\begin{aligned}\sum_{d=1}^{\min(k,j)} L_{kd} U_{dj} &= \frac{a_k}{a_1} a_j + \sum_{d=2}^{\min(k,j)} \frac{a_k}{a_d} \frac{a_j (a_{d-1} - a_d)}{a_{d-1}} \\ &= a_k a_j \left[\frac{1}{a_1} + \sum_{d=2}^{\min(k,j)} \left(\frac{1}{a_d} - \frac{1}{a_{d-1}} \right) \right] = \frac{a_k a_j}{a_{\min(k,j)}},\end{aligned}$$

which, by Lemma 4.3, equals $a_{\max(k,j)}$, as expected.

Define the matrix $T = [T_{kj}]$ with

$$T_{kj} = \begin{cases} 1 & \text{if } k \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$T_{kj}^{-1} = \begin{cases} (-1)^{k+j} & \text{if } 0 \leq k - j \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the proofs related to inverse matrices L^{-1}, U^{-1}, A_N^{-1} and B_N^{-1} follow from Proposition 4.2.

In order to prove the LU-decomposition of $(M_1)_N^{-1}$, it is sufficient to show that $(M_1)_N = B_N^{-1} \bullet A_N^{-1}$. Consider

$$\sum_{d=\max(k,j)}^{N} B_{kd}^{-1} A_{dj}^{-1} = \sum_{d=\max(k,j)}^{N-1} (a_d - a_{d+1}) + a_N = a_{\max(k,j)},$$

as desired.

For the Cholesky decomposition, consider

$$\sum_{d=1}^{\min(k,j)} C_{kd} C_{jd} = \frac{a_k a_j}{a_1} + \sum_{d=2}^{\min(k,j)} \frac{a_k a_j}{a_d a_{d-1}} (a_{d-1} - a_d) = a_{\max(k,j)},$$

which completes the proof (Note that it can be also derived by the help of Corollary 4.1).

Finally, in order to prove the inverse matrix, we have three cases: $j = 1$, $1 < j < N$ and $j = N$. For these cases, consider the following equalities, respectively.

$$\sum_{d=1}^{N}(M_1)_{kd}(M_1^{-1})_{d1} = \frac{a_{\max(k,1)}}{a_1 - a_2} + \frac{a_{\max(k,2)}}{a_2 - a_1} = [k = 1],$$
$$\sum_{d=1}^{N}(M_1)_{kd}(M_1^{-1})_{dj} = \frac{a_{\max(k,j-1)}}{a_j - a_{j-1}} + \frac{a_{\max(k,j)}(a_{j-1} - a_{j+1})}{(a_{j+1} - a_j)(a_j - a_{j-1})} + \frac{a_{\max(k,j+1)}}{a_{j+1} - a_j} = [k = j],$$
$$\sum_{d=1}^{N}(M_1)_{kd}(M_1^{-1})_{dN} = \frac{a_{\max(k,N-1)}}{a_N - a_{N-1}} + \frac{a_{N-1}a_{\max(k,N)}}{a_N(a_{N-1} - a_N)} = [k = N].$$

By all of them, the proof is complete. □

Corollary 4.5. *For $N \geq 2$,*

$$\det(M_1)_N = a_N \prod_{d=1}^{N-1}(a_d - a_{d+1})$$

and $\det(M_1)_1 = a_1$.

Proof. Since $\det(M_1)_N = \prod_{d=1}^{N} U_{dd}$, it is immediately seen. □

For example, let T_1 be the matrix defined by $[\max(k, j)]_{1 \leq k,j \leq N}$. Then

$$\det T_1 = (-1)^{N-1}N.$$

One can easily obtain many special and nice examples. The evaluation of the determinants of these kinds of matrices by using other methods needs more effort.

Remark 4.1. *If the sequence $\{a_n\}$ is positive and decreasing, then the matrix M_1 is a positive definite matrix, which can be easily seen by Corollary 4.5. On the other hand, the sequence $\{a_n\}$ is negative and increasing, then the matrix M_1 is a negative definite matrix.*

Now we shall give the results for the reciprocal Max matrix $\mathcal{M}_1$ without proof because all of them could be seen by choosing reciprocal term in the results for the matrix M_1. Studying with reciprocals could sometimes be more challenging. For this reason, we list the results for the quick access. We have the following results for the LU-decomposition, inverse matrix , Cholesky decomposition and LU-decomposition of the inverse matrix $\mathcal{M}_1^{-1}$, respectively.

Corollary 4.6. *For $k, j \geq 1$,*

$$\mathcal{L}_{kj} = \begin{cases} \dfrac{a_j}{a_k} & \text{if } k \geq j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{U}_{kj} = \begin{cases} \dfrac{1}{a_j} & \text{if } k = 1, \\ \dfrac{a_k - a_{k-1}}{a_j a_k} & \text{if } j \geq k > 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \leq k, j \leq N$, $(\mathcal{M}_1^{-1})_N$ is the symmetric tridiagonal matrix defined by

$$(\mathcal{M}_1^{-1})_{kj} = \begin{cases} \dfrac{a_1 a_2}{a_2 - a_1} & \text{if } k = j = 1, \\ \dfrac{a_k^2 (a_{k+1} - a_{k-1})}{(a_{k+1} - a_k)(a_k - a_{k-1})} & \text{if } 1 \neq k = j \neq N, \\ \dfrac{a_N^2}{a_N - a_{N-1}} & \text{if } k = j = N, \\ \dfrac{a_k a_j}{a_{k-1} - a_k} & \text{if } k = j + 1. \end{cases}$$

For $k, j \geq 1$, $\mathcal{C}$ is the lower triangular matrix defined by

$$\mathcal{C}_{kj} = \begin{cases} \dfrac{\sqrt{a_1}}{a_k} & \text{if } j = 1, \\ \dfrac{\sqrt{a_j - a_{j-1}}}{a_k} & \text{if } j > 1. \end{cases}$$

For $1 \leq k, j \leq N$,

$$\mathcal{A}_{kj} = \begin{cases} (-1)^{k+j} & \text{if } 0 \leq k - j \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{B}_{kj} = \begin{cases} (-1)^{k+j} \dfrac{a_{k+1} a_k}{a_{k+1} - a_k} & \text{if } 0 \leq j - k \leq 1, \\ a_N & \text{if } k = j = N, \\ 0 & \text{otherwise.} \end{cases}$$

4.4.2 Min Matrices and Their Reciprocal Analogues

Firstly, we list the LU-decomposition, inverse matrix, Cholesky decomposition of the matrix M_2 and LU-decomposition of the inverse matrix M_2^{-1}, respectively. We omit the results related to L^{-1}, U^{-1}, A^{-1} and B^{-1} here. They could be easily obtained as in the proof of the matrix M_1 by the help of Proposition 4.2.

Theorem 4.19. *Take the matrix M_2, for $k, j \geq 1$,*

$$L_{kj} = \begin{cases} 1 & \text{if } k \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{kj} = \begin{cases} a_1 & \text{if } k = 1, \\ a_k - a_{k-1} & \text{if } j \geq k > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$(M_2^{-1})_{kj} = \begin{cases} \dfrac{a_2}{a_1\,(a_2 - a_1)} & \text{if } k = j = 1, \\ \dfrac{(a_{k+1} - a_{k-1})}{(a_{k+1} - a_k)\,(a_k - a_{k-1})} & \text{if } 1 \neq k = j \neq N, \\ \dfrac{1}{a_N - a_{N-1}} & \text{if } k = j = N, \\ \dfrac{1}{a_{k-1} - a_k} & \text{if } k = j + 1, \end{cases}$$

$$C_{kj} = \begin{cases} \sqrt{a_1} & \text{if } j = 1, \\ \sqrt{a_j - a_{j-1}} & \text{if } j > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{kj} = \begin{cases} (-1)^{k+j} \dfrac{a_j}{a_k} & \text{if } 0 \leq k - j \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{kj} = \begin{cases} (-1)^{k+j} \dfrac{a_{k+1}}{a_j\,(a_{k+1} - a_k)} & \text{if } 0 \leq j - k \leq 1, \\ \dfrac{1}{a_N} & \text{if } k = j = N, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the inverse matrix M_2^{-1} is a symmetric tridiagonal matrix of order N.

Corollary 4.7. *Take the matrix $\mathcal{M}_2$, for $k, j \geq 1$, we have*

$$\mathcal{L}_{kj} = \begin{cases} 1 & \text{if } k \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{U}_{kj} = \begin{cases} \dfrac{1}{a_1} & \text{if } k = 1, \\ \dfrac{a_{k-1} - a_k}{a_k a_{k-1}} & \text{if } j \geq k > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\mathcal{M}_2^{-1})_{kj} = \begin{cases} \dfrac{a_1^2}{a_1 - a_2} & \text{if } k = j = 1, \\ \dfrac{a_k^2 (a_{k-1} - a_{k+1})}{(a_{k+1} - a_k)(a_k - a_{k-1})} & \text{if } 1 \neq k = j \neq N, \\ \dfrac{a_N a_{N-1}}{a_{N-1} - a_N} & \text{if } k = j = N, \\ \dfrac{a_k a_j}{a_k - a_{k-1}} & \text{if } k = j + 1, \end{cases}$$

$$\mathcal{C}_{kj} = \begin{cases} \dfrac{1}{\sqrt{a_1}} & \text{if } j = 1, \\ \dfrac{1}{a_j a_{j-1}} \sqrt{a_j a_{j-1} (a_{j-1} - a_j)} & \text{if } j > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_{kj} = \begin{cases} (-1)^{k+j} \dfrac{a_k}{a_j} & \text{if } 0 \leq k - j \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_{kj} = \begin{cases} (-1)^{k+j} \dfrac{a_k a_j}{a_j (a_k - a_{k+1})} & \text{if } 0 \leq j - k \leq 1, \\ a_N & \text{if } k = j = N, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, note that the inverse matrix $\mathcal{M}_2^{-1}$ is a symmetric tridiagonal matrix of order N.

Proof. By Lemma 4.3, we can write

$$a_{\min(k,j)} = \frac{a_k a_j}{a_{\max(k,j)}} \text{ and } \frac{1}{a_{\min(k,j)}} = \frac{a_{\max(k,j)}}{a_k a_j}.$$

So all claimed results follow by Proposition 4.2 and the results for the matrices M_1 and $\mathcal{M}_1$. □

By the LU-decomposition of the matrix M_2, we have the following corollary.

Corollary 4.8. *For $N \geq 1$,*

$$\det(M_2)_N = a_1 \prod_{d=1}^{N-1} (a_{d+1} - a_d).$$

Remark 4.2. *By the above corollary, it is seen that if a_1 is a positive real number and the sequence $\{a_n\}$ is increasing, then the matrix M_2 is a positive definite matrix. Conversely, if a_1 is a negative real number and the sequence $\{a_n\}$ is decreasing, then the matrix M_2 is a negative definite matrix.*

4.4.3 Further Applications

Recall the Lehmer matrix M defined by

$$M_{kj} = \frac{\min(k,j)}{\max(k,j)}.$$

By Lemma 4.3, one can write the (k,j)th entry of it as:

$$\frac{\min(k,j)}{\max(k,j)} = \frac{k \times j}{(\max(k,j))^2} = \frac{k \times j}{\max(k^2, j^2)}.$$

By using Proposition 4.2 and the results for the matrix $\mathcal{M}_1$ for $a_n = n^2$, i.e. $\max(k^2, j^2) = a_{\max(k,j)}$, it is easily rediscovered the LU-decomposition, inverse matrix and Cholesky decomposition of the Lehmer matrix. Also the results of [91, 92] for some recursive analogues of the Lehmer matrix can be retrieved by using similar approach.

Moreover, our results give us an idea to find a sequential generalization of the Lehmer matrix. For example, we define the matrix $H = [H_{kj}]$ for any positive and strictly increasing sequence $\{a_n\}$ by

$$H_{kj} = \frac{\min(a_k, a_j)}{\max(a_k, a_j)} = \frac{a_k a_j}{\max(a_k^2, a_j^2)}.$$

Thus by our results for the matrix $\mathcal{M}_1$ with the sequence $\{a_n^2\}$ and Proposition 4.2, the LU-decomposition, inverse and Cholesky decomposition of the matrix H could be derived but we omit the details here due to the similarities with the following corollary. Note that if the sequence $\{a_n\}$ is decreasing than we may also obtain sequential generalization of the Lehmer matrix by using the results for the matrix $\mathcal{M}_2$.

Up to now, any reciprocal analogue of the Lehmer matrix has not been studied yet. The following corollary will be the first reciprocal-sequential generalization of the Lehmer matrix.

Corollary 4.9. *Let $\{a_n\}$ be a positive and strictly increasing sequence and $\mathcal{H} = [\mathcal{H}_{kj}]$ be the matrix defined by*

$$\mathcal{H}_{kj} = \frac{\max(a_k, a_j)}{\min(a_k, a_j)}.$$

Then

$$\mathcal{L}_{kj} = \begin{cases} \dfrac{a_k}{a_j} & \text{if } k \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{U}_{kj} = \begin{cases} \dfrac{a_j}{a_1} & \text{if } k = 1, \\ \dfrac{a_j\,(b_{k-1} - b_k)}{a_k b_{k-1}} & \text{if } j \geq k > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{C}_{kj} = \begin{cases} \dfrac{a_k}{\sqrt{b_1}} & \text{if } j = 1, \\ \dfrac{a_k}{b_j b_{j-1}} \sqrt{b_j b_{j-1}\,(b_{j-1} - b_j)} & \text{if } j > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{H}^{-1}_{kj} = \begin{cases} \dfrac{b_1}{b_1 - b_2} & \text{if } k = j = 1, \\ \dfrac{b_k\,(b_{k-1} - b_{k+1})}{(b_{k+1} - b_k)\,(b_k - b_{k-1})} & \text{if } 1 \neq k = j \neq N, \\ \dfrac{b_{N-1}}{b_{N-1} - b_N} & \text{if } k = j = N, \\ \dfrac{a_k a_j}{b_k - b_{k-1}} & \text{if } k = j + 1, \end{cases}$$

where $\mathcal{H}^{-1}$ is a symmetric tridiagonal matrix of order N and $b_n = a_n^2$.

Proof. Since $\{a_n\}$ is a positive and strictly increasing, by Lemma 4.3, we have

$$\mathcal{H}_{kj} = \frac{\max(a_k, a_j)}{\min(a_k, a_j)} = \frac{a_k a_j}{b_{\min(k,j)}}.$$

So the proof follows by the results given for the matrix $\mathcal{M}_2$ with the sequence $\{b_n\} = \{a_n^2\}$ and Proposition 4.2. □

Note that when $a_n = n$, we get the reciprocal analogue of the usual Lehmer matrix. Besides, one can also obtain related results when the sequence $\{a_n\}$ is decreasing by the help of the results for the matrix $\mathcal{M}_1$.

Now we give an interesting and useful idea. Although, one can't directly use our main results to derive some results related to some certain kind of matrices, our results allow us to guess these results with less effort. We shall give an example to show this idea.

Corollary 4.10. *For a positive integer r, define the matrix $\mathcal{F} = [\mathcal{F}_{kj}]$ by*

$$\mathcal{F}_{kj} = \begin{cases} a_{\max(k,j)} & \text{if } k \geq j - r, \\ 0 & \text{otherwise.} \end{cases},$$

for a given sequence $\{a_n\}$. *Then for* $k, j \geq 1$, *the LU-decomposition of the matrix* $\mathcal{F}$ *is*

$$\mathcal{L}_{kj} = \begin{cases} \dfrac{a_k}{a_j} & \text{if } k \geq j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{U}_{kj} = \begin{cases} a_j & \text{if } k = 1 \text{ and } j \leq r+1, \\ a_j & \text{if } j > r+1 \text{ and } k = j-r, \\ \dfrac{a_j\,(a_{k-1} - a_k)}{a_{k-1}} & \text{if } k + r - 1 > j \geq k > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, for $N = 8$ and $r = 3$, the matrix $\mathcal{F}$ takes the form

$$\mathcal{F}_8 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 \\ a_2 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & 0 \\ a_3 & a_3 & a_3 & a_4 & a_5 & a_6 & 0 & 0 \\ a_4 & a_4 & a_4 & a_4 & a_5 & a_6 & a_7 & 0 \\ a_5 & a_5 & a_5 & a_5 & a_5 & a_6 & a_7 & a_8 \\ a_6 & a_6 & a_6 & a_6 & a_6 & a_6 & a_7 & a_8 \\ a_7 & a_7 & a_7 & a_7 & a_7 & a_7 & a_7 & a_8 \\ a_8 & a_8 & a_8 & a_8 & a_8 & a_8 & a_8 & a_8 \end{bmatrix}.$$

As seen, the matrix $\mathcal{F}$ is obtained from the matrix M_1 by deleting the entries after rth superdiagonal. (Note that similar example can be obtained for the matrix which is obtained by applying the same process to the matrix M_2).

Proof. We should show that

$$\mathcal{F}_{kj} = \sum_{d=1}^{\min(k,j)} \mathcal{L}_{kd}\mathcal{U}_{dj}.$$

The proof for the case $j \leq r+1$ can be similarly done as in the proof of the results for the matrix M_1. Now consider for $j > r+1$ and $k > j-r$,

$$\begin{aligned} \sum_{d=1}^{\min(k,j)} \mathcal{L}_{kd}\mathcal{U}_{dj} &= \frac{a_k a_j}{a_{j-r}} + \sum_{d=j-r+1}^{\min(k,j)} \mathcal{L}_{kd}\mathcal{U}_{dj} = \frac{a_k a_j}{a_{j-r}} + a_k a_j \sum_{d=j-r+1}^{\min(k,j)} \left(\frac{1}{a_d} - \frac{1}{a_{d-1}}\right) \\ &= \frac{a_k a_j}{a_{\min(k,j)}} = a_{\max(k,j)}. \end{aligned}$$

And the final case $j > r+1$ and $k \leq j-r$ can be easily computed as 0, which completes the proof. □

It is possible to obtain similar results for the matrix which is derived by deleting the entries of Max matrix (or Min matrix) after rth subdiagonal by the help of our results. In the light of all these, our results are very useful to define new families of combinatorial matrices.

As a conclusion remark, our results cover the results which will be valid for the matrices $[\max(a_k, a_j)]_{k,j\geq 1}$ and $[\min(a_k, a_j)]_{k,j\geq 1}$ when the sequence $\{a_n\}$ is increasing or decreasing. Note that the increasing case is given in [5] for only finite order matrices with a different approach. Unfortunately, if a sequence $\{c_n\}$ is neither increasing nor decreasing, such as unimodal sequences, then our results don't work for the matrices $[\max(c_k, c_j)]_{k,j\geq 1}$ and $[\min(c_k, c_j)]_{k,j\geq 1}$ and we could not find explicit results for such matrices.

4.5 A Non-symmetric Variant of the Filbert Matrix

As mentioned in Section 3.2.4, in this section we will study a new non-symmetric variant of Filbert matrix. We define our main matrix $M = [M_{kj}]_{k,j\geq 0}$ by

$$M_{kj} = \frac{1 - xq^{\lambda k - \mu j}}{1 - xq^{\lambda k + \mu j}},$$

where λ and μ are positive integers and x is a real number such that $x \neq q^{-\lambda k - \mu j}$ for all $k, j \geq 0$. Here we would like to point out indexes of the entries of the matrix M start from 0. Otherwise related matrix would be singular that makes no sense to study.

We will derive explicit formulæ for the LU-decomposition, inverse matrices L^{-1} and U^{-1}, and inverse of the matrix M. Our approach is mainly to guess the relevant quantities. Afterwards, we will provide proofs of these formulæ. It is worthwhile to note that, although all the sum identities we need to prove seem to be Gosper-summable, the q-Zeilberger algorithm does not work for the general parameters λ and μ. The algorithm can only compute the specialized sums for some fixed special numerical values of λ and μ. But it is not enough to prove general results. For this reason, we will use some traditional ways which cause to guess some new sum identities with one additional parameter. Finally, as applications, we will give some particular results related to the generalized Fibonacci and Lucas numbers as non-symmetric variants of Filbert and Lilbert matrices for the special choices of q and x.

4.5.1 Main Results

Here, we will list the LU-decomposition of the matrix M, inverse matrices L^{-1}, U^{-1}, determinant and inverse matrix M^{-1}. In Section 4.5.2, we will provide the proofs of these results. Recall that the matrix M would be never a symmetric matrix for any choice of the parameters. For this reason, we can not talk about the Cholesky decomposition.

Theorem 4.20. *For $k, j \geq 0$,*

$$L_{kj} = \frac{(xq^{\lambda j+\mu};q^{\mu})_j (q^{\lambda(k-j+1)};q^{\lambda})_j}{(xq^{\lambda k+\mu};q^{\mu})_j (q^{\lambda};q^{\lambda})_j}$$

and

$$U_{kj} = \begin{cases} \dfrac{1-xq^{-\mu j}}{1-xq^{\mu j}} & \text{if } k=0, \\ q^{-\mu j+(\lambda+\mu)\binom{k}{2}} x^k (1+q^{\mu j}) \dfrac{(q^{\mu(j-k+1)};q^{\mu})_k (q^{\lambda};q^{\lambda})_k}{(xq^{\mu j};q^{\lambda})_{k+1} (xq^{\lambda k+\mu};q^{\mu})_{k-1}} & \text{if } k>0. \end{cases}$$

Theorem 4.21. *For $N > 1$,*

$$\det M_N = x^{\binom{N}{2}} q^{-\mu\binom{N}{2}} \frac{(-q^{\mu};q^{\mu})_{N-1} (xq^{\lambda};q^{\lambda})_{N-1}}{(xq^{\lambda+\mu};q^{\lambda+\mu})_{N-1}} \prod_{d=1}^{N-1} q^{(\lambda+\mu)\binom{d}{2}} \frac{(q^{\mu};q^{\mu})_d (q^{\lambda};q^{\lambda})_d}{(xq^{\mu d};q^{\lambda})_d (xq^{\lambda d};q^{\mu})_d}$$

and $\det M_1 = 1$.

Theorem 4.22. *For $k, j \geq 0$,*

$$L^{-1}_{kj} = (-1)^{k+j} q^{\lambda\binom{k-j}{2}} \frac{(xq^{\lambda j+\mu};q^{\mu})_{k-1} (q^{\lambda(k-j+1)};q^{\lambda})_j}{(xq^{\lambda k+\mu};q^{\mu})_{k-1} (q^{\lambda};q^{\lambda})_j}$$

and

$$U^{-1}_{kj} = \begin{cases} 1 & \text{if } k=j=0, \\ \dfrac{q^{-\lambda\binom{j}{2}} (-1)^{j+1} (xq^{\lambda j+\mu};q^{\mu})_j}{x^j (q^{\lambda};q^{\lambda})_j} & \\ \quad \times \displaystyle\sum_{t=1}^{j} \frac{q^{\mu(\binom{t+1}{2}+t-tj)} (-1)^t (1-xq^{-\mu t})(xq^{\mu t};q^{\lambda})_j}{(1-xq^{\mu t})(1-q^{2\mu t})(q^{\mu};q^{\mu})_{j-t} (q^{\mu};q^{\mu})_{t-1}} & \text{if } j \geq 1 \text{ and } k=0, \\ (-1)^{k+j} \dfrac{q^{-\lambda\binom{j}{2}+\mu(\binom{k+1}{2}+k-kj)}}{x^j (1-q^{2\mu k})} \dfrac{(xq^{\mu k};q^{\lambda})_j (xq^{\lambda j+\mu};q^{\mu})_j}{(q^{\lambda};q^{\lambda})_j (q^{\mu};q^{\mu})_{j-k} (q^{\mu};q^{\mu})_{k-1}} & \text{if } j \geq k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For the inverse matrix M_N^{-1}, we have the following result.

Theorem 4.23. *For $1 \le k < N$ and $0 \le j < N$,*

$$M_{kj}^{-1} = \frac{(-1)^{k+j}}{x^{N-1}} \frac{q^{\lambda\binom{j}{2}+\mu\binom{k+1}{2}-(N-2)(\lambda j+\mu k)}}{(1-xq^{\mu k+\lambda j})(1-q^{2\mu k})} \times \frac{(xq^{\lambda j+\mu};q^{\mu})_{N-1}(xq^{\mu k};q^{\lambda})_N}{(q^{\mu};q^{\mu})_{N-k-1}(q^{\lambda};q^{\lambda})_{N-j-1}(q^{\lambda};q^{\lambda})_j(q^{\mu};q^{\mu})_{k-1}}$$

and for $0 \le j < N$,

$$M_{0j}^{-1} = [j=0] + (-1)^{j+1} q^{\lambda\binom{j}{2}-\lambda(N-2)j} \frac{x^{N-1}(xq^{\lambda j+\mu};q^{\mu})_{N-1}}{(q^{\lambda};q^{\lambda})_{N-j-1}(q^{\lambda};q^{\lambda})_j} \times \sum_{t=1}^{N-1} \frac{1-xq^{-\mu t}}{1-xq^{\mu t}} \frac{(-1)^t q^{\mu\binom{t+1}{2}-\mu(N-2)t}}{1-q^{2\mu t}} \frac{(xq^{\mu t};q^{\lambda})_N}{(1-xq^{\mu t+\lambda j})(q^{\mu};q^{\mu})_{N-t-1}(q^{\mu};q^{\mu})_{t-1}}.$$

Here note that the entries except the first row of the matrices U^{-1} and M^{-1} can be nicely factorized but the first row of these matrices can't be factorized. The guessing procedure of these entries is extremely time consuming.

Since our matrix is not symmetric, the results related to its transposed matrix $M^T = \left[\frac{1-xq^{\lambda j-\mu k}}{1-xq^{\lambda j+\mu k}}\right]_{k,j\ge 0}$ may yield new results for a new matrix family. By Propositions 4.2 and 4.3 and the fact $(M^T)^{-1} = (M^{-1})^T$, we have the following theorem for the matrix M^T.

Theorem 4.24. *For $k, j \ge 0$,*

$$L_{kj} = \begin{cases} \dfrac{1-xq^{-\mu k}}{1-xq^{\mu k}} & \text{if } j = 0, \\ q^{\mu(j-k)} \dfrac{(1+q^{\mu k})}{(1+q^{\mu j})} \dfrac{(q^{\mu(k-j+1)};q^{\mu})_j (xq^{\mu j};q^{\lambda})_{j+1}}{(xq^{\mu k};q^{\lambda})_{j+1}(q^{\mu};q^{\mu})_j} & \text{if } j > 0 \end{cases}$$

and

$$U_{kj} = q^{-\mu k+(\lambda+\mu)\binom{k}{2})} x^k (1+q^{\mu k}) \frac{(q^{\mu};q^{\mu})_k (q^{\lambda(j-k+1)};q^{\lambda})_k}{(xq^{\lambda j+\mu};q^{\mu})_k (xq^{\mu k};q^{\lambda})_k}.$$

For $k, j \ge 0$,

$$L_{kj}^{-1} = \begin{cases} 1 & \text{if } k = j = 0, \\ \dfrac{q^{-\mu k+\mu\binom{k}{2}}(-1)^{k+1}(1+q^{\mu k})(q^{\mu};q^{\mu})_k}{(xq^{\mu k};q^{\lambda})_k} \\ \quad \times \displaystyle\sum_{t=1}^{k} \frac{q^{\mu(\binom{t+1}{2}+t-tk)}(-1)^t(1-xq^{-\mu t})(xq^{\mu t};q^{\lambda})_k}{(1-xq^{\mu t})(1-q^{2\mu t})(q^{\mu};q^{\mu})_{k-t}(q^{\mu};q^{\mu})_{t-1}} & \text{if } k \ge 1 \text{ and } j = 0, \\ \dfrac{(-1)^{k+j}(1+q^{\mu k})}{q^{-\mu\binom{k}{2}-\mu(\binom{j+1}{2}+j-k-kj)}(1-q^{2\mu j})} \dfrac{(q^{\mu(k-j+1)};q^{\mu})_j (xq^{\lambda k+\mu};q^{\mu})_k}{(q^{\mu};q^{\mu})_{j-1}(xq^{\lambda j+\mu};q^{\mu})_{k-1}} & \text{if } k \ge j \ge 1, \\ 0 & \text{otherwise} \end{cases}$$

and for $j \geq k$,

$$U^{-1}_{kj} = \frac{(-1)^{k+j} q^{\mu j - \mu\binom{j}{2} + \lambda\binom{k+1}{2} - \lambda k j}}{x^j (1+q^{\mu j})} \frac{(xq^{\mu j};q^\lambda)_{j+1} (xq^{\lambda k+\mu};q^\mu)_{j-1}}{(q^\mu;q^\mu)_j (q^\lambda;q^\lambda)_{j-k} (q^\lambda;q^\lambda)_k}$$

and 0 *otherwise. Finally, for* $1 \leq j < N$ *and* $0 \leq k < N$,

$$\begin{aligned}(M^T)^{-1}_{kj} = \frac{(-1)^{k+j}}{x^{N-1}} & \frac{q^{\lambda\binom{k}{2} + \mu\binom{j+1}{2} - (N-2)(\lambda k + \mu j)}}{(1-xq^{\mu j+\lambda k})(1-q^{2\mu j})} \\ & \times \frac{(xq^{\lambda k+\mu};q^\mu)_{N-1} (xq^{\mu j};q^\lambda)_N}{(q^\mu;q^\mu)_{N-j-1} (q^\lambda;q^\lambda)_{N-k-1} (q^\lambda;q^\lambda)_k (q^\mu;q^\mu)_{j-1}}\end{aligned}$$

and for $0 \leq k < N$,

$$\begin{aligned}(M^T)^{-1}_{k0} = [k=0] + (-1)^{k+1} q^{\lambda\binom{k}{2} - \lambda(N-2)k} & \frac{x^{N-1} (xq^{\lambda k+\mu};q^\mu)_{N-1}}{(q^\lambda;q^\lambda)_{N-k-1} (q^\lambda;q^\lambda)_k} \\ \times \sum_{t=1}^{N-1} \frac{1-xq^{-\mu t}}{1-xq^{\mu t}} & \frac{(-1)^t q^{\mu\binom{t+1}{2} - \mu(N-2)t}}{1-q^{2\mu t}} \frac{(xq^{\mu t};q^\lambda)_N}{(1-xq^{\mu t+\lambda k}) (q^\mu;q^\mu)_{N-t-1} (q^\mu;q^\mu)_{t-1}}.\end{aligned}$$

Now we can proceed with the proofs of the above theorems.

4.5.2 Proofs

Define the following four sums:

$$S_1(K) = \sum_{d=K}^{\min(k,j)} q^{(\lambda+\mu)\binom{d}{2}} x^d (1-xq^{d(\lambda+\mu)}) \frac{(q^{\lambda(k-d+1)};q^\lambda)_d (q^{\mu(j-d+1)};q^\mu)_{d-1}}{(xq^{\lambda k+\mu};q^\mu)_d (xq^{\mu j};q^\lambda)_{d+1}},$$

$$S_2(K) = \sum_{d=j}^{K} (-1)^d q^{\lambda\binom{d-j}{2}} (1-xq^{d(\lambda+\mu)}) \frac{(xq^{\lambda j+\mu};q^\mu)_{d-1} (q^{\lambda(k-d)+\lambda};q^\lambda)_d}{(xq^{\lambda k+\mu};q^\mu)_d (q^\lambda;q^\lambda)_{d-j}},$$

$$S_3(K) = \sum_{d=k}^{K} (-1)^d q^{\mu\binom{d}{2} - \mu k d} (1-xq^{d(\lambda+\mu)}) \frac{(xq^{\mu k};q^\lambda)_d (q^{\mu(j-d)+\mu};q^\mu)_d}{(xq^{\mu j};q^\lambda)_{d+1} (q^\mu;q^\mu)_{d-k}}$$

and

$$S_4(K) = \sum_{d=\max(k,j)}^{K} q^{-\mu k d - \lambda j d} x^{-d} (1-xq^{d(\lambda+\mu)}) \frac{(xq^{\mu k};q^\lambda)_d (xq^{\lambda j+\mu};q^\mu)_{d-1}}{(q^\mu;q^\mu)_{d-k} (q^\lambda;q^\lambda)_{d-j}}.$$

We provide the following lemmas for later use.

Lemma 4.4.

$$S_1(K) = x^K q^{(\lambda+\mu)\binom{K}{2}} \frac{(q^{\lambda(k-K+1)};q^\lambda)_K (q^{\mu(j-K+1)};q^\mu)_{K-1}}{(1-xq^{\lambda k+\mu j}) (xq^{\mu j};q^\lambda)_K (xq^{\lambda k+\mu};q^\mu)_{K-1}}.$$

Proof. We will use the backward induction method. Let us denote the summand term by s_d for brevity.

Firstly, assume that $k \geq j$ so when $K = j$ the claim is obvious. Similarly for the case $j > k$, the initial claim is clear.

The backward induction step amounts to show that

$$\begin{aligned}
&S_1(K-1)\\
&= S_1(K) + s_{K-1}\\
&= x^K q^{(\lambda+\mu)\binom{K}{2}} \frac{(q^{\lambda(k-K+1)};q^\lambda)_K (q^{\mu(j-K+1)};q^\mu)_{K-1}}{(1-xq^{\lambda k+\mu j})(xq^{\mu j};q^\lambda)_K (xq^{\lambda k+\mu};q^\mu)_{K-1}}\\
&+ q^{(\lambda+\mu)\binom{K-1}{2}} x^{K-1}(1-xq^{(K-1)(\lambda+\mu)}) \frac{(q^{\lambda(k-K+2)};q^\lambda)_{K-1} (q^{\mu(j-K+2)};q^\mu)_{K-2}}{(xq^{\lambda k+\mu};q^\mu)_{K-1} (xq^{\mu j};q^\lambda)_K}\\
&= x^{K-1} q^{(\lambda+\mu)\binom{K-1}{2}} \frac{(q^{\lambda(k-K+2)};q^\lambda)_{K-1} (q^{\mu(j-K+2)};q^\mu)_{K-2}}{(1-xq^{\lambda k+\mu j})(xq^{\mu j};q^\lambda)_K (xq^{\lambda k+\mu};q^\mu)_{K-1}}\\
&\times \big(xq^{(\lambda+\mu)(K-1)}(1-q^{\lambda j-\lambda(K-1)})(1-q^{\mu j-\mu k+\mu}) + (1-xq^{\lambda k+\mu j})(1-xq^{(K-1)(\lambda+\mu)})\big).
\end{aligned}$$

After some simplifications, the expression in the last line can be rewritten as

$$(1-xq^{\lambda k+\mu(K-1)})(1-xq^{\mu j+\lambda(K-1)}).$$

Finally,

$$S_1(K-1) = x^{K-1} q^{(\lambda+\mu)\binom{K-1}{2}} \frac{(q^{\lambda(k-K+2)};q^\lambda)_{K-1} (q^{\mu(j-K+2)};q^\mu)_{K-2}}{(1-xq^{\lambda k+\mu j})(xq^{\mu j};q^\lambda)_{K-1} (xq^{\lambda k+\mu};q^\mu)_{K-2}},$$

which completes the proof. □

Lemma 4.5. *For $k > j$,*

$$S_2(K) = (-1)^K q^{\lambda\binom{K-j+1}{2}} \frac{(xq^{\lambda j+\mu};q^\mu)_K (q^{\lambda(k-K)};q^\lambda)_{K+1}}{(1-q^{\lambda(k-j)})(xq^{\lambda k+\mu};q^\mu)_K (q^\lambda;q^\lambda)_{K-j}}.$$

Proof. This time we will use the usual induction method. Similarly, we denote the summand term by s_d. The initial case $K = j$ is easily verified. So, the induction step amounts to show that

$$S_2(K+1) = S_2(K) + s_{K+1}.$$

Consider

$$S_2(K) + s_{K+1} = (-1)^K q^{\lambda\binom{K-j+1}{2}} \frac{(xq^{\lambda j+\mu};q^\mu)_K (q^{\lambda(k-K)};q^\lambda)_{K+1}}{(1-q^{\lambda(k-j)})(xq^{\lambda k+\mu};q^\mu)_K (q^\lambda;q^\lambda)_{K-j}}$$

$$
\begin{aligned}
&+(-1)^{K+1}q^{\lambda\binom{K+1-j}{2}}(1-xq^{(K+1)(\lambda+\mu)})\frac{(xq^{\lambda j+\mu};q^{\mu})_K(q^{\lambda(k-K)};q^{\lambda})_{K+1}}{(xq^{\lambda k+\mu};q^{\mu})_{K+1}(q^{\lambda};q^{\lambda})_{K+1-j}}\\
&=(-1)^{K+1}q^{\lambda\binom{K-j+1}{2}}\frac{(xq^{\lambda j+\mu};q^{\mu})_K(q^{\lambda(k-K)};q^{\lambda})_{K+1}}{(1-q^{\lambda(k-j)})(xq^{\lambda k+\mu};q^{\mu})_{K+1}(q^{\lambda};q^{\lambda})_{K+1-j}}\\
&\times\big((1-xq^{(K+1)(\lambda+\mu)})(1-q^{\lambda(k-j)})-(1-xq^{\lambda k+\mu(K+1)})(1-q^{\lambda(K+1-j)})\big)\\
&=(-1)^{K+1}q^{\lambda\binom{K-j+1}{2}}\frac{(xq^{\lambda j+\mu};q^{\mu})_K(q^{\lambda(k-K)};q^{\lambda})_{K+1}}{(1-q^{\lambda(k-j)})(xq^{\lambda k+\mu};q^{\mu})_{K+1}(q^{\lambda};q^{\lambda})_{K+1-j}}\\
&\times q^{\lambda(K-j+1)}(1-xq^{\lambda j+\mu(K+1)})(1-q^{\lambda(k-K)-\lambda}),
\end{aligned}
$$

which is equal to $S_2(K+1)$, as desired. □

Lemma 4.6. *For $j>k$,*

$$
S_3(K)=(-1)^K q^{\mu\binom{K}{2}+\mu(K(1-k)-k)}\frac{(xq^{\mu k};q^{\lambda})_{K+1}(q^{\mu(j-k+1)};q^{\mu})_k(q^{\mu(j-K)};q^{\mu})_{K-k}}{(xq^{\mu j};q^{\lambda})_{K+1}(q^{\mu};q^{\mu})_{K-k}}.
$$

The proof of Lemma 4.6 can be similarly done as in the proof of Lemma 4.5. We do not give it to avoid repetition.

Lemma 4.7.

$$
S_4(K)=q^{-\mu Kk-\lambda Kj}x^{-K}\frac{(xq^{\lambda j+\mu};q^{\mu})_K(xq^{\mu k};q^{\lambda})_{K+1}}{(1-xq^{\mu k+\lambda j})(q^{\mu};q^{\mu})_{K-k}(q^{\lambda};q^{\lambda})_{K-j}}.
$$

Proof. Consider the case $k\geq j$,

$$
\begin{aligned}
S_4(k)&=\sum_k^k q^{-\mu k^2-\lambda jk}x^{-k}(1-xq^{k(\lambda+\mu)})\frac{(xq^{\mu k};q^{\lambda})_k(xq^{\lambda j+\mu};q^{\mu})_{k-1}}{(q^{\lambda};q^{\lambda})_{k-j}}\\
&=q^{-\mu k^2-\lambda kj}x^{-k}\frac{(xq^{\lambda j+\mu};q^{\mu})_k(xq^{\mu k};q^{\lambda})_{k+1}}{(1-xq^{\mu k+\lambda j})(q^{\lambda};q^{\lambda})_{k-j}},
\end{aligned}
$$

as claimed. Similarly, the case $k<j$ is obvious. Thus the initial claim is completed. Let's denote the summand by s_d. Then $S_4(K)+s_{K+1}$ equals

$$
\begin{aligned}
&q^{-\mu kK-\lambda jK}x^{-K}\frac{(xq^{\lambda j+\mu};q^{\mu})_K(xq^{\mu k};q^{\lambda})_{K+1}}{(1-xq^{\mu k+\lambda j})(q^{\mu};q^{\mu})_{K-k}(q^{\lambda};q^{\lambda})_{K-j}}\\
&+q^{-\mu k(K+1)-\lambda j(K+1)}x^{-K-1}(1-xq^{(K+1)(\lambda+\mu)})\frac{(xq^{\lambda j+\mu};q^{\mu})_K(xq^{\mu k};q^{\lambda})_{K+1}}{(q^{\mu};q^{\mu})_{K+1-k}(q^{\lambda};q^{\lambda})_{K+1-j}}\\
=&q^{-\mu k(K+1)-\lambda j(K+1)}x^{-K-1}\frac{(xq^{\lambda j+\mu};q^{\mu})_K(xq^{\mu k};q^{\lambda})_{K+1}}{(1-xq^{\mu k+\lambda j})(q^{\mu};q^{\mu})_{K-k}(q^{\lambda};q^{\lambda})_{K-j}}\\
&+\big(xq^{\mu k+\lambda j}(1-q^{\mu(K+1-k)})(1-q^{\lambda(K+1-j)})+(1-xq^{\mu k+\lambda j})(1-xq^{(K+1)(\lambda+\mu)})\big).
\end{aligned}
$$

After simplifications, the last line of the above equation is

$$
(1-xq^{\lambda j+\mu(K+1)})(1-xq^{\mu k+\lambda(K+1)}).
$$

Finally, $S_4(K)+s_{K+1}=S_4(K+1)$, which completes the proof. □

Now we can give the proofs of our main results.

For the LU-decomposition of the matrix M, we have to prove that

$$\sum_{0\le d\le \min(k,j)} L_{kd}U_{dj} = M_{kj}.$$

By Lemma 4.4, we obtain

$$\begin{aligned}\sum_{0\le d\le \min(k,j)} L_{kd}U_{dj} &= \frac{1-xq^{-\mu j}}{1-xq^{\mu j}} + q^{-\mu j}(1-q^{2\mu j})S_1(1)\\ &= \frac{1-xq^{-\mu j}}{1-xq^{\mu j}} + xq^{-\mu j}\frac{(1-q^{2\mu j})(1-q^{\lambda k})}{(1-xq^{\lambda k+\mu j})(1-xq^{\mu j})}\\ &= \frac{1-xq^{\lambda k-\mu j}}{1-xq^{\lambda k+\mu j}},\end{aligned}$$

which completes the proof. As mentioned before, the q-Zeilberger algorithm can only compute the related sum for the special numerical values of λ and μ. For example, when $\lambda = \mu = 1$, the algorithm computes the sum $S_1(1)$ as

$$\frac{x(1-q^k)}{(1-xq^j)(1-xq^{k+j})}.$$

Then the claim is done for the case $\lambda = \mu = 1$. But in general we can not use this algorithm to prove the claim. That is why we need the previous lemmas.

For L and L^{-1}, it is obvious that $L_{kk}L^{-1}_{kk} = 1$. For $k > j$,

$$\sum_{j\le d\le k} L_{kd}L^{-1}_{dj} = \frac{(-1)^j}{(q^\lambda;q^\lambda)_j}S_2(k),$$

which equals 0 by Lemma 4.5. So we conclude

$$\sum_{j\le d\le k} L_{kd}L^{-1}_{dj} = [k=j],$$

as desired.

Before moving on, notice that the matrices U^{-1} and M^{-1} can be also written as follows:

$$U^{-1} = PR \text{ and } M_N^{-1} = P_N E_N,$$

where the matrix P is defined by

$$P_{00} = 1 \text{ and } P_{0j} = -\frac{1-xq^{-\mu j}}{1-xq^{\mu j}} \quad \text{for } j > 0,$$

$$P_{kj} = [k=j] \quad \text{for } j \ge 0 \text{ and } k \ge 1,$$

and

$$R_{00} = 1 \text{ and } R_{0j} = 0 \text{ for } j > 0 \text{ and } R_{kj} = U_{kj}^{-1} \text{ for } j \geq k \geq 1,$$
$$E_{00} = 1 \text{ and } E_{0j} = 0 \text{ for } 0 < j < N, \text{ and, } E_{kj} = M_{kj}^{-1} \text{ otherwise.}$$

It is easily seen that the inverse matrix P^{-1} is computed as

$$P_{00}^{-1} = 1 \text{ and } P_{0j}^{-1} = \frac{1 - xq^{-\mu j}}{1 - xq^{\mu j}} \text{ for } j > 0,$$
$$P_{kj}^{-1} = [k = j] \text{ for } j \geq 0 \text{ and } k \geq 1.$$

In order to show that $U^{-1}U = I$, we will show that $PRU = I$. Consider the product matrix RU. The first row of this matrix is the same as the first row of the matrix U. Then for $k \geq 1$, obviously $R_{kk}U_{kk} = 1$, so when $k \neq j$ we have

$$\sum_{k \leq d \leq j} R_{kd}U_{dj} = (-1)^k q^{-\mu j + \frac{1}{2}\mu k(k+3)} \frac{1 + q^{\mu j}}{(1 - q^{2\mu k})(q^{\mu}; q^{\mu})_{k-1}} S_3(j) = 0,$$

which gives $RU = P^{-1}$, so the claim follows.

Finally, for the inverse matrix M_N^{-1}, we use the fact $M_N^{-1} = U_N^{-1}L_N^{-1} = P_N R_N L_N^{-1}$. The first row of the multiplication $R_N L_N^{-1}$ is $[j = 0]$ for $0 \leq j \leq N - 1$. For $k \geq 1$, by Lemma 4.7, we obtain

$$\sum_{\max(k,j) \leq d \leq N-1} R_{kd}L_{dj}^{-1} = \frac{(-1)^{k+j} q^{\mu\binom{k+1}{2} + \lambda\binom{j}{2}}}{(1 - q^{2\mu j})(q^{\lambda}; q^{\lambda})_j (q^{\mu}; q^{\mu})_{k-1}} S_4(N-1) = M_{kj}^{-1}.$$

So $R_N L_N^{-1} = E_N$, which completes the proof.

As a consequence of the LU-decomposition, the determinant of M_N is easily evaluated as the product of the diagonal entries of the matrix U. So the claim follows after some simplifications. The results for the matrix M^T follow by Proposition 4.3 after performing some simplifications.

4.5.3 Applications

In this subsection, we will give some applications of our main results. For example, consider the matrix $\mathcal{F}$, defined by

$$\mathcal{F}_{kj} = \frac{U_{\lambda k - \mu j + r}}{U_{\lambda k + \mu j + r}}$$

for positive integers λ, μ and r, where $\{U_n\}$ is the generalized Fibonacci sequence. By (2.6), the entries of the matrix $\mathcal{F}$ can be rewritten as

$$\mathcal{F}_{kj} = q^{\mu j}(-1)^{\mu j}\frac{1-q^{\lambda k-\mu j+r}}{1-q^{\lambda k+\mu j+r}},$$

where $q = \beta/\alpha$. As seen, it is not directly obtained from the matrix M. However for $x = q^r$ and $q = \beta/\alpha$, we can write

$$\mathcal{F} = M \bullet D(q^{\mu n}(-1)^{\mu n}),$$

where $D(a_n)$ is the diagonal matrix defined as before. So by Proposition 4.2, we can easily derive all related results for the matrix $\mathcal{F}$ from the results of the matrix M.

Note that an interesting feature of the matrix $\mathcal{F}$ is that it includes some zero terms as entries. Especially, when $\lambda = \mu = 1$, then the entries on the rth superdiagonal are all zero. Similarly, we can find the results for the transposed matrix $\mathcal{F}^T$. This transposed matrix have zeros on the rth subdiagonal when $\lambda = \mu = 1$.

We find the LU-decomposition of the matrix $\mathcal{F}$ and inverse matrices L^{-1}, U^{-1} and $\mathcal{F}^{-1}$ as follows:

Corollary 4.11. *For $k, j \geq 0$,*

$$\mathcal{L}_{kj} = \frac{\left(\prod_{d=1}^{j} U_{\lambda j+\mu d+r}\right)\left(\prod_{d=1}^{j} U_{\lambda(k+1)-\lambda d}\right)}{\left(\prod_{d=1}^{j} U_{\lambda k+\mu d+r}\right)\left(\prod_{d=1}^{j} U_{\lambda d}\right)}, \tag{4.5}$$

$$\mathcal{U}_{kj} = \begin{cases} \dfrac{U_{-\mu j+r}}{U_{\mu j+r}} & \text{if } k = 0, \\ (-1)^{\mu j+(\lambda+\mu)\binom{k}{2}+rk} U_{2\mu j}\dfrac{\left(\prod_{d=1}^{k-1} U_{\mu j-\mu d}\right)\left(\prod_{d=1}^{k} U_{\lambda d}\right)}{\left(\prod_{d=1}^{k+1} U_{\mu j+\lambda(d-1)+r}\right)\left(\prod_{d=1}^{k-1} U_{\lambda k+\mu d+r}\right)} & \text{if } k > 0, \end{cases} \tag{4.6}$$

$$\mathcal{L}^{-1}_{kj} = (-1)^{k+j+\lambda\binom{k-j}{2}}\frac{\left(\prod_{d=1}^{k-1} U_{\lambda j+\mu d+r}\right)\left(\prod_{d=1}^{j} U_{\lambda(k+1)-\lambda d}\right)}{\left(\prod_{d=1}^{k-1} U_{\lambda k+\mu d+r}\right)\left(\prod_{d=1}^{j} U_{\lambda d}\right)},$$

and $\mathcal{U}_{kj}^{-1} =$

$$\begin{cases} 1 & \text{if } k = j = 0, \\ (-1)^{k+j+\lambda\binom{j}{2}+rj+\mu kj+\mu\binom{k}{2}} \dfrac{1}{U_{2\mu k}} \dfrac{\left(\prod_{d=1}^{j} U_{\mu k+\lambda(d-1)+r}\right)\left(\prod_{d=1}^{j} U_{\lambda j+\mu d+r}\right)}{\left(\prod_{d=1}^{j} U_{\lambda d}\right)\left(\prod_{d=1}^{j-k} U_{\mu d}\right)\left(\prod_{d=1}^{k-1} U_{\mu d}\right)} & \text{if } j \geq k \geq 1, \end{cases}$$

and for $j \geq 1$,

$$\mathcal{U}_{0j}^{-1} = (-1)^{j+1+rj+\lambda\binom{j}{2}} \frac{\left(\prod_{d=1}^{j} U_{\lambda j+\mu d+r}\right)}{\left(\prod_{d=1}^{j} U_{\lambda d}\right)} \times \sum_{t=1}^{j} (-1)^{t+\mu tj+\mu\binom{t}{2}} \frac{U_{-\mu t+r}}{U_{2\mu t} U_{\mu t+r}} \frac{\left(\prod_{d=1}^{j} U_{\mu t+\lambda(d-1)+r}\right)}{\left(\prod_{d=1}^{j-t} U_{\mu d}\right)\left(\prod_{d=1}^{t-1} U_{\mu d}\right)}$$

and 0 *otherwise.*

For the inverse matrix of order N*, we have for* $1 \leq k < N$ *and* $0 \leq j < N$,

$$\mathcal{F}_{kj}^{-1} = \frac{(-1)^{k+j+\lambda\binom{j}{2}+\mu\binom{k+1}{2}+N(\lambda j+\mu k)+r(N-1)}}{U_{2\mu k} U_{\mu k+\lambda j+r}} \times \frac{\left(\prod_{d=1}^{N} U_{\mu k+\lambda(d-1)+r}\right)\left(\prod_{d=1}^{N-1} U_{\lambda j+\mu d+r}\right)}{\left(\prod_{d=1}^{N-k-1} U_{\mu d}\right)\left(\prod_{d=1}^{N-j-1} U_{\lambda d}\right)\left(\prod_{d=1}^{j} U_{\lambda d}\right)\left(\prod_{d=1}^{k-1} U_{\mu d}\right)}$$

and for $0 \leq j < N$,

$$\mathcal{F}_{0j}^{-1} = [j = 0] - \sum_{t=1}^{N-1} \frac{U_{-\mu t+r}}{U_{\mu t+r}} \mathcal{F}_{tj}^{-1}.$$

Proof. We only give the proof of the LU-decomposition. The others are very similar and the application of Proposition 4.2.

Since when $q = \beta/\alpha$ and $x = q^r$,

$$\mathcal{F}_{kj} = (-1)^{\mu j} q^{\mu j} M_{kj},$$

by Proposition 4.2 when $s_k = 1$ and $m_j = (-1)^{\mu j} q^{\mu j}$ and Theorem 4.20, we obtain the LU-decomposition of the matrix $\mathcal{F}$ as follows

$$\mathcal{L}_{kj} = \frac{(q^{\lambda j+\mu+r}; q^{\mu})_j (q^{\lambda(k-j+1)}; q^{\lambda})_j}{(q^{\lambda k+\mu+r}; q^{\mu})_j (q^{\lambda}; q^{\lambda})_j}$$

and

$$\mathcal{U}_{kj} = \begin{cases} \dfrac{1-q^{-\mu j+r}}{1-q^{\mu j+r}} & \text{if } k = 0, \\ (-1)^{\mu j} q^{(\lambda+\mu)\binom{k}{2}+rk}(1+q^{\mu j}) \dfrac{(q^{\mu(j-k+1)}; q^{\mu})_k (q^{\lambda}; q^{\lambda})_k}{(q^{\mu j+r}; q^{\lambda})_{k+1} (q^{\lambda k+\mu+r}; q^{\mu})_{k-1}} & \text{if } k > 0, \end{cases}$$

where $q = \beta/\alpha$. These are the q-forms of the results given by (4.5) and (4.6). Thus the proof of the LU-decomposition is completed. □

The matrix $\mathcal{F}$ is a new non-symmetric variant of the Filbert matrix.

Similarly, we can obtain a non-symmetric variant of the Lilbert matrix. So we define the matrix $\mathcal{T}$ with

$$\mathcal{T}_{kj} = \frac{V_{\lambda k-\mu j+r}}{V_{\lambda k+\mu j+r}} = q^{\mu j}(-1)^{\mu j} \frac{1+q^{\lambda k-\mu j+r}}{1+q^{\lambda k+\mu j+r}}$$

for positive integers λ, μ and integer r and $q = \beta/\alpha$, as the Lucas analogue of the matrix M, where $\{V_n\}$ is generalized Lucas sequence.

If we choose $x = -q^r$ in our main results and apply Proposition 4.2 to our main results, after converting the results to the generalized Fibonacci or Lucas numbers, we have the following corollary for the matrix $\mathcal{T}$.

Corollary 4.12. *For $k, j \geq 0$,*

$$\mathcal{L}_{kj} = \frac{\left(\prod_{d=1}^{j} V_{\lambda j+\mu d+r}\right)\left(\prod_{d=1}^{j} U_{\lambda(k+1)-\lambda d}\right)}{\left(\prod_{d=1}^{j} V_{\lambda k+\mu d+r}\right)\left(\prod_{d=1}^{j} U_{\lambda d}\right)},$$

$$\mathcal{U}_{kj} = \begin{cases} \dfrac{V_{-\mu j+r}}{V_{\mu j+r}} & \text{if } k = 0, \\ \Delta^k (-1)^{\mu j+(\lambda+\mu)\binom{k}{2}+r(k+1)} U_{2\mu j} \dfrac{\left(\prod_{d=1}^{k-1} U_{\mu j-\mu d}\right)\left(\prod_{d=1}^{k} U_{\lambda d}\right)}{\left(\prod_{d=1}^{k+1} V_{\mu j+\lambda(d-1)+r}\right)\left(\prod_{d=1}^{k-1} V_{\lambda k+\mu d+r}\right)} & \text{if } k > 0, \end{cases}$$

where Δ defined as in equation (2.3).

$$\mathcal{L}_{kj}^{-1} = (-1)^{k+j+\lambda\binom{k-j}{2}} \frac{\left(\prod_{d=1}^{k-1} V_{\lambda j+\mu d+r}\right)\left(\prod_{d=1}^{j} U_{\lambda(k+1)-\lambda d}\right)}{\left(\prod_{d=1}^{k-1} V_{\lambda k+\mu d+r}\right)\left(\prod_{d=1}^{j} U_{\lambda d}\right)},$$

and $\mathcal{U}_{kj}^{-1} =$

$$\begin{cases} 1 & \text{if } k=j=0, \\ \dfrac{(-1)^{k+\lambda\binom{j}{2}+rj+\mu kj+\mu\binom{k}{2}}}{\Delta^j U_{2\mu k}} \dfrac{\left(\prod_{d=1}^{j} V_{\mu k+\lambda(d-1)+r}\right)\left(\prod_{d=1}^{j} V_{\lambda j+\mu d+r}\right)}{\left(\prod_{d=1}^{j} U_{\lambda d}\right)\left(\prod_{d=1}^{j-k} U_{\mu d}\right)\left(\prod_{d=1}^{k-1} U_{\mu d}\right)} & \text{if } j \geq k \geq 1, \end{cases}$$

and for $j \geq 1$

$$\mathcal{U}_{0j}^{-1} = \frac{(-1)^{rj+\lambda\binom{j}{2}+1}}{\Delta^j} \frac{\left(\prod_{d=1}^{j} V_{\lambda j+\mu d+r}\right)}{\left(\prod_{d=1}^{j} U_{\lambda d}\right)} \times \sum_{t=1}^{j} (-1)^{t+\mu tj+\mu\binom{t}{2}} \frac{V_{-\mu t+r}}{U_{2\mu t} V_{\mu t+r}} \frac{\left(\prod_{d=1}^{j} V_{\mu t+\lambda(d-1)+r}\right)}{\left(\prod_{d=1}^{j-t} U_{\mu d}\right)\left(\prod_{d=1}^{t-1} U_{\mu d}\right)}$$

and 0 *otherwise.*

For the inverse matrix, we have for $1 \leq k < N$ *and* $0 \leq j < N$,

$$\mathcal{T}_{kj}^{-1} = \frac{(-1)^{k+j+\lambda\binom{j}{2}+\mu\binom{k+1}{2}-N(\mu k+\lambda j)+(r+1)(N-1)}}{\Delta^{N-1} U_{2\mu k} V_{\mu k+\lambda j+r}} \times \frac{\left(\prod_{d=1}^{N} V_{\mu k+\lambda(d-1)+r}\right)\left(\prod_{d=1}^{N} V_{\lambda j+\mu d+r}\right)}{\left(\prod_{d=1}^{N-k-1} U_{\mu d}\right)\left(\prod_{d=1}^{N-j-1} U_{\lambda d}\right)\left(\prod_{d=1}^{j} U_{\lambda d}\right)\left(\prod_{d=1}^{k-1} U_{\mu d}\right)}$$

and for $0 \leq j < N$,

$$\mathcal{T}_{0j}^{-1} = [j=0] - \sum_{t=1}^{N-1} \frac{V_{-\mu t+r}}{V_{\mu t+r}} \mathcal{T}_{tj}^{-1}.$$

More specially, by choosing $x = q^r$ such that r is a positive integer and performing the limit $q \to 1$ in our main results, we obtain the related results for the matrix $\mathcal{H} = [\mathcal{H}_{kj}]_{k,j\geq 0}$ as a non-symmetric variant of the Hilbert matrix with entries

$$\mathcal{H}_{kj} = \frac{\lambda k - \mu j + r}{\lambda k + \mu j + r}.$$

So we list the results for the matrix $\mathcal{H}$ below.

Corollary 4.13. *For $k, j \geq 0$,*

$$\mathcal{L}_{kj} = \frac{\left(\prod_{d=1}^{j}[\lambda j + \mu d + r]\right)\left(\prod_{d=1}^{j}\lambda[k-d+1]\right)}{\left(\prod_{d=1}^{j}[\lambda k + \mu d + r]\right)\left(\prod_{d=1}^{j}\lambda d\right)},$$

$$\mathcal{U}_{kj} = \begin{cases} \dfrac{r-\mu j}{r+\mu j} & \text{if } k = 0, \\ 2 \times \dfrac{\left(\prod_{d=1}^{k}\mu[j-d+1]\right)\left(\prod_{d=1}^{k}\lambda d\right)}{\left(\prod_{d=1}^{k+1}[\mu j + \lambda(d-1) + r]\right)\left(\prod_{d=1}^{k-1}[\lambda k + \mu d + r]\right)} & \text{if } k > 0, \end{cases}$$

$$\mathcal{L}_{kj}^{-1} = (-1)^{k+j}\frac{\left(\prod_{d=1}^{k-1}[\lambda j + \mu d + r]\right)\left(\prod_{d=1}^{j}\lambda[k-d+1]\right)}{\left(\prod_{d=1}^{k-1}[\lambda k + \mu d + r]\right)\left(\prod_{d=1}^{j}\lambda d\right)}$$

and

$$\mathcal{U}_{kj}^{-1} = \begin{cases} 1 & \text{if } k = j = 0, \\ (-1)^{j+1} \dfrac{\left(\prod_{d=1}^{j} [\lambda j + \mu d + r]\right)}{\left(\prod_{d=1}^{j} \lambda d\right)} & \\ \quad \times \sum_{t=1}^{j} \dfrac{(-1)^t (r-\mu t)}{(r+\mu t)(2\mu t)} \dfrac{\left(\prod_{d=1}^{j} [\mu t + \lambda(d-1) + r]\right)}{\left(\prod_{d=1}^{j-t} \mu d\right)\left(\prod_{d=1}^{t-1} \mu d\right)} & \text{if } j \geq 1 \text{ and } k = 0, \\ \dfrac{(-1)^{k+j}}{(2\mu k)} \dfrac{\left(\prod_{d=1}^{j} [\mu k + \lambda(d-1) + r]\right)\left(\prod_{d=1}^{j} [\lambda j + \mu d + r]\right)}{\left(\prod_{d=1}^{j} \lambda d\right)\left(\prod_{d=1}^{j-k} \mu d\right)\left(\prod_{d=1}^{k-1} \mu d\right)} & \text{if } j \geq k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for $1 \leq k < N$ *and* $0 \leq j < N$,

$$\mathcal{H}_{kj}^{-1} = \frac{(-1)^{k+j}}{(\mu k + \lambda j + r)(2\mu k)} \frac{\left(\prod_{d=1}^{N-1} [\lambda j + \mu d + r]\right)\left(\prod_{d=1}^{N} [\mu k + \lambda(d-1) + r]\right)}{\left(\prod_{d=1}^{N-k-1} \mu d\right)\left(\prod_{d=1}^{N-j-1} \lambda d\right)\left(\prod_{d=1}^{j} \lambda d\right)\left(\prod_{d=1}^{k-1} \mu d\right)}$$

and for $0 \leq j < N$,

$$\mathcal{H}_{0j}^{-1} = [j = 0] - \sum_{t=1}^{N-1} \frac{r - \mu t}{r + \mu t} \mathcal{H}_{tj}^{-1}.$$

4.6 A Nonlinear Generalization of the Filbert Matrix

As mentioned in Section 3.2.4, so far authors studied many generalizations and variants of the Filbert matrix. In previous section, we have also obtained a new non-symmetric variant of it. But no one has considered a generalization where the indexes of the recursive sequence is in nonlinear form yet. In this section, we will introduce a new nonlinear generalization of the Filbert matrix. We define the matrix M as a *nonlinear* generalization of the Filbert matrix with indexes in geometric progression for some parameters by

$$M_{kj} = \frac{1}{U_{\lambda(k+r)^n + \mu(j+s)^m + c}},$$

where U_n is nth generalized Fibonacci number. Moreover, as Lilbert analogue, for some parameters, we define the matrix T with entries

$$T_{kj} = \frac{1}{V_{\lambda(k+r)^n+\mu(j+s)^m+c}},$$

where V_n is the nth generalized Lucas number. Our study [6] is accepted for publication.

Note that when $n = m = 1$, our results will cover all Filbert-like matrices except the matrices whose entries include the products of the generalized Fibonacci or Lucas numbers.

For the matrices M and T, we derive explicit formulæ for the inverse matrix, LU-decomposition and inverse matrices L^{-1}, U^{-1} as well as we present the Cholesky decomposition when the matrices are symmetric. Later, we will give the q-forms of these results. Actually, although the results related to the q-forms are more general, i.e. when $q = \beta/\alpha$ gives the matrices M and T, we prefer to give Fibonacci and Lucas forms first. Because they seem nicer and manipulating them is easier.

Note that any mechanic summation methods or q-Zeilberger algorithm will not work here due to the non-hypergeometric terms. This is another reason of presenting Fibonacci and Lucas form first. In order to prove our results we will use some traditional methods as in previous section.

Throughout this section, we assume that λ, μ, n and m are positive integers, r, s and c are any integers such that $\lambda(k+r)^n + \mu(j+s)^m + c > 0$ for all positive integers k and j.

4.6.1 A Nonlinear Filbert Matrix

For the matrix M, we will give explicit formulæ for its inverse, LU-decomposition, the inverse matrices L^{-1} and U^{-1} as well as we present its Cholesky decomposition when the matrix is symmetric, that is, the case $r = s$, $n = m$ and $\lambda = \mu$.

We obtain the LU-decomposition:

Theorem 4.25. *For $k, j \geq 1$,*

$$L_{kj} = \frac{\left(\prod_{t=1}^{j} U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)}{\left(\prod_{t=1}^{j} U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^n-\lambda(t+r)^n}\right)}$$

and

$$U_{kj} = (-1)^{(\lambda+\mu)\binom{k}{2}+(\lambda r+\mu s+c)(k+1)} \frac{\left(\prod_{t=1}^{k-1} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)\left(\prod_{t=1}^{k-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{k-1} U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{k} U_{\mu(j+s)^m+\lambda(t+r)^n+c}\right)}.$$

Similar to the previous sections, the determinant of the matrix M can be derived, as well. We also determine the inverse matrices L^{-1} and U^{-1}:

Theorem 4.26. *For $k, j \geq 1$,*

$$L_{kj}^{-1} = (-1)^{(\lambda+1)(k+j)+\lambda\binom{k-j+1}{2}} \frac{\left(\prod_{t=1}^{k-j-1} U_{\lambda(k+r)^n-\lambda(t+j+r)^n}\right)}{\left(\prod_{t=1}^{k-j-1} U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)} \times \frac{\left(\prod_{t=1}^{k-1} U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)}{\left(\prod_{t=1}^{k-1} U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^n-\lambda(t+r)^n}\right)}$$

and

$$U_{kj}^{-1} = (-1)^{\lambda\binom{j+1}{2}+\mu\binom{k+1}{2}+k(\mu j+1)+j(\lambda+1)+(\lambda r+\mu s+c)(j+1)} \times \frac{\left(\prod_{t=1}^{j-1} U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{j} U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{k-1} U_{\mu(k+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-k} U_{\mu(j+s+1-t)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^n-\lambda(t+r)^n}\right)}.$$

Now we give the explicit expression for the inverse matrix M_N^{-1}.

Theorem 4.27. *For $1 \leq k, j \leq N$,*

$$M_{kj}^{-1} = (-1)^{k+j+\lambda\binom{j+1}{2}+\mu\binom{k+1}{2}+N(\lambda j+\mu k+c+\lambda r+\mu s)+c+\lambda r+\mu s} \times \frac{1}{U_{\lambda(j+r)^n+\mu(k+s)^m+c}\left(\prod_{t=1}^{k-1} U_{\mu(k+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^n-\lambda(t+r)^n}\right)} \times \frac{\left(\prod_{t=1}^{N} U_{\lambda(t+r)^n+\mu(k+s)^m+c}\right)\left(\prod_{t=1}^{N} U_{\mu(t+s)^m+\lambda(j+r)^n+c}\right)}{\left(\prod_{t=1}^{N-k} U_{\mu(N+s+1-t)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{N-j} U_{\lambda(N+r+1-t)^n-\lambda(j+r)^n}\right)}.$$

Finally, we provide the Cholesky decomposition of the matrix M when it is symmetric, that is $r = s$, $n = m$ and $\lambda = \mu$.

Theorem 4.28. *For $k, j \geq 1$,*

$$C_{kj} = \frac{\prod_{t=1}^{j-1} U_{\lambda(k+r)^n - \lambda(t+r)^n}}{\prod_{t=1}^{j} U_{\lambda(k+r)^n + \lambda(t+r)^n + c}} \sqrt{(-1)^{c(j+1)} U_{2\lambda(j+r)^n + c}}.$$

Note that when $n = m = 1$, $\lambda = \mu = 1$, $r = s = 0$ and $c = -1$ with $p = 1$, the matrix M is reduced to Filbert matrix and so we obtain the results of [64]. Similarly, when $n = m = 1$ and $r = s = 0$ our results cover the results of [70]. For the cases $n > 1$ or $m > 1$, our results are all new.

Proofs

Define the following four sums:

$$S_5(K) = \sum_{d=K}^{\min(k,j)} (-1)^{(\lambda+\mu)\binom{d}{2} + (\lambda r + \mu s + c)(d+1)} U_{\lambda(d+r)^n + \mu(d+s)^m + c} \times \frac{\left(\prod_{t=1}^{d-1} U_{\lambda(k+r)^n - \lambda(t+r)^n}\right)\left(\prod_{t=1}^{d-1} U_{\mu(j+s)^m - \mu(t+s)^m}\right)}{\left(\prod_{t=1}^{d} U_{\lambda(k+r)^n + \mu(t+s)^m + c)}\right)\left(\prod_{t=1}^{d} U_{\mu(j+s)^m + \lambda(t+r)^n + c}\right)},$$

$$S_6(K) = \sum_{d=j}^{K} (-1)^{(\lambda+1)(d+j) + \lambda\binom{d-j+1}{2}} U_{\lambda(d+r)^n + \mu(d+s)^m + c} \frac{\left(\prod_{t=1}^{d-j-1} U_{\lambda(d+r)^n - \lambda(t+j+r)^n}\right)}{\left(\prod_{t=1}^{d-j-1} U_{\lambda(t+j+r)^n - \lambda(j+r)^n}\right)} \times \frac{\left(\prod_{t=1}^{d-1} U_{\lambda(k+r)^n - \lambda(t+r)^n}\right)\left(\prod_{t=1}^{d-1} U_{\lambda(j+r)^n + \mu(t+s)^m + c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(d+r)^n - \lambda(t+r)^n}\right)}{\left(\prod_{t=1}^{d-1} U_{\lambda(d+r)^n - \lambda(t+r)^n}\right)\left(\prod_{t=1}^{d} U_{\lambda(k+r)^n + \mu(t+s)^m + c}\right)},$$

$$S_7(K) = \sum_{d=\max(k,j)}^{K} (-1)^{k\mu d + \lambda d j + (\lambda r + \mu s + c)d} U_{\lambda(d+r)^n + \mu(d+s)^m + c} \frac{\left(\prod_{t=1}^{d-j-1} U_{\lambda(d+r)^n - \lambda(t+j+r)^n}\right)}{\left(\prod_{t=1}^{d-j-1} U_{\lambda(t+j+r)^n - \lambda(j+r)^n}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{d-1} U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{d-1} U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(d+r)^n-\lambda(t+r)^n}\right)}{\left(\prod_{t=1}^{d-k} U_{\mu(d+s+1-t)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{d-1} U_{\lambda(d+r)^n-\lambda(t+r)^n}\right)}$$

and

$$S_8(K) = \sum_{d=k}^{K} (-1)^{k\mu d+\mu\binom{d}{2}+d} U_{\lambda(d+r)^n+\mu(d+s)^m+c} \times \frac{\left(\prod_{t=1}^{d-1} U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{d-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{d-k} U_{\mu(d+s+1-t)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{d} U_{\mu(j+s)^m+\lambda(t+r)^n+c}\right)}.$$

We need the following lemmas for later use.

Lemma 4.8.

$$S_5(K) = \frac{(-1)^{(\lambda+\mu)\binom{K}{2}+(\lambda r+\mu s+c)(K+1)}\left(\prod_{t=1}^{K-1} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{U_{\lambda(k+r)^n+\mu(j+s)^m+c}\left(\prod_{t=1}^{K-1} U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m+\lambda(t+r)^n+c}\right)}.$$

Proof. We will use the backward induction method. For brevity, denote the summand term by s_d. First, assume that $k \geq j$ so when $K = j$ the claim is obvious. Similarly for the case $j > k$, the claim is clear. The backward induction step amounts to show that

$$S_5(K-1) = S_5(K) + s_{K-1}.$$

By the definition of $S_5(K)$ and s_{K-1}, consider the RHS of the above equality

$$\frac{(-1)^{(\lambda+\mu)\binom{K}{2}+(\lambda r+\mu s+c)(K+1)}}{U_{\lambda(k+r)^n+\mu(j+s)^m+c}} \frac{\left(\prod_{t=1}^{K-1} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-1} U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m+\lambda(t+r)^n+c}\right)}$$
$$+ (-1)^{(\lambda+\mu)\binom{K-1}{2}+(\lambda r+\mu s+c)K} U_{\lambda(K-1+r)^n+\mu(K-1+s)^m+c}$$
$$\times \frac{\left(\prod_{t=1}^{K-2} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)\left(\prod_{t=1}^{K-2} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-1} U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m+\lambda(t+r)^n+c}\right)}$$

$$= \frac{(-1)^{(\lambda+\mu)\binom{K-1}{2}+(\lambda r+\mu s+c)K}}{U_{\lambda(k+r)^n+\mu(j+s)^m+c}} \frac{\left(\prod_{t=1}^{K-2} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)\left(\prod_{t=1}^{K-2} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-1} U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m+\lambda(t+r)^n+c}\right)}$$

$$\times \left[(-1)(\lambda+\mu)(K-1)+(\lambda r+\mu s+c)U_{\lambda(k+r)^n-\lambda(K-1+r)^n}U_{\mu(j+s)^m-\mu(K-1+s)^m}\right.$$

$$\left.+U_{\lambda(K-1+r)^n+\mu(K-1+s)^m+c}U_{\lambda(k+r)^n+\mu(j+s)^m+c}\right].$$

By using the fact $U_n = (-1)^{n-1}U_{-n}$, the last expression in the bracket is rewritten as

$$\begin{aligned}&(-1)^{\lambda(k+r)+\mu(j+s)+c}U_{\lambda(K-1+r)^n-\lambda(k+r)^n}U_{\mu(K-1+s)^m-\mu(j+s)^m}\\&+U_{\lambda(K-1+r)^n+\mu(K-1+s)^m+c}U_{\lambda(k+r)^n+\mu(j+s)^m+c}\end{aligned} \tag{4.7}$$

and then by using the identity (2.4) for $m \to \mu(j+s)^m + \lambda(K-1+r)^n + c$, $n \to \lambda(k+r)^n + \mu(K-1+s)^m + c$ and $k \to \mu(K-1+s)^m - \mu(j+s)^m$, the expression in (4.7) equals

$$U_{\lambda(k+r)^n+\mu(K-1+s)^m+c}U_{\mu(j+s)^m+\lambda(K-1+r)^n+c}.$$

Finally, we write

$$S_5(K)+s_{K-1} = \frac{(-1)^{(\lambda+\mu)\binom{K-1}{2}+(\lambda r+\mu s+c)K}}{U_{\lambda(k+r)^n+\mu(j+s)^m+c}} \times \frac{\left(\prod_{t=1}^{K-2} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)\left(\prod_{t=1}^{K-2} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-2} U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-2} U_{\mu(j+s)^m+\lambda(t+r)^n+c}\right)},$$

which completes the proof. □

Lemma 4.9. *For $k > j$,*

$$S_6(K) = \frac{(-1)^{\lambda\binom{K-j}{2}+(\lambda+1)(K+j)}}{U_{\lambda(k+r)^n-\lambda(j+r)^n}} \frac{\left(\prod_{t=1}^{K} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K} U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)}.$$

Proof. Denote the summand term by s_d. By using induction, the case $K = j$ is obvious. So the induction step amounts to show that

$$S_6(K+1) = S_6(K) + s_{K+1}.$$

So after some simplifications, $S_6(K)+s_{K+1}$ equals

$$\frac{(-1)^{\lambda\binom{K+1-j}{2}+(K+1+j)}}{U_{\lambda(k+r)^n-\lambda(j+r)^n}}\frac{\left(\prod_{t=1}^{K}U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)\left(\prod_{t=1}^{K}U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K+1}U_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-j+1}U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)}$$
$$\times\left[U_{\lambda(k+r)^n-\lambda(j+r)^n}U_{\lambda(K+1+r)^n+\mu(K+1+s)^m+c}\right.$$
$$\left.-U_{\lambda(k+r)^n+\mu(K+1+s)^m+c}U_{\lambda(K+1+r)^n-\lambda(j+r)^n}\right].$$

Since again by (2.4), the last expression in the bracket equals

$$U_{\lambda(k+r)^n-\lambda(K+1+r)^n}U_{\lambda(j+r)^n+\mu(K+1+s)^m+c},$$

the claim follows. □

Lemma 4.10.

$$S_7(K)=\frac{(-1)^{K(\lambda j+\mu k+c+\lambda r+\mu s)}}{U_{\lambda(j+r)^n+\mu(k+s)^m+c}}\frac{\left(\prod_{t=1}^{K}U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{K}U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K-k}U_{\mu(t+k+s)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{K-j}U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)}.$$

Proof. If $j\geq k$, the case $K=j$ easily follows. If $k>j$, then

$$S_7(k)=\frac{(-1)^{k(\mu k+\lambda j+\lambda r+\mu s+c)}}{U_{\lambda(k+r)^n-\lambda(j+r)^n}}\frac{\left(\prod_{t=1}^{k}U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{k-1}U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{k-j-1}U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)}$$
$$=\frac{(-1)^{k(\mu k+\lambda j+\lambda r+\mu s+c)}}{U_{\lambda(j+r)^n+\mu(k+s)^m+c}}\frac{\left(\prod_{t=1}^{k}U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{k}U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{k-j}U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)}.$$

So the first step of induction is complete. For the next step, we have

$$S_7(K+1)=\frac{(-1)^{K(\lambda j+\mu k+c+\lambda r+\mu s)}}{U_{\lambda(j+r)^n+\mu(k+s)^m+c}}\frac{\left(\prod_{t=1}^{K}U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{K}U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K-k}U_{\mu(t+k+s)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{K-j}U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)}$$
$$+(-1)^{(\mu k+\lambda j+\lambda r+\mu s+c)(K+1)}U_{\lambda(K+1+r)^n+\mu(K+1+s)^m+c}$$

$$\times \frac{\left(\prod_{t=1}^{K} U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K+1-k} U_{\mu(t+k+s)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{K+1-j} U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)},$$

which, after some simplifications, equals

$$\frac{(-1)^{(\mu k+\lambda j+\lambda r+\mu s+c)(K+1)}}{U_{\lambda(j+r)^n+\mu(k+s)^m+c}} \frac{\left(\prod_{t=1}^{K} U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K+1-k} U_{\mu(t+k+s)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{K+1-j} U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)}$$

$$\times \left[(-1)^{\lambda(j+r)+\mu(k+s)+c} U_{\mu(K+1+s)^m-\mu(k+s)^m} U_{\lambda(K+1+r)^n-\lambda(j+r)^n}\right.$$

$$\left.+U_{\lambda(K+1+r)^n+\mu(K+1+s)^m+c} U_{\lambda(j+r)^n+\mu(k+s)^m+c}\right].$$

By the identity (2.4) with appropriate parameters, the last expression in the bracket equals

$$U_{\mu(k+s)^m+\lambda(K+1+r)^n+c} U_{\lambda(j+r)^n+\mu(K+1+s)^m+c},$$

so the proof follows by induction. □

Lemma 4.11. *For $j > k$,*

$$S_8(K) = \frac{(-1)^{K\mu k+\mu\binom{K}{2}+K}}{U_{\lambda(k+r)^n+\mu(j+s)^m+c}} \frac{\left(\prod_{t=1}^{K} U_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{K} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-k} U_{\mu(K+s+1-t)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{K} U_{\mu(j+s)^m+\lambda(t+r)^n+c}\right)}.$$

Proof of Lemma 4.11 can be done by induction similar to the previous two lemmas.

Now we shall give the proofs of the main results for the matrix M.

For the matrices L and L^{-1}, it is obviously seen that $L_{kk}L_{kk}^{-1} = 1$. For $k > j$, by Lemma 4.9

$$\sum_{j\le d\le k} L_{kd}L_{dj}^{-1} = S_6(k) = 0,$$

so we conclude

$$\sum_{j\le d\le k} L_{kd}L_{dj}^{-1} = [k = j],$$

as desired.

For the matrices U and U^{-1}, $U_{kk}^{-1}U_{kk}=1$ is clear. In order to show the case $j>k$, by Lemma 4.11, we have

$$\sum_{k\leq d\leq j} U_{kd}^{-1}U_{dj} = \frac{(-1)^{\mu\binom{k+1}{2}+k}}{\left(\prod_{t=1}^{k-1} U_{\mu(k+s)^m-\mu(t+s)^m}\right)} S_8(j) = 0.$$

Thus $U^{-1}\bullet U = I$, as claimed.

For the LU-decomposition, we have to prove that

$$\sum_{1\leq d\leq \min(k,j)} L_{kd}U_{dj} = M_{kj}.$$

By Lemma 4.8, we obtain

$$\sum_{1\leq d\leq \min(k,j)} L_{kd}U_{dj} = S_5(1) = \frac{1}{U_{\lambda(k+r)^n+\mu(j+s)^m+c}},$$

which completes the proof.

For the inverse matrix M_N^{-1}, we use the fact $M_N^{-1} = U_N^{-1}\bullet L_N^{-1}$. Consider

$$\begin{aligned}\sum_{\max(k,j)\leq d\leq N} U_{kd}^{-1}L_{dj}^{-1} &= \frac{(-1)^{\mu\binom{k+1}{2}+k+\lambda r+\mu s+c+j+\lambda\binom{j+1}{2}}}{\left(\prod_{t=1}^{k-1} U_{\mu(k+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^n-\lambda(t+r)^n}\right)} S_7(N)\\ &= (M_N^{-1})_{kj}.\end{aligned}$$

The Cholesky decomposition follows by Corollary 4.1. Thus all proofs are complete.

4.6.2 A Nonlinear Lilbert Matrix

Now we give the LU-decomposition of the matrix T, inverse matrices L^{-1} and U^{-1}, inverse matrix T^{-1} and its Cholesky decomposition when $r=s$, $n=m$ and $\lambda=\mu$, respectively. We don't give the proofs of these results because they may be done very similar to the proofs of the previous subsection. One needs only very small and proper changes in lemmas given above.

Theorem 4.29. *For $k,j\geq 1$,*

$$L_{kj} = \frac{\left(\prod_{t=1}^{j} V_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)}{\left(\prod_{t=1}^{j} V_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^n-\lambda(t+r)^n}\right)}$$

and

$$U_{kj} = (-1)^{(\lambda+\mu)\binom{k}{2}+(\lambda r+\mu s+c+1)(k+1)}\Delta^{k-1} \times \frac{\left(\prod_{t=1}^{k-1} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)\left(\prod_{t=1}^{k-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{k-1} V_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{k} V_{\mu(j+s)^m+\lambda(t+r)^n+c}\right)},$$

where Δ *is defined as before.*

Theorem 4.30. *For* $k, j \geq 1$,

$$L_{kj}^{-1} = (-1)^{(\lambda+1)(k+j)+\lambda\binom{k-j+1}{2}} \frac{\left(\prod_{t=1}^{k-j-1} U_{\lambda(k+r)^n-\lambda(t+j+r)^n}\right)}{\left(\prod_{t=1}^{k-j-1} U_{\lambda(t+j+r)^n-\lambda(j+r)^n}\right)} \times \frac{\left(\prod_{t=1}^{k-1} V_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(k+r)^n-\lambda(t+r)^n}\right)}{\left(\prod_{t=1}^{k-1} V_{\lambda(k+r)^n+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^n-\lambda(t+r)^n}\right)}$$

and

$$U_{kj}^{-1} = (-1)^{\lambda\binom{j+1}{2}+\mu\binom{k+1}{2}+k(\mu j+1)+j(\lambda+1)+(\lambda r+\mu s+c+1)(j+1)}\Delta^{1-j} \times \frac{\left(\prod_{t=1}^{j-1} V_{\mu(k+s)^m+\lambda(t+r)^n+c}\right)\left(\prod_{t=1}^{j} V_{\lambda(j+r)^n+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{k-1} U_{\mu(k+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-k} U_{\mu(j+s+1-t)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^n-\lambda(t+r)^n}\right)}.$$

Theorem 4.31. *For* $1 \leq k, j \leq N$,

$$M_{kj}^{-1} = \frac{1}{\Delta^{N-1} V_{\lambda(j+r)^n+\mu(k+s)^m+c}} \frac{(-1)^{k+j+\lambda\binom{j+1}{2}+\mu\binom{k+1}{2}+N(\lambda j+\mu k)+(N+1)(c+\lambda r+\mu s+1)}}{\left(\prod_{t=1}^{k-1} U_{\mu(k+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^n-\lambda(t+r)^n}\right)} \times \frac{\left(\prod_{t=1}^{N} V_{\lambda(t+r)^n+\mu(k+s)^m+c}\right)\left(\prod_{t=1}^{N} V_{\mu(t+s)^m+\lambda(j+r)^n+c}\right)}{\left(\prod_{t=1}^{N-k} U_{\mu(N+s+1-t)^m-\mu(k+s)^m}\right)\left(\prod_{t=1}^{N-j} U_{\lambda(N+r+1-t)^n-\lambda(j+r)^n}\right)}.$$

Theorem 4.32. *For $k, j \geq 1$,*

$$C_{kj} = \frac{\prod_{t=1}^{j-1} U_{\lambda(k+r)^n - \lambda(t+r)^n}}{\prod_{t=1}^{j} V_{\lambda(k+r)^n + \lambda(t+r)^n + c}} \sqrt{(-1)^{(c+1)(j+1)} \Delta^{j-1} V_{2\lambda(j+r)^n + c}}.$$

Here note that when $n = m = 1$ and $r = s = 0$ the above results are reduced to the results of [70]. Similarly, when $n = m = 1$, $\lambda = \mu = 1$, $r = s = 0$ and $c = -1$ with $p = 1$ the matrix M is reduced to the usual Lilbert matrix. For the cases $n > 1$ or $m > 1$, our results are all new.

4.6.3 q-Analogues

We present the q-analogues of the results of Sections 4.6.1 and 4.6.2. The results for the matrices M and T given previously come out as corollaries of the below results for the special choice of $q = \beta/\alpha$, so that the results, will be provided after a while, are more general. Nevertheless, we prefer to give first the results related to the matrices M and T because they look nicer. We don't provide the proofs of the results of this section. They could be similarly done by finding the q-forms of the lemmas given before.

We denote the q-analogues of the matrices M and T by $\mathcal{M}$ and $\mathcal{T}$:

$$\mathcal{M}_{kj} = \mathbf{i}^{-[\lambda(k+r)^n + \mu(j+s)^m + c - 1]} q^{\frac{1}{2}[\lambda(k+r)^n + \mu(j+s)^m + c - 1]} \frac{1-q}{1-q^{\lambda(k+r)^n + \mu(j+s)^m + c}}$$

and

$$\mathcal{T}_{kj} = \mathbf{i}^{-[\lambda(k+r)^n + \mu(j+s)^m + c]} q^{\frac{1}{2}[\lambda(k+r)^n + \mu(j+s)^m + c]} \frac{1}{1+q^{\lambda(k+r)^n + \mu(j+s)^m + c}},$$

respectively.

For the convenience, we will define a generalization of the q-Pochhammer symbol with two additional parameters in which one of them is in geometric progression as follows

$$(a;q)_n^{(r,m)} := \left(1 - aq^{(1+r)^m}\right)\left(1 - aq^{(2+r)^m}\right) \ldots \left(1 - aq^{(n+r)^m}\right) = \prod_{t=1}^{n} \left(1 - aq^{(t+r)^m}\right)$$

with $\left(a;q^{(r,m)}\right)_0 = 1$, where a is a real number, r is an integer and n, m are positive integers. As examples, we note that

$$(1;q)_n^{(0,2)} = (1-q)(1-q^4)\ldots(1-q^{n^2}),$$

$$(a;q^2)_n^{(1,2)} = (1-aq^8)(1-aq^{18})\dots(1-aq^{2(n+1)^2}),$$
$$(-q;q)_n^{(-1,3)} = (1+q)(1+q^2)(1+q^9)\cdots(1+q^{(n-1)^3+1}),$$
$$(a;q^\lambda)_n^{(0,1)} = (1-aq^\lambda)(1-aq^{2\lambda})\dots(1-aq^{n\lambda}) = (aq^\lambda;q^\lambda)_n.$$

So the relationship between the usual and general q-Pochhammer symbol is

$$(x;q)_n = (x;q)_n^{(-1,1)}.$$

As the q-analogues of the results related to the matrix M, we present the following theorem for the matrix $\mathcal{M}$.

Theorem 4.33. *For the matrix $\mathcal{M}$ and $k, j \geq 1$,*

$$\mathcal{L}_{kj} = q^{\frac{1}{2}\lambda[(k+r)^n-(j+r)^n]}\mathbf{i}^{\lambda[(j+r)^n-(k+r)^n]}\frac{\left(q^{\lambda(j+r)^n+c};q^\mu\right)_j^{(s,m)}\left(q^{\lambda(k+r)^n};q^{-\lambda}\right)_{j-1}^{(r,n)}}{\left(q^{\lambda(k+r)^n+c};q^\mu\right)_j^{(s,m)}\left(q^{\lambda(j+r)^n};q^{-\lambda}\right)_{j-1}^{(r,n)}},$$

$$\mathcal{U}_{kj} = q^{-\frac{1}{2}[\lambda(k+r)^n+\mu(j+s)^m+c+1]+k[\lambda(k+r)^n+\mu(j+s)^m+c]}\mathbf{i}^{[-\lambda(k+r)^n-\mu(j+s)^m-c+1]}$$
$$\times\frac{(1-q)\left(q^{-\lambda(k+r)^n};q^\lambda\right)_{k-1}^{(r,n)}\left(q^{-\mu(j+s)^m};q^\mu\right)_{k-1}^{(s,m)}}{\left(q^{\lambda(k+r)^n+c};q^\mu\right)_{k-1}^{(s,m)}\left(q^{\mu(j+s)^m+c};q^\lambda\right)_k^{(r,n)}},$$

$$\mathcal{L}_{kj}^{-1} = (-1)^{\lambda(k+j)+1}\mathbf{i}^{\lambda[(k+r)^n-(j+r)^n]}q^{\lambda(j-k)[(j+r)^n-(k+r)^n]+\frac{1}{2}\lambda[(j+r)^n-(k+r)^n]}$$
$$\times\frac{\left(q^{-\lambda(k+r)^n};q^\lambda\right)_{k-j-1}^{(j+r,n)}\left(q^{\lambda(j+r)^n+c};q^\mu\right)_{k-1}^{(s,m)}\left(q^{\lambda(k+r)^n};q^{-\lambda}\right)_{j-1}^{(r,n)}}{\left(q^{-\lambda(j+r)^n};q^\lambda\right)_{k-j-1}^{(j+r,n)}\left(q^{\lambda(k+r)^n+c};q^\mu\right)_{k-1}^{(s,m)}\left(q^{\lambda(j+r)^n};q^{-\lambda}\right)_{j-1}^{(r,n)}},$$

$$\mathcal{U}_{kj}^{-1} = (-1)^{\mu(k+s)+\lambda(j+r)+c}\mathbf{i}^{[-\mu(k+s)^m-\lambda(j+r)^n-c-1]}q^{\left(\frac{1}{2}-j\right)[\mu(k+s)^m+\lambda(j+r)^n+c]+\frac{1}{2}}$$
$$\times\frac{\left(q^{\mu(k+s)^m+c};q^\lambda\right)_{j-1}^{(r,n)}\left(q^{\lambda(j+r)^n+c};q^\mu\right)_j^{(s,m)}}{(1-q)\left(q^{-\mu(k+s)^m};q^\mu\right)_{k-1}^{(s,m)}\left(q^{-\lambda(j+r)^n};q^\lambda\right)_{j-1}^{(r,n)}\left(q^{-\mu(k+s)^m};q^\mu\right)_{j-k}^{(k+s,m)}},$$

$$(\mathcal{M}_N^{-1})_{kj} = (-1)^{\mu k+\lambda j+\lambda r+\mu s+c}\mathbf{i}^{[-\mu(k+s)^m-\lambda(j+r)^n-c-1]}q^{\left(\frac{1}{2}-N\right)[\mu(k+s)^m+\lambda(j+r)^n+c+1]+\frac{1}{2}}$$
$$\times\frac{\left(q^{\mu(k+s)^m+c};q^\lambda\right)_N^{(r,n)}\left(q^{\lambda(j+r)^n+c};q^\mu\right)_N^{(s,m)}}{(1-q)(1-q^{\lambda(j+r)^n+\mu(k+s)^m+c})\left(q^{-\mu(k+s)^m};q^\mu\right)_{k-1}^{(s,m)}\left(q^{-\lambda(j+r)^n};q^\lambda\right)_{j-1}^{(r,n)}}$$
$$\times\frac{1}{\left(q^{-\mu(k+s)^m};q^\mu\right)_{N-k}^{(k+s,m)}\left(q^{-\lambda(j+r)^n};q^\lambda\right)_{N-j}^{(j+r,n)}}$$

and when $r=s$, $n=m$ and $\lambda=\mu$,

$$\mathcal{C}_{kj} = (-1)^{\lambda\left[jr+k+\binom{j}{2}\right]+j+1}\mathbf{i}^{[\lambda(k+r)^n-\lambda(j+r)^n+1-cj]}$$

$$\times q^{\frac{1}{2}[\lambda(j+r)^n-\lambda(k+r)^n+cj-1]+\lambda(k+r)^n j}\frac{\left(q^{-\lambda(k+r)^n};q^{\lambda}\right)_{j-1}^{(r,n)}}{\left(q^{\lambda(k+r)^n+c};q^{\lambda}\right)_{j}^{(r,n)}}$$
$$\times\sqrt{(1-q)q^{-\lambda(j+r)^n+\frac{1-c}{2}}(-1)^{\lambda(j+r)^n+c(j+1)+\frac{c-1}{2}}(1-q^{2\lambda(j+r)^n+c})}.$$

Similarly, as the q-analogue of the matrix T, we present the following theorem.

Theorem 4.34. *For the matrix $\mathcal{T}$ and $k,j\geq 1$,*

$$\mathcal{L}_{kj}=q^{\frac{1}{2}\lambda[(k+r)^n-(j+r)^n]}\mathbf{i}^{\lambda[(j+r)^n-(k+r)^n]}$$
$$\times\frac{\left(-q^{\lambda(j+r)^n+c};q^{\mu}\right)_{j}^{(s,m)}\left(q^{\lambda(k+r)^n};q^{-\lambda}\right)_{j-1}^{(r,n)}}{\left(-q^{\lambda(k+r)^n+c};q^{\mu}\right)_{j}^{(s,m)}\left(q^{\lambda(j+r)^n};q^{-\lambda}\right)_{j-1}^{(r,n)}},$$

$$\mathcal{U}_{kj}=q^{(k-1)[\lambda(k+r)^n+\mu(j+s)^m+c]+\frac{1}{2}[\lambda(k+r)^n+\mu(j+s)^m+c]}(-1)^{k-1}\mathbf{i}^{[-\lambda(k+r)^n-\mu(j+s)^m-c]}$$
$$\times\frac{\left(q^{-\lambda(k+r)^n};q^{\lambda}\right)_{k-1}^{(r,n)}\left(q^{-\mu(j+s)^m};q^{\mu}\right)_{k-1}^{(s,m)}}{\left(-q^{\lambda(k+r)^n+c};q^{\mu}\right)_{k-1}^{(s,m)}\left(-q^{\mu(j+s)^m+c};q^{\lambda}\right)_{k}^{(r,n)}},$$

$$\mathcal{L}_{kj}^{-1}=(-1)^{\lambda(k-j)+1}\mathbf{i}^{\lambda[(k+r)^n-(j+r)^n]}q^{\lambda(j-k)[(j+r)^n-(k+r)^n]+\frac{1}{2}\lambda[(j+r)^n-(k+r)^n]}$$
$$\times\frac{\left(q^{-\lambda(k+r)^n};q^{\lambda}\right)_{k-j-1}^{(j+r,n)}\left(-q^{\lambda(j+r)^n+c};q^{\mu}\right)_{k-1}^{(s,m)}\left(q^{\lambda(k+r)^n};q^{-\lambda}\right)_{j-1}^{(r,n)}}{\left(q^{-\lambda(j+r)^n};q^{\lambda}\right)_{k-j-1}^{(j+r,n)}\left(-q^{\lambda(k+r)^n+c};q^{\mu}\right)_{k-1}^{(s,m)}\left(q^{\lambda(j+r)^n};q^{-\lambda}\right)_{j-1}^{(r,n)}},$$

$$\mathcal{U}_{kj}^{-1}=q^{\left(\frac{1}{2}-j\right)[\mu(k+s)^m+\lambda(j+r)^n+c]}(-1)^{j-1}\mathbf{i}^{[-\mu(k+s)^m-\lambda(j+r)^n-c]}$$
$$\times\frac{\left(-q^{\mu(k+s)^m+c};q^{\lambda}\right)_{j-1}^{(r,n)}\left(-q^{\lambda(j+r)^n+c};q^{\mu}\right)_{j}^{(s,m)}}{\left(q^{-\mu(k+s)^m};q^{\mu}\right)_{k-1}^{(s,m)}\left(q^{-\lambda(j+r)^n};q^{\lambda}\right)_{j-1}^{(r,n)}\left(q^{-\mu(k+s)^m};q^{\mu}\right)_{j-k}^{(k+s,m)}},$$

$$(\mathcal{T}_N^{-1})_{kj}=q^{\left(\frac{1}{2}-N\right)[\lambda(j+r)^n+\mu(k+s)^m+c]}(-1)^{\lambda j+\mu k+c+\lambda r+\mu s+N+1}\mathbf{i}^{[-\lambda(j+r)^n-\mu(k+s)^m-c]}$$
$$\times\frac{\left(-q^{\mu(k+s)^m+c};q^{\lambda}\right)_{N}^{(r,n)}\left(-q^{\lambda(j+r)^n+c};q^{\mu}\right)_{N}^{(s,m)}}{\left(1+q^{\lambda(j+r)^n+\mu(k+s)^m+c}\right)\left(q^{-\mu(k+s)^m};q^{\mu}\right)_{N-k}^{(k+s,m)}\left(q^{-\lambda(j+r)^n};q^{\lambda}\right)_{N-j}^{(j+r,n)}}$$
$$\times\frac{1}{\left(q^{-\mu(k+s)^m};q^{\mu}\right)_{k-1}^{(s,m)}\left(q^{-\lambda(j+r)^n};q^{\lambda}\right)_{j-1}^{(r,n)}}$$

and when $r=s$, $n=m$ and $\lambda=\mu$,

$$\mathcal{C}_{kj}=(-1)^{j+1+\lambda\binom{j}{2}+\lambda r(j-1)}\mathbf{i}^{[-\lambda(k+r)^n-\lambda(j+r)^n-cj]}q^{\frac{1}{2}[\lambda(k+r)^n+\lambda(j+r)^n+cj]+\lambda(j-1)(k+r)^n}$$
$$\times\frac{\left(q^{-\lambda(k+r)^n};q^{\lambda}\right)_{j-1}^{(r,n)}}{\left(-q^{\lambda(k+r)^n+c};q^{\lambda}\right)_{j}^{(r,n)}}\sqrt{q^{-\lambda(j+r)^n-\frac{c}{2}}(-1)^{\lambda(j+r)+(c+1)(j+1)+\frac{c}{2}}(1+q^{2\lambda(j+r)^n+c})}.$$

Note that the determinants of each matrices studied in this section can be evaluated by the multiplication of the diagonal elements of the related matrix U. We don't state them because they are overlong.

If you look at the q-forms of the matrices $\mathcal{M}$ and $\mathcal{T}$, you may realize that there is some separable parts. So one may give some simple formulæ for these matrices by using Proposition 4.2. However we would prefer to present our results in this form because nonlinear generalizations of the Filbert and Lilbert matrices can directly obtain by choosing $q = \beta/\alpha$.

At the end of this section, we will give a nonlinear generalization of the Hilbert matrix as a corollary of Theorem 4.33 and Proposition 4.2. Namely, when $q \to 1$ the entries of the matrix $\mathcal{M}$ takes the form

$$\lim_{q\to 1} \mathcal{M}_{kj} = \mathbf{i}^{[-\lambda(k+r)^n - \mu(j+s)^m - c + 1]} \frac{1}{\lambda(k+r)^n + \mu(j+s)^m + c}.$$

Since the factor in front of the ratio is separable with regards to the variables k and j, by Proposition 4.2 and Theorem 4.33, one could derive the results for the matrix $\hat{M}$ with entries for $k, j \geq 1$,

$$\hat{M}_{kj} = \frac{1}{\lambda(k+r)^n + \mu(j+s)^m + c},$$

which is a nonlinear generalization of the Hilbert matrix.

We only state the LU-decomposition of the matrix $\hat{M}$ by the following corollary. The others can be similarly derived.

Corollary 4.14. *For $k, j \geq 1$,*

$$\hat{L}_{kj} = \frac{\left(\prod_{t=1}^{j} [\lambda(j+r)^n + \mu(t+s)^m + c]\right)\left(\prod_{t=1}^{j-1} [\lambda(k+r)^n - \lambda(t+r)^n]\right)}{\left(\prod_{t=1}^{j} [\lambda(k+r)^n + \mu(t+s)^m + c]\right)\left(\prod_{t=1}^{j-1} [\lambda(j+r)^n - \lambda(t+r)^n]\right)}$$

and

$$\hat{U}_{kj} = \frac{\left(\prod_{t=1}^{k-1} [\lambda(t+r)^n - \lambda(k+r)^n]\right)\left(\prod_{t=1}^{k-1} [\mu(t+s)^m - \mu(j+s)^m]\right)}{\left(\prod_{t=1}^{k-1} [\lambda(k+r)^n + \mu(t+s)^m + c]\right)\left(\prod_{t=1}^{k} [\mu(j+s)^m + \lambda(t+r)^n + c]\right)}.$$

4.7 Hessenberg Determinants via Generating Functions

As mentioned in Section 3.2.5, the determinants of the Hessenberg matrices are investigated by several authors. In [7], author introduced a new method to compute the determinant of a special class of the Hessenberg matrices via generating functions. In this section, we extend his method to new three classes of the Hessenberg matrices. Another extension of it to the convolution-like matrices could be found in [93]. This method is based on to determine the relationships between determinants of the Hessenberg matrices whose entries are terms of some certain number sequences and generating functions of these sequences. As an application of our main results, we give an elegant method to compute the determinants of the Hessenberg matrices whose entries consist of the terms of the higher order linear recursive sequences, which based on to find an adjacency-factor matrix. Our results cover many previous results about determinants of the Hessenberg matrices. The obtained results are presented in [8].

In Section 4.7.1, we introduce these three classes of the Hessenberg matrices and show how to compute determinants of these matrices. Also we provide many useful examples to understand the method well.

We would like to remind that we use n instead of N for the order of the matrix throughout this section.

4.7.1 Main Results

Let $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 1}$ be any sequences. Denote their generating functions as $B(x) = \sum_{k\geq 0} b_k x^k$ and $C(x) = \sum_{k\geq 1} c_k x^k$, respectively. The capital calligraphic letter denotes the generating function of the determinant of the related matrix.

To generalize the result of [7], we define the Hessenberg matrix $H_n(r,s)$ of order $n+1$:

For arbitrary nonzero real numbers r and s,

$$H_n(r,s) := \begin{bmatrix} b_0 & r & & & & & 0 \\ b_1 & c_1 & s & & & & \\ b_2 & c_2 & c_1 & r & & & \\ b_3 & c_3 & c_2 & c_1 & s & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ b_{n-1} & c_{n-1} & c_{n-2} & \cdots & \cdots & c_1 & d_n(r,s) \\ b_n & c_n & c_{n-1} & \cdots & \cdots & c_2 & c_1 \end{bmatrix}, \tag{4.8}$$

where

$$d_n(r,s) = \begin{cases} r & \text{if } n \text{ is even,} \\ s & \text{if } n \text{ is odd.} \end{cases}$$

Briefly, we use H_n instead of $H_n(r,s)$ if there is no restrictions on r and s. The case $r = s = 1$ is reduced to matrix considered in [7].

To compute the determinant of H_n via generating function method, we have the following result:

Theorem 4.35. *If*

$$H(x) = \frac{B(x)\left(C(-x) + \frac{r+s}{2}\right) - B(-x)\left(\frac{r-s}{2}\right)}{C(x)C(-x) + \left(\frac{r+s}{2}\right)(C(x) + C(-x)) + rs},$$

then

(i) for even n such that $n = 2t$,

$$\det H_n = r^{t+1}s^t h_n,$$

(ii) for odd n such that $n = 2t+1$,

$$\det H_n = -r^{t+1}s^{t+1}h_n,$$

where $H(x)$ is the generating function of $\{h_n\}_{n\geq 0}$.

Proof. We consider the infinite linear system of equations

$$\begin{bmatrix} r & & & & & 0 \\ c_1x & sx & & & & \\ c_2x^2 & c_1x^2 & rx^2 & & & \\ c_3x^3 & c_2x^3 & c_1x^3 & sx^3 & & \\ c_4x^4 & c_3x^4 & c_2x^4 & c_1x^4 & rx^4 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1x \\ b_2x^2 \\ b_3x^3 \\ b_4x^4 \\ \vdots \end{bmatrix}. \tag{4.9}$$

Here we write

$$\begin{aligned} rh_0 &= b_0 \\ c_1h_0x + sh_1x &= b_1x \\ c_2h_0x^2 + c_1h_1x^2 + rh_2x^2 &= b_2x^2 \\ c_3h_0x^3 + c_2h_1x^3 + c_1h_2x^3 + sh_3x^3 &= b_3x^3 \\ \vdots &= \vdots \end{aligned}$$

By summing both side of the above equalities and (2.13), we obtain

$$H(x)C(x) + r\sum_{k\geq 0} h_{2k}x^{2k} + s\sum_{k\geq 0} h_{2k+1}x^{2k+1} = B(x). \tag{4.10}$$

By the relations (2.14), the above equation could be rewritten as

$$H(x)\left[C(x) + \frac{r+s}{2}\right] + H(-x)\left[\frac{r-s}{2}\right] = B(x).$$

Taking $(-x)$ instead of x, we get

$$H(-x)\left[C(-x) + \frac{r+s}{2}\right] + H(x)\left[\frac{r-s}{2}\right] = B(-x).$$

Solving these two equations in terms of $H(x)$, we get

$$H(x) = \frac{B(x)\left(C(-x) + \frac{r+s}{2}\right) - B(-x)\left(\frac{r-s}{2}\right)}{C(x)C(-x) + \left(\frac{r+s}{2}\right)\left(C(x) + C(-x)\right) + rs},$$

as desired.

Now we examine the relationship between the sequences $\{h_n\}_{n\geq 0}$ and $\{\det H_n\}_{n\geq 0}$. If we consider the system (4.9) for the only first $n+1$ equations and take $x = 1$, the system (4.9) turns to

$$\begin{bmatrix} r & & & & & 0 \\ c_1 & s & & & & \\ c_2 & c_1 & r & & & \\ c_3 & c_2 & c_1 & s & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ c_n & c_{n-1} & c_{n-2} & \cdots & \vdots & d_{n+1}(r,s) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix},$$

where $d_n(r,s)$ is defined as before.

By Cramer's rule (see page 24 in [43]), we obtain $h_n = \dfrac{\det H_n}{r^{t+1}s^t}$ for even n such that $n = 2t$ and $h_n = -\dfrac{\det H_n}{r^{t+1}s^{t+1}}$ for odd n such that $n = 2t+1$, which completes the proof. □

We want to note some important and useful special cases of Theorem 4.35 with the following corollaries:

Corollary 4.15. *For the matrix $H(1,1)$, we have that $h_n = (-1)^n \det H_n$ and the generating function of the sequence $\{\det H_n(1,1)\}_{n\geq 0}$ is*

$$\mathcal{H}(x) = \frac{B(-x)}{1+C(-x)}.$$

This result was firstly given in [7]. We refer to it for some examples.

Corollary 4.16. *For the matrix $H(-1,-1)$, we have that $h_n = -\det H_n$ and the generating function of the sequence $\{\det H_n(-1,-1)\}_{n\geq 0}$ is*

$$\mathcal{H}(x) = \frac{B(x)}{1-C(x)}. \tag{4.11}$$

Let's give some examples.

Example 4.1. *For $n \geq 0$, we have that*

$$\begin{vmatrix} F_1 & -1 & & & & & 0 \\ F_2 & 1 & -1 & & & & \\ F_3 & 1 & 1 & -1 & & & \\ F_4 & 0 & 1 & 1 & -1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ F_n & 0 & 0 & \cdots & \cdots & 1 & -1 \\ F_{n+1} & 0 & 0 & \cdots & \cdots & 1 & 1 \end{vmatrix} = \sum_{k=0}^{n} F_{k+1}F_{n+1-k}.$$

Proof. If $b_n = F_{n+1}$ and $\{c_n\}_{n\geq 1} = \{1,1,0,\ldots\}$, then $B(x) = \frac{1}{1-x-x^2}$ and $C(x) = x+x^2$. So the generating function of $\{\det H_n(-1,-1)\}_{n\geq 0}$ by Corollary 4.16, is $\frac{1}{(1-x-x^2)^2}$, which is the generating function of $\{\sum_{k=0}^{n} F_{k+1}F_{n+1-k}\}_{n\geq 0}$, as well. Thus the proof is complete by Theorem 2.5. □

Example 4.2. *For $n \geq 0$, we have that*

$$\begin{vmatrix} L_0 & -1 & & & & & 0 \\ L_1 & -F_1 & -1 & & & & \\ L_2 & -F_2 & -F_1 & -1 & & & \\ L_3 & -F_3 & -F_2 & -F_1 & -1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ L_{n-1} & -F_{n-1} & -F_{n-2} & \cdots & \cdots & -F_1 & -1 \\ L_n & -F_n & -F_{n-1} & \cdots & \cdots & -F_2 & -F_1 \end{vmatrix} = \begin{cases} 2 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since $b_n = L_n$ and $\{c_n\}_{n\geq 1} = \{-F_n\}_{n\geq 1}$, $B(x) = \frac{2-x}{1-x-x^2}$ and $C(x) = \frac{-x}{1-x-x^2}$. By Corollary 4.16, the generating function of $\{\det H_n(-1,-1)\}_{n\geq 0}$ is

$$\mathcal{H}(x) = \frac{B(x)}{1-C(x)} = \frac{2-x}{1-x^2},$$

which gives the periodic sequence $\{2,-1,2,-1,\ldots\}_{n\geq 0}$. □

Let $\{b_n\}$ be any sequence and $\{c_n\}_{n\geq 1} = \{1,0,0,\ldots\}$. Since $\frac{1}{1-x}B(x)$ is the generating function of the sum of the first nth term of the sequence $\{b_n\}$, by Corollary 4.16 and Theorem 2.5, we have

$$\det H_n(-1,-1) = \sum_{k=0}^{n} b_k.$$

For example,

$$\begin{vmatrix} 1 & -1 & & & & & 0 \\ \frac{1}{2} & 1 & -1 & & & & \\ \frac{1}{3} & 0 & 1 & -1 & & & \\ \frac{1}{4} & 0 & 0 & 1 & -1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ \frac{1}{n} & 0 & 0 & \cdots & \cdots & 1 & -1 \\ \frac{1}{n+1} & 0 & 0 & \cdots & \cdots & 0 & 1 \end{vmatrix} = \mathscr{H}_{n+1},$$

where $\mathscr{H}_n$ stands for nth harmonic number, which is $\sum_{k=1}^{n} \frac{1}{k}$.

Since the permanental and determinantal relationships between the matrices $H_n(1,1)$ and $H_n(-1,-1)$ are

$$\det H_n(1,1) = \mathrm{per} H_n(-1,-1) \text{ and } \mathrm{per} H_n(1,1) = \det H_n(-1,-1),$$

one can easily derive some permanental relations for the Hessenberg matrices by the help of the above corollaries.

Corollary 4.17. *If*

$$H(x) = \frac{C(-x)B(x) - B(-x)}{C(x)C(-x) - 1},$$

then we have

$$\det H_n(1,-1) = (-1)^{\binom{n}{2}} h_n.$$

We shall give an example:

Example 4.3. *If we take* $\{c_n\} = \{(-1)^n F_{n-1}\}$ *and define the sequence* $\{b_n\}_{n\geq 0}$ *as* $b_{2n} = -b_{2n+1} = F_{2n+2}$*, then for even* n *such that* $n = 2k$*, the matrix* $H_n(1,-1)$ *takes the form*

$$H_{2k}(1,-1) = \begin{bmatrix} F_2 & 1 & & & & & 0 \\ -F_2 & 0 & -1 & & & & \\ F_4 & F_1 & 0 & 1 & & & \\ -F_4 & -F_2 & F_1 & 0 & -1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ -F_{2k} & -F_{2k-2} & F_{2k-3} & \cdots & \cdots & 0 & -1 \\ F_{2k+2} & F_{2k-1} & -F_{2k-2} & \cdots & \cdots & F_1 & 0 \end{bmatrix}$$

and for odd n *such that* $n = 2k-1$*, the matrix* $H_n(1,-1)$ *takes the form*

$$H_{2k-1}(1,-1) = \begin{bmatrix} F_2 & 1 & & & & & 0 \\ -F_2 & 0 & -1 & & & & \\ F_4 & F_1 & 0 & 1 & & & \\ -F_4 & -F_2 & F_1 & 0 & -1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ F_{2k} & F_{2k-2} & -F_{2k-3} & \cdots & \cdots & 0 & -1 \\ -F_{2k} & -F_{2k-2} & F_{2k-2} & \cdots & \cdots & F_1 & 0 \end{bmatrix}.$$

So that

$$\det H_{2k}(1,-1) = (-1)^k F_{2k+1} \text{ and } \det H_{2k-1}(1,-1) = (-1)^k F_{2k}.$$

Proof. The generating functions of $\{b_n\}_{n\geq 0}$ and $\{c_n\}$ are $B(x) = \frac{1-x}{(1+x-x^2)(1-x-x^2)}$ and $C(x) = \frac{x^2}{1+x-x^2}$, respectively. So we get $H(x) = \frac{1}{1-x-x^2}$. By Corollary 4.17, the claim follows. □

Corollary 4.18. *For* $d \neq 0$*, if*

$$H(x) = \frac{B(x)}{C(x)+d},$$

then

$$\det H_n(d,d) = (-1)^n d^{n+1} h_n$$

and the generating function of $\{\det H_n(d,d)\}_{n\geq 0}$ *is*

$$\mathcal{H}(x) = d \cdot H(-dx).$$

Example 4.4. *If* $b_n = -(\mathscr{H}_n + 1)$ *with* $b_0 = -1$ *and* $c_n = \frac{2}{n}$, *then*

$$\begin{vmatrix} -1 & 2 & & & & & 0 \\ -(\mathscr{H}_1+1) & 2 & 2 & & & & \\ -(\mathscr{H}_2+1) & 1 & 2 & 2 & & & \\ -(\mathscr{H}_3+1) & \frac{2}{3} & 1 & 2 & 2 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ -(\mathscr{H}_{n-1}+1) & \frac{2}{n-1} & \frac{2}{n-2} & \cdots & \cdots & 2 & 2 \\ -(\mathscr{H}_n+1) & \frac{2}{n} & \frac{2}{n-1} & \cdots & \cdots & 1 & 2 \end{vmatrix} = (-1)^{n-1}2^n.$$

Proof. If we take $d = 2$, $b_n = -(\mathscr{H}_n + 1)$ with $b_0 = -1$ and $c_n = \frac{2}{n}$ in Corollary 4.18, then we get

$$B(x) = \frac{\ln(1-x) - 1}{1-x} \quad \text{and} \quad C(x) = \ln(1-x)^{-2}.$$

Thus $H(x) = \frac{1}{2x-2}$ and $\det H_n = 2(-2)^n h_n$, which give us $\det H_n = (-1)^{n-1}2^n$, as claimed. □

When $c_0 = d$, by Corollary 4.18, we obtain $H(x) = \frac{B(x)}{C(x)}$, where $C(x) = \sum_{k\geq 0} c_k x^k$. For example, if we choose $B(x) = x + 4x^2 + x^3$ and $C(x) = (1-x)^4$, then

$$\begin{vmatrix} 0 & 1 & & & & & & 0 \\ 1 & -4 & 1 & & & & & \\ 4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ 0 & 1 & -4 & 6 & \ddots & \ddots & & \\ 0 & 0 & 1 & -4 & \ddots & -4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \cdots & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 \end{vmatrix}_{(n+1)\times(n+1)} = (-1)^n n^3.$$

Now we recall an already known result given in [89]. But we will give an alternative and simpler proof for it.

Corollary 4.19. *If $\{c_n\}_{n\geq 0}$ is any sequence such that $c_0 \neq 0$, then we have*

$$\begin{vmatrix} c_1 & c_0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{vmatrix}_{n\times n} = [x^n] \frac{c_0}{C(-c_0 x)},$$

where $C(x) = \sum_{k\geq 0} c_k x^k$ and $[\circ]$ is the coefficient extraction operator, i.e. $[x^n] \sum_{k\geq 0} a_k x^k = a_n$.

Proof. First we consider an equal determinant to the claimed determinant by the following equality

$$\begin{vmatrix} c_1 & c_0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{vmatrix}_{n\times n} = \begin{vmatrix} 1 & c_0 & 0 & 0 & \cdots & 0 \\ 0 & c_1 & c_0 & 0 & \cdots & 0 \\ 0 & c_2 & c_1 & c_0 & \cdots & 0 \\ 0 & c_3 & c_2 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{vmatrix}_{(n+1)\times(n+1)}.$$

The value of the determinant on the RHS of the above equation could be easily found by Corollary 4.18. So that the claimed result directly follows. □

Let's give an example related to Theorem 4.35.

Example 4.5. *Let $\{b_n\}_{n\geq 0}$ be the alternating of the sequence A135491 in* [94]. *Then for $n = 2k$,*

$$\begin{vmatrix} b_0 & 1 & & & & & 0 \\ b_1 & 1 & -3 & & & & \\ b_2 & 1 & 1 & 1 & & & \\ b_3 & 1 & 1 & 1 & -3 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ b_{2k-1} & 0 & 0 & \cdots & \cdots & 1 & -3 \\ b_{2k} & 0 & 0 & \cdots & \cdots & 1 & 1 \end{vmatrix} = (-3)^k T_{2k+2}.$$

Similarly, for $n = 2k+1$, determinant of the corresponding Hessenberg matrix is equal to $-T_{2k+3}(-3)^{k+1}$, where T_n stands for the nth Tribonacci number defined in Table 1.

Proof. The generating functions of $\{b_n\}_{n\geq 0}$ and $\{c_n\}$ are $B(x) = \frac{1-x+x^2-x^3}{1+x-x^2+x^3}$ and $C(x) = x + x^2 + x^3$, respectively. By Theorem 4.35, when $r = 1$ and $s = -3$, we obtain $\det H_n = T_{n+2}(-3)^k$ for $n = 2k$ and $\det H_n = -T_{n+2}(-3)^{k+1}$ for $n = 2k+1$, as desired. □

Up to now, we consider certain Hessenberg matrices whose superdiagonal are constant or two periodic. Now we give a general idea for Hessenberg matrices with arbitrary superdiagonal entries. To show how this idea will be applied, we present two Hessenberg matrices whose superdiagonals will consist of the terms of two special sequences, $\{n\}$ and $\{2^{n-1}\}$, respectively.

Let $\{b_n\}_{n\geq 0}$, $\{c_n\}$ and $\{d_n\}_{n\geq 0}$ such that $d_n \neq 0$ for all $n \geq 0$ be any sequences. First define the Hessenberg matrix H_n of order $n+1$ as

$$H_n := \begin{bmatrix} b_0 & d_0 & & & & & 0 \\ b_1 & c_1 & d_1 & & & & \\ b_2 & c_2 & c_1 & d_2 & & & \\ b_3 & c_3 & c_2 & c_1 & d_3 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ b_{n-1} & c_{n-1} & c_{n-2} & \cdots & \cdots & c_1 & d_{n-1} \\ b_n & c_n & c_{n-1} & \cdots & \cdots & c_2 & c_1 \end{bmatrix}.$$

Now consider the following infinite linear system of equations

$$\begin{bmatrix} d_0 & & & & & & 0 \\ c_1x & d_1x & & & & & \\ c_2x^2 & c_1x^2 & d_2x^2 & & & & \\ c_3x^3 & c_2x^3 & c_1x^3 & d_3x^3 & & & \\ c_4x^4 & c_3x^4 & c_2x^4 & c_1x^4 & d_4x^4 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1x \\ b_2x^2 \\ b_3x^3 \\ b_4x^4 \\ \vdots \end{bmatrix},$$

which gives us the relation

$$H(x)C(x) + \sum_{k=0}^{\infty} h_k d_k x^k = B(x), \tag{4.12}$$

where $C(x) = \sum_{k\geq 1} c_k x^k$. If we restricted this infinite system to the first $n+1$ equations with $x = 1$, then by Cramer's rule we have

$$h_n = \frac{(-1)^n \det H_n}{\prod_{k=0}^{n} d_k}.$$

Unfortunately, the series $\sum_{k=0}^{\infty} h_k d_k x^k$ in (4.12), which is the generating function of the Hadamard product of the sequences $\{h_n\}_{n\geq 0}$ and $\{d_n\}_{n\geq 0}$, can not be always computed explicitly in terms of the generating functions $H(x)$ and $D(x)$. Nevertheless, it is possible to compute it for some special cases. So that one can compute the determinant of these type of matrices via generating function method. Now we present two special example to show how we can use the idea mentioned above.

Theorem 4.36. *If* $\{d_n\}_{n\geq 0} = \{n+1\}$, *then*

$$xH(x)\left(e^{\int \frac{C(x)}{x}dx}\right) = \int e^{\int \frac{C(x)}{x}dx} B(x)dx + C,$$

with

$$\det H_n = (-1)^n (n+1)! h_n,$$

where C *is a constant.*

Proof. By (4.12), we have

$$H(x)C(x) + \sum_{k=0}^{\infty} h_k (k+1) x^k = B(x),$$

which, equivalently, gives us

$$H(x)C(x) + (xH(x))' = B(x).$$

By taking $y = x \cdot H(x)$, we get the first order linear differential equation

$$y\frac{C(x)}{x} + y' = B(x).$$

The solution of this differential equation is

$$y = \left(e^{\int \frac{C(x)}{x}dx}\right)^{-1} \left(\int e^{\int \frac{C(x)}{x}dx} B(x)dx + C\right),$$

which completes the proof. Note that the constant C will be determined according to the initial value $y(0) = 0$. □

Example 4.6. *For* $n \geq 0$, *we have*

$$\begin{vmatrix} 1 & 1 & & & & & 0 \\ 3 & 1 & 2 & & & & \\ 5 & 1 & 1 & 3 & & & \\ 7 & 1 & 1 & 1 & 4 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ 2n-1 & 1 & 1 & \cdots & \cdots & 1 & n \\ 2n+1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{vmatrix} = (-1)^n (n+1)!.$$

Proof. Since $b_n = 2n+1$ and $c_n = 1$, we obtain $B(x) = \frac{x+1}{(x-1)^2}$ and $C(x) = \frac{x}{1-x}$. So we get

$$\int \frac{1}{1-x}dx = -\ln(x-1) \text{ and } e^{\int \frac{C(x)}{x}dx} = \frac{1}{x-1}.$$

By Theorem 4.36, we have that

$$xH(x)\frac{1}{x-1} = \int \frac{x+1}{(x-1)^3}dx + C$$
$$xH(x)\frac{1}{x-1} = -\frac{x}{(x-1)^2} + C.$$

For $x = 0$, we find that $C = 0$ and so

$$H(x) = \frac{1}{1-x},$$

which gives $\det H_n = (-1)^n(n+1)!$. □

For the case $b_n = c_{n+1}$, i.e. $B(x) = \frac{C(x)}{x}$, the relation given in Theorem 4.36 turns

$$xH(x) = 1 + C\left(e^{\int \frac{C(x)}{x}dx}\right)^{-1}.$$

Now we present the other interesting special case with an example which could be produced by the help of relation (4.12).

Example 4.7. *For $n \geq 0$,*

$$\begin{vmatrix} 1 & 1 & & & & & 0 \\ 3 & 1 & 2 & & & & \\ 4 & 1 & 1 & 4 & & & \\ \frac{10}{3} & \frac{1}{2} & 1 & 1 & 8 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ \frac{2^{n-2}(n+1)}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & \cdots & 1 & 2^{n-1} \\ \frac{2^{n-1}(n+2)}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \cdots & 1 & 1 \end{vmatrix} = \frac{(-1)^n 2^{\binom{n+1}{2}}}{n!}.$$

Proof. Since $b_n = \frac{2^{n-1}(n+2)}{n!}$ and $c_n = \frac{1}{(n-1)!}$, their generating functions are $B(x) = e^{2x}(x+1)$ and $C(x) = xe^x$, respectively. By (4.12), we have

$$xe^x H(x) + H(2x) = e^{2x}(x+1).$$

Hence we find that $H(x) = e^x$, which gives $h_n = \frac{1}{n!}$. Finally, from the relation $h_n = \frac{(-1)^n \det H_n}{2^{\binom{n+1}{2}}}$, we obtain claimed result. □

Now we consider different two classes of Hessenberg determinants, which have not been studied before. We start with the first one: For any nonzero real d, we define a Hessenberg matrix of order $n+1$ as follows:

$$H_n := \begin{bmatrix} b_0 & d & & & & & 0 \\ b_1 & c_1 & d & & & & \\ b_2 & c_2 & d_1 & d & & & \\ b_3 & c_3 & d_2 & d_1 & d & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ b_{n-1} & c_{n-1} & d_{n-2} & \cdots & \cdots & d_1 & d \\ b_n & c_n & d_{n-1} & \cdots & \cdots & d_2 & d_1 \end{bmatrix}.$$

Theorem 4.37. *If*

$$H(x) = \frac{B(x) + h_0 D(x) - h_0 C(x)}{D(x) + d} \quad \textit{with } h_0 = b_0/d, \tag{4.13}$$

where $B(x)$ and $C(x)$ defined as before and $D(x) = \sum_{k\geq 1} d_k x^k$, then

$$\det H_n = (-1)^n d^{n+1} h_n$$

and the generating function of $\{\det H_n\}_{n\geq 0}$ is

$$\mathcal{H}(x) = d \cdot H(-dx).$$

Proof. Similar to the proof of Theorem 4.35, we have the following infinite linear system of equations

$$\begin{bmatrix} d & & & & & 0 \\ c_1 x & dx & & & & \\ c_2 x^2 & d_1 x^2 & dx^2 & & & \\ c_3 x^3 & d_2 x^3 & d_1 x^3 & dx^3 & & \\ c_4 x^4 & d_3 x^4 & d_2 x^4 & d_1 x^4 & dx^4 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 x \\ b_2 x^2 \\ b_3 x^3 \\ b_4 x^4 \\ \vdots \end{bmatrix}.$$

By summing the equations come from this infinite linear system of equations and adding $h_0 D(x)$ to the both sides of it, we obtain

$$h_0 C(x) + H(x)D(x) + h_0 H(x) = B(x) + h_0 D(x),$$

which gives

$$H(x) = \frac{B(x) + h_0 D(x) - h_0 C(x)}{D(x) + d},$$

as desired. Finally, if we restrict this linear system of equations to the first $(n+1)$ equations and take $x = 1$, then by Cramer's rule, we get $h_n = \frac{(-1)^n \det H_n}{d^{n+1}}$, as claimed. □

As an example,

Example 4.8. *For $n > 0$,*

$$\begin{vmatrix} P_1 & 1 & & & & & 0 \\ P_2 & F_2 & 1 & & & & \\ P_3 & F_3 & P_2 & 1 & & & \\ P_4 & F_4 & P_3 & P_2 & 1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ P_n & F_n & P_{n-1} & \cdots & \cdots & P_2 & 1 \\ P_{n+1} & F_{n+1} & P_n & \cdots & \cdots & P_3 & P_2 \end{vmatrix} = (-1)^n F_{n-1},$$

where F_n and P_n are the nth Fibonacci and Pell numbers, given in Table 1.

Proof. It is a consequence of Theorem 4.37. When $d = 1$, $B(x) = \sum_{k\geq 0} P_{k+1} x^k = \frac{1}{1-2x-x^2}$, $C(x) = \sum_{k\geq 1} F_{k+1} x^k = \frac{x+x^2}{1-x-x^2}$ and $D(x) = \sum_{k\geq 1} P_{k+1} x^k = \frac{2x+x^2}{1-2x-x^2}$, then

$$H(x) = \frac{1 - 2x - x^2}{1 - x - x^2},$$

which completes the proof. □

Now we define the second class of Hessenberg matrices of order $n+1$, whose columns are periodic after first column, as follows:

$$H_n = \begin{bmatrix} b_0 & d & & & & & & 0 \\ b_1 & c_1 & d & & & & & \\ b_2 & c_2 & d_1 & d & & & & \\ b_3 & c_3 & d_2 & c_1 & d & & & \\ b_4 & c_4 & d_3 & c_2 & d_1 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & d & \\ b_{n-1} & c_{n-1} & d_{n-2} & c_{n-3} & d_{n-4} & \cdots & s(n,1) & d \\ b_n & c_n & d_{n-1} & c_{n-2} & d_{n-3} & \cdots & s(n,2) & s(n+1,1) \end{bmatrix},$$

where

$$s(n,k) = \begin{cases} c_k & \text{if } n \text{ is even,} \\ d_k & \text{if } n \text{ is odd.} \end{cases}$$

We have the following theorem for the generating function of the determinant of the just above matrix.

Theorem 4.38. *If*

$$H(x) = \frac{B(x)(C(-x)+D(-x)+2) - B(-x)(C(x)-D(x))}{C(x)(1+D(-x)) + D(x)(1+C(-x)) + (C(-x)+D(-x)) + 2d},$$

then

$$\det H_n = (-1)^n d^{n+1} h_n$$

and the generating function of $\{\det H_n\}_{n\geq 0}$ *is*

$$\mathcal{H}(x) = d \cdot H(-dx).$$

Proof. Similar to the previous theorems, if we consider the infinite linear system of equations, then we obtain

$$C(x)\sum_{k\geq 0} h_{2k}x^{2k} + D(x)\sum_{k\geq 0} h_{2k+1}x^{2k+1} + dH(x) = B(x). \tag{4.14}$$

By the formulæ (2.14), the equation (4.14) is written as

$$H(x)\left(\frac{C(x)+D(x)}{2}+1\right) + H(-x)\left(\frac{C(x)-D(x)}{2}\right) = B(x),$$

which, by solving in terms of $H(x)$, gives us

$$H(x) = \frac{B(x)(C(-x)+D(-x)+2) - B(-x)(C(x)-D(x))}{C(x)(1+D(-x)) + D(x)(1+C(-x)) + (C(-x)+D(-x)) + 2d},$$

as desired. When we restricted the infinite system of equations to the first $n+1$ equations with $x=1$, we complete the proof by Cramer's rule. □

Example 4.9. *For even n, we have*

$$\begin{vmatrix} L_0 & 1 & & & & & & 0 \\ L_1 & F_1 & 1 & & & & & \\ L_2 & F_2 & L_0 & 1 & & & & \\ L_3 & F_3 & L_1 & F_1 & 1 & & & \\ L_4 & F_4 & L_2 & F_2 & L_0 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 1 & \\ L_{n-1} & F_{n-1} & L_{n-3} & F_{n-3} & L_{n-5} & \cdots & F_1 & 1 \\ L_n & F_n & L_{n-2} & F_{n-2} & L_{n-4} & \cdots & F_2 & L_0 \end{vmatrix} = 2^{\frac{n}{2}} + 1.$$

If $n = 2k+1$, the determinant of the corresponding matrix is equal to 2^k.

Proof. Since $b_n = L_n$, $c_n = F_n$ and $d_n = L_{n-1}$, we have $B(x) = \frac{2-x}{1-x-x^2}$, $C(x) = \frac{x}{1-x-x^2}$ and $D(x) = \frac{2x-x^2}{1-x-x^2}$. Hence, for $d = 1$ by Theorem 4.38, we obtain

$$\begin{aligned} A(x) =& \frac{-x-3x^2+x^3+2}{(x-1)(x+1)(2x^2-1)} = \frac{1}{1-x^2} + \frac{1+x}{1-2x^2} \\ =& \sum_{k=0}^{\infty} x^{2k} + \sum_{k=0}^{\infty} 2^k x^{2k} + \sum_{k=0}^{\infty} 2^k x^{2k+1} \\ =& \sum_{k=0}^{\infty} (2^k+1)x^{2k} + \sum_{k=0}^{\infty} 2^k x^{2k+1}, \end{aligned}$$

as claimed. □

Remark 4.3. *This generating function method works for only Hessenberg matrices. If a matrix has nonzero two superdiagonal bands, then the corresponding infinite linear system of equations is inconsistent. So we can't apply the same steps for this matrix. In that case, we may reduce this matrix to a Hessenberg matrix by applying some row or column operations. Then we may use Theorems 4.37 or 4.35. If the number of nonzero superdiagonals is increased, then computing their determinants via generating functions would become harder and more complicated.*

4.7.2 A Matrix Method to Compute a Class of Hessenberg Determinants

Now we give a new and simple method to compute a class of Hessenberg determinants whose entries consist of the terms of the higher order linear recursive sequence with constant coefficients.

Consider the following lower Hessenberg matrix of order n for nonzero real r:

$$E_n(r) = \begin{bmatrix} u_1 & r & & & & & 0 \\ u_2 & u_1 & r & & & & \\ u_3 & u_2 & u_1 & r & & & \\ u_4 & u_3 & u_2 & u_1 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ u_{n-1} & u_{n-2} & u_{n-3} & u_{n-4} & \cdots & u_1 & r \\ u_n & u_{n-1} & u_{n-2} & u_{-3} & \cdots & u_2 & u_1 \end{bmatrix},$$

where the terms u_n's are defined as in (2.1). Briefly, we use E_n instead of $E_n(r)$.

Indeed one can compute the determinant of the matrix E_n by using the results of Section 4.7.1. Here we would like to present a new and simple method to compute $\det E_n$. For this, we define an adjacency-factor matrix related to the matrix E_n: Define a lower triangular adjacency-factor matrix M of order n as

$$M_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -p_{i-j} & \text{if } 1 \leq i - j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the matrix M is of the form

$$M = \begin{bmatrix} 1 & & & & & & 0 \\ -p_1 & 1 & & & & & \\ -p_2 & -p_1 & 1 & & & & \\ \vdots & -p_2 & \ddots & \ddots & & & \\ -p_k & & \ddots & & & & \\ & \ddots & & \ddots & \ddots & \ddots & \\ 0 & & -p_k & \cdots & -p_2 & -p_1 & 1 \end{bmatrix}.$$

Then we obtain that

$$ME_n = \widehat{E}_n,$$

where

$$\left(\widehat{E}_n\right)_{ij} = \begin{cases} r & \text{if } j = i + 1, \\ b_i & \text{if } j = 1 \text{ and } i \leq k, \\ d_{i-j+1} & \text{if } i \geq j > 1 \text{ and } i - j \leq k - 1, \\ 0 & \text{otherwise,} \end{cases}$$

with

$$b_m = u_m - \sum_{l=1}^{m-1} u_{m-l} p_l \text{ and } d_m = u_m - \sum_{l=1}^{m-1} u_{m-l} p_l - r p_m,$$

for $1 \leq m \leq k$.

Here since $\det M = 1$, we have $\det E_n = \det \widehat{E}_n$. Afterwards, we prefer to compute the value of the determinant of the matrix $\widehat{E}_n$ instead of the matrix E_n because the matrix $\widehat{E}_n$ is a banded matrix with bandwidth $(k+1)$ and includes many zeros and so it gives us advantage to choose the matrix $\widehat{E}_n$ rather than E_n regard to use of the results of Section 4.7.1. That means one can easily apply the results given in Section 4.7.1 to the matrix $\widehat{E}_n$ with less computation.

For example, when $r=-1$, by Corollary 4.16, we have that

$$\sum_{i\geq 0}\det E_{i+1}(-1)x^i=\sum_{i\geq 0}\det \widehat{E}_{i+1}(-1)x^i=\frac{\sum_{i=1}^{k}b_ix^{i-1}}{1-\sum_{i=1}^{k}d_ix^i}. \tag{4.15}$$

In [88], author computed the determinant of the matrix $E_n(-1)$ when $k=2$ by using the cofactor expansion and he only gave complicated formula for the case $k=3$ without proof. Our method is simpler to determine those formulæ and also one can find related formula for larger k with less effort.

As a special case, if we consider the recurrence relation of the sequence $\{u_n\}$ defined in (2.1) with the initials $u_{-k+2}=u_{-k+3}=\cdots=u_{-1}=u_0=0$ and $u_1=1$, then we have

$$b_1=1 \text{ and } b_i=0 \text{ for } 1<i\leq k,$$
$$d_1=1+p_1 \text{ and } d_i=p_i \text{ for } 1<i\leq k.$$

Hence the generating function of the determinant of the matrix $E_{n+1}(-1)$ is written as

$$\frac{1}{1-(1+p_1)x-p_2x^2-\cdots-p_kx^k}. \tag{4.16}$$

Now we give an example to show how to use the method described above.

Example 4.10. *For positive integer m, define the sequence $\{u_n\}$ with $u_n=\binom{m+n-1}{m}$ and construct the following matrix $G_n(m)$ of order n*

$$G_n(m):=\begin{bmatrix}
\binom{m}{m} & -1 & & & & & 0\\
\binom{m+1}{m} & \binom{m}{m} & -1 & & & & \\
\binom{m+2}{m} & \binom{m+1}{m} & \binom{m}{m} & \ddots & & & \\
\binom{m+3}{m} & \binom{m+2}{m} & \binom{m+1}{m} & \cdots & -1 & & \\
\cdots & \cdots & \cdots & \cdots & \binom{m}{m} & -1 & \\
\binom{m+n-2}{m} & \binom{m+n-3}{m} & \binom{m+n-4}{m} & \vdots & \binom{m+1}{m} & \binom{m}{m} & -1\\
\binom{m+n-1}{m} & \binom{m+n-2}{m} & \binom{m+n-3}{m} & \vdots & \binom{m+2}{m} & \binom{m+1}{m} & \binom{m}{m}
\end{bmatrix}.$$

Then

$$\det G_{n+1}(m)=\sum_{k=0}^{n}\binom{(m+1)n+m(1-k)}{k}.$$

Proof. We should find the recurrence relation for the sequence $\{u_n\}$. From [13], we recall the Equation 5.24: For $l\geq 0$ and integers m, n,

$$\sum_k\binom{l}{m+k}\binom{s+k}{n}(-1)^k=(-1)^{l+m}\binom{s-m}{n-l},$$

which is a variant of the Vandermonde's identity. If we choose $l \to m+1$, $m \to 1$, $s \to m-n$ and $n \to m$ in the equation above, then we obtain

$$\begin{aligned}\sum_{k=-1}^{m}(-1)^k\binom{m+1}{k+1}\binom{n-k-1}{m} &= \sum_{k=-1}^{m}(-1)^{k+m}\binom{m+1}{k+1}\binom{m-n+k}{m}\\ &=\binom{n-m-1}{-1}=0.\end{aligned}$$

So, we can deduce that

$$\sum_{k=0}^{m}(-1)^k\binom{m+1}{k+1}\binom{n-k-1}{m}=\binom{n}{m}.$$

If we take $n = n+m-1$, then we get the recurrence relation of order $m+1$ for the sequence $\{u_n\}$:

$$u_n=\sum_{k=0}^{m}(-1)^k\binom{m+1}{k+1}u_{n-k-1},$$

with $u_{-m+1}=u_{-m+2}=\cdots=u_{-1}=u_0=0$ and $u_1=1$. By our method, we see that the adjacency-factor matrix for the matrix $G_n(m)$ is

$$M_{ij}=(-1)^{i-j}\binom{m+1}{i-j}.$$

Thus by (4.16), we find the generating function of the sequence $\{\det G_{n+1}(m)\}_{n\geq 0}$ as follows

$$\frac{1}{1-\left(1+\binom{m+1}{1}\right)x+\binom{m+1}{2}x^2-\cdots-(-1)^m\binom{m+1}{m+1}x^{m+1}}=\frac{1}{(1-x)^{m+1}-x}.$$

In other words, we have that

$$[x^n]\frac{1}{(1-x)^{m+1}-x}=\det G_{n+1}(m). \tag{4.17}$$

To prove this claim, by Theorem 2.5 it is sufficient to show that

$$\sum_{n\geq 0}\sum_{k=0}^{n}\binom{(m+1)n+m(1-k)}{k}x^n=\frac{1}{(1-x)^{m+1}-x}.$$

Consider,

$$\begin{aligned}\sum_{n\geq 0}\sum_{k=0}^{n}\binom{(m+1)n+m(1-k)}{k}x^n &= \sum_{k\geq 0}\sum_{n\geq k}\binom{(m+1)n+m(1-k)}{k}x^n\\ &=\sum_{n\geq 0}x^n\sum_{k\geq 0}\binom{m+n+mn+k}{k}x^k\end{aligned}$$

$$= \frac{1}{(1-x)^{m+1}} \sum_{n\geq 0} \left(\frac{x}{(1-x)^{m+1}} \right)^n$$
$$= \frac{1}{(1-x)^{m+1} - x},$$

which completes the proof. □

As a special case for $m = 1$, we get

$$\begin{vmatrix} 1 & -1 & & & & 0 \\ 2 & 1 & -1 & & & \\ 3 & 2 & \ddots & \ddots & & \\ \cdots & \cdots & \cdots & 1 & -1 & 0 \\ n-1 & n-2 & \vdots & 2 & 1 & -1 \\ n & n-1 & \vdots & 3 & 2 & 1 \end{vmatrix} = \sum_{k=0}^{n-1} \binom{2n-k-1}{k} = F_{2n},$$

which could be also found in [88].

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Publications

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2. Arıkan T., Kılıç E., A class of non-symmetric band determinants with the Gaussian q-binomial coefficients, *Quaestiones Mathematicae*, 40(5), 645–660, **2017.**

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