PURE DIRECT INJECTIVE OBJECTS IN GROTHENDIECK CATEGORIES

GROTHENDIECK KATEGORİLERDE SAF DİREKT İNJEKTİF NESNELER

ALİYE YİĞİT

ASSOC. PROF. DR. SULTAN EYLEM TOKSOY
Supervisor

Submitted to
Graduate School of Science and Engineering of Hacettepe University
as a Partial Fulfillment to the Requirements
for the Award of the Degree of Master of Science
in Mathematics

May 2022

ABSTRACT

PURE DIRECT INJECTIVE OBJECTS IN GROTHENDIECK CATEGORIES

Aliye Yiğit

Master of Science, Mathematics

Supervisor: Assoc. Prof. Dr. Sultan Eylem TOKSOY

May 2022, 76 pages

We study generalizations of the concept of direct-injectivity (respectively pure-direct-injectivity) from module categories to abelian categories (respectively Grothendieck categories). We examine for which categories or under what conditions direct-injective objects are injective or quasi-injective. Also we examine for which categories or under what conditions pure-direct-injective objects are injective, quasi-injective, pure-injective or direct-injective. We investigate classes all of whose objects are direct-injective (respectively pure-direct-injective). We also give applications of some results to module categories and comodule categories.

Keywords: pure subobjects, direct-injective objects, pure-direct-injective objects, abelian categories, Grothendieck categories

ÖZET

GROTHENDIECK KATEGORİLERDE SAF DİREKT İNJEKTİF NESNELER

Aliye Yiğit

Yüksek Lisans, Matematik

Danışman: Assoc. Prof. Dr. Sultan Eylem TOKSOY

Mayıs 2022, 76 sayfa

Bu tezde direkt-injektif (sırasıyla saf-direkt-injektif) kavramlarının modül kategoriden abel kategorilere (sırasıyla Grothendieck kategorilere) genelleştirilmesi üzerine çalıştık. Hangi kategorilerde ya da hangi koşullar altında direkt-injektif nesnelerin injektif ya da yarı-injektif olduğunu inceledik. Ayrıca hangi kategorilerde ya da hangi koşullar altında saf-direkt-injektif nesnelerin injektif, yarı-injektif, saf-injektif ya da direkt-injektif olduğunu inceledik. Bütün nesneleri direkt-injektif (sırasıyla saf-direkt-injektif) olan sınıfları belirledik. Ayrıca bazı sonuçlarımızın modül kategorilere ve eşmodül kategorilere uygulamalarını verdik.

Keywords: saf alt nesneler, direkt-injektif nesneler, saf-direkt-injektif nesneler, abel kategoriler, Grothendieck kategoriler

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor Assoc. Prof. Dr. Sultan Eylem TOKSOY for her invaluable supervision, continuous support and guidance. She is the true role model for me.

Also, I would like to thank the valued thesis committee: Prof. Dr. Ayşe Çiğdem ÖZCAN, Prof. Dr. Evrim AKALAN, Assoc. Prof. Dr. Bülent SARAÇ and Assoc. Prof. Dr. Oğuz YAYLA, for their efforts and attention.

Also, I would like to thank my best friends Eylül ÖZDEMİR and Zeynep Dilhan ÇELİK, for their excellent encouragement and caring, my sister Ayçelen YİĞİT for always being there for me.

Finally, I would like to thank my dad Adnan YİĞİT and my mom Betül YİĞİT for your endless support. You have always stood behind me.

Aliye YİĞİT

May 2022, Ankara

CONTENTS

Page

ABSTRACT i
ÖZET ii
ACKNOWLEDGEMENTS iii
CONTENTS iv
ABBREVIATIONS vi
1. INTRODUCTION 1
2. PRELIMINARIES 4
2.1 Categories 4
2.2 Functors 7
2.3 Natural transformations 10
2.4 Special morphisms 10
2.5 Products and Coproducts 18
2.6 Kernels and Cokernels 21
2.7 Exact sequences 23
2.8 Pullbacks and Pushouts 25
2.9 Grothendieck categories 30
2.10 Injective objects 34
2.11 Pure subobjects 37
2.12 Pure-injective objects 38
3. DIRECT INJECTIVE OBJECTS IN ABELIAN CATEGORIES 42
3.1 Direct-injective objects 42
3.2 Classes all of whose objects are direct-injective 46
4. PURE DIRECT INJECTIVE OBJECTS IN GROTHENDIECK CATEGORIES 48
4.1 Pure-direct-injective objects 48
4.2 Classes all of whose objects are pure-direct-injective 55
4.3 Relative pure-direct-injective objects 57
5. APPLICATIONS 60

5.1 Modules and Comodules 60
6. CONCLUSION 63
6.1 Direct-injective Objects 63
6.2 Pure-direct-injective Objects 64

ABBREVIATIONS

$\mathcal{A}$	A category
$\mathcal{O}\text{b}(\mathcal{A})$	All objects of a category $\mathcal{A}$
$\text{Mor}\,(M, N)$	All morphisms from M to N
$\circ$	The composition function
$E(M)$	Injective envelope of an object M of a category $\mathcal{A}$
$PE(M)$	Pure injective envelope of an object M of a category $\mathcal{A}$
$\text{Im}\, f$	The image of morphism f
$\text{Coim}\, f$	The coimage of morphism f
$\text{Ker}\, f$	The kernel of a morphism f
$\text{Coker}\, f$	The cokernel of a morphism f
Set	The category of sets
Grp	The category of groups
Top	The category of topological spaces
$Ring$	The category of rings with identity
Ab	The category of abelian groups
R-Mod	The category of left R-modules
Mod-R	The category of right R-modules
$DivAb$	The category of divisible abelian groups
$\mathcal{A}^{op}$	The opposite category of a category $\mathcal{A}$
C	A coalgebra
$\mathcal{M}^C$	The category of right comodules
$\mathcal{E}$	The class of short exact sequence of objects of an abelian category $\mathcal{A}$
$\lim F$	The limit of functor F
$\text{colim}\, F$	The colimit of functor F
$\varinjlim X_i$	The direct limit of direct system X_i
$\bigoplus M_i$	The coproduct of the family of objects of a category $\mathcal{A}$

$\prod M_i$	The product of the family of objects of a category $\mathcal{A}$
$\triangle$	The diagonal functor
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{Q}_p$	A normal subgroup of $\mathbb{Q}$

1. INTRODUCTION

A right R-module M is said to be *direct-injective* if every submodule A of M with A isomorphic to a direct summand of M is a direct summand of M. Direct-injective modules were introduced by Nicholson in [1] and further studies on direct-injective modules were done by Chae and Kwon in [2], Xue in [3] and Zhizhong in [4]. The notion of extending module was generalized to purely extending by Fuchs in [5] and basic characterisations were given by Clark in [6]. Motivated by their work the notion of pure-direct-injective modules were introduced and studied by Maurya, Das and Alagöz in [7]. Namely, a right R-module is said to be *pure-direct-injective* if every pure submodule A of which with A isomorphic to a direct summand is a direct summand.

In this work we study generalizations of these notions to abelian categories and Grothendieck categories, namely direct-injective objects and pure-direct-injective objects respectively. Some generalizations of direct-injective modules to abelian categories were studied by Crivei and Kör in [8] and Crivei and Keskin Tütüncü in [9]. An object M of an abelian category $\mathcal{A}$ is said to be *direct-injective* if every subobject A of M with A isomorphic to a direct summand of M is a direct summand. Let M and N be objects of an abelian category $\mathcal{A}$. M is called N-*injective* if for every subobject A of N any homomorphism from A to M can be extended to a homomorphism from N to M. M is said to be *quasi-injective* if it is M-injective. We have the following implications.

$$\text{injective} \Rightarrow \text{quasi-injective} \Rightarrow \text{direct-injective}$$

An object M of a Grothendieck category $\mathcal{A}$ is said to be *pure-injective* if M is relatively injective for every pure short exact sequence in $\mathcal{A}$ and it is said to be *pure-direct-injective* if every pure subobject A of M with A isomorphic to a direct summand of M is a direct summand. We also have the following implications.

$$\text{injective} \Rightarrow \text{pure-injective} \Rightarrow \text{pure-direct-injective}$$

In Chapter 2 some definitions and lemmas which will be used in the next sections of the paper are recalled. It is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is regular if and only if every pure-injective object of $\mathcal{A}$ is injective if and only if every pure-injective object of $\mathcal{A}$ is absolutely pure (Theorem 2.12.11).

In Chapter 3 the concept of direct-injectivity is generalized to abelian categories. It is obtained that the class of direct-injective objects of an abelian category $\mathcal{A}$ with enough injectives need not be closed under subobjects and taking finite coproducts (Corollary 3.1.5 and Corollary 3.1.6). It is shown that the coproduct of two direct-injective objects of an abelian category $\mathcal{A}$ with enough injectives is direct-injective if and only if every direct-injective object of $\mathcal{A}$ is injective (Corollary 3.1.7). It is proved that an abelian category $\mathcal{A}$ with enough injective objects is (cosemi)hereditary if and only if every (finitely cogenerated) quotient object of an injective object of $\mathcal{A}$ is direct-injective (Theorem 3.1.11). It is shown for a locally finitely presented Grothendieck category $\mathcal{A}$ that $\mathcal{A}$ is regular if and only if every pure-injective object of $\mathcal{A}$ is a quasi-injective (Proposition 3.1.14). Also classes all of whose objects are direct-injective are investigated. It is proved that an abelian category $\mathcal{A}$ is spectral if and only if $\mathcal{A}$ has enough injectives and every object of $\mathcal{A}$ is direct-injective if and only if $\mathcal{A}$ has enough injectives and every subobject of a direct-injective object of $\mathcal{A}$ is direct-injective (Theorem 3.2.2). It is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is semisimple if and only if every object of $\mathcal{A}$ is direct-injective if and only if the coproduct of two direct-injective objects of $\mathcal{A}$ is direct-injective (Theorem 3.2.7).

In Chapter 4 the concept of pure-direct-injectivity is generalized to Grothendieck categories. It is obtained that the class of pure-direct-injective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ need not be closed under pure subobjects and taking finite coproducts (Corollary 4.1.5 and Corollary 4.1.6). It is proved that the coproduct of two pure-direct-injective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ is pure-direct-injective if and only if every pure-direct-injective object of $\mathcal{A}$ is pure-injective (Corollary 4.1.7). Also it is proved that a locally finitely presented Grothendieck category $\mathcal{A}$ is regular if and only if every pure-direct-injective object of $\mathcal{A}$ is direct-injective (Theorem 4.1.9). It is obtained for a locally finitely presented Grothendieck category $\mathcal{A}$ that $\mathcal{A}$ is regular and the coproduct of two pure-direct-injective objects is pure-direct-injective if and only if every pure-direct-injective object of $\mathcal{A}$ is injective (Proposition 4.1.11). As a result of this, it is given for a locally finitely presented regular Grothendieck category $\mathcal{A}$ that if the coproduct of two pure-direct-injective objects is pure-direct-injective, then every pure-direct-injective object of $\mathcal{A}$ is quasi-injective (Corollary 4.1.12). It is obtained that if every pure-direct-injective object of a locally finitely presented Grothendieck category $\mathcal{A}$ is quasi-injective, then $\mathcal{A}$ is regular (Proposition 4.1.13). It is proved for a locally finitely presented Grothendieck category $\mathcal{A}$ whose class of pure-injective objects is closed under extensions that $\mathcal{A}$ is pure hereditary if and only if every quotient of a pure-injective object of $\mathcal{A}$ is pure-direct-injective (Theorem 4.1.17). It is obtained that the class of pure-direct-injective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ need not be closed under taking quotients (Corollary 4.1.18). It is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is pure-semisimple if and only if every object of $\mathcal{A}$ is pure-injective if and only if every object of $\mathcal{A}$ is pure-direct-injective if and only if every subobject of a pure-direct-injective object of $\mathcal{A}$ is pure-direct-injective (Theorem 4.2.6). Relative pure-direct-injective objects of a Grothendieck category $\mathcal{A}$ are defined and it is shown for objects M, N_1, N_2 of $\mathcal{A}$ that if M is pure-direct-$N_1 \oplus N_2$-injective, then M is pure-direct-N_1-injective and M is pure-direct-N_2-injective.

In Chapter 5 applications of some of our results to module and comodule categories are given. It is obtained that a coalgebra C over a field is hereditary if and only if every factor comodule of an injective right C-comodule is direct-injective (Corollary 5.1.4). As a result of Theorem 3.2.2 it is obtained for comodule categories that a coalgebra C over a field is cosemisimple if and only if every right C-comodule is direct-injective if and only if every subcomodule of a direct-injective right C-comodule is direct-injective (Corollary 5.1.6). Also as a result of Theorem 4.1.9 it is obtained that a coalgebra C over a field is cosemisimple if and only if every pure-injective right C-comodule is injective if and only if every pure-direct-injective right C-comodule is direct-injective (Corollary 5.1.8).

2. PRELIMINARIES

In this chapter some preliminary information which will be needed is given. Definitions, Examples, Propositions and Theorems which are not cited here can be found in [10], [11] and [12].

2.1 Categories

Definition 2.1.1. A *category* $\mathcal{A}$ consists of

(1) a collection $\mathcal{O}\mathrm{b}(\mathcal{A})$ of objects;

(2) a collection $\mathrm{Mor}_{\mathcal{A}}(A, B)$ of moprhisms $f : A \longrightarrow B$ for each $A, B \in \mathcal{O}\mathrm{b}(\mathcal{A})$;

(3) a function $\circ : \mathrm{Mor}_{\mathcal{A}}(A, B) \times \mathrm{Mor}_{\mathcal{A}}(B, C) \longrightarrow \mathrm{Mor}_{\mathcal{A}}(A, C)$ which is called the *composition* and assigns a morphism $g \circ f \in \mathrm{Mor}_{\mathcal{A}}(A, C)$ to every pair (f, g) where $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$ and $g \in \mathrm{Mor}_{\mathcal{A}}(B, C)$ for each $A, B, C \in \mathcal{O}\mathrm{b}(\mathcal{A})$

such that the following conditions are satisfied.

(i) for each $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$, $g \in \mathrm{Mor}_{\mathcal{A}}(B, C)$, $h \in \mathrm{Mor}_{\mathcal{A}}(C, D)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(ii) for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ and for each $f \in \mathrm{Mor}_{\mathcal{A}}(B, A)$, $g \in \mathrm{Mor}_{\mathcal{A}}(A, C)$ there exists a morphism $1_A \in \mathrm{Mor}_{\mathcal{A}}(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$. The morphism 1_A is called the *identity morphism*.

Example 2.1.1. *[12, Example 1.3.2] The canonical example of a category is the* category of sets*, denoted as Set, which can be described as follows:*
Objects. All sets X.
Morphisms. All functions between sets $f : X \longrightarrow Y$.

Example 2.1.2. *[12, Example 1.3.3] The second canonical example is the* category of groups*, denoted as* Grp*. This category can be described as follows:*

Objects. All groups $(G, \cdot)$*. Here,* $\cdot : G \times G \longrightarrow G$ *is the group operation.*

Morphisms. All group homomorphisms $\varphi : (G, \cdot) \longrightarrow (H, \cdot)$*. Specifically, set functions* $\varphi : G \longrightarrow H$ *where* $\varphi(g \cdot g') = \varphi(g) \cdot \varphi(g')$*.*

Example 2.1.3. *[12, Example 1.3.4] The third canonical example is the* category of topological spaces*, denoted* Top*. We describe this category as follows:*

Objects. All topological space (X, τ) *where* τ *is a topology on the set* X*.*

Morphisms. All continuous functions $f : (X, \tau) \longrightarrow (Y, \tau')$*.*

Example 2.1.4. *[12, Example 1.3.9] Let* $Ring$ *be the category described as follows:*

Objects. Unital rings $(R, +, \cdot)$*. That is, ring* R *with a multiplicative identity* 1 *which is not equal to its additive identity* 0*.*

Morphisms. (Unit preserving) Ring homomorphisms $\varphi : R \longrightarrow R'$*. That is, functions* $\varphi : R \longrightarrow R'$ *such that*

$\varphi(a + b) = \varphi(a) + \varphi(b)$

$\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$

$\varphi(0_R) = 0_{R'}$ *and* $\varphi(1_R) = 1_{R'}$*. This category is called the* category of rings.

Example 2.1.5. *[12, Example 1.3.8] The category of abelian groups* Ab *can be described as follows:*

Objects. All abelian groups $(G, .)$

Morphisms. Group homomorphisms.

Example 2.1.6. *Let* R *be a ring. The category of left* R*-modules* R*-Mod can be described as follows:*

Objects. All left R*-modules.*

Morphisms. Module homomorphisms.

Example 2.1.7. *Let* R *be a ring. The category of right* R*-modules Mod-*R *can be described as follows:*

Objects. All right R*-modules.*

Morphisms. Module homomorphisms.

Example 2.1.8. *The category $DivAb$ of divisible abelian groups can be described as follows:*
Objects. All divisible abelian groups.
Morphisms. All group homomorphisms.

Definition 2.1.2. A category $\mathcal{A}$ is called a *small* category if the class of objects and the class of morphisms are sets.

Definition 2.1.3. The *opposite category* of a category $\mathcal{A}$, which is denoted by $\mathcal{A}^{op}$, is defined as follows:
The objects of the category $\mathcal{A}^{op}$ coincide with the objects of category $\mathcal{A}$, that is $\mathcal{O}\mathrm{b}(\mathcal{A}) = \mathcal{O}\mathrm{b}(\mathcal{A}^{op})$.
The composition $\beta \circ \alpha$ in $\mathcal{A}^{op}$ is defined as the composition $\alpha \circ \beta$ in $\mathcal{A}$.

Definition 2.1.4. A *preadditive category* is a category $\mathcal{A}$ such that for each pair of objects A, B there exists an abelian group operation $+$ on the set $\mathrm{Mor}_{\mathcal{A}}(A, B)$ such that

$$\circ : \mathrm{Mor}_{\mathcal{A}}(A, B) \times \mathrm{Mor}_{\mathcal{A}}(B, C) \longrightarrow \mathrm{Mor}_{\mathcal{A}}(A, C)$$

$$(f, g) \mapsto g \circ f$$

is bilinear. That is, given morphisms $f, g : A \longrightarrow B$ and $h, k : B \longrightarrow C$ we have that

$$(h + k) \circ f = (h \circ f) + (k \circ f)$$

$$h \circ (g + f) = (h \circ g) + (h \circ f)$$

Example 2.1.9. *[13] The category Ab of abelian groups is a preadditive category. Indeed, if X, Y are two abelian groups, then the set $\mathrm{Mor}(X, Y)$ has canonically an abelian group structure: for $f, g \in \mathrm{Mor}_{Ab}(X, Y)$ we put $f+g : X \longrightarrow Y$ for the homomorphism of groups defined by $(f + g)(x) = f(x) + g(x)$.*

Example 2.1.10. *[13] The category $\mathcal{A}^{op}$, the dual of a preadditive category, is also preadditive.*

Definition 2.1.5. [11] A category $\mathcal{A}'$ is said to be *subcategory* of a category $\mathcal{A}$ under the following conditions:

(1) $\mathcal{A}' \subset \mathcal{A}$.

(2) $\mathrm{Mor}_{\mathcal{A}'}(A, B) \subset \mathrm{Mor}_{\mathcal{A}}(A, B)$ for all $A, B \in \mathcal{A}'$.

(3) The composition of any two morphisms in $\mathcal{A}'$ is the same as their composition in $\mathcal{A}$.

(4) 1_A is the same in $\mathcal{A}'$ as in $\mathcal{A}$ for all $A \in \mathcal{A}'$.

Furthermore, we say that $\mathcal{A}'$ is a *full subcategory* of $\mathcal{A}$ if for each pair of objects A, B of $\mathcal{A}'$ we have $\mathrm{Mor}_{\mathcal{A}}(A, B) = \mathrm{Mor}_{\mathcal{A}'}(A, B)$.

Example 2.1.11. *The category of abelian groups Ab is a full subcategory of the category of groups Grp.*

2.2 Functors

Throughout the composition of morphisms $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$, $g \in \mathrm{Mor}_{\mathcal{A}}(B, C)$ will be denoted by gf instead of $g \circ f$.

Definition 2.2.1. A *covariant functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ is a mapping which assigns each object A of $\mathcal{A}$ to an object $F(A)$ of $\mathcal{B}$ and each morphism $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ to a morphism $T(f) \in \mathrm{Mor}_{\mathcal{B}}(T(A), T(B))$ in $\mathcal{B}$ such that the following conditions are satisfied.

(1) If $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$ and $g \in \mathrm{Mor}_{\mathcal{A}}(B, C)$, then $F(gf) = F(g)F(f)$.

(2) $F(1_A) = 1_{F(A)}$ holds for every object A of $\mathcal{A}$.

Definition 2.2.2. A *contravariant functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ is a mapping which assigns each object A of $\mathcal{A}$ to an object $F(A)$ of $\mathcal{B}$ and each morphism $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ to a morphism $F(f) \in \mathrm{Mor}_{\mathcal{B}}(F(B), F(A))$ in $\mathcal{B}$ such that the following conditions are satisfied.

(1) If $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$ and $g \in \mathrm{Mor}_{\mathcal{A}}(B, C)$, then $F(gf) = F(f)F(g)$.

(2) $F(1_A) = 1_{F(A)}$ holds for every object A of $\mathcal{A}$.

Example 2.2.1. *The functor $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ such that $1_{\mathcal{A}}(A) = A$ for all $A \in \mathcal{A}$ and $1_{\mathcal{A}}(\alpha) = \alpha$ for all morphisms α in $\mathcal{A}$ is called the* identity *functor on $\mathcal{A}$.*

Example 2.2.2. *If $\mathcal{A}'$ is a subcategory of $\mathcal{A}$, then the covariant functor $I : \mathcal{A}' \longrightarrow \mathcal{A}$ such that $I(A) = A$ for all $A \in \mathcal{A}'$ and $I(\alpha) = \alpha$ for all morphisms α in $\mathcal{A}'$ is called the* inclusion *functor of $\mathcal{A}'$ in $\mathcal{A}$.*

Definition 2.2.3. [12, Definition 1.8.1] Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be functors. Then the *composite functor* $GF : \mathcal{A} \longrightarrow C$ is a functor which assigns to each object A of $\mathcal{A}$ an object $G(F(A))$ of $\mathcal{C}$ and assigns to each morphism $f : A \longrightarrow B$ in $\mathcal{A}$ a morphism $G(F(f)) : G(F(A)) \longrightarrow G(F(B))$ in $\mathcal{C}$.

Definition 2.2.4. [12, Definition 1.8.3] Let $\mathcal{A}$ and $\mathcal{B}$ be categories and suppose $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor. Then F is said to be *forgetful* whenever F does not preserve the axioms and structure present in the objects of $\mathcal{A}$.

Example 2.2.3. *[12, Example 1.8.4] Consider a group $(G, \cdot)$ with $\cdot$ binary operation. In some sense, groups are simply sets with added structure, while group homomorphisms are simply functions that respect group structure. Hence we can create a map between Grp and Set that forgets this structure:*

$$(G, \cdot) \longrightarrow G$$

$$\varphi : (G, \cdot) \longrightarrow (H, +)$$

$$\varphi : G \mapsto H$$

We can demonstrate that this process is functorial. Observe that $1_G : (G, \cdot) \longrightarrow (G, \cdot)$ is the identity homomorphism, then one can readily note that $1_G(g) = g$ for all $g \in G$, so that it is also an identity function on the underlying set G. Therefore $F(1_G) = 1_{F(G)}$. Next if $\varphi : G \longrightarrow H$ and $\psi : H \longrightarrow K$ are group homomorphisms, then $F(\psi\varphi)$ is the underlying *function* $(\psi\varphi) : G \longrightarrow K$. *Note however that for each* $g \in G$,

$$F(\psi\varphi)(g) = \psi(\varphi(g)) = F(\psi)F(\varphi)(g) \Rightarrow F(\psi\varphi) = F(\psi)F(\varphi)$$

Hence, we see that we have a forgetful functor $F : Grp \longrightarrow Set$ *which leaves behind group operations and moreover regards every group homomorphism as a function.*

Example 2.2.4. *[12, Example 1.8.5] Let* $(R, +, \cdot)$ *be a ring. Recall that* $(R, +)$ *is an abelian group. Hence we can forget the structure of* $\cdot : R \times R \longrightarrow R$ *and treat every ring as an abelian group. Then this defines a forgetful functor* $F : Ring \longrightarrow Ab$ *which simply maps a ring to its abelian group. This works on the morphisms, since every ring homomorphism* $\varphi : (R, +, \cdot) \longrightarrow (S, +, \cdot)$ *is a group homomorphism* $\varphi : (R, +) \longrightarrow (S, +)$ *on the abelian groups.*

Definition 2.2.5. [12, Definition 1.8.8] Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. Then F is said to be

(1) *full* if for all A, B, every morphism $g : F(A) \longrightarrow F(B)$ in $\mathcal{B}$ is the image of some $f : A \longrightarrow B$ in $\mathcal{A}$,

(2) *faithful* if for all $f_1, f_2 : A \longrightarrow B$ in $\mathcal{A}$ with $F(f_1) = F(f_2)$ implies that $f_1 = f_2$,

(3) *fully faithful* if it is both full and faithful.

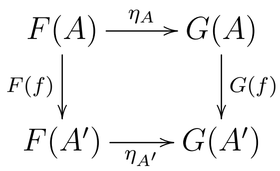
Definition 2.2.6. [12, Definition 1.8.13] A category $\mathcal{A}$ is called *concrete category* if there is a faithful functor $F : \mathcal{A} \longrightarrow Set$.

Definition 2.2.7. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. F is said to *preserve* the property P if the image of a morphism (or an object or a diagram) under $F : \mathcal{A} \longrightarrow \mathcal{B}$ which has the property P in $\mathcal{A}$ has also the same property P in $\mathcal{B}$.

Definition 2.2.8. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. F is said to *reflect* the property P if whenever the image of a morphism (or an object or a diagram) of $\mathcal{A}$ under $F : \mathcal{A} \longrightarrow \mathcal{B}$ has a property P in $\mathcal{B}$, already has that property P in $\mathcal{A}$.

2.3 Natural transformations

Definition 2.3.1. [10] Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{A} \longrightarrow \mathcal{B}$ be two covariant functors. Suppose that for every object $A \in \mathcal{A}$ we have a morphism $\eta_A : F(A) \longrightarrow G(A)$ in $\mathcal{B}$ such that for every morphism $\alpha : A \longrightarrow A'$ in $\mathcal{A}$ the diagram below is commutative.



Then we call η a *natural transformation* from F to G and we write $\eta : F \longrightarrow G$.

Example 2.3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant functor. $1_F : F \longrightarrow F$ is a natural transformation.*

Example 2.3.2. *Consider the categories Grp and Set. Take the covariant functors $F : Grp \longrightarrow Set$ and $G : Set \longrightarrow Grp$. For $FG : Set \longrightarrow Set$ and $1_{Set} : Set \longrightarrow Set$, $\eta : 1_{Set} \longrightarrow FG$ is also a natural transformation.*

2.4 Special morphisms

Definitions, Examples, Propositions and Theorems which are not cited here can be found in [14] and [15].

Definition 2.4.1. A morphism $f : A \longrightarrow B$ in a category $\mathcal{A}$ is called an *isomorphism* provided that there exists a morphism $g : B \longrightarrow A$ with $gf = 1_A$ and $fg = 1_B$. In this case g is called the *inverse* of f and f is said to be *invertible*.

Remark 2.4.2. A morphism f in a category $\mathcal{A}$ is an isomorphism if and only if f is an isomorphism in $\mathcal{A}^{op}$.

Proposition 2.4.3. *If $f : A \longrightarrow B$, $g : B \longrightarrow A$ and $h : B \longrightarrow A$ are morphisms in a category $\mathcal{A}$ such that $gf = 1_A$ and $fh = 1_B$, then $g = h$.*

Proof. Let $f : A \longrightarrow B$, $g : B \longrightarrow A$ and $h : B \longrightarrow A$ are morphisms such that $gf = 1_A$ and $fh = 1_B$. Then $h = 1_A h = gfh = g1_B = g$. $\square$

Corollary 2.4.4. *If g_1 and g_2 are inverses of a morphism f in a category $\mathcal{A}$, then $g_1 = g_2$.*

Proof. Let $f : A \longrightarrow B$ be a morphism and g_1 and g_2 be inverses of f in $\mathcal{A}$. Since g_1 is an inverse of f, $fg_1 = 1_B$ and $g_1 f = 1_A$. Since g_2 is inverse of f, $fg_2 = 1_B$ and $g_2 f = 1_A$. Therefore $g_1 = 1_A g_1 = g_2 f g_1 = g_2 1_B = g_2$. $\square$

Remark 2.4.5. Inverse of a morphism f is unique and it is denoted by f^{-1}.

Proposition 2.4.6. *If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are isomorphisms, then so is gf and $(gf)^{-1} = f^{-1}g^{-1}$.*

Proof. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are isomorphisms. Since $f : A \longrightarrow B$ is an isomorphism, there exists $f^{-1} : B \longrightarrow A$ such that $ff^{-1} = 1_B$ and $f^{-1}f = 1_A$. And also since $g : B \longrightarrow C$ is an isomorphism, there exists $g^{-1} : C \longrightarrow B$ such that $gg^{-1} = 1_C$ and $g^{-1}g = 1_B$. Then $(gf)(f^{-1}g^{-1}) = g(ff^{-1})g^{-1} = gg^{-1} = 1_C$ and $(f^{-1}g^{-1})(gf) = f^{-1}(g^{-1}g)f = f^{-1}1_B f = 1_A$. Since the inverse of gf is unique, $(gf)^{-1} = f^{-1}g^{-1}$. $\square$

Definition 2.4.7. Two objects A and B of a category $\mathcal{A}$ are said to be *isomorphic* provided that there is an isomorphism $f : A \longrightarrow B$ between them.

Proposition 2.4.8. *Every functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ preserves isomorphisms.*

Proof. Let $f : A \longrightarrow B$ is an isomorphism in a category $\mathcal{A}$. Then $F(f)F(f^{-1}) = F(ff^{-1}) = F(1_B) = 1_{F(B)}$. Similarly, $F(f^{-1})F(f) = F(f^{-1}f) = F(1_A) = 1_{F(A)}$. So $F(f)$ is an isomorphism in $\mathcal{B}$. $\square$

Definition 2.4.9. A morphism $\alpha : A \longrightarrow B$ in a category $\mathcal{A}$ is said to be *monomorphism* (or *monic*) if $\alpha f = \alpha g$ implies that $f = g$ for all pair of morphisms f, g.

In other words, α is monic if it is left cancellable.

Definition 2.4.10. A morphism $\alpha : A \longrightarrow B$ in a category $\mathcal{A}$ is said to be *epimorphism* (or *epic*) if $f, g : B \longrightarrow C$ the equality $f\alpha = g\alpha$ implies that $f = g$ for all pair of morphisms f, g.
In other words, α is epic if it is right cancellable.

Proposition 2.4.11. *Suppose A and B are sets. Then the following conditions are equivalent for a morphism $f : A \longrightarrow B$.*

(1) f is one-to-one.

(2) f is monic.

Proof. (1) $\Rightarrow$ (2) Let $f : A \longrightarrow B$, $g, h : C \longrightarrow A$ be functions of sets and $fg = fh$. Therefore $(fg)(x) = (fh)(x)$, that is $f(g(x)) = f(h(x))$ for all $x \in \mathcal{A}$. Since f is one-to-one, $g(x) = h(x)$. Then $g(x) = h(x)$ for all $x \in \mathcal{A}$ and therefore $g = h$. So f is monic.
(2) $\Rightarrow$ (1) Let us take $C = \{a\}$ and define the functions $g, h : C \longrightarrow A$ by $g(a) = x$ and $h(a) = y$ for some $x, y \in A$. Suppose that $f(x) = f(y)$. Since $x = g(a)$ and $y = h(a)$, $f(g(a)) = f(x) = f(y) = f(h(a))$, that is $(fg)(a) = (fh)(a)$. Since f is left cancellable, $g(a) = h(a)$, i.e. $x = y$. Thus f is one-to-one. □

Remark 2.4.12. In a concrete category every one-to-one morphism is monic but the converse of this need not be true.

Example 2.4.1. *Take the divisible group $(\mathbb{Q}, +)$ and normal subgroup $\mathbb{Z}$ of $\mathbb{Q}$. Consider the quotient group $\mathbb{Q}/\mathbb{Z} = \{p/q + \mathbb{Z} : p/q \in \mathbb{Q}\}$. We know that $\mathbb{Q}/\mathbb{Z}$ is also divisible. Take the homomorphism $h : \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ which is defined by $h(x) = x + \mathbb{Z}$ for all $x \in \mathbb{Q}$. This h is not one-to-one because $h(1) = 1 + \mathbb{Z} = \mathbb{Z} = 2 + \mathbb{Z} = h(2)$ but $1 \neq 2$. We want to show that h is a monomorphism, that is left cancellable. Let $f, g : A \longrightarrow \mathbb{Q}$ be two homomorphisms such that $f \neq g$ where A is a divisible abelian group. So for at least one $a \in A$, $f(a) \neq g(a)$. Then $f(a) - g(a) \neq 0$. We can write $f(a) - g(a) = r/s$ where $r/s \in \mathbb{Q}$ and $s \neq -1, 1$. Since A is divisible, for some $r \in \mathbb{Z}^+$ there exists an element b of A such that $a = rb$. Then* *we can write* $r(f(b)-g(b)) = r(f(b)) - r(g(b)) = f(rb) - g(rb) = f(a) - g(a) = r/s$. *So* $f(b) - g(b) = 1/s$. *Therefore* $h(f(b)) \neq h(g(b))$ *if otherwise, that is* $h(f(b)) = h(g(b))$, *then* $h(f(b) - g(b)) = 0 + \mathbb{Z}$. *Thus* $1/s + \mathbb{Z} = h(1/s) = 0 + \mathbb{Z} = \mathbb{Z}$ *and therefore* $1/s \in \mathbb{Z}$. *But this is a contradiction since* $s \neq -1, 1$.

Proposition 2.4.13. *Suppose A and B are sets. Then the following conditions are equivalent for a morphism $f : A \longrightarrow B$.*

(1) f is onto.

(2) f is epic.

Proof. (1) $\Rightarrow$ (2) Let $f : A \longrightarrow B$ be an epimorphism. Take functions $g, h : B \longrightarrow \{0,1\}$ defined as follows:

$$g(x) = \begin{cases} x, & \text{if } x \in \operatorname{Im}(f) \\ 0, & \text{if } otherwise \end{cases}$$

$$h(x) = \begin{cases} x, & \text{if } x \in \operatorname{Im}(f) \\ 1, & \text{if } otherwise \end{cases}$$

So $g(f(y)) = f(y) = h(f(y))$ for all $y \in A$ and therefore $gf = hf$. Since f is epic, $g = h$. If $\operatorname{Im}(f) = f(A) \subsetneqq B$, then there exists an element x such that $x \in B$ but $x \neq \operatorname{Im}(f)$. Since $g = h$, then $0 = g(x) = h(x) = 1$. This is a contradiction. So $f(A) = B$. Thus f is onto.
(2) $\Rightarrow$ (1) Suppose $f : A \longrightarrow B$ is onto. Let $g, h : B \longrightarrow C$ be two functions such that $gf = hf$. Since f is onto, there exists an element a such that $f(a) = b$ for all $b \in B$. $g(b) = g(f(a)) = gf(a) = hf(a) = h(f(a)) = h(b)$ for all $b \in B$. So $g = h$, that is f is an epimorphism. □

Remark 2.4.14. In every concrete category every onto morphism is epi but the converse of this need not be true.

Example 2.4.2. *Take $\mathcal{A}$ = Ring and take inclusion homomorphism $i : \mathbb{Z} \longrightarrow \mathbb{Q}$. Clearly, i is not onto. We want to show that i is epi. Let $g, h : \mathbb{Q} \longrightarrow R$ be* *two ring homomorphisms. Suppose $gi = hi$. Since $gi = hi$, $g(m) = gi(m) = hi(m) = h(m)$ for all $m \in \mathbb{Z}$. So $g = h$. $g(m/n) = g(mn^{-1}1) = g(m)g(n^{-1})g(1) = h(m)g(n^{-1})h(1) = h(m)g(n^{-1})h(nn^{-1}) = h(m)g(n^{-1})h(n)h(n^{-1}) = h(m)g(n^{-1})g(n)h(n^{-1}) = h(m)g(nn^{-1})h(n^{-1}) = h(m)g(1)h(n^{-1}) = h(m)h(1)h(n^{-1}) = h(m1n^{-1}) = h(m/n)$ for every $m/n \in \mathbb{Q}$. Thus $g = h$.*

Theorem 2.4.15. *Let f and g be two morphisms in a category $\mathcal{A}$.*

(1) If f and g are monic, then fg is monic.

(2) If f and g are epic, then fg is epic.

(3) If fg is monic, then g is monic.

(4) If fg is epic, then f is epic.

Proof. (1) Suppose $(fg)h = (fg)k$ for two morphisms h, k in $\mathcal{A}$. Then we have $f(gh) = (fg)h = (fg)k = f(gk)$. Since f is left cancellable, $gh = gk$ and since g is left cancellable, $h = k$.
(2) Suppose $h(fg) = k(fg)$ for two morphisms h, k in $\mathcal{A}$. Then we have $(hf)g = h(fg) = k(fg) = (kf)g$. Since g is right cancellable, $hf = kf$ and since f is right cancellable, $h = k$.
(3) Since fg is monic, fg is left cancellable. Let $gh = gk$ for two morphisms h, k in $\mathcal{A}$. Suppose $f(gh) = f(gk)$. So we have $(fg)h = f(gh) = f(gk) = (fg)k$. Since fg is left cancellable, $h = k$. Thus g is monic.
(4) Since fg is epic, fg is right cancellable. Let $hf = kf$ for two morphisms h, k in $\mathcal{A}$. Suppose $(hf)g = (kf)g$. So we have $h(fg) = (hf)g = (kf)g = k(fg)$. Since fg is right cancellable, $h = k$. Thus f is epic. □

Definition 2.4.16. [15, Definition 7.19] A morphism $\alpha : A \longrightarrow B$ said to be *section* if there is a morphism $\beta : B \longrightarrow A$ such that $\beta\alpha = 1_A$.

Proposition 2.4.17. *[15, Proposition 7.21] Let $\mathcal{A}$ be a category, $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be morphisms in $\mathcal{A}$. Then the following hold.*

(1) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are sections, then gf is a section.

(2) If gf is a section, then f is a section.

Proof. (1) Given h with $hf = 1_A$ and k with $kg = 1_B$, we have

$$(hk)(gf) = h(kg)f = h(1_B)f = hf = 1_A.$$

(2) Given h with $h(gf) = 1_A$, we have $(hg)f = 1_A$. □

Proposition 2.4.18. *[15, Proposition 7.22] Every functor preserves sections.*

Proof. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and f is a section. If $hf = 1_A$, then $F(h)F(f) = F(hf) = F(1_A) = 1_{F(A)}$. □

Proposition 2.4.19. *[15, Proposition 7.23] Every fully faithful functor reflects sections.*

Proof. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and f is a section. Given $h : F(B) \longrightarrow F(A)$ with $hF(f) = 1_{F(A)}$, there is a morphism $k : B \longrightarrow A$ with $h = F(k)$ by fullness. Thus $F(kf) = F(k)F(f) = hF(f) = 1_{F(A)} = F(1_A)$, so that $kf = 1_A$ by faithfulness. □

Proposition 2.4.20. *Suppose A and B are sets. Then the following conditions are equivalent for a morphism $f : A \longrightarrow B$.*

(1) f is one-to-one.

(2) f is a section.

Proof. (1) $\Rightarrow$ (2) Let $f : A \longrightarrow B$ be a one-to-one map. For $b \in \mathrm{Im}(f)$ there is an element $x \in A$ such that $f(x) = b$. Denote this element by $\hat{b}$.

$$g(b) = \begin{cases} \hat{b}, & \text{if } b \in \mathrm{Im}(f) \\ \alpha, & \text{if } otherwise \end{cases}$$

So for every $a \in A$, $(gf)(a) = g(f(a)) = f\hat{(a)} = a$. Thus $gf = 1_A$.
$(2) \Rightarrow (1)$ Suppose $gf = 1_A$ for some $g : B \longrightarrow A$. And suppose $fh = fk$ for two functions h, k in Set. Then

$$h = 1_A h = (gf)h = g(fh) = g(fk) = (gf)k = 1_A k = k.$$

Thus f is monic. □

Remark 2.4.21. In every concrete category every section is one-to-one but the converse of this statement need not be true.

Example 2.4.3. *Take the category of left R-modules $\mathcal{A}$ = R-Mod. Consider $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(n) = 2n$ for all $n \in \mathbb{Z}$. This f is clearly one-to-one. If we take $\mathbb{Z}$ as a set, since f is one-to-one there is a function $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $gf = 1_{\mathbb{Z}}$. Because in Set every one-to-one function is a section. But this g is not a module homomorphism. If it was a module homomorphism, then for all $n \in \mathbb{Z}$, $2g(n) = g(n) + g(n) = g(n + n) = g(2n) = g(f(n)) = gf(n) = n$. If $n = 1$, $2g(1) = 1$ and so $g(1) = 1/2 \notin \mathbb{Z}$.*

Definition 2.4.22. [15, Definition 7.24] A morphism $\alpha : A \longrightarrow B$ said to be *retraction* if there is a morphism $\beta : B \longrightarrow A$ such that $\alpha\beta = 1_B$.

Proposition 2.4.23. *[15, Proposition 7.27] Let $\mathcal{A}$ be a category, $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be morphisms in $\mathcal{A}$. Then the following hold.*

(1) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are retractions, then gf is a retraction.

(2) If gf is a retraction, then g is a retraction.

Proof. (1) Given h with $fh = 1_B$ and k with $gk = 1_C$, we have

$$(gf)(hk) = g(fh)k = g(1_B)k = gk = 1_C.$$

(2) Given h with $(gf)h = 1_C$, we have $g(fh) = 1_C$. □

Proposition 2.4.24. *[15, Proposition 7.28] Every functor preserves retractions.*

Proof. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and f is a retraction. If $gf = 1_B$, then $F(g)F(f) = F(gf) = F(1_B) = 1_{F(B)}$. $\square$

Proposition 2.4.25. *[15, Proposition 7.29] Every fully faithful functor reflects retractions.*

Proof. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and f is a retraction. Given $h : F(B) \longrightarrow F(A)$ with $F(f)h = 1_{F(B)}$, by fullness there is $k : B \longrightarrow A$ with $h = F(k)$. Thus $F(fk) = F(f)F(k) = F(f)h = 1_{F(B)}$, so that by faithfulness $fk = 1_B$. $\square$

Proposition 2.4.26. *Suppose A and B are sets. Then the following conditions are equivalent for a morphism $f : A \longrightarrow B$.*

(1) f is onto.

(2) f is a retraction.

Proof. (1) $\Rightarrow$ (2) Let $f : A \longrightarrow B$ be onto. So for all $b \in B$, there is an element $a \in A$ such that $f(a) = b$. Denote this element by a_b. Define $g : B \longrightarrow A$ by $g(b) = a_b$ for all $b \in B$. So for all $b \in B$, $fg(b) = f(g(b)) = f(a_b) = b$. Therefore $fg = 1_B$. Thus f is a retraction.
(2) $\Rightarrow$ (1) Suppose $f : A \longrightarrow B$ is a retraction, that is there exists a morphism $g : B \longrightarrow A$ such that $fg = 1_B$. Let $hf = kf$ for two morphisms h, k in Set. Then

$$h = h1_B = h(fg) = (hf)g = (kf)g = k(fg) = k1_B = k.$$

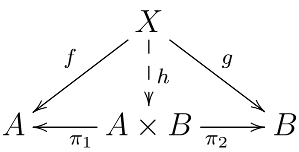
So f is an epimorphism. $\square$

Remark 2.4.27. In every concrete category every retraction is onto but the converse of this statement need not be true.

Example 2.4.4. *Take Ab. Consider $\mathbb{Q}_p = \{x \in \mathbb{Q} \mid \exists k \in \mathbb{Z}, \exists n \in \mathbb{N}, x = k/p^n\}$. $\mathbb{Q}_p$ is a subgroup of $\mathbb{Q}$ and $\mathbb{Z}$ is a subgroup of $\mathbb{Q}_p$. $\mathbb{Q}_p$ is a normal subgroup of $\mathbb{Q}$ and $\mathbb{Z}$ is a normal* *subgroup of* $\mathbb{Q}_p$ *in Ab. So* $\mathbb{Q}_p/\mathbb{Z}$ *can be constructed and this group is an abelian group too. Consider* $f : \mathbb{Q}_p/\mathbb{Z} \longrightarrow \mathbb{Q}_p/\mathbb{Z}$ *defined by* $f(x+\mathbb{Z}) = px + \mathbb{Z}$. *For all* $b + \mathbb{Z} \in \mathbb{Q}_p/\mathbb{Z}$, *there is an element* $k \in \mathbb{Z}$, $n \in \mathbb{N}$ *such that* $b = kp^{-n}$. *Also since there is an element* $x = kp^{-n-1} + \mathbb{Z} \in \mathbb{Q}_p/\mathbb{Z}$ *such that* $f(kp^{-n-1} + \mathbb{Z}) = pkp^{-n-1} + \mathbb{Z} = b + \mathbb{Z}$, f *is onto. Since* f *is onto in Set, it is a retraction. So there exists a function* $h : \mathbb{Q}_p/\mathbb{Z} \longrightarrow \mathbb{Q}_p/\mathbb{Z}$ *such that* $fh = 1_{\mathbb{Q}_p/\mathbb{Z}}$. *But this function is not a group homomorphism If it was, then* $p^{-1} + \mathbb{Z} = (fh)(p^{-1}) = f(h(p^{-1})) = ph(p^{-1}) = h(p^{-1}) + ... + h(p^{-1}) = h(p^{-1} + ... + p^{-1} + \mathbb{Z}) = h(pp^{-1}+\mathbb{Z}) = h(1+\mathbb{Z}) = 0+\mathbb{Z}$. *Therefore* $h(0+\mathbb{Z}) = 0+\mathbb{Z}$ *because group homomorphisms preserve the identity and when* p *is prime* $p^{-1}+\mathbb{Z} = 0+\mathbb{Z}$ *so* $1/p \in \mathbb{Z}$. *This is a contradiction. So* h *is not a group homomorphism. As a result there is no homomorphism* $h : \mathbb{Q}_p/\mathbb{Z} \longrightarrow \mathbb{Q}_p/\mathbb{Z}$ *such that* $fh = 1_{\mathbb{Q}_p/\mathbb{Z}}$. *So* f *is not a retraction.*

2.5 Products and Coproducts

Definition 2.5.1. [12, Definition 3.3.4] Let $\mathcal{A}$ be a category and A and B are objects of $\mathcal{A}$. The *product* of A and B is an object $A \times B$ equipped with morphisms $\pi_1 : A \times B \longrightarrow A$ and $\pi_2 : A \times B \longrightarrow B$ with the following universal property: for an object X of $\mathcal{A}$ with morphisms $f : X \longrightarrow A$, $g : X \longrightarrow B$, there exists a unique morphism $h : X \longrightarrow A \times B$ such that the diagram below commutes.



Example 2.5.1. *[12, Example 3.3.1] Let* $(G, \cdot)$ *and* $(H, \triangle)$ *be two groups with operations* $\cdot : G \times G \longrightarrow G$ *and* $\triangle$: $H \times H \longrightarrow H$. *The product group of* G, H *is the group*

$$(G \times H, *) = \{(g, h) \,|\, g \in G, h \in H\}$$

defined by $(g,h) * (g_1,h_1) = (g \cdot g_1, h \vartriangle h_1)$. *This product satisfies the required universal property, hence* Grp *has products. If* G, H *are abelian groups, then the product group is replaced with the direct sum. In this case, the product is denote by* $(G \oplus H, *)$.

Example 2.5.2. *[12, Example 3.3.3] Let* X, Y *be two sets in Set category. The cartesian product of* $X \times Y$ *defined as the set*

$$X \times Y = \{(x,y) \,|\, x \in X, y \in Y\}$$

and projection functions by

$$\begin{aligned} \pi_X &: X \times Y \longrightarrow X \quad \pi_X(a,b) = a \\ \pi_Y &: X \times Y \longrightarrow Y \quad \pi_Y(a,b) = b. \end{aligned}$$

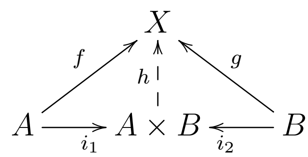
Then given any set W *with functions* $f : W \longrightarrow X$ *and* $g : W \longrightarrow Y$, *we can define* $h : W \longrightarrow X \times Y$ *by* $h(a,b) = (f(a), g(b))$, *which satisfies the required universal property. So* Set *has products.*

Example 2.5.3. *[12, Example 3.3.6] Consider the category* $Ring$ *of rings with identity. We can create products as follows: let* $(R,+,\cdot)$ *and* $(S,+,\cdot)$ *be two rings with zeros* 0_R *and* 0_S, *and units* 1_R, 1_S. *Then the product ring of* R, S *is the ring*

$$(R \times S, +, \cdot) = \{(r,s) \,|\, r \in R, s \in S\}$$

where for all pairs (r_1,s_1) *and* (r_2,s_2) *in* $R \times S$, *we define the ring operation as follows:* $(r_1,s_1)+(r_2,s_2) = (r_1+r_2, s_1+s_2)$ *and* $(r_1,s_1)\cdot(r_2,s_2) = (r_1 \cdot r_2, s_1 \cdot s_2)$. *This product satisfies the required universal property, hence* $Ring$ *has products.*

Definition 2.5.2. [12, Definition 3.4.3] Let $\mathcal{A}$ be a category and A and B are objects of $\mathcal{A}$. The *coproduct* of A and B is an object $A \Pi B$ of $\mathcal{A}$ which is equipped with morphisms $i_1 : A \longrightarrow A \Pi B$ and $i_2 : B \longrightarrow A \Pi B$ with the following universal property: for any object X of $\mathcal{A}$ with a pair of morphisms $f : A \longrightarrow X$ and $g : B \longrightarrow X$, then there exists a unique morphism $h : A \Pi B \longrightarrow X$ such that the diagram below commutes.



Example 2.5.4. *[12, Example 3.4.1] Let $(G, \cdot)$ and $(H, \circ)$ be two groups with group operations $\cdot : G \times G \longrightarrow G$ and $\circ : H \times H \longrightarrow H$. The* coproduct *of G and H is the group*

$$(G * H, \star) = \{g_1 h_1 g_2 h_2 ... g_i h_i \,|\, g_j \in G, h_j \in H\}$$

with the following operation. If $g_1 h_1 ... g_j h_j$ and $g'_1 h'_1 ... g'_k h'_k$ are two elements of $G * H$, then

$$(g_1 h_1 ... g_j h_j) \star (g'_1 h'_1 ... g'_k h'_k) = g_1 h_1 ... g_j h_j g'_1 h'_1 ... g'_k h'_k$$

We require the group operation to obey the following two rules. Let $g_1 h_1 ... g_j h_j \in G * H$. If $g \in G$, then

$$g \star (g_1 h_1 ... g_j h_j) = (g.g_1) h_1 ... g_j h_j$$

If $h \in H$, then

$$(g_1 h_1 ... g_j h_j) h = g_1 h_1 ... g_j (h_j \circ h)$$

Definition 2.5.3. [11, Chapter I, p.24] Let $\{(A_i)\}_{i \in I}$ be a family of objects in an arbitrary category $\mathcal{A}$. A *product* for the family of morphisms $\{p_i : A \longrightarrow A_i\}_{i \in I}$, called *projections*, with the property that for any family $\{\alpha_i : A' \longrightarrow A_i\}_{i \in I}$ there is a unique morphism $\alpha : A' \longrightarrow A$ such that $p_i \alpha = \alpha_i$ for all $i \in I$. The object A will be denoted by $\prod_{i \in I} A_i$.

Definition 2.5.4. [11, Chapter I, p.26] The *coproduct* of the family $\{(A_i)\}_{i \in I}$ in an arbitrary category $\mathcal{A}$ is defined dually to the product. Thus the coproduct is a family of morphisms $\{u_i : A_i \longrightarrow A\}_{i \in I}$ called *injections*, such that for each family of morphisms $\{\alpha_i : A_i \longrightarrow A'\}_{i \in I}$ we have a unique morphism $\alpha : A \longrightarrow A'$ with $\alpha u_i = \alpha_i$ for all $i \in I$. The object A will be denoted by $\bigoplus_{i \in I} A_i$.

Remark 2.5.5. If finite products and finite coproducts exist in a category $\mathcal{A}$, then there exists a canonical morphism

$$f : A_1 \Pi A_2 \Pi ... \Pi A_n \longrightarrow A_1 \times A_2 \times ... \times A_n$$

of the coproduct to the product.
In R-Mod and Ab, finite products and finite coproducts of a family of objects $\{A_i\}$, where $i = 1, \ldots, n$, are isomorphic to each other and called the *direct sum*, denoted by $\bigoplus_{i=1}^{n} A_i$.

2.6 Kernels and Cokernels

Definition 2.6.1. Let $\mathcal{A}$ be a category. An object T of $\mathcal{A}$ is said to be *terminal* if for each object A there exists exactly one morphism $f : A \longrightarrow T$ with codomain T. An object I of $\mathcal{A}$ is said to be *initial* if for each object A there exists exactly one morphism $f : I \longrightarrow A$ with domain I. An object Z of $\mathcal{A}$ is a *zero object* if it is both initial and terminal. Given any two objects A, B there exists exactly one morphism $f : Z \longrightarrow A$ with domain Z and exactly one morphism $g : B \longrightarrow Z$ with codomain Z. Hence, for any two objects A, B there exists a morphism through the zero object between them, namely given by fg, called the *zero morphism* from B to A.

Remark 2.6.2. Initial, terminal and hence zero objects of a category $\mathcal{A}$ are unique up to an isomorphism.

Example 2.6.1. *Let T be a set with exactly one element. Since for any set X in Set there exists one and only one function $f : X \longrightarrow T$ mapping every element of X to the single element of T, T is an initial object in Set. On the other hand, for any set X we can write a function $f : \emptyset \longrightarrow X$, so $\emptyset$ is an initial object in Set. In fact, it is the only initial object since for any other initial object Y there would have to be a morphism $g : Y \longrightarrow \emptyset$ from Y to $\emptyset$.*

Definition 2.6.3. [11, Chapter I, p.14] Let $\mathcal{A}$ be a category with a zero object and let $\alpha : A \longrightarrow B$. Then a morphism $u : K \longrightarrow A$ is called the *kernel* of α if $\alpha u = 0$ and if for every morphism $u' : K' \longrightarrow A$ such that $\alpha u' = 0$ there is a unique morphism $\gamma : K' \longrightarrow K$ such that $u\gamma = u'$.

Proposition 2.6.4. *[16, Proposition 1.1.14] Let $\mathcal{A}$ be a category and $f : A \longrightarrow B$ be a morphism. If $i : K \longrightarrow A$ and $i' : K' \longrightarrow A$ are both kernels of the morphism f, then there is a unique isomorphism $j : K \longrightarrow K'$ such that $i'j = i$.*

Proof. Since $i' : K' \longrightarrow A$ is a kernel, there is a unique morphism $j : K \longrightarrow K'$ such that $i'j = i$. Since $i : K \longrightarrow A$ is also a kernel, there is a unique morphism $j' : K' \longrightarrow K$ such that $ij' = i'$. Then $i'jj' = ij' = i'$. Since $i' : K' \longrightarrow A$ is a kernel and $i'jj' = i' = i'1_{K'}$, it follows that $jj' = 1_{K'}$. Similarly, $ij'j = i'j = i$. Since $i : K \longrightarrow A$ is a kernel and $ij'j = i = i1_K$, it follows that $j'j = 1_K$. This proves that j is an isomorphism. □

Proposition 2.6.5. *[16, Proposition 1.1.15] A kernel of a morphism is always a monomorphism.*

Proof. Let $i : K \longrightarrow A$ be a kernel of a morphism $f : A \longrightarrow B$. Let $g : X \longrightarrow K$ and $h : X \longrightarrow K$ be morphisms such that $ig = ih$. Then $fig = 0_{X,B} = fih$. Since $i : K \longrightarrow A$ is a kernel of f and $ig = ih$, $g = h$. This shows that i can be left cancellable. Hence i is a monomorphism. □

Definition 2.6.6. [11, Chapter I, p.15] Let $\mathcal{A}$ be a category with a zero object and let $\alpha : A \longrightarrow B$. Then a morphism $u : B \longrightarrow E$ is called the *cokernel* of α if $u\alpha = 0$ and if for every morphism $u' : B \longrightarrow E'$ such that $u'\alpha = 0$ there is a unique morphism $\gamma : E \longrightarrow E'$ such that $\gamma u = u'$.

Proposition 2.6.7. *[16, Proposition 1.1.14] Let $\mathcal{A}$ be a category and $f : A \longrightarrow B$ be a morphism. If $v : B \longrightarrow K$ and $v' : B \longrightarrow K'$ are both cokernels of the morphism f, then there is a unique isomorphism $j : K \longrightarrow K'$ such that $jv = v'$.*

Proof. Since $v : B \longrightarrow K$ is a cokernel of f, there is a unique morphism $j : K \longrightarrow K'$ such that $jv = v'$. Since $v' : B \longrightarrow K'$ is also cokernel of f, there is a unique morphism $j' : K' \longrightarrow K$ such that $j'v' = v$. Then $jj'v' = jv = v'$. Since $v' : B \longrightarrow K'$ is a cokernel and $jj'v' = v'$, it follows that $jj' = 1_{K'}$. Similarly, $j'jv = j'v' = v$. Since $v : B \longrightarrow K$ is a cokernel and $j'jv = v = v1_K$, it follows that $j'j = 1_K$. This proves that j is an isomorphism. □

Proposition 2.6.8. *[16, Proposition 1.1.15] A cokernel of a morphism is always an epimorphism.*

Proof. Let $i : B \longrightarrow K$ be a cokernel of $f : A \longrightarrow B$. Let $g : K \longrightarrow X$ and $h : K \longrightarrow X$ be morphisms such that $gi = hi$. Then $gif = 0_{A,X} = hif$. Since $i : B \longrightarrow K$ is a cokernel of f and $gi = hi$, $g = h$. This shows that i can be right cancellable. Hence i is an epimorphism. □

2.7 Exact sequences

Definition 2.7.1. [11, Chapter I, p.16] If $A' \longrightarrow A$ is the kernel of some morphism, then A' is called a *normal* subobject of A. If every monomorphism in a category is normal, then the category is said to be *normal*.

Definition 2.7.2. [11, Chapter I, p.16] If $A \longrightarrow A''$ is the cokernel of some morphism, then A'' is called a *conormal* quotient object of A and if every epimorphism in a category is conormal, then the category is said to be *conormal*.

Definition 2.7.3. [11, Chapter I, p.18] Let $\mathcal{A}$ be a normal and conormal category with kernels and cokernels. If every morphism $\alpha : A \longrightarrow B$ can be written as a composition

$$A \xrightarrow{v} I \xrightarrow{q} B$$

where q is an epimorphism and v is a monomorphism, then $\mathcal{A}$ is called an *exact category*.

Definition 2.7.4. [11, Chapter I, p.18] Let $\mathcal{A}$ be an exact category. A sequence of morphisms

$$\dots \longrightarrow A_{i-1} \xrightarrow{\alpha_{i-1}} A_i \xrightarrow{\alpha_i} A_{i+1} \xrightarrow{\alpha_{i+1}} A_{i+2} \xrightarrow{\alpha_{i+2}} \dots$$

in $\mathcal{A}$ is called an *exact sequence* if $\mathrm{Ker}(\alpha_{i+1}) = \mathrm{Im}(\alpha_i)$ as subobjects of A_{i+1} for every i.

Proposition 2.7.5. *[11, Proposition 15.1] Let $\mathcal{A}$ be an exact category. Then the following statements hold.*

(1) $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ *is exact in* $\mathcal{A}$ *if and only if* $A \xleftarrow{\alpha} B \xleftarrow{\beta} C$ *is exact in* $\mathcal{A}^{op}$.

(2) $0 \longrightarrow A \xrightarrow{\alpha} B$ *is exact if and only if* α *is a monomorphism.*

(3) $A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if* α *is an epimorphism.*

(4) $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if* α *is an isomorphism.*

Proof. (1) Consider

$$A \xrightarrow{q} I \xrightarrow{v} B \xrightarrow{r} J \xrightarrow{w} C$$

where $v = \mathrm{im}(\alpha)$ and $w = \mathrm{im}(\beta)$. Then $r = \mathrm{coim}(\beta)$. If $A \longrightarrow B \longrightarrow C$ is exact, then $v = \ker(\beta)$ and hence also $v = \ker(r)$. Therefore $r = \mathrm{coker}(v)$ and hence also $r = \mathrm{coker}(\alpha)$. Then $r = \ker(\alpha)$ as well as $r = \mathrm{im}(\beta)$ in the dual category and so $A \longleftarrow B \longleftarrow C$ is exact.
(2) If α is a monomorphism, then its $\mathrm{Ker}(\alpha) = 0$ and so $0 \longrightarrow A \longrightarrow B$ is exact. Conversely, if $0 \longrightarrow A \longrightarrow B$ is exact, then $\mathrm{Ker}(\alpha) = 0$. Let $A \xrightarrow{q} I \xrightarrow{v} B$ be a factorization of α as an epimorphism followed by a monomorphism. Then $q = \mathrm{coker}(\alpha)$. Since the latter is 0, q must be an isomorphism. But then $\alpha = vq$ must be a monomorphism.
(3) Follows from (1) and (2).
(4) Since a normal category is balanced that is, every morphism which is both a monomorphism and an epimorphism is also an isomorphism, the proof of this part follows from (2) and (3). $\square$

Definition 2.7.6. [11, Chapter I, p.19] Let $\mathcal{A}$ be an exact category. A sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is said to be *short exact sequence* if and only if α is a monomorphism, β is an epimorphism and $\alpha = \ker(\beta)$ equivalently $\beta = \mathrm{coker}(\alpha)$.

Definition 2.7.7. [11, Chapter I, p.32] Let $\mathcal{A}$ be an exact category. Then a short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

splits if β is a retraction.

Corollary 2.7.8. *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be a short exact sequence. The following assertions are equivalent:*

(1) f *is a section.*

(2) g *is a retraction.*

(3) A *is a direct summand of* B*.*

Then B *is canonically isomorphic with the direct sum* $A \oplus C$*.*

Proof. (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Let $e : B \longrightarrow A$ be a retraction of f and $p = fe$. Then $p^2 = p$. Every element $b \in B$ can be decomposed as $b = (b - p(b)) + p(b)$. Since $e(b - p(b)) = e(b) - ep(b) = e(b) - efe(b) = e(b) - e(b) = 0$, $b - p(b) \in Ker(e)$ and $p(b) \in \mathrm{Im}(f)$. This decomposition is unique since if $b = f(a)$ and $e(b) = 0$, $0 = e(b) = e(f(a)) = a$. This shows that $B \cong \mathrm{Im}(f) \oplus Ker(e)$ is a direct sum and $f : A \longrightarrow B$ is the canonical inclusion of $\mathrm{Im}(f)$. By exactness, $Ker(e) \cong \mathrm{Im}(f)$ and hence $B \cong A \oplus C$.
(3) $\Rightarrow$ (1) Clear. □

2.8 Pullbacks and Pushouts

Definition 2.8.1. [11, Chapter I, p.9] Given two morphisms $\alpha_1 : A_1 \longrightarrow A$ and $\alpha_2 : A_2 \longrightarrow A$ with a common codomain, a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\beta_2} & A_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ A_1 & \xrightarrow[\alpha_1]{} & A \end{array}$$

is called a *pullback* for α_1 and α_2 if for every pair of morphisms ${\beta_1}' : P' \longrightarrow A_1$ and ${\beta_2}' : P' \longrightarrow A_2$ such that $\alpha_1{\beta_1}' = \alpha_2{\beta_2}'$, there exists a unique morphism $\gamma : P' \longrightarrow P$ such that ${\beta_1}' = \beta_1\gamma$ and ${\beta_2}' = \beta_2\gamma$.

Definition 2.8.2. [11, Chapter I, p.9] Given two morphism $\alpha_1 : A \longrightarrow A_1$ and $\alpha_2 : A \longrightarrow A_2$ with a common domain, a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & A_1 \\ {\scriptstyle\alpha_2}\downarrow & & \downarrow{\scriptstyle\beta_1} \\ A_2 & \xrightarrow[\beta_2]{} & R \end{array}$$

is called a *pushout* for α_1 and α_2 if for every pair of morphism ${\beta_1}' : A_1 \longrightarrow R'$ and ${\beta_2}' : A_2 \longrightarrow R'$ such that ${\beta_1}'\alpha_1 = {\beta_2}'\alpha_2$, there exists a unique morphism $\gamma : R \longrightarrow R'$ such that ${\beta_1}' = \gamma\beta_1$ and ${\beta_2}' = \gamma\beta_2$.

Proposition 2.8.3. *[11, Proposition 7.1] In the pullback diagram*

$$\begin{array}{ccc} P & \xrightarrow{\beta_2} & A_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ A_1 & \xrightarrow[\alpha_1]{} & A \end{array}$$

if α_1 is a monomorphism, then β_2 is also a monomorphism.

Proof. Suppose that $\beta_2 f = \beta_2 g$ for some morphisms f and g. Then $\alpha_1\beta_1 f = \alpha_2\beta_2 f = \alpha_2\beta_2 g = \alpha_1\beta_1 g$. Since α_1 is a monomorphism, $\beta_1 f = \beta_1 g$. Therefore $f = g$ by uniqueness of factorization through the pullback. This shows that β_2 is a monomorphism. □

Proposition 2.8.4. *In the pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & A_1 \\ {\scriptstyle\alpha_2}\downarrow & & \downarrow{\scriptstyle\beta_1} \\ A_2 & \xrightarrow[\beta_2]{} & R \end{array}$$

if α_1 is an epimorphism, then β_2 is also an epimorphism.

Proof. Suppose that $f\beta_2 = g\beta_2$. Then $f\beta_1\alpha_1 = f\beta_2\alpha_2 = g\beta_2\alpha_2 = g\beta_1\alpha_1$. Since α_1 is an epimorphism, $f\beta_1 = g\beta_1$. Therefore by uniqueness of factorization through the pushout we have $f = g$. This shows that β_2 is an epimorphism. □

Proposition 2.8.5. *[11, Proposition 7.2] If each square in the diagram*

$$\begin{array}{ccccc} P & \longrightarrow & Q & \longrightarrow & B' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & I & \longrightarrow & B \end{array}$$

is a pullback and $B' \longrightarrow B$ is a monomorphism, then the outer rectangle is a pullback.

Proof. Given morphisms $X \longrightarrow A$ and $X \longrightarrow B'$ such that $X \longrightarrow A \longrightarrow I \longrightarrow B = X \longrightarrow B' \longrightarrow B$ we must find a unique morphism $X \longrightarrow P$ such that $X \longrightarrow P \longrightarrow A = X \longrightarrow A$ and $X \longrightarrow P \longrightarrow Q \longrightarrow B' = X \longrightarrow B'$. Now since the right-hand square is a pullback, there is a unique morphism $X \longrightarrow Q$ such that $X \longrightarrow Q \longrightarrow I = X \longrightarrow A \longrightarrow I$ and $X \longrightarrow Q \longrightarrow B' = X \longrightarrow B'$. Since the left-hand square is a pullback we have a unique morphism $X \longrightarrow P$ such that $X \longrightarrow P \longrightarrow A = X \longrightarrow A$ and $X \longrightarrow P \longrightarrow Q = X \longrightarrow Q$. Then the morphism $X \longrightarrow P$ satisfies the required conditions. Since $B' \longrightarrow B$ is a monomorphism, $P \longrightarrow A$ is a monomorphism by Proposition 2.8.3 and the uniqueness of the morphism $X \longrightarrow P$ follows from this. □

Proposition 2.8.6. *If each square in the diagram*

$$\begin{array}{ccccc} A & \longrightarrow & I & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & P & \longrightarrow & Q \end{array}$$

is a pushout and $A \longrightarrow A'$ is an epimorphism, then the outer rectangle is a pushout.

Proof. Given morphisms $B \longrightarrow X$ and $A' \longrightarrow X$ such that $A \longrightarrow I \longrightarrow B \longrightarrow X = A \longrightarrow A' \longrightarrow X$ we must find a unique morphism $Q \longrightarrow X$ such that $B \longrightarrow Q \longrightarrow X = B \longrightarrow X$ and $A' \longrightarrow P \longrightarrow Q \longrightarrow X = A' \longrightarrow X$. Now since the right-hand square is a pushout we have a unique morphism $Q \longrightarrow X$ such that $B \longrightarrow Q \longrightarrow X = B \longrightarrow X$ and $P \longrightarrow Q \longrightarrow X = P \longrightarrow X$. Since the left-hand square is a pushout we have a unique morphism $P \longrightarrow X$ such that $I \longrightarrow P \longrightarrow X = I \longrightarrow B \longrightarrow X$ and $A' \longrightarrow P \longrightarrow$

$X = A' \longrightarrow X$. The morphism $Q \longrightarrow X$ satisfies the required conditions. $A \longrightarrow A'$ is an epimorphism, $B \longrightarrow Q$ is an epimorphism by Proposition 2.8.4 and the uniqueness of the morphism follows. □

Proposition 2.8.7. *[11, Proposition 13.1] Consider a commutative diagram*

$$\begin{array}{ccccc} K & \xrightarrow{\gamma} & P & \xrightarrow{\beta_2} & A_2 \\ \| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \alpha_2} \\ K & \xrightarrow{u} & A_1 & \xrightarrow{\alpha_1} & A \end{array}$$

where the right-hand square is a pullback, u is the kernel of α_1 and γ is the morphism into the pullback induced by two morphisms $u : K \longrightarrow A_1$ and $0 : K \longrightarrow A_2$. Then γ is the kernel of β_2.

Proof. Since $u = \beta_1\gamma$ and u is a monomorphism, then γ must be a monomorphism. Also $\beta_2\gamma = 0$ by construction of γ. Now let $v : X \longrightarrow P$ be such that $\beta_2 v = 0$. Then $0 = \alpha_2\beta_2 v = \alpha_1\beta_1 v$ and since u is the kernel of α_1, $w : X \longrightarrow K$ such that $uw = \beta_1 v$. Then we see that $\gamma w = v$ since each of these morphisms gives the same thing when composed with both β_1 and β_2. This proves that γ is the kernel of β_2. □

Proposition 2.8.8. *[11, Proposition 13.2] Consider a diagram*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

where $B' \longrightarrow B$ is the kernel of some morphism $B \longrightarrow B''$. Then the diagram can be extended to a pullback if and only if $A' \longrightarrow A$ is the kernel of the composition $A \longrightarrow B \longrightarrow B''$.

Proof. ($\Rightarrow$) Suppose there is a morphism $A' \longrightarrow B'$ such that $A' \longrightarrow A \longrightarrow B = A' \longrightarrow B' \longrightarrow B$. If there is another A'' and there are morphisms $A'' \longrightarrow A$ and $A'' \longrightarrow B'$ such that $A'' \longrightarrow A \longrightarrow B = A'' \longrightarrow B' \longrightarrow B$, then there is a unique morphism $A'' \longrightarrow A'$

such that $A'' \longrightarrow A' \longrightarrow B' = A'' \longrightarrow B'$ and $A'' \longrightarrow A' \longrightarrow A = A'' \longrightarrow A$. Since u is the kernel of $B \longrightarrow B''$, $B' \longrightarrow B \longrightarrow B''$ is zero. $A' \longrightarrow A \longrightarrow B \longrightarrow B' = A' \longrightarrow B' \longrightarrow B \longrightarrow B''$ is zero and $A'' \longrightarrow A \longrightarrow B \longrightarrow B'' = A'' \longrightarrow B' \longrightarrow B \longrightarrow B''$ is zero. So there is a unique morphism $A'' \longrightarrow A$ such that $A'' \longrightarrow A' \longrightarrow A = A'' \longrightarrow A$. Hence $A' \longrightarrow A$ is the kernel of $A \longrightarrow B \longrightarrow B''$.

($\Leftarrow$) Suppose that $A' \longrightarrow A$ is the kernel $A \longrightarrow B \longrightarrow B''$. Then $A' \longrightarrow A \longrightarrow B \longrightarrow B''$ is zero. Since $B' \longrightarrow B$ is the kernel of $B \longrightarrow B''$, there is a unique morphism $A' \longrightarrow B'$ such that $A' \longrightarrow B' \longrightarrow B = A' \longrightarrow A \longrightarrow B$. Suppose that $X \longrightarrow A \longrightarrow B = X \longrightarrow B' \longrightarrow B$. Then $X \longrightarrow A \longrightarrow B \longrightarrow B'$ is zero. Hence there is a unique morphism $X \longrightarrow A'$ such that $X \longrightarrow A' \longrightarrow A = X \longrightarrow A$. Then also $X \longrightarrow A' \longrightarrow B' \longrightarrow B = X \longrightarrow A' \longrightarrow A \longrightarrow B = X \longrightarrow B' \longrightarrow B$ and since $B' \longrightarrow B$ is a monomorphism $X \longrightarrow A' \longrightarrow B' = X \longrightarrow B'$. So the diagram can be extended to a pullback. □

Proposition 2.8.9. *Consider the diagram*

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \\ B' & \longrightarrow & C' \end{array}$$

where $C \longrightarrow C'$ is the cokernel of some morphism $A \longrightarrow B$. Then the diagram can be extended to a pushout if and only if $B' \longrightarrow C'$ is the cokernel of the composition $A \longrightarrow B \longrightarrow B'$.

Proof. ($\Rightarrow$) Suppose there is a morphism $C \longrightarrow C'$ such that $B \longrightarrow C \longrightarrow C' = B \longrightarrow B' \longrightarrow C'$. If there is another C'' and there are morphisms $C \longrightarrow C''$ and $B' \longrightarrow C''$ such that $B \longrightarrow C \longrightarrow C'' = B \longrightarrow B' \longrightarrow C''$, then there is a unique morphism $C' \longrightarrow C''$ such that $C \longrightarrow C' \longrightarrow C'' = C \longrightarrow C''$. Since u is the cokernel of $A \longrightarrow B$, $A \longrightarrow B \longrightarrow C$ is zero. $A \longrightarrow B \longrightarrow B' \longrightarrow C' = A \longrightarrow B \longrightarrow C \longrightarrow C'$ is zero and $A \longrightarrow B \longrightarrow B' \longrightarrow C'' = A \longrightarrow B \longrightarrow C \longrightarrow C''$ is zero. So there is a unique $C' \longrightarrow C''$ such that $B' \longrightarrow C' \longrightarrow C'' = B' \longrightarrow C''$. Hence $B' \longrightarrow C'$ is the cokernel of $A \longrightarrow B \longrightarrow B'$.

($\Leftarrow$) Suppose that $B' \longrightarrow C'$ is the cokernel $A \longrightarrow B \longrightarrow B'$. Then $A \longrightarrow B \longrightarrow$

$B' \longrightarrow C'$ is zero. Since $B \longrightarrow C$ is the kernel of $A \longrightarrow B$, there is a unique morphism $C \longrightarrow C'$ such that $B \longrightarrow C \longrightarrow C' = B \longrightarrow B' \longrightarrow C'$. Suppose that $B \longrightarrow C \longrightarrow C'' = B \longrightarrow B' \longrightarrow C''$. Then $B \longrightarrow B' \longrightarrow C' \longrightarrow C''$ is zero. Hence there is a unique morphism $C \longrightarrow C''$ such that $C \longrightarrow C' \longrightarrow C'' = C \longrightarrow C''$. Then also $B \longrightarrow B' \longrightarrow C' \longrightarrow C'' = B \longrightarrow C \longrightarrow C' \longrightarrow C'' = B \longrightarrow B' \longrightarrow C''$ and since $B' \longrightarrow C'$ is an epimorphism $B' \longrightarrow C' \longrightarrow C'' = B' \longrightarrow C''$. So the diagram can be extended to a pushout. □

2.9 Grothendieck categories

Definition 2.9.1. [10] A category $\mathcal{A}$ is said to be an *abelian category* if it is a preadditive category satisfying the following conditions.

(1) $\mathcal{A}$ has a zero object.

(2) $\mathcal{A}$ has a binary products.

(3) Every morphism in $\mathcal{A}$ has a cokernel and a kernel.

(4) Every monic morphism is a kernel and every epi is a cokernel.

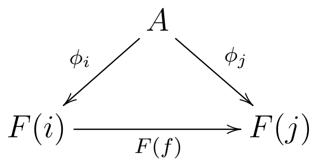
Example 2.9.1. *Let R be a ring. Then the category of right R-modules Mod-R is an abelian category.*

Remark 2.9.2. Let $\mathcal{A}$ be an abelian category. For every morphism $f : M \longrightarrow N$ in $\mathcal{A}$ we have the following notation and analysis:

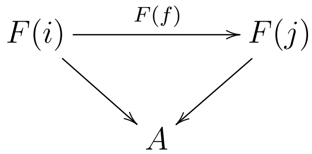
$$\begin{array}{ccccccc} \mathrm{Ker}(f) & \xrightarrow{\ker(f)} & M & \xrightarrow{f} & N & \xrightarrow{\mathrm{coker}(f)} & \mathrm{Coker}(f) \\ & & \downarrow{\scriptstyle \mathrm{coim}(f)} & & \uparrow{\scriptstyle \mathrm{im}(f)} & & \\ & & \mathrm{Coim}(f) & \xrightarrow[\overline{f}]{} & \mathrm{Im}(f) & & \end{array}$$

where $\overline{f}$ is an isomorphism.

Definition 2.9.3. Let $\mathcal{A}$ be a category, A be an object of $\mathcal{A}$. For an arbitrary category $\mathcal{J}$, let $F : \mathcal{J} \longrightarrow \mathcal{A}$ be a functor. A family of morphisms $\phi_i : A \longrightarrow F(i)$, $i \in \mathcal{J}$, such that for each morphism $f : i \longrightarrow j$ in $\mathcal{J}$ the following diagram commutes



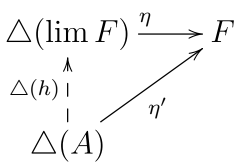
is called a *cone with A over F* and it is denoted by $Cone(A|F)$. Dually a family of morphisms $\phi : F(i) \longrightarrow A$, $i \in J$, such that for every morphism $f : i \longrightarrow j$ in J the following diagram commutes



is called a *cone with F over A* and it is denoted by $Cone(F|A)$.

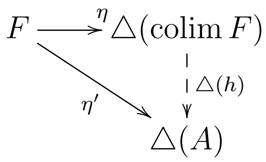
Definition 2.9.4. Let $\mathcal{A}$ and $\mathcal{J}$ be categories. A functor $\triangle : \mathcal{A} \longrightarrow Fun(\mathcal{J}, \mathcal{A})$ which sends an object A of $\mathcal{A}$ to the functor $\triangle(A) : \mathcal{J} \longrightarrow \mathcal{A}$ under which each $i \in \mathcal{J}$ is mapped to A and every morphism in $\mathcal{J}$ is mapped to the identity of $\mathcal{A}$ is said to be the *diagonal functor*.

Definition 2.9.5. Let $F : \mathcal{J} \longrightarrow \mathcal{A}$ be a functor. The *limit* of F is an object $\lim F$ equipped with a natural transformation $\eta : \triangle(\lim F) \longrightarrow F$ such that $(\lim F, u)$ is universal from $\triangle$ to $\lim F$. This means that for any other pair $(A, \eta' : \triangle(A) \longrightarrow F)$ with η' a natural transformation and $A \in \mathcal{A}$, there exists a unique morphism $h : A \longrightarrow \lim F$ in $\mathcal{A}$ such that the following diagram is commutative.



The morphism $\eta : \triangle(\lim F) \longrightarrow F$ forms a cone with $\lim F$ over F via family of morphisms $\eta_i : \lim F \longrightarrow F(i)$ for all $i \in I$.

Definition 2.9.6. Let $F : \mathcal{J} \longrightarrow \mathcal{A}$ be a functor. The *colimit* of F is an object $\operatorname{colim} F$ equipped with a natural transformation $\eta : F \longrightarrow \triangle(\operatorname{colim} F)$ such that $(\operatorname{colim} F, \eta : F \longrightarrow \triangle(\operatorname{colim} F))$ is universal from F to $\triangle$. This means that for any pair $(A, \eta' : F \longrightarrow \triangle(A))$ with η' a natural transformation where $A \in \mathcal{A}$, there exists a unique morphism $h : \operatorname{colim} F \longrightarrow A$ in $\mathcal{A}$ such that the following diagram is commutative.



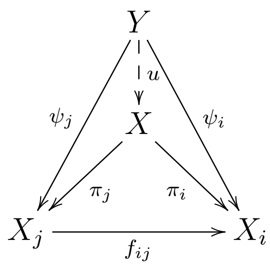
Definition 2.9.7. Let $(I, \leq)$ be a poset, let $(A_i)_{i \in I}$ be a family of objects and $f_{ij} : A_j \longrightarrow A_i$ for all $i \leq j$ be a family of morphisms. If we have the following properties

(1) f_{ii} is the identity on A_{ii}

(2) $f_{ik} = f_{ij} = f_{jk}$ for all $i \leq j \leq k$

then the pair $((A_i)_{i \in I}, (f_{ij})_{i \leq j})$ is called an *inverse system* of objects and morphisms on I.

Definition 2.9.8. Let (X_i, f_{ij}) be an inverse system of objects and morphisms in a category $\mathcal{A}$. The *inverse limit* of this system is an object X, which is denoted by $\varprojlim X_i$, in $\mathcal{A}$ together with morphisms $\pi_i : X \longrightarrow X_i$ satisfying $\pi_i = f_{ij}\pi_j$ for all $i \leq j$. The pair (X, π_i) must be universal in the sense that for any other such pair (Y, ψ_i) there exists a unique morphism $u : Y \longrightarrow X$ such that the following diagram is commutative for all i, j..



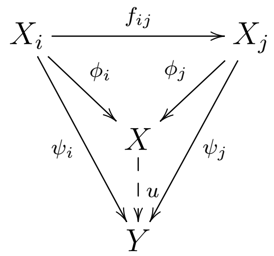
Definition 2.9.9. Let $(I, \leq)$ be a directed set. Let $\{A_i \,|\, i \in I\}$ be a family of objects indexed by I and $f_{ij} : A_i \longrightarrow A_j$ be a morphism for all $i \leq j$ with the following properties

(1) f_{ii} is the identity on I.

(2) $f_{ik} = f_{ij} = f_{jk}$ for all $i \leq j \leq k$.

Then the pair $((A_i)_{i \in I}, (f_{ij})_{i \leq j})$ is called a *direct system* over I.

Definition 2.9.10. Let (X_i, f_{ij}) be a direct system of objects and morphisms in a category $\mathcal{A}$. The *direct limit* of this system is an object X, which is denoted by $X = \varinjlim X_i$, in $\mathcal{A}$ with morphisms $\phi : X_i \longrightarrow X$ such that $\phi_i = \phi_j f_{ij}$ for all $i \leq j$. The direct limit (X, ϕ_i) of the direct system (X_i, f_{ij}) is universal in the sense that for any other such pair (Y, ψ_i) there is a unique morphism $u : X \longrightarrow Y$ such that $u\phi_i = \psi_i$ for each $i \in I$. That is the following diagram is commutative for all i, j.



Definition 2.9.11. A family of objects $\{U_i\}_{i \in I}$ is called a *family of generators* for a category $\mathcal{A}$ if for every pair of distinct morphisms $\alpha, \beta : A \longrightarrow B$ there is a morphism $u : U_i \longrightarrow A$ for some i such that $\alpha u \neq \beta u$. An object U in $\mathcal{A}$ is called a *generator* for $\mathcal{A}$ if $\{U\}$ is a family of generators for $\mathcal{A}$.

Definition 2.9.12. [17] A category $\mathcal{A}$ is called a *Grothendieck category* if

(1) $\mathcal{A}$ is an abelian category.

(2) Every (possibly infinite) family of objects in $\mathcal{A}$ has a coproduct.

(3) Direct limits are exact in $\mathcal{A}$.

(4) $\mathcal{A}$ has a generator, i.e. there is an object U in $\mathcal{A}$ for every object X of $\mathcal{A}$ there is an epimorphism $U^{(I)} \to X$, where $U^{(I)}$ denotes a coproduct of copies of U.

Example 2.9.2. *[17] Let R be a ring. Then the category of right R-modules Mod-R is a Grothendieck category.*

Definition 2.9.13. [18] Let $\mathcal{C}$ be a coalgebra over a field. The category of right comodules over the coalgebra $\mathcal{C}$ denoted by $\mathcal{M}^{\mathcal{C}}$. The objects are all right $\mathcal{C}$-comodules and the morphisms between two objects are the morphisms of comodules. We will also denote the morphisms in $\mathcal{M}^{\mathcal{C}}$ from M to N by $\mathcal{C}om_{\mathcal{C}}(M, N)$. Similarly, the category of left $\mathcal{C}$-comodules will be denoted by ${}^{\mathcal{C}}\mathcal{M}$.

Example 2.9.3. *[18, Corollary 2.2.8] The category $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category.*

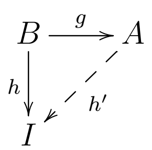
Definition 2.9.14. [19] An object M of a category $\mathcal{A}$ is said to be *finitely generated* if whenever $M = \sum M_i$ for a family $(M_i)_I$ of subobject of M, there is an $i \in I$ such that $M = M_i$.

Definition 2.9.15. [19] An object M of a category $\mathcal{A}$ is said to be *finitely presented* if it is finitely generated and every epimorphism $L \longrightarrow M$ when L is finitely generated has a finitely generated kernel.

Definition 2.9.16. [19] A category $\mathcal{A}$ is said to be *locally finitely generated (presented)* if it has a family of finitely generated (presented) generators.

2.10 Injective objects

Definition 2.10.1. [10] In any category an object I is called *injective* if every morphism $h : B \longrightarrow I$ from I factors through every monic $g : B \longrightarrow A$, as $h'g = h$ for some h'.



Theorem 2.10.2. *[20] Let $\mathcal{A}$ be an abelian category and $(Q_i)_{i\in I}$ a family of objects of $\mathcal{A}$ such that the product $Q = \prod Q_i$ exists. Then Q is injective if and only if each Q_i is injective.*

Proof. ($\Rightarrow$) Let $\pi_i : Q \longrightarrow Q_i$ be the canonical projections and $\alpha_i : Q_i \longrightarrow Q$ morphisms such that $\pi_i\alpha_i = 1_{Q_i}$. If Q is injective and $f : X' \longrightarrow X$ is a monomorphism in $\mathcal{A}$ and $g : X' \longrightarrow Q_i$ is a morphism in $\mathcal{A}$, then $\alpha_i g : X' \longrightarrow Q$. There exists $h : X \longrightarrow Q$ with $hf = \alpha_i g$ and therefore $\pi_i hf = \pi_i\alpha_i g = 1_{Q_i} g = g$. Hence Q_i is injective.
($\Leftarrow$) Assume that Q_i is injective for any $i \in I$, and let $g : X' \longrightarrow Q$ be a morphism in $\mathcal{A}$. Let $h_i : X \longrightarrow Q_i$ be such that $\pi_i g = h_i f$. From the definition of products it follows that there exists a unique morphism $h : X \longrightarrow Q$ such that $\pi_i h = h_i$. Then $\pi_i(hf) = (\pi_i h)f = h_i f = \pi_i g$. So $hf = g$. Thus Q is injective. □

Definition 2.10.3. Let $\mathcal{A}$ be an abelian category, a short exact sequence which satisfies one of the equivalent conditions of Corollary 2.7.8 is a split exact sequence.

Lemma 2.10.4. *Let $\mathcal{A}$ be an abelian category and M be an object of $\mathcal{A}$. Then the following conditions are equivalent.*

(1) M is an injective object of $\mathcal{A}$.

(2) Every short exact sequence starting with M, splits.

Proof. (1) $\Rightarrow$ (2) Let

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

be a short exact sequence. Since M is injective, for every morphism $g : M \longrightarrow M$, there exists a morphism $h : N \longrightarrow M$ such that $hf = g$. Since $1_M \in \mathrm{Mor}_{\mathcal{A}}(M, M)$, there exists a morphism $h : N \longrightarrow M$ such that $hf = 1_M$. So the sequence splits.
(2) $\Rightarrow$ (1) Let

$$0 \longrightarrow K \longrightarrow N \longrightarrow L \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$ and $f : K \longrightarrow M$ be a morphism. Then by pushout we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & L & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow[\gamma]{} & G & \xrightarrow[\delta]{} & L & \longrightarrow & 0 \end{array}$$

We know that $\beta = \delta g$. Since β is an epimorphism, δ is also an epimorphism. Thus

$$0 \longrightarrow M \longrightarrow G \longrightarrow L \longrightarrow 0$$

is a short exact sequence. By assumption

$$0 \longrightarrow M \longrightarrow G \longrightarrow L \longrightarrow 0$$

splits. So there exists a morphism $h : G \longrightarrow M$ such that $h\gamma = 1_M$. Put $\varphi = hg$. Then $\varphi\alpha = hg\alpha = h\gamma f = 1_M f = f$. So M is an injective. $\square$

Definition 2.10.5. The category $\mathcal{A}$ is said to have *enough injectives* if every object is a subobject of an injective object.

Definition 2.10.6. Let $\mathcal{A}$ be an abelian category and $\alpha : L \longrightarrow M$ be a monomorphism. If every $\varphi : M \longrightarrow N$ is a monomorphism when $\varphi\alpha$ is a monomorphism, then α is called an *essential monomorphism*.

Definition 2.10.7. Let $\mathcal{A}$ be an abelian category. An essential monomorphism $\alpha : L \longrightarrow M$ is *maximal* if every monomorphism $\varphi : M \longrightarrow L$ with $\varphi\alpha$ essential is an isomorphism.

Definition 2.10.8. Let $\mathcal{A}$ be an abelian category. An essential monomorphism $\alpha : L \longrightarrow M$ with M injective is called an *injective envelope*.

Definition 2.10.9. [13] A category $\mathcal{A}$ is said to have *injective envelope* if every object of $\mathcal{A}$ has an injective envelope.

2.11 Pure subobjects

Definition 2.11.1. [19, Definition, p.353] A short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in a Grothendieck category $\mathcal{A}$ is said to be *pure* if every finitely presented object is relatively projective to it. In this case L is a pure subobject of M.

Definition 2.11.2. [19, Definition, p.354] Also an object M of a Grothendieck category $\mathcal{A}$ is said to be *flat* if every short exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

is pure.

Definition 2.11.3. [21, p. 160] Consider a class $\mathcal{E}$ of short exact sequence of objects an abelian category $\mathcal{A}$ such that every sequence isomorphic to a sequence in $\mathcal{E}$ is also in $\mathcal{E}$. The corresponding class of monomorphism is denoted by $\mathcal{E}_m$ and epimorphism is denoted by $\mathcal{E}_e$. $\mathcal{E}$ is called a *proper class* if it satisfies the following conditions.

(1) Every short exact sequence is in $\mathcal{E}$.

(2) If α,$\beta \in \mathcal{E}_m$, then $\beta\alpha \in \mathcal{E}_m$ if defined.

(3) If α,$\beta \in \mathcal{E}_e$, then $\beta\alpha \in \mathcal{E}_e$ if defined.

(4) If $\beta\alpha \in \mathcal{E}_m$, then $\alpha \in \mathcal{E}_m$.

(5) If $\beta\alpha \in \mathcal{E}_e$, then $\beta \in \mathcal{E}_e$.

Lemma 2.11.4. *[19, Lemma 6(i)] The class $\mathcal{P}ure$ of pure short exact sequences in a Grothendieck category $\mathcal{A}$ forms a proper class.*

Definition 2.11.5. [22] A Grothendieck category $\mathcal{A}$ is said to be *regular* if every object M of $\mathcal{A}$ is regular in the sense that every short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is pure in $\mathcal{A}$.

Theorem 2.11.6. *[19, Theorem 4] Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following statements are equivalent.*

(1) $\mathcal{A}$ is regular.

(2) All objects are flat.

(3) All short exact sequences are pure.

(4) All finitely presented objects are projective.

2.12 Pure-injective objects

Definition 2.12.1. Let $\mathcal{A}$ be a Grothendieck category and M be an object of $\mathcal{A}$. M is called a *pure-injective* object if it is relatively injective to every pure short exact sequence in $\mathcal{A}$.

Proposition 2.12.2. *[23, Proposition 4.1] Let $\mathcal{A}$ be a Grothendieck category and $\{A_i\}_{i\in I}$ a family of objects of $\mathcal{A}$. Then $A = \prod_{i\in I} A_i$ is pure-injective if and only if each A_i is pure-injective.*

Lemma 2.12.3. *Let $\mathcal{A}$ be a Grothendieck category and M be an object of $\mathcal{A}$. Then the following conditions are equivalent.*

(1) M is pure-injective object of $\mathcal{A}$.

(2) Every pure exact sequence starting with M splits.

Proof. (1) $\Rightarrow$ (2) Let

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

be a pure exact sequence in $\mathcal{A}$. Since M is pure-injective, for every morphism $g : M \longrightarrow M$ there exists a morphism $h : N \longrightarrow M$ such that $hf = g$. Since $1_M \in \mathrm{Mor}_{\mathcal{A}}(M, M)$, there exists $h : N \longrightarrow M$ such that $hf = 1_M$. So the sequence splits.

(2) $\Rightarrow$ (1) Let

$$0 \longrightarrow K \longrightarrow N \longrightarrow L \longrightarrow 0$$

be a pure exact sequence in $\mathcal{A}$ and $f : K \longrightarrow M$ be a morphism. Then by pushout we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow[\gamma]{} & G & \xrightarrow[\delta]{} & L & \longrightarrow & 0 \end{array}$$

We know that $\delta g = \beta$. Since $\mathcal{P}$ure is a proper class by Lemma 2.11.4, $\delta \in \mathcal{P}ure_e$. Now the sequence

$$0 \longrightarrow M \longrightarrow G \longrightarrow L \longrightarrow 0$$

splits by assumption. So there exists a morphism $e : G \longrightarrow M$ such that $e\gamma = 1_M$. Put $h = eg$. Then $h\alpha = eg\alpha = e\delta f = 1_M f = f$. So M is pure-injective. $\square$

Definition 2.12.4. The category $\mathcal{A}$ is said to have *enough pure-injectives* if every object is a pure subobject of a pure-injective object, that is for every object M of $\mathcal{A}$ there is a pure monomorphism $f : M \longrightarrow P$ with P pure-injective.

Theorem 2.12.5. *[24, Theorem 4.1] Every locally finitely presented Grothendieck category $\mathcal{A}$ has enough pure-injective objects.*

The following definitions are particular cases of [21, Definitions, p.162] when $\mathcal{P}$ure is considered instead of a proper class $\mathcal{E}$.

Definition 2.12.6. Let $\mathcal{A}$ be a Grothendieck category. A pure monomorphism $\alpha : L \longrightarrow M$ is *pure-essential* if every $\varphi : M \longrightarrow N$, such that $\varphi\alpha$ is a pure monomorphism, is a monomorphism.

Definition 2.12.7. Let $\mathcal{A}$ be a Grothendieck category. A pure-essential monomorphism $\alpha : L \longrightarrow M$ is *maximal* if every monomorphism $\varphi : M \longrightarrow N$ with $\varphi\alpha$ pure-essential is an isomorphism.

Definition 2.12.8. Let $\mathcal{A}$ be a Grothendieck category. A pure-essential monomorphism $\alpha : L \longrightarrow M$ with M pure-injective is called a *pure-injective envelope*.

Corollary 2.12.9. *[24, Corollary 4.5] Every object of a locally finitely presented Grothendieck category $\mathcal{A}$ has a pure-injective envelope.*

Definition 2.12.10. An object M of Grothendieck category is said to be *absolutely pure* if it is a pure subobject of every object containing it.

Theorem 2.12.11. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is regular.

(2) Every pure-injective object of $\mathcal{A}$ is injective.

(3) Every pure-injective object of $\mathcal{A}$ is absolutely pure.

Proof. $(1) \Rightarrow (2)$ Since $\mathcal{A}$ is regular, every object of $\mathcal{A}$ is flat by [19, Theorem 4]. Therefore every short exact sequence is pure exact. This implies that every pure-injective object is injective.
$(2) \Rightarrow (3)$ Clear.
$(3) \Rightarrow (1)$ Let M be an object of $\mathcal{A}$. Since there are enough pure-injective objects in $\mathcal{A}$ by [24, Theorem 4.1], there exists a pure exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow P/M \longrightarrow 0$$

with P pure-injective and therefore P is absolutely pure by assumption. Then M is also absolutely pure by [25, Lemma 8.1]. This means that every object of $\mathcal{A}$ is absolutely pure and therefore every short exact sequence in $\mathcal{A}$ is pure. So $\mathcal{A}$ is regular. $\square$

3. DIRECT INJECTIVE OBJECTS IN ABELIAN CATEGORIES

3.1 Direct-injective objects

Definition 3.1.1. Let $\mathcal{A}$ be an abelian category. An object M of $\mathcal{A}$ is said to be *direct-injective* if every subobject A of M with A isomorphic to a direct summand of M is a direct summand.

Proposition 3.1.2. *Let $\mathcal{A}$ be an abelian category. Then the following conditions are equivalent for an object M of $\mathcal{A}$.*

(1) Given a direct summand N of M with the inclusion map $i : N \longrightarrow M$ and any monomorphism $f : N \longrightarrow M$ there is an endomorphism $g : M \longrightarrow M$ of M such that $gf = i$.

(2) M is direct-injective.

(3) Any monomorphism $f : N \longrightarrow M$ with N a direct summand of M splits.

Proof. $(1) \Rightarrow (2)$ Let N be a direct summand of M and K be a subobject of M which is isomorphic to N. Let $f : N \longrightarrow K$ be that isomorphism. There is an endomorphism $g : M \longrightarrow M$ such that $gf = i$ where $i : N \longrightarrow M$ is the inclusion monomorphism by assumption. Let $h = \pi g$ where $\pi : M \longrightarrow N$ is the canonical projection. Then $hf = \pi gf = \pi i = 1_N$. So $f : N \longrightarrow K$ splits. Thus K is a direct summand of M.
$(2) \Rightarrow (3)$ Let $f : N \longrightarrow M$ be a monomorphism with N a direct summand of M. We know that $\mathrm{Im}(f) \cong N$ and N is a direct summand of M. Since M is direct-injective, $\mathrm{Im}(f)$ is also a direct summand of M. Thus f splits.
$(3) \Rightarrow (1)$ Since $f : N \longrightarrow M$ is a splitting monomorphism, there is a morphism $\alpha : M \longrightarrow N$ such that $\alpha f = 1_N$. Let $g = i\alpha : M \longrightarrow M$. Then $gf = i\alpha f = i1_N = i$. □

Lemma 3.1.3. *Let $\mathcal{A}$ be an abelian category. If $N \oplus M$ is direct-injective, then an exact sequence*

$$0 \longrightarrow N \xrightarrow{f} M \longrightarrow T \longrightarrow 0$$

of objects and morphisms of $\mathcal{A}$ splits.

Proof. Let $i_1 : N \longrightarrow N \oplus M$ and $i_2 : M \longrightarrow N \oplus M$ be inclusion maps. Since $N \oplus M$ is direct-injective and $i_2 f : N \longrightarrow N \oplus M$ is a monomorphism, there exists an endomorphism $h : N \oplus M \longrightarrow N \oplus M$ such that $hi_2 f = i_1$ by Proposition 3.1.2. Put $g = p_1 h i_2$ where $p_1 : N \oplus M \longrightarrow N$ is the canonical projection map, then we have $gf = p_1 h i_2 f = p_1 i_1 = 1_N$. So f splits. □

Theorem 3.1.4. *Let $\mathcal{A}$ an abelian category with enough injective objects, M be an object of $\mathcal{A}$ and $f : M \longrightarrow K$ be a monomorphism from M to an injective object K of $\mathcal{A}$. Then M is injective if and only if $K \oplus M$ is direct-injective.*

Proof. ($\Rightarrow$) Assume that M is injective. So $K \oplus M$ is injective by Theorem 2.10.2. Since every injective object is direct-injective, $M \oplus K$ is direct-injective.
($\Leftarrow$) Assume that $M \oplus K$ is direct-injective. Then the following short exact sequence

$$0 \longrightarrow M \longrightarrow K \longrightarrow K/\operatorname{Im}(f) \longrightarrow 0$$

splits by Lemma 3.1.3. So M is injective. □

Corollary 3.1.5. *Let $\mathcal{A}$ be an abelian category with enough injective objects. Then the class of direct-injective objects of $\mathcal{A}$ need not be closed under subobjects.*

Proof. Let M be an object which is not direct-injective. Then its injective envelope $E(M)$ is direct-injective and M is a subobject of $E(M)$ which is not direct-injective. □

Corollary 3.1.6. *Let $\mathcal{A}$ be an abelian category with enough injective objects. Then the class of direct-injective objects of $\mathcal{A}$ need not be closed under taking finite coproducts.*

Proof. Let M be an object which is direct-injective but not injective. Since $\mathcal{A}$ has enough injective objects, there is an exact sequence

$$0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0$$

where $E(M)$ denotes the injective envelope of M. Since M is not injective, $M \oplus E(M)$ can not be direct-injective by Theorem 3.1.4. □

Corollary 3.1.7. *Let $\mathcal{A}$ be an abelian category having enough injective objects. The coproduct of two direct-injective objects of $\mathcal{A}$ is direct-injective if and only if every direct-injective object of $\mathcal{A}$ is injective.*

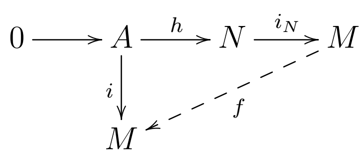
Proof. (⇒) Let M be a direct-injective object in $\mathcal{A}$. Since $\mathcal{A}$ has enough injective objects, there exists an exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow N \longrightarrow 0$$

with I injective. Since I is injective and every injective object is direct-injective, I is direct-injective. Then $M \oplus I$ is direct-injective by assumption and therefore M is injective by Theorem 3.1.4.
(⇐) It is clear by Theorem 2.10.2. □

Proposition 3.1.8. *Direct summands of direct-injective objects of an abelian category $\mathcal{A}$ are direct-injective.*

Proof. Let M be a direct-injective object of $\mathcal{A}$ and let N be a direct summand of M. We consider any direct summand A of N. Let $i_N : N \longrightarrow M$ and $i_A : A \longrightarrow N$ be the inclusion maps and let $h : A \longrightarrow N$ be a monomorphism. Then $g = i_N h : A \longrightarrow M$ is a monomorphism.



Since M is direct-injective, there exists an endomorphism $f : M \longrightarrow M$ such that $fi_N h = i$ where $i : A \longrightarrow M$ is a monomorphism. Put $f' = \pi_N f i_N$. Then $f'h = \pi_N f i_N h = i_A$. $\square$

Definition 3.1.9. An abelian category $\mathcal{A}$ is said to be *hereditary* if and only if every subobject of a projective object is projective if and only if every quotient object of an injective object is injective.

Definition 3.1.10. An abelian category $\mathcal{A}$ is said to be *semihereditary* if every finitely generated subobject of a projective object is projective and *cosemihereditary* if every finitely cogenerated quotient object of an injective object is injective.

Theorem 3.1.11. *Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ has enough injectives. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is (cosemi)hereditary.

(2) Every (finitely cogenerated) quotient object of an injective object of $\mathcal{A}$ is direct-injective.

Proof. $(1) \Rightarrow (2)$ Clear.
$(2) \Rightarrow (1)$ Let I be an injective object and J be a subobject of I. Let $\pi : I \longrightarrow I/J$ be the canonical epimorphism. Being a quotient object of $E(I/J) \oplus I$, $E(I/J) \oplus I/J$ is direct-injective by assumption, where $E(I/J)$ denotes the injective envelope of I/J. Therefore I/J is injective by Theorem 3.1.4. $\square$

Definition 3.1.12. An object M of an abelian category $\mathcal{A}$ is called N*-injective* if for every subobject A of N any homomorphism from A to M can be extended to a homomorphism from N to M.

Definition 3.1.13. An object M of an abelian category $\mathcal{A}$ is called a *quasi-injective* object if it is M-injective.

Proposition 3.1.14. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is regular.

(2) Every pure-injective object of $\mathcal{A}$ is a quasi-injective.

Proof. $(1) \Rightarrow (2)$ Since $\mathcal{A}$ is regular, every pure-injective object of $\mathcal{A}$ is injective by Theorem 2.12.11. Since every injective object is quasi-injective, every pure-injective object of $\mathcal{A}$ is quasi-injective.

$(2) \Rightarrow (1)$ Let M be a pure-injective object of $\mathcal{A}$. Since $\mathcal{A}$ has enough injective objects, there is an injective object I and a monomorphism $f : M \longrightarrow I$. Since I is injective, it is pure-injective. So $M \oplus I$ is pure-injective. Now $M \oplus I$ is quasi-injective by assumption. Since every quasi-injective object is direct-injective, $M \oplus I$ is direct-injective. We have an exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow I/M \longrightarrow 0$$

with I injective and $M \oplus I$ is direct-injective. Thus M is injective by Theorem 3.1.4. Hence $\mathcal{A}$ is regular by Theorem 2.12.11. $\square$

3.2 Classes all of whose objects are direct-injective

Definition 3.2.1. An abelian category $\mathcal{A}$ is called a *spectral category* if every short exact sequence in $\mathcal{A}$ splits.

Theorem 3.2.2. *Let $\mathcal{A}$ be an abelian category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is spectral.

(2) $\mathcal{A}$ has enough injectives and every object of $\mathcal{A}$ is direct-injective.

(3) $\mathcal{A}$ has enough injectives and every subobject of a direct-injective object of $\mathcal{A}$ is direct-injective.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ Clear.

$(3) \Rightarrow (1)$ Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough injective objects, M has the injective envelope $E(M)$. Since $E(M) \oplus E(M)$ is direct-injective, $M \oplus E(M)$ is also direct-injective by assumption. Therefore M is injective by Theorem 3.1.4. Thus $\mathcal{A}$ is spectral. □

Definition 3.2.3. [17] Let $\mathcal{A}$ be a Grothendieck category. An object M of $\mathcal{A}$ is *simple* if it is non-zero and has no other subobjects than 0 and M.

Definition 3.2.4. [17] Let $\mathcal{A}$ be a Grothendieck category. An object M of $\mathcal{A}$ is *semisimple* if it is a coproduct of simple subobject.

Definition 3.2.5. A Grothendieck category $\mathcal{A}$ is said to be *semisimple* if every object of $\mathcal{A}$ is semisimple.

Proposition 3.2.6. *[17, Proposition 6.7, Chapter V] A locally finitely generated Grothendieck category $\mathcal{A}$ is semisimple if and only if it is spectral.*

Theorem 3.2.7. *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is semisimple.

(2) Every object of $\mathcal{A}$ is direct-injective.

(3) The coproduct of two direct-injective objects of $\mathcal{A}$ is direct-injective.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ Clear.
$(3) \Rightarrow (1)$ Since every quasi-injective object is direct-injective, the coproduct of two quasi-injective objects of $\mathcal{A}$ is quasi-injective by assumption. Then $\mathcal{A}$ is semisimple by [26, Corollary 2.3]. □

4. PURE DIRECT INJECTIVE OBJECTS IN GROTHENDIECK CATEGORIES

4.1 Pure-direct-injective objects

Definition 4.1.1. Let $\mathcal{A}$ be a Grothendieck category. An object M of $\mathcal{A}$ is said to be *pure-direct-injective* if every pure subobject A of M with A isomorphic to a direct summand of M is a direct summand of M.

Proposition 4.1.2. *[27, Proposition 3.2] Let $\mathcal{A}$ be a Grothendieck category. Then the following conditions are equivalent for an object M of $\mathcal{A}$.*

(1) Given a direct summand N of M with the inclusion $i : N \longrightarrow M$ and any monomorphism $f : N \longrightarrow M$ with $\mathrm{Im}(f)$ a pure subobject of M there is an endomorphism $g : M \longrightarrow M$ of M such that $gf = i$.

(2) M is pure-direct-injective.

(3) Any monomorphism $f : N \longrightarrow M$ with N a direct summand of M and $\mathrm{Im}(f)$ pure in M splits.

Proof. (1) $\Rightarrow$ (2) Let K be a pure subobject of M which is isomorphic to N. Let $f : N \longrightarrow K$ be that isomorphism. There is an endomorphism $g : M \longrightarrow M$ such that $gf = i$ where $i : N \longrightarrow M$ is the inclusion monomorphism by assumption. Let $h = \pi g$ where $\pi : M \longrightarrow N$ is the canonical projection. Then $hf = \pi g f = \pi i = 1_N$. So $f : N \longrightarrow K$ splits. Thus K is a direct summand of M.
(2) $\Rightarrow$ (3) Let N be a direct summand of M and $f : N \longrightarrow M$ be a monomorphism with $\mathrm{Im}(f)$ pure in M. Then, since $\mathrm{Im}(f) \cong N$, $\mathrm{Im}(f)$ is a direct summand of M and f splits.
(3) $\Rightarrow$ (1) Let N be a direct summand of M, $i : N \longrightarrow M$ be the inclusion map and $f : N \longrightarrow M$ be a monomorphism with $\mathrm{Im}(f)$ pure in M. Since $f : N \longrightarrow M$ is a splitting

monomorphism by assumption, there is a morphism $e : M \longrightarrow N$ such that $ef = 1_N$. Let $g = ie : M \longrightarrow M$. Then $gf = ief = i1_N = i$. $\square$

Lemma 4.1.3. *Let $\mathcal{A}$ be a Grothendieck category. If $N \oplus M$ is pure-direct-injective, then a pure short exact sequence*

$$0 \longrightarrow N \xrightarrow{f} M \longrightarrow K \longrightarrow 0$$

of objects and morphisms of $\mathcal{A}$ splits.

Proof. Let

$$0 \longrightarrow N \xrightarrow{f} M \longrightarrow K \longrightarrow 0$$

be a pure exact sequence in $\mathcal{A}$. Suppose that $N \oplus M$ is pure-direct-injective in $\mathcal{A}$. Let $i_1 : N \longrightarrow N \oplus M$ and $i_2 : M \longrightarrow N \oplus M$ be the inclusion maps. Since $N \oplus M$ is pure-direct-injective, there exists an endomorphism $h : N \oplus M \longrightarrow N \oplus M$ such that $hi_2f = i_1$ by Proposition 4.1.2. Define $g : M \longrightarrow N$ by $g = \pi_1 hi_2$ where $\pi_1 : N \oplus M \longrightarrow N$ is the canonical projection map. Then $gf = \pi_1 hi_2 f = \pi_1 i_1 = 1_N$. Thus the sequence splits. $\square$

Theorem 4.1.4. *Let $\mathcal{A}$ be a Grothendieck category and*

$$0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$$

be a pure exact sequence in $\mathcal{A}$ with M pure-injective. Then $N \oplus M$ is pure-direct-injective if and only if N is pure-injective.

Proof. ($\Rightarrow$) Assume that

$$0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$$

is a pure exact sequence with M pure-injective and $N \oplus M$ is pure-direct-injective. Then the sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$$

splits by Lemma 4.1.3. So N is a direct summand of a pure-injective object M. Thus N is pure-injective by Proposition 2.12.2.

($\Leftarrow$) Suppose N is pure-injective. Then $N \oplus M$ is pure-injective by Proposition 2.12.2 and therefore $N \oplus M$ is pure-direct-injective since pure-injective objects are pure-direct-injective. $\square$

Recall that every locally finitely presented Grothendieck category has enough pure-injective objects (see [28, Theorem 4.1] or [29, Corollary 1.5]).

Corollary 4.1.5. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the class of pure-direct-injective objects of $\mathcal{A}$ need not be closed under pure subobjects.*

Proof. Let M be an object of $\mathcal{A}$ which is not pure-direct-injective. But its pure-injective envelope $PE(M)$ is pure-direct-injective and M is a pure subobject of $PE(M)$ which is not pure-direct-injective. $\square$

Corollary 4.1.6. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the class of pure-direct-injective objects of $\mathcal{A}$ need not be closed under finite coproducts.*

Proof. Let M be an object which is pure-direct-injective but not pure-injective. Since $\mathcal{A}$ has enough pure-injective objects by [28, Theorem 4.1], there is a pure exact sequence

$$0 \longrightarrow M \longrightarrow PE(M) \longrightarrow PE(M)/M \longrightarrow 0$$

where $PE(M)$ denotes the pure-injective envelope of M. Since M is not pure-injective, $M \oplus PE(M)$ can not be pure-direct-injective by Theorem 4.1.4. $\square$

Corollary 4.1.7. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then every pure-direct-injective object of $\mathcal{A}$ is pure-injective if and only if the coproduct of two pure-direct-injective objects of $\mathcal{A}$ is pure-direct-injective.*

Proof. ($\Rightarrow$) It is clear by [23, Proposition 4.1].

($\Leftarrow$) Let M be a pure-direct-injective object in $\mathcal{A}$. Since every locally finitely presented

Grothendieck category $\mathcal{A}$ has enough pure-injective objects by [29, Corollary 1.5], there exists a pure exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$$

with P pure-injective. Since P is pure-injective and every pure-injective object is pure-direct-injective, P is pure-direct-injective. Now $M \oplus P$ is pure-direct-injective by assumption. Thus M is pure-injective by Theorem 4.1.4. $\square$

Proposition 4.1.8. *Direct summands of pure-direct-injective objects of a Grothendieck category $\mathcal{A}$ are pure-direct-injective.*

Proof. Let M be a pure-direct-injective object of $\mathcal{A}$ and N be a direct summand of M. Let K be a pure subobject of N which is isomorphic to a direct summand L of N. Since N is a direct summand of M, it is a pure subobject of M. So K is a pure subobject of M by Lemma 2.11.4. Therefore K is a direct summand of M. Thus K is a direct summand of N. Hence N is pure-direct-injective. $\square$

Theorem 4.1.9. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is regular.

(2) Every pure-direct-injective object of $\mathcal{A}$ is direct-injective.

Proof. $(1) \Rightarrow (2)$ Clear since every short exact sequence in a regular category $\mathcal{A}$ is pure exact by [19, Theorem 4].

$(2) \Rightarrow (1)$ Let M be a pure-injective object of $\mathcal{A}$. Since $\mathcal{A}$ has enough injective objects, there is an injective object I and a monomorphism $f : M \longrightarrow I$. Since I is injective, it is pure-injective. So $M \oplus I$ is pure-injective by Proposition 2.12.2 and therefore it is pure-direct-injective. Now $M \oplus I$ is direct-injective by assumption. We have an exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow I/M \longrightarrow 0$$

with I injective and $M \oplus I$ direct-injective. Thus M is injective by Theorem 3.1.4. Hence $\mathcal{A}$ is regular by Theorem 2.12.11. □

Remark 4.1.10. Pure-direct-injective objects of locally finitely presented regular Grothendieck category need not be injective.

Example 4.1.1. *Let F be a field. Consider the F-subalgebra*

$$R =< \bigoplus_{i=1}^{\infty} F_i, 1 >$$

of $\prod_{i=1}^{\infty} F_i$ generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1 where $F_i = F$ for $n = 1, 2, ...$ Then R is a von Neumann regular ring. Also R is continuous by [30, Example 3]. But R is not self-injective.

Proposition 4.1.11. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is regular and the coproduct of two pure-direct-injective objects is pure-direct-injective.

(2) Every pure-direct-injective object of $\mathcal{A}$ is injective.

Proof. $(1) \Rightarrow (2)$ Let M be a pure-direct-injective object of $\mathcal{A}$. Since every locally finitely presented Grothendieck category $\mathcal{A}$ has enough pure-injective objects by [28, Theorem 4.1], there exists a pure exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$$

with P pure-injective. Now $M \oplus P$ is pure-direct-injective by assumption. Then M is pure-injective by Theorem 4.1.4 and therefore M is injective by Theorem 2.12.11.
$(2) \Rightarrow (1)$ Let M be a pure-injective object of $\mathcal{A}$. Then M is injective by assumption and therefore $\mathcal{A}$ is regular by Theorem 2.12.11. Since the coproduct of two injective objects is injective, the coproduct of two pure-direct-injective objects is pure-direct-injective. □

Corollary 4.1.12. *Let $\mathcal{A}$ be a locally finitely presented regular Grothendieck category. If the coproduct of two pure-direct-injective objects is pure-direct-injective, then every pure-direct-injective object of $\mathcal{A}$ is quasi-injective.*

Proposition 4.1.13. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. If every pure-direct-injective object of $\mathcal{A}$ is quasi-injective, then $\mathcal{A}$ is regular.*

Proof. Let M be a pure-injective object of $\mathcal{A}$. Since there are enough injectives in $\mathcal{A}$, M has an injective envelope $E(M)$. So $M \oplus E(M)$ is pure-injective and therefore it is pure-direct-injective. Now $M \oplus E(M)$ is quasi-injective by assumption. Then $M \oplus E(M)$ is direct-injective. Thus M is direct-injective by Theorem 3.1.4. Hence $\mathcal{A}$ is regular by Theorem 4.1.9. □

Definition 4.1.14. A class $\mathcal{C}$ of objects of a category is said to be *closed under extensions* if $A, M/A \in \mathcal{C}$ implies that $M \in \mathcal{C}$.

Definition 4.1.15. A Grothendieck category $\mathcal{A}$ is said to be *pure-hereditary* if epimorphic image of an injective object is pure-injective.

Proposition 4.1.16. *Let $\mathcal{A}$ be a Grothendieck category. If the class of pure-injective objects of $\mathcal{A}$ is closed under extensions, then the following conditions are equivalent.*

(1) $\mathcal{A}$ is pure hereditary.

(2) Every quotient of a pure-injective object of $\mathcal{A}$ is pure-injective.

Proof. (1) $\Rightarrow$ (2) Let M be a pure-injective object of $\mathcal{A}$ and K be a subobject of M. Since $\mathcal{A}$ has enough injectives, there exists an exact sequence

$$0 \longrightarrow K \longrightarrow I \longrightarrow L \longrightarrow 0$$

with I injective. We have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & G & \longrightarrow & M/K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathcal{A}$ is pure-hereditary, L is pure-injective because it is an epimorphic image of an injective object. Since the class of pure-injective objects closed under extensions by assumption and M and L are pure-injective, G is pure-injective. Since I is injective,

$$0 \longrightarrow I \longrightarrow G \longrightarrow M/K \longrightarrow 0$$

splits by Lemma 2.10.4. Therefore M/K is pure injective.

$(2) \Rightarrow (1)$ Let M be an injective object of $\mathcal{A}$ and K be a subobject of M. Since $\mathcal{A}$ has enough injective objects, there is an injective envelope $E(M/K)$ of M/K. Since $E(M/K) \oplus M/K$ is a quotient object of the pure-injective object $E(M/K) \oplus M$, $E(M/K) \oplus M/K$ is pure-injective by assumption. We know that every pure-injective object is pure-direct-injective and therefore $E(M/K) \oplus M/K$ is pure-direct-injective. Then the following pure exact sequence splits by Lemma 4.1.3.

$$0 \longrightarrow M/K \longrightarrow E(M/K) \longrightarrow E(M/K)/(M/K) \longrightarrow 0$$

Thus M/K is pure-injective by Lemma 2.12.3. Hence $\mathcal{A}$ is pure-hereditary. □

Theorem 4.1.17. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. If the class of pure-injective objects of $\mathcal{A}$ is closed under extensions, then the following conditions are* *equivalent.*

(1) $\mathcal{A}$ is pure hereditary.

(2) Every quotient of a pure-injective object of $\mathcal{A}$ is pure-injective.

(3) Every quotient of a pure-injective object of $\mathcal{A}$ is pure-direct-injective.

Proof. $(1) \Rightarrow (2)$ Clear by Proposition 4.1.16.
$(2) \Rightarrow (3)$ Clear since every pure-injective object is pure-direct-injective.
$(3) \Rightarrow (1)$ Let M be an injective object of $\mathcal{A}$ and N be a subobject of M. Since $\mathcal{A}$ has enough pure-injective objects by [28, Theorem 4.1], there exists pure-injective envelope $PE(M/N)$ of M/N. Since it is a quotient object of a pure-injective object, $PE(M/N) \oplus M/N$ is pure-direct-injective. Therefore the following pure exact sequence splits

$$0 \longrightarrow M/N \longrightarrow PE(M/N) \longrightarrow PE(M/N)/(M/N) \longrightarrow 0$$

by Lemma 4.1.3. Thus M/N is pure-injective by Lemma 2.12.3. Hence $\mathcal{A}$ is pure-hereditary. □

Corollary 4.1.18. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the class of pure-direct-injective objects of $\mathcal{A}$ need not be closed under taking quotients.*

4.2 Classes all of whose objects are pure-direct-injective

Recall that an abelian category $\mathcal{A}$ is called a *spectral category* if every short exact sequence in $\mathcal{A}$ splits. We can give the following immediate result without proof.

Proposition 4.2.1. *Let $\mathcal{A}$ be a spectral Grothendieck category. Then the following conditions are equivalent.*

(1) Every object in $\mathcal{A}$ is pure-direct-injective.

(2) Every pure-direct-injective object of $\mathcal{A}$ is injective.

(3) Every pure-direct-injective object of $\mathcal{A}$ is direct-injective.

Definition 4.2.2. A Grothendieck category $\mathcal{A}$ is called *pure-semisimple* if it is locally finitely presented and each of its objects is pure-projective, that is every object of $\mathcal{A}$ is projective relative to every pure exact sequence in $\mathcal{A}$.

Remark 4.2.3. If a Grothendieck category $\mathcal{A}$ is pure-semisimple, then every object is pure-injective by [28, Theorem 2].

Remark 4.2.4. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. $\mathcal{A}$ is called *pure-semisimple* if it has pure global dimension zero, which means that each of its objects is a direct summand of a coproduct of finitely presented objects (see [24]). $\mathcal{A}$ is pure-semisimple if and only if it satisfies the pure noetherian property a coproduct of any family of pure-injective objects in $\mathcal{A}$ is pure-injective (see [29, Theorem 1.9]).

Lemma 4.2.5. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is pure-semisimple.

(2) Every pure exact sequence in $\mathcal{A}$ splits.

Proof. $(1) \Rightarrow (2)$ Every object of $\mathcal{A}$ is pure-projective by Definition 4.2.2. Since every pure exact sequence ending with a pure-projective object splits which can be proved dually to Lemma 2.12.3, every pure exact sequence splits.
$(2) \Rightarrow (1)$ Suppose that every pure exact sequence in $\mathcal{A}$ splits. Let M be an object of $\mathcal{A}$. We want to show that M is pure-projective. Since every locally finitely presented Grothendieck category has enough pure-projective objects(see [19, Lemma 6 (ii)]), there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. By assumption this sequence splits and therefore M is pure-projective by [23, Proposition 4.1]. This means that $\mathcal{A}$ is pure-semisimple. □

Theorem 4.2.6. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is pure-semisimple.

(2) Every object of $\mathcal{A}$ is pure-injective.

(3) Every object of $\mathcal{A}$ is pure-direct-injective.

(4) Every subobject of a pure-direct-injective object of $\mathcal{A}$ is pure-direct-injective.

Proof. $(1) \Rightarrow (2)$ Clear by Theorem 4.2.5.
$(2) \Rightarrow (3)$ Clear since every pure-injective object is pure-direct-injective.
$(3) \Rightarrow (4)$ Clear by assumption.
$(4) \Rightarrow (3)$ Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough pure-injective objects by [29, Corollary 1.5], there exists a pure exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow L \longrightarrow 0$$

with P pure-injective. Since P is pure-injective, it is pure-direct-injective and therefore M is pure-direct-injective by assumption.
$(3) \Rightarrow (1)$ Let

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

be a pure exact sequence in $\mathcal{A}$. By assumption $M \oplus N$ is pure-direct-injective. So the sequence splits by Lemma 4.1.3. Thus $\mathcal{A}$ is pure-semisimple by Theorem 4.2.5. □

4.3 Relative pure-direct-injective objects

Definition 4.3.1. Let $\mathcal{A}$ be a Grothendieck category and M, N be objects of $\mathcal{A}$. An object M of $\mathcal{A}$ is called *pure-direct-N-injective* if every pure subobject A of N which is A isomorphic to a direct summand of M is a direct summand of N.

Lemma 4.3.2. *Let M and N be objects of a Grothendieck category $\mathcal{A}$. Then M is pure-direct-N-injective if and only if for every monomorphism $f : P \longrightarrow N$ with P a direct summand of M and $\mathrm{Im}(f)$ pure in N and for every morphism $g : P \longrightarrow M$, there exists a morphism $h : N \longrightarrow M$ such that $hf = g$.*

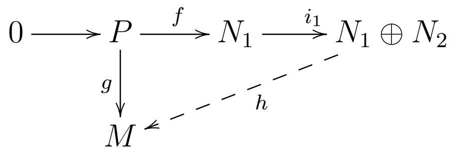
Proof. ($\Rightarrow$) Let M be a pure-direct-N-injective object of $\mathcal{A}$. Let $f : P \longrightarrow N$ be a monomorphism with P a direct summand of M and $\mathrm{Im}(f)$ pure in N. Let $g : P \longrightarrow M$ be a morphism. Since M is pure-direct-N-injective, $\mathrm{Im}(f)$ is a direct summand of N. Thus f splits. Since $f : P \longrightarrow N$ is a splitting monomorphism, there is a morphism $\psi : N \longrightarrow P$ such that $\psi f = 1_P$. Let $h = g\psi : N \longrightarrow M$. Then $hf = g\psi f = g1_P = g$.
($\Leftarrow$) Let K be a pure subobject of N which is isomorphic to a direct summand L of M. Let $\varphi : L \longrightarrow K$ be that isomorphism. Now $i_1\varphi : L \longrightarrow N$ is monomorphism where $i_1 : K \longrightarrow N$ is the inclusion map. Then there exists a morphism $h : N \longrightarrow M$ such that $hi_1\varphi = i_2$ where $i_2 : L \longrightarrow M$ is the inclusion map. If $\pi : M \longrightarrow L$ is the projection map, then define $\psi = \pi h : N \longrightarrow L$. Now $\psi i_1\varphi = \pi h i_1 \varphi = \pi i_2 = 1_L$. So $i_1\varphi$ splits. Thus K is a direct summand of N. So M is pure-direct-N-injective. □

Proposition 4.3.3. *Let $\mathcal{A}$ be a Grothendieck category and M, N_1, N_2 be objects of $\mathcal{A}$. If M is pure-direct-$N_1 \oplus N_2$-injective, then M is pure-direct-N_1-injective and M is pure-direct-N_2-injective.*

Proof. Let $f : P \longrightarrow N_1$ be a pure monomorphism with P a direct summand of M.

$$0 \longrightarrow P \longrightarrow N_1 \longrightarrow N_1 \oplus N_2$$

Since $i_1 f$ is a composition of two pure monomorphisms, $i_1 f$ is also a pure monomorphism by [19, Lemma 6(i)], where $i_1 : N_1 \longrightarrow N_1 \oplus N_2$ is the inclusion map. Since M is pure-direct-$N_1 \oplus N_2$-injective, every diagram as follows can be completed commutatively, i.e. there is a unique morphism $h : N_1 \oplus N_2 \longrightarrow M$ such that $hi_1 f = g$.



Now put $\psi = hi_1$. Then $g = hi_1 f = \psi f$. So M is pure-direct-N_1-injective. Similarly, it can be shown that M is pure-direct-N_2-injective. □

5. APPLICATIONS

In this chapter we give applications of some of our results to module and comodule categories.

5.1 Modules and Comodules

Remark 5.1.1. Let R be a unitary ring and Mod(R) be the category of right R-modules. Mod(R) is a locally finitely generated Grothendieck category with enough injectives and enough projectives. Mod(R) is hereditary if and only if the ring R is right hereditary.

Then we have the following corollary of Theorem 3.1.11 for module categories.

Corollary 5.1.2. *[3, Theorem 4] Let R be a unitary ring. Then the following conditions are equivalent.*

(1) R is right hereditary.

(2) Every factor module of an injective right R-module is direct-injective.

Remark 5.1.3. Let C be a coalgebra over a field and $\mathcal{M}^C$ be the category of right C-comodules. $\mathcal{M}^C$ is a locally finitely generated Grothendieck category. Then it has enough injectives. The category $\mathcal{M}^C$ is hereditary if and only if C is a (left and right) hereditary coalgebra (see [31]).

Then we have the following corollary of Theorem 3.1.11 for comodule categories.

Corollary 5.1.4. *Let C be a coalgebra over a field. Then the following conditions are equivalent.*

(1) $\mathcal{C}$ is hereditary.

(2) Every factor comodule of an injective right $\mathcal{C}$-comodule is direct-injective.

Remark 5.1.5. Let C be a coalgebra over a field. Then the category $\mathcal{M}^C$ of right C-comodules is spectral if and only if $\mathcal{M}^C$ is semisimple if and only if C is cosemisimple.

Now we have the following result of Theorem 3.2.2 for comodule categories.

Corollary 5.1.6. *Let C be a coalgebra over a field. Then the following conditions are equivalent.*

(1) C is cosemisimple.

(2) Every right C-comodule is direct-injective.

(3) Every subcomodule of a direct-injective right C-comodule is direct-injective.

Proof. C is a cosemisimple coalgebra if and only if every right C-comodule is injective in the category $\mathcal{M}^C$ (see [18, Theorem 3.1.5]). □

We have the following corollary of Theorem 4.1.9 for module categories.

Corollary 5.1.7. *[7, Proposition 19] Let R be a unitary ring. Then the following conditions are equivalent.*

(1) R is a von Neumann regular ring.

(2) Every pure-injective right R-module is injective.

(3) Every pure-direct-injective right R-module is direct-injective.

Also we have the following corollary of Theorem 4.1.9 for comodule categories.

Corollary 5.1.8. *Let C be a coalgebra over a field. Then the following statements are equivalent.*

(1) C is cosemisimple.

(2) Every pure-injective right C-comodule is injective.

(3) Every pure-direct-injective right C-comodule is direct-injective.

Proof. C is cosemisimple if and only if every right C-comodule is injective if and only if every right C-comodule is projective by [18, Theorem 3.1.5]. If a coalgebra C over a field is cosemisimple, then the category of right C-comodules $\mathcal{M}^C$ is regular. Conversely, if $\mathcal{M}^C$ is regular, then every right C-comodule K is FP-injective, that is every short exact sequence of the form $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ is pure (see [19]). The category of right C-comodules $\mathcal{M}^C$ coincides with the category $\sigma[{}^*_C\,\mathcal{C}]$ of submodules of C-generated left C^*-modules (see [19, Section 2.5]). Since $\mathcal{M}^C$ is locally noetherian, every FP-injective right C-comodule is injective by [22, 35.7]. Therefore, every right C-comodule is injective. Hence, C is cosemisimple by [19, Theorem 3.1.5]. □

6. CONCLUSION

In this thesis we study direct-injective objects in abelian categories and pure-direct-injective objects in Grothendieck categories. Also we give applications of some of our results to module and comodule categories. Our results given in Chapter 3. and Chapter 4. are new in abelian categories and Grothendieck categories respectively. We can list some of the most important results as follows:

6.1 Direct-injective Objects

Theorem 6.1.1. *Let $\mathcal{A}$ an abelian category with enough injective objects, M be an object of $\mathcal{A}$ and $f : M \longrightarrow K$ be a monomorphism from M to an injective object K of $\mathcal{A}$. Then M is injective if and only if $K \oplus M$ is direct-injective.*

Theorem 6.1.2. *Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ has enough injectives. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is (cosemi)hereditary.

(2) Every (finitely cogenerated) quotient object of an injective object of $\mathcal{A}$ is direct-injective.

Proposition 6.1.3. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is regular.

(2) Every pure-injective object of $\mathcal{A}$ is a quasi-injective.

Theorem 6.1.4. *Let $\mathcal{A}$ be an abelian category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is spectral.

(2) $\mathcal{A}$ has enough injectives and every object of $\mathcal{A}$ is direct-injective.

(3) $\mathcal{A}$ has enough injectives and every subobject of a direct-injective object of $\mathcal{A}$ is direct-injective.

Theorem 6.1.5. *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is semisimple.

(2) Every object of $\mathcal{A}$ is direct-injective.

(3) The coproduct of two direct-injective objects of $\mathcal{A}$ is direct-injective.

6.2 Pure-direct-injective Objects

Theorem 6.2.1. *Let $\mathcal{A}$ be a Grothendieck category and*

$$0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$$

be a pure exact sequence in $\mathcal{A}$ with M pure-injective. Then $N \oplus M$ is pure-direct-injective if and only if N is pure-injective.

Theorem 6.2.2. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is regular.

(2) Every pure-direct-injective object of $\mathcal{A}$ is direct-injective.

Proposition 6.2.3. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is regular and the coproduct of two pure-direct-injective objects is pure-direct-injective.

(2) Every pure-direct-injective object of $\mathcal{A}$ is injective.

Theorem 6.2.4. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. If the class of pure-injective objects of $\mathcal{A}$ is closed under extensions, then the following conditions are equivalent.*

(1) $\mathcal{A}$ is pure hereditary.

(2) Every quotient of a pure-injective object of $\mathcal{A}$ is pure-injective.

(3) Every quotient of a pure-injective object of $\mathcal{A}$ is pure-direct-injective.

Proposition 6.2.5. *Let $\mathcal{A}$ be a spectral Grothendieck category. Then the following conditions are equivalent.*

(1) Every object in $\mathcal{A}$ is pure-direct-injective.

(2) Every pure-direct-injective object of $\mathcal{A}$ is injective.

(3) Every pure-direct-injective object of $\mathcal{A}$ is direct-injective.

Theorem 6.2.6. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) $\mathcal{A}$ is pure-semisimple.

(2) Every pure exact sequence in $\mathcal{A}$ splits.

REFERENCES

[1] W. K. Nicholson. Semiregular modules and rings. *Canad. J. Math.*, XXVIII (5):1105–1120, **1976**.

[2] Y. Chae and J. Kwon. On direct injective modules. *Kyungpook Math. J.*, 20 (2):189–191, **1980**.

[3] W. Xue. Characterization of rings using direct-projective modules and direct-injective modules. *J. Pure Appl. Algebra*, 87:99–104, **1993**.

[4] C. Zhizhong. Direct injective modules. *Acta Mathematica Sinica*, 9 (3):307–310, **1993**.

[5] L. Fuchs. *Notes on generalized continuous modules*. -, **1995**.

[6] J. Clark. On purely extending modules. In *Abelian Groups and Modules: Proceedings of the international conference in Dublin*, pages 353–358. **1998**.

[7] S. K. Maurya, S. Das, and Y. Alagöz. Pure-direct-injective modules. *Lobachevskii Journal of Mathematics*, **2021**.

[8] S. Crivei and A. Kör. Rickart and dual rickart objects in abelian categories. *Appl. Categor. Struct.*, 24:797–824, **2016**.

[9] S. Crivei and D. Keskin Tütüncü. Weak rickart and dual weak rickart objects in abelian categories. *Commun. Algebra*, 46:2912–2926, **2018**.

[10] S. Mac Lane. *Categories for the Working Mathematicians*. Graduate Texts in Mathematics Vol. 5. Springer-Verlag, **1971**.

[11] B. Mitchell. *Theory of Categories*. Pure and Applied Mathematics A Series of Monographs and Text-books. Academic Press, **1965**.

[12] L. Trujillo. *Introductory category theory textbook*. Draft, **2019**.

[13] N. Popescu. *Abelian categories with applications to rings and modules*. Pure and Applied Mathematics A Series of Monographs and Text-books. Academic Press, **1973**.

[14] E. Ulualan. *Kategori Teori I.* Ders Notları. MATH Lecture Notes and Hobbies, **-**.

[15] J. Adamek, H. Herrlich, and G. E. Strecker. *Abstract and Concrete Categories*. Pure and Applied Mathematics. John Wiley and Sons, **1990**.

[16] R. Lal. *Algebra 3: Homological Algebra and Its Applications*. Springer, **2021**.

[17] B. Stenström. An introduction to methods of ring theory. In *Rings of Quotients*, Berlin, Heidelberg, New York, pages Springer–Verlag. **1975**.

[18] S. Dăscălescu, C. Năstăsescu, and Ş. Raianu. *Hoph Algebras: An Introduction*. Marcel Dekker, **2001**.

[19] B. T. Stenström. Purity in functor categories. *J. Algebra*, 8:352–361, **1968**.

[20] I. Bucur. *Introduction to the Theory of Categories and Functors*. Pure and Applied Mathematics Monograph. John Wiley and Sons, **1968**.

[21] B. T. Stenström. Pure submodules. *Arkiv För Matematik*, 7 (10):159–171, **1966**.

[22] R. Wisbauer. *Foundations of Module and Ring Theory*. Gordon and Breach Science Publisher, **1991**.

[23] D. J. Fieldhouse. Pure theories. *Math. Ann.*, 184:1–18, **1970**.

[24] D. Simson. On pure global dimension of locally finitely presented grothendieck categories. *Fund. Math.*, 96 (2):91–116, **1977**.

[25] G. A. Garkusha. Grothendieck categories. *(Russian) Algebra i Analiz*, 13 (2):1–68, **2001**.

[26] A. Koehler. Quasi-projective covers and direct sums. *Proc. Amer. Math. Soc*, 24:655–658, **1970**.

[27] S. E. Toksoy. Pure-direct-objects in categories: transfer via functors. *submitted*, **-**.

[28] D. Simson. On colimits of injectives in grothendieck categories. *Ark. Mat.*, 12:161–165, **1974**.

[29] D. Simson. On pure semi-simple grothendieck categories i. *Fund. Math.*, 100 (3):211–222, **1978**.

[30] Y. Utumi. On continuous regular rings and semisimple self-injective rings. *Canad. J. Math.*, 12:597–605, **1960**.

[31] C. Năstăsescu, B. Torrecillas, and Y. H. Zhang. Hereditary coalgebras. *Commun. Algebra*, 24 (4):1521–1528, **1966**.