PURE DIRECT PROJECTIVE OBJECTS IN GROTHENDIECK CATEGORIES

GROTHENDIECK KATEGORİLERDE SAF DİREKT PROJEKTİF NESNELER

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ABSTRACT

PURE DIRECT PROJECTIVE OBJECTS IN GROTHENDIECK CATEGORIES

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We study generalizations of the concept of direct-projectivity (respectively pure-direct-projectivity) from module categories to abelian categories (respectively Grothendieck categories). We examine for which categories or under what conditions direct-projective objects are projective. Also we examine for which categories or under what conditions pure-direct-projective objects are projective, quasi-projective, pure-projective, flat or direct-projective. We investigate classes all of whose objects are direct-projective (respectively pure-direct-projective). We also give applications of some results to module categories and comodule categories.

Keywords: pure subobjects, direct-projective objects, pure-direct-projective objects, abelian categories, Grothendieck categories

ÖZET

GROTHENDIECK KATEGORİLERDE SAF DİREKT PROJEKTİF NESNELER

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Bu tezde direkt-projektif (sırasıyla saf-direkt-projektif) kavramlarının abel kategorilere (sırasıyla Grothendieck kategorilere) genelleştirilmesi üzerine çalıştık. Hangi kategorilerde ya da hangi koşullar altında direkt-projektif nesnelerin projektif olduğunu inceledik. Ayrıca hangi kategorilerde ya da hangi koşullar altında saf-direkt-projektif nesnelerin projektif, yarı-projektif, saf-projektif, düz ya da direkt-projektif olduğunu inceledik. Bütün nesneleri direkt-projektif (sırasıyla saf-direkt-projektif) olan sınıfları belirledik. Ayrıca bazı sonuçlarımızın modül kategorilere ve eşmodül kategorilere uygulamalarını verdik.

Keywords: saf alt nesneler, direkt-projektif nesneler, saf-direkt-projektif nesneler, abel kategoriler, Grothendieck kategoriler

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ABBREVIATIONS

$\mathcal{C}$	A category
$\mathcal{O}\mathrm{b}(\mathcal{C})$	All objects of a category $\mathcal{C}$
$\mathrm{Mor}\,(M,N)$	All morphisms from M to N
$\circ$	The composition function
$E(M)$	Injective envelope of an object M
$PE(M)$	Pure-injective envelope of an object M
$\mathrm{Im}\,f$	The image of morphism f
$\mathrm{Coim}\,f$	The coimage of morphism f
$\mathrm{Ker}\,f$	The kernel of a morphism f
$\mathrm{Coker}\,f$	The cokernel of a morphism f
Set	The category of sets
Grp	The category of groups
Top	The category of topological spaces
$Ring$	The category of rings with identity
Ab	The category of abelian groups
R-Mod	The category of left R-modules
$DivAb$	The category of divisible abelian groups
Fld	The category of all fields
$\mathcal{C}^{op}$	The opposite category of a category $\mathcal{C}$
Cat	The category of all small categories
C	a coalgebra
$\mathcal{M}^C$	The category of right C-comodules
$\mathcal{E}$	The class of short exact sequence of objects of an abelian category $\mathcal{A}$
$\lim F$	The limit of functor F
$\mathrm{colim}\,F$	The colimit of functor F
$\varinjlim X_i$	The direct limit of direct system $\{X_i\}$

$\bigoplus M_i$	The coproduct of the family of objects of a category $\mathcal{A}$
$\prod M_i$	The product of the family of objects of a category $\mathcal{A}$
$\triangle$	The diagonal functor

1. INTRODUCTION

A right R-module M is said to be *direct-projective* if every submodule A of M with M/A isomorphic to a direct summand of M is a direct summand of M. Direct-projective modules were introduced by Nicholson in [1] and further studies on direct-projective modules were done by Tiwary and Bharadwaj in [2] and by Hausen in [3]. The notion of extending module was generalized to purely extending module by Fuchs in [4] and basic characterisations were given by Clark in [5]. Motivated by their work the notion of pure-direct-projective modules were introduced and studied by Alizade and Toksoy in [6]. Namely, a right R-module is said to be *pure-direct-projective* if every pure submodule A of which with M/A isomorphic to a direct summand is a direct summand.

In this work we study generalizations of these notions to abelian categories and Grothendieck categories, namely direct-projective objects and pure-direct-projective objects respectively. Some generalizations of direct-projective modules to abelian categories were studied by Crivei and Kör in [7] and Crivei and Keskin Tütüncü in [8]. An object M of an abelian category $\mathcal{A}$ is said to be *direct-projective* if every subobject A of M with M/A isomorphic to a direct summand of M is a direct summand. Let M and N be objects of an abelian category $\mathcal{A}$. M is called N*-projective* if given any epimorphism from N to an object L of $\mathcal{A}$, any homomorphism from M to L can be lifted to a homomorphism from M to N. M is said to be *quasi-projective* if it is M-projective. The following implications hold.

projective $\Rightarrow$ quasi-projective $\Rightarrow$ direct-projective

An object M of a Grothendieck category $\mathcal{A}$ is said to be *pure-projective* if M is relatively projective for every pure short exact sequence in $\mathcal{A}$ and it is said to be *pure-direct-projective* if every pure subobject A of M with M/A isomorphic to a direct summand of M is a direct summand. We also have the following implications.

projective $\Rightarrow$ pure-projective $\Rightarrow$ pure-direct-projective

Since every direct summand is a pure subobject, every direct-projective object is pure-direct-projective.

In Chapter 2 some definitions and lemmas which will be used in the next sections of the paper are recalled. It is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is regular if and only if every pure-projective object of $\mathcal{A}$ is projective if and only if every pure-projective object of $\mathcal{A}$ is flat (Theorem 2.12.6).

In Chapter 3 the concept of direct-projectivity is generalized to abelian categories. It is obtained that the class of direct-projective objects of an abelian category $\mathcal{A}$ with enough projective objects need not be closed under factor objects and taking finite coproducts (Corollary 3.1.5 and Corollary 3.1.6). It is shown that the coproduct of two direct-projective objects of an abelian category $\mathcal{A}$ with enough projective objects is direct-projective if and only if every direct-projective object of $\mathcal{A}$ is projective (Corollary 3.1.7). It is proved that an abelian category $\mathcal{A}$ with enough projective objects is (semi) hereditary if and only if every (finitely generated) subobject of a projective object is direct-projective (Theorem 3.1.10). It is shown for a locally finitely presented Grothendieck category $\mathcal{A}$ that $\mathcal{A}$ is regular if and only if every pure-projective object of $\mathcal{A}$ is a quasi-projective (Proposition 3.1.13). Also classes all of whose objects are direct-projective are investigated. It is proved that an abelian category $\mathcal{A}$ is spectral if and only if $\mathcal{A}$ is perfect and every object of $\mathcal{A}$ is direct-projective if and only if $\mathcal{A}$ is perfect and every factor object of a direct-projective object of $\mathcal{A}$ is direct-projective (Theorem 3.2.5). It is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is semisimple if and only if $\mathcal{A}$ has enough projectives and every object of $\mathcal{A}$ is direct-projective if and only if $\mathcal{A}$ has enough projectives and the coproduct of two direct-projective objects is direct-projective (Theorem 3.2.9).

In Chapter 4 the concept of pure-direct-projectivity is generalized to Grothendieck categories. It is obtained that the class of pure-direct-projective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ need not be closed under pure factors and taking finite coproducts (Corollary 4.1.5 and Corollary 4.1.6). It is proved that the coproduct of two pure-direct-projective objects of a locally finitely presented Grothendieck category

$\mathcal{A}$ is pure-direct-projective if and only if every pure-direct-projective object of $\mathcal{A}$ is pure-projective (Corollary 4.1.7). It is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is regular if and only if every pure-direct-projective object of $\mathcal{A}$ is flat (Theorem 4.1.9). Also it is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is regular if and only if every pure-direct-projective object of $\mathcal{A}$ is direct-projective (Theorem 4.1.10). It is obtained for a locally finitely presented Grothendieck category $\mathcal{A}$ that $\mathcal{A}$ is regular and the coproduct of two pure-direct-projective objects is pure-direct-projective if and only if every pure-direct-projective object of $\mathcal{A}$ is projective (Proposition 4.1.11). As a result of this, it is given for a locally finitely presented regular Grothendieck category $\mathcal{A}$ that if the coproduct of two pure-direct-projective objects is pure-direct-projective, then every pure-direct-projective object of $\mathcal{A}$ is quasi-projective (Corollary 4.1.12). It is obtained that if every pure-direct-projective object of a locally finitely presented Grothendieck category $\mathcal{A}$ is quasi-projective, then $\mathcal{A}$ is regular (Proposition 4.1.13). It is proved for a flat object M of a locally finitely presented Grothendieck category $\mathcal{A}$ that M is pure-direct-projective if and only if its direct-projective (Proposition 4.1.14). It is shown for a locally finitely presented Grothendieck category $\mathcal{A}$ with enough projective objects whose class of pure-injective objects is closed under extensions that $\mathcal{A}$ is pure hereditary if and only if every subobject of any projective object of $\mathcal{A}$ is pure-direct-projective if and only if every subobject of any pure-projective object of $\mathcal{A}$ is pure-direct-projective (Proposition 4.1.20). It is obtained that the class of pure-direct-projective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ need not be closed under subobjects (Corollary 4.1.21). It is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is pure-semisimple if and only if every object of $\mathcal{A}$ is pure-projective if and only if every object of $\mathcal{A}$ is pure-direct-projective if and only if every pure quotient of a pure-direct-projective object of $\mathcal{A}$ is pure-direct-projective (Theorem 4.2.6).

In Chapter 5 applications of some of our results to module and comodule categories are given. It is obtained that a coalgebra C over a field is hereditary if and only if every subcomodule of a projective right C-comodule is direct-projective (Corollary 5.1.4). As a result of Theorem 3.2.5 it is obtained for comodule categories that a coalgebra C over a field is cosemisimple if and only if C is right semiperfect and every right C-comodule is direct-projective if and only if C is right semiperfect and every factor comodule of a direct-projective right C-comodule is direct-projective (Corollary 5.1.6). Also as a result of Theorem 4.1.10 it is obtained that a coalgebra C over a field is cosemisimple if and only if every pure-projective right C-comodule is projective if and only if every pure-direct-projective right C-comodule is direct-projective (Corollary 5.1.8).

2. PRELIMINARIES

In this chapter some preliminary information which will be needed is given. Definitions, Examples, Propositions and Theorems which are not cited can be found in [9], [10] and [11].

2.1 Categories

Definition 2.1.1. A *category* $\mathcal{C}$ consists of

(1) a collection $\mathcal{O}\mathrm{b}(\mathcal{C})$ of objects;

(2) a collection $\mathrm{Mor}_{\mathcal{C}}(A,B)$ of morphisms $f : A \longrightarrow B$ for any objects A, B;

(3) a function $\circ$: $\mathrm{Mor}_{\mathcal{C}}(A,B) \times \mathrm{Mor}_{\mathcal{C}}(B,C) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(A,C)$ which is called the *composition* and assigns a morphism $g \circ f \in \mathrm{Mor}_{\mathcal{C}}(A,C)$ to every pair (f,g) where $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(B,C)$ for each $A, B, C \in \mathcal{O}\mathrm{b}(\mathcal{C})$

such that the following conditions are satisfied.

(i) Composition is associative: for each quadruple A, B, C, D of objects, if $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$, $g \in \mathrm{Mor}_{\mathcal{C}}(B,C)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(C,D)$, then $(f \circ g) \circ h = f \circ (g \circ h)$,

(ii) Composition satisfies left and right unit laws: for each $A \in \mathcal{O}\mathrm{b}(\mathcal{C})$ and for each $f \in \mathrm{Mor}_{\mathcal{C}}(B,A)$, $g \in \mathrm{Mor}_{\mathcal{A}}(A,C)$ there exists a morphism $1_A \in \mathrm{Mor}_{\mathcal{A}}(A,A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$. The morphism 1_A is called the *identity morphism*.

Example 2.1.1. *Category of sets, denoted by Set, can be described as follows*

- *Objects: All sets X.*
- *Morphisms: All functions between sets.*

Example 2.1.2. *Category of groups, denoted by Grp, we can describe as,*

- *Objects: All groups $(G, \cdot)$ where $\cdot : G \times G \longrightarrow G$ is the group operation on G.*

- *Morphisms: All group homomorphisms $\phi : (G, \cdot) \longrightarrow (H, \cdot')$.*

Example 2.1.3. *Another canonical example of a category is the category of topological spaces Top, can be described as,*

- *Objects: All topological spaces (X, τ) where τ is a topology on the set X.*
- *Morphisms: All continuous functions $f : (X, \tau) \longrightarrow (Y, \tau')$.*

Example 2.1.4. *We can describe the category of rings $Ring$ as follows*

- *Objects: All unital rings $(R, +, \cdot)$.*
- *Morphisms: Ring homomorphisms $\varphi : R \longrightarrow R'$.*

Example 2.1.5. *Let R be a ring. The category of all left R-modules R-Mod is described as follows*

- *Objects: All left R-modules $(M, +, \cdot)$.*
- *Morphisms: Module homomorphisms $\varphi : R \longrightarrow R'$.*

Example 2.1.6. *The category $DivAb$ of divisible abelian groups consists of*

- *Objects: All divisible abelian groups $(G, \cdot)$.*
- *Mophisms: All group homomorphisms $\psi : G \longrightarrow G'$.*

Definition 2.1.2. Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{op}$ is defined as the category whose objects are the same with $\mathcal{C}$, but with reversed morphisms. In other words, morphism $f : A \longrightarrow B$ in $\mathcal{C}^{op}$ is the same with the morphism $f : B \longrightarrow A$ in $\mathcal{C}$. Composition $f \circ g$ of two morphisms in $\mathcal{C}^{op}$ defined by $g \circ f$ in $\mathcal{C}$.

Definition 2.1.3. A category $\mathcal{C}$ is said to be

- *finite* if it has only finitely many objects and morphisms.

- *locally finite* if the collection $\mathrm{Mor}(A, B)$ of morphisms is finite for any pair A, B of objects.
- *small* if the collection of objects and collections of morphisms assemble into a set.
- *locally small* if the collection $\mathrm{Mor}(A, B)$ of morphisms is a set for every pair A, B of objects.
- *large* if $\mathcal{C}$ is not locally small.

Definition 2.1.4. Let $\mathcal{C}$ be a category. A category $\mathcal{S}$ is called a *subcategory* of $\mathcal{C}$ if it satisfies the following conditions.

(1) Composition operator of $\mathcal{S}$ is the same with the composition operator of $\mathcal{C}$.

(2) The objects and morphisms of $\mathcal{S}$ is contained in the collection of objects and morphisms of $\mathcal{C}$.
Furthermore, we say that $\mathcal{S}$ is a *full subcategory* of $\mathcal{C}$ if

(3) For each pair of objects A, B of $\mathcal{S}$ we have $\mathrm{Mor}_{\mathcal{C}}(A, B) = \mathrm{Mor}_{\mathcal{S}}(A, B)$.

Example 2.1.7. *The category of abelian groups Ab can be described as*

- *Objects: Abelian groups $(G, +)$.*
- *Morphisms: Group homomorphisms $\varphi : G \longrightarrow G'$.*

Then Ab is a full subcategory of Grp.

Definition 2.1.5. A category $\mathcal{C}$ is called a *preadditive category* or an *Ab-category* if for each pair of objects A and B there is an abelian group operation $+$ on the set $\mathrm{Mor}_{\mathcal{C}}(A, B)$ such that

$$\circ\colon\ \mathrm{Mor}_{\mathcal{C}}(A, B) \times \mathrm{Mor}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(A, C)$$
$$(f, g) \longmapsto g \circ f$$

is bilinear, i.e. following equations hold.

$$(h+k)\circ f = (h\circ f) + (k\circ f)$$
$$h\circ (g+f) = (h\circ g) + (h\circ f).$$

Example 2.1.8. *The category Ab of abelian groups is an Ab-category with usual group action* $+$ *on morphisms: Let* $f, g : A \longrightarrow B$ *be two group homomorphisms. Then* $f+g : A \longrightarrow B$ *defined by* $(f+g)(a) = f(a) +_B f(a)$ *is again a group homomorphism and bilinearity of composition is satisfied.*

2.2 Functors

Throughout the composition of morphisms $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Mor}_{\mathcal{C}}(B, C)$ will be denoted by gf instead of $g \circ f$.

Definition 2.2.1. A *covariant functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ is a mapping which assigns each object A of $\mathcal{A}$ to an object $F(A)$ of $\mathcal{B}$ and each morphism $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ to a morphism $T(f) \in \mathrm{Mor}_{\mathcal{B}}(T(A), T(B))$ in $\mathcal{B}$ such that the following conditions are satisfied.

(1) If $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$ and $g \in \mathrm{Mor}_{\mathcal{A}}(B, C)$, then $F(gf) = F(g)F(f)$.

(2) $F(1_A) = 1_{F(A)}$ holds for every object A of $\mathcal{A}$.

Definition 2.2.2. A *contravariant functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ is a mapping which assigns each object A of $\mathcal{A}$ to an object $F(A)$ of $\mathcal{B}$ and each morphism $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ to a morphism $F(f) \in \mathrm{Mor}_{\mathcal{B}}(F(B), F(A))$ in $\mathcal{B}$ such that the following conditions are satisfied.

(1) If $f \in \mathrm{Mor}_{\mathcal{A}}(A, B)$ and $g \in \mathrm{Mor}_{\mathcal{A}}(B, C)$, then $F(gf) = F(f)F(g)$.

(2) $F(1_A) = 1_{F(A)}$ holds for every object A of $\mathcal{A}$.

Example 2.2.1. *Every category $\mathcal{C}$ is equipped with identity functor $1_{\mathcal{C}}$, which acts as identity map on both objects and morphisms.*

Example 2.2.2. *The power set functor $\mathcal{P} : Set \longrightarrow Set$ is a functor which maps each set $X \in \mathcal{O}\mathrm{b}(Set)$ to the power set $P(X)$ of X and each morphism $f : X \longrightarrow Y$ to $\mathcal{P}(f) : P(X) \longrightarrow P(Y)$ which is defined by $\mathcal{P}(f)(S) = f(S)$ for all $S \in P(X)$.*

The class of morphisms $\mathrm{Mor}_{\mathcal{C}}(A, B)$ of a category $\mathcal{C}$ is sometimes denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

Example 2.2.3. *Let $\mathcal{C}$ be a locally small category. Then for each object C of $\mathcal{C}$ we define* covariant hom functor *by*

$$\mathrm{Hom}(A, -) : \mathcal{C} \longrightarrow Set$$

which defined on objects by $C \longrightarrow \mathrm{Hom}(A, C)$ and on morphisms by $(f : C \longrightarrow C') \longrightarrow f^* : \mathrm{Hom}(A, C) \longrightarrow \mathrm{Hom}(A, C')$ where f^* is defined pointwise by $f^*(\varphi) = f\varphi$.

Definition 2.2.3. [11, Definition 1.8.1] Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories and $F : \mathcal{A} \longrightarrow \mathcal{B}, G : \mathcal{B} \longrightarrow \mathcal{C}$ be functors. Then the *composite functor GF* is defined as follows

$$\begin{aligned} GF \colon \mathcal{A} &\longrightarrow \mathcal{C} \\ A &\longmapsto G(F(A)) \\ (f : A \longrightarrow B) &\longmapsto G(F(f)) \in \mathrm{Hom}_{\mathcal{C}}(G(F(A)), G(F(B))). \end{aligned}$$

Now we can see functors as morphisms of categories and since for any category $\mathcal{C}$ the identity functor $1_{\mathcal{C}}$ exists, we may construct the category of all categories Cat. Unfortunately, to avoid paradoxes like Russell's paradox we will have to restrict objects of Cat to small categories, since otherwise Cat would have to be an object of itself.

Example 2.2.4. *Category of all small categories Cat, can be described as follows*

- *Objects: All small categories $\mathcal{C}$.*

- *Morphisms: Functors* $F : \mathcal{C} \longrightarrow \mathcal{C}'$.

Definition 2.2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is said to be an *isomorphism* if there exists a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$ hold.

Definition 2.2.5. A functor which simply "forgets" some or all of the structure on objects is called a *forgetful functor*.

Example 2.2.5. *The functor* $U : Grp \longrightarrow Set$ *mapping each group to its underlying set of elements and group homomorphisms to themselves as just functions is a forgetful functor, "forgetting" the algebraic group structure on objects of* Grp.

Definition 2.2.6. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between categories $\mathcal{C}$ and $\mathcal{D}$. F is then said to be

- *full* if for all objects A, B of $\mathcal{C}$ and every morphism $g : F(A) \longrightarrow F(B)$ there exists a morphism $f : A \longrightarrow B$ such that $F(f) = g$.
- *faithful* if for all objects A, B and morphisms $f_1, f_2 : A \longrightarrow B$, $F(f_1) = F(f_2)$ implies $f_1 = f_2$.

F is called a *fully faithful* functor if it is both full and faithful.

Definition 2.2.7. A category $\mathcal{C}$ is said to be *concrete* if there exits a faithful functor $F : \mathcal{C} \longrightarrow Set$.

Definition 2.2.8. The *product category* $\mathcal{C} \times \mathcal{D}$ of categories $\mathcal{C}$ and $\mathcal{D}$ is defined as follows

- Objects: All pairs (C, D) where $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$ and $D \in \mathcal{O}\mathrm{b}(\mathcal{D})$
- Morphisms: All pairs (f, g) with $f \in \mathrm{Mor}(\mathcal{C})$ and $g \in \mathrm{Mor}(\mathcal{D})$.

Then the composition of morphisms $(f, g), (f', g')$ in $\mathcal{C} \times \mathcal{D}$ is then defined by

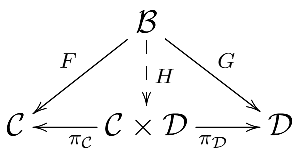
$$(f', g')(f, g) = (f'f, g'g)$$

provided compositions $f'f, g'g$ exist.

We also define the *projection functors* $\pi_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{C}$ and $\pi_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{D}$ on objects (C, D) and on morphisms (f, g) by

$$\pi_{\mathcal{C}}(C, D) = C, \quad \pi_{\mathcal{C}}(f, g) = f$$
$$\pi_{\mathcal{D}}(C, D) = D, \quad \pi_{\mathcal{D}}(f, g) = g$$

Furthermore, projection functors satisfy the following property. Let $\mathcal{B}$ be a category. For any pair of functors $F : \mathcal{B} \longrightarrow \mathcal{C}$ and $G : \mathcal{B} \longrightarrow \mathcal{D}$, there exists a functor $H : \mathcal{B} \longrightarrow \mathcal{C} \times \mathcal{D}$ such that $\pi_{\mathcal{C}} H = F$ and $\pi_{\mathcal{D}} H = G$ hold.



Definition 2.2.9. [11] Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ and $G : \mathcal{D} \longrightarrow \mathcal{D}'$ be two functors. We define the *product functor* $F \times G : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{C}' \times \mathcal{D}$ as

- On objects: If (C, D) is an object of $\mathcal{C} \times \mathcal{D}$ then $F \times G(C, D) = (F(C), G(D))$,
- On morphisms: If (f, g) is a morphisms of $\mathcal{C} \times \mathcal{D}$ then $F \times G(f, g) = (F(f), G(g))$.

Furthermore, composition of product functors is defined by

$$(GG')(F \times F') = (GF) \times (G'F').$$

Definition 2.2.10. Functors whose domain is a product category are called bifunctors.

Definition 2.2.11. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. F is said to *preserve* the property P if the image of a morphism (or an object or a diagram) under $F : \mathcal{A} \longrightarrow \mathcal{B}$ which has the property P in $\mathcal{A}$ has also the same property P in $\mathcal{B}$.

Definition 2.2.12. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. F is said to *reflect* the property P if whenever the image of a morphism (or an object or a diagram) of $\mathcal{A}$ under $F : \mathcal{A} \longrightarrow \mathcal{B}$ has a property P in $\mathcal{B}$, already has that property P in $\mathcal{A}$.

2.3 Natural transformations

Definition 2.3.1. Let $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ be two functors from category $\mathcal{A}$ to category $\mathcal{B}$. Suppose that for every object $A \in \mathcal{A}$ we have a morphism $\eta_A : F(A) \longrightarrow G(A)$ in $\mathcal{B}$ such that for every morphism $\alpha : A \longrightarrow A'$ in $\mathcal{A}$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_N} & G(A) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow{\eta_{A'}} & G(A') \end{array}$$

is commutative. Then we call η a *natural transformation* from F to G and write $\eta : F \longrightarrow G$.

Example 2.3.1. *For any functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$, $1_F : F \longrightarrow F$ is a natural transformation.*

2.4 Special morphisms

Definition 2.4.1. Let $f : A \longrightarrow B$ be a morphism in a category $\mathcal{C}$. Then f is said to be

- *monomorphism* (or *monic*) if $fg_1 = fg_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : C \longrightarrow A$ where C is an arbitrary object.
- *epimorphism* (or *epic*) if $g_1 f = g_2 f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : A \longrightarrow D$ where D is arbitrary.
- *split monomorphism* (or *section*) if there exists a morphism $g : B \longrightarrow A$ such that $gf = 1_A$.
- *split epimorphism* (or *retraction*) if there exists a morphism $g : B \longrightarrow A$ such that $fg = 1_B$.

Example 2.4.1. *In Set, monomorphisms coincide exactly with injective functions: let $f : X \longrightarrow Y$ in Set be injective. Then for any $g_1, g_2 : Z \longrightarrow X$ we have*

$$f(g_1(z)) = f(g_2(z)) \Rightarrow g_1(z) = g_2(z)$$

for all $z \in Z$. Conversely, given $f(a) = f(b)$, we can choose $g_1, g_2 : Z \longrightarrow X$ to be defined by $g_1(z) = a$ for all $z \in Z$ and $g_2(z) = b$ for all $z \in Z$, hence

$$f(g_1(z)) = f(a) = f(b) = f(g_2(z)) \Rightarrow a = g_1(z) = g_2(z) = b.$$

Conversely, let $f : A \longrightarrow B$ be a monomorphism and let $f(x) = f(y)$. Define $g_1, g_2 : C \longrightarrow A$ by $g_1(c) = x$ and $g_2(c) = y$ for all $c \in C$. Then $fg_1 = fg_2$ and hence $x = g_1(c) = g_2(c) = y$.
Similarly, epimorphisms coincide with surjective functions.

Example 2.4.2. *Let Fld be the category of all fields with field homomorphisms as morphisms. Then, every nonzero morphism is a monomorphism since otherwise, kernel of it would be non-trivial, contradicting the fact that only nontrivial ideal of a field is itself.*

Definition 2.4.2. A morphism $f : A \longrightarrow B$ between two objects A, B is said to be *isomorphism* if there exists a morphism $g : B \longrightarrow A$ such that $fg = 1_A$ and $gf = 1_B$. The morphism g is then called the *inverse* of f and it is denoted by $g = f^{-1}$.

Proposition 2.4.3. *Let g_1, g_2 be inverses of a morphism $f : A \longrightarrow B$ in a category $\mathcal{A}$. Then $g_1 = g_2$.*

Proof. Follows simply by $g_1 = 1_A g_1 = g_2 f g_1 = g_2 1_B = g_2$. □

Proposition 2.4.4. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Then F preserves isomorphisms, sections and retractions. That is if $f : A \longrightarrow B$ is*

(1) *an isomorphism, then $F(f)$ is an isomorphism in $\mathcal{D}$.*

(2) *a section, then $F(f)$ is a section in $\mathcal{D}$.*

(3) *a retraction, then* $F(f)$ *is a retraction in* $\mathcal{D}$.

Proof. (1) Let $f : A \longrightarrow B$ be an isomorphism. Let $g : B \longrightarrow A$ be the inverse of f. Then

$$1_{F(A)} = F(1_A) = F(gf) = F(g)F(f),$$

$$1_{F(B)} = F(1_B) = F(fg) = F(f)F(g).$$

Hence $F(f)$ is an isomorphism.

(2) Let $f : A \longrightarrow B$ be a section. Then there exists a morphism $g : B \longrightarrow A$ such that $gf = 1_A$. Hence

$$1_{F(A)} = F(1_A) = F(gf) = F(g)F(f).$$

(3) Let $f : A \longrightarrow B$ be a retraction. Then there exists a morphism $g : B \longrightarrow A$ such that $fg = 1_B$. Therefore

$$1_{F(B)} = F(1_B) = F(fg) = F(f)F(g).$$

□

Proposition 2.4.5. *Let* $\mathcal{C}$ *be a category.*

(1) *If* f *and* g *are monomorphisms (respectively epimorphisms), then* fg *is a monomorphism (respectively an epimorphism).*

(2) *If* fg *is a monomorphism (respectively an epimorphism), then* f *(*g*) is a monomorphism (respectively an epimorphism)*

Proof. (1) Suppose f and g are monomorphisms. If $(fg)h = (fg)k$ for two morphisms h, k in $\mathcal{C}$, then $f(gh) = (fg)h = (fg)k = f(gk)$. Since f is left cancellable, $gh = gk$ and since g is left cancellable, $h = k$. So fg is a monomorphism.

Suppose now f and g are epimorphism. If $h(fg) = k(fg)$ for two morphisms h, k in $\mathcal{C}$, then

$(hf)g = h(fg) = k(fg) = (kf)g$. Since g is right cancellable, $hf = kf$ and since f is right cancellable, $h = k$. So fg is an epimorphism.

(2) Suppose that fg is monic. Then fg is left cancellable. Let $gh = gk$ for two morphisms h, k in $\mathcal{C}$. Suppose $f(gh) = f(gk)$. So $(fg)h = f(gh) = f(gk) = (fg)k$. Since fg is left cancellable, $h = k$. Thus g is monic.

Suppose now fg is epic. Then fg is right cancellable. Let $hf = kf$ for two morphisms h, k in $\mathcal{C}$. Suppose $(hf)g = (kf)g$. So $h(fg) = (hf)g = (kf)g = k(fg)$. Since fg is right cancellable, $h = k$. Thus f is epic. □

Definition 2.4.6. Let $\mathcal{C}$ be a category.

- An object T of $\mathcal{C}$ is said to be *terminal* if for each object C there exists exactly one morphism $f : C \longrightarrow T$ with codomain T.
- An object I of $\mathcal{C}$ is said to be *initial* if for each object C there exists exactly one morphism $f : I \longrightarrow C$ with domain I.
- An object Z of $\mathcal{C}$ is a *zero object* if it is both initial and terminal, that is, given any two objects A, B there exists exactly one morphism $f : Z \longrightarrow A$ with domain Z and exactly one morphism $g : B \longrightarrow Z$ with codomain Z.

  Hence, for any two objects A, B there exists a morphism through the zero object between them, namely given by fg, called the *zero morphism* from B to A.

Remark 2.4.7. Initial, terminal and hence zero objects of a category $\mathcal{C}$ are unique up to a isomorphism.

Example 2.4.3. *Let T be a set with exactly one element. Since for any set X in Set there exists one and only one function $f : X \longrightarrow T$ mapping every element of X to the single element of T, T is an initial object in Set. On the other hand, for any set X we can write a function $f : \emptyset \longrightarrow X$, so $\emptyset$ is an initial object in Set. In fact, it is the only initial object since for any other initial object Y there would have to be a morphism $g : Y \longrightarrow \emptyset$ from Y to $\emptyset$.*

Example 2.4.4. *In Top the terminal object is point-space and the initial object is empty space.*

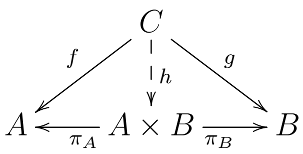
Example 2.4.5. *Trivial group $G = \{0\}$ is a zero object in Grp.*

2.5 Products and Coproducts

Definition 2.5.1. Let $\mathcal{C}$ be a category and A, B be objects of $\mathcal{C}$. The *product* of A and B is an object $A \times B$ of $\mathcal{C}$ equipped with morphisms

$$\pi_A : A \times B \longrightarrow A \quad \pi_B : A \times B \longrightarrow B$$

satisfying the following universal property: for any object C of $\mathcal{C}$ with morphisms $f : C \longrightarrow A$ and $g : C \longrightarrow B$ there exists a morphism $h : C \longrightarrow A \times B$ such that $f = \pi_A h$ and $g = \pi_B h$, i.e. the following diagram commutes.



Example 2.5.1. *In Set, for any pair of objects A, B we can create cartesian product $A \times B$ defined by*

$$A \times B = \{(a, b) | a \in A, b \in B\},$$

and projection functions by

$$\pi_A : A \times B \longrightarrow A \quad \pi_A(a, b) = a$$
$$\pi_B : A \times B \longrightarrow B \quad \pi_B(a, b) = b.$$

Then given any set C with functions $f : C \longrightarrow A$ and $g : C \longrightarrow B$, we can define $h : C \longrightarrow A \times B$ by $h(a, b) = (f(a), g(b))$, which satisfies the required universal property. So Set has products.

Example 2.5.2. *Let $(R, +_R, \cdot_R)$ and $(S, +_S, \cdot_S)$ be two rings in $Ring$. We can form the product ring of R and S to be the ring*

$$(R \times S, +, \cdot) = \{(r, s) | r \in R, s \in S\}$$

where for all pairs $(r_1, s_1), (r_2, s_2)$ in $R \times S$ the ring operations defined as

- $(r_1, s_1) + (r_2, s_2) = (r_1 +_R r_2, s_1 +_S s_2)$
- $(r_1, s_1) \cdot (r_2, s_2) = (r_1 \cdot_R r_2, s_1 \cdot_S s_2)$.

This product satisfies the required universal property, hence $Ring$ has products.

Proposition 2.5.2. *[9, Proposition 1, p.73] Let $\mathcal{C}$ be a category with a terminal object T and let the product object $A \times B$ for each pair of objects A and B exists in $\mathcal{C}$. Then*

(i) *$\mathcal{C}$ has finite products.*

(ii) *There exists a bifunctor $\Pi : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ such that $\Pi(A, B) = A \times B$.*

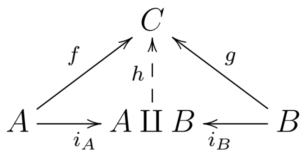
(iii) *For any three objects A, B and C of $\mathcal{C}$, we have the isomorphism*

$$(A \times B) \times C \cong A \times (B \times C).$$

Definition 2.5.3. Let $\mathcal{C}$ be a category and let A and B be two objects of $\mathcal{C}$. The *coproduct* of A and B is an object $A \amalg B$ which is equipped with morphisms

$$i_A : A \longrightarrow A \amalg B \quad i_B : B \longrightarrow A \amalg B \tag{1}$$

with the following universal property: For any object C with a pair of morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow C$ there exists a unique morphism $h : A \amalg B \longrightarrow C$ such that the following diagram commutes.



Example 2.5.3. *Let A_1 and A_2 be two sets. The disjoint union $A_1 \sqcup A_2$ of A_1 and A_2 is defined by*

$$A_1 \sqcup A_2 = \bigcup_{i=1}^{2} \{(x, i) | x \in A_i\}.$$

Furthermore, let the functions $i_1 : A_1 \longrightarrow A \sqcup B$ and $i_2 : A_2 \longrightarrow A \sqcup B$ be defined as follows

$$i_1(x) = (x, 1) \quad i_2(y) = (y, 2).$$

Then the disjoint union together with the functions i_1, i_2 satisfies the required universal property, hence it is the coproduct of the category of sets Set.

Definition 2.5.4. [10, Chapter I, p.24] Let $\{(A_i)\}_{i\in I}$ be a family of objects in an arbitrary category $\mathcal{A}$. A *product* for the family of morphisms $\{p_i : A \longrightarrow A_i\}_{i\in I}$, called *projections*, with the property that for any family $\{\alpha_i : A' \longrightarrow A_i\}_{i\in I}$ there is a unique morphism $\alpha : A' \longrightarrow A$ such that $p_i\alpha = \alpha_i$ for all $i \in I$. The object A will be denoted by $\prod_{i\in I} A_i$.

Definition 2.5.5. [10, Chapter I, p.26] The *coproduct* of the family $\{(A_i)\}_{i\in I}$ in an arbitrary category $\mathcal{A}$ is defined dually to the product. Thus the coproduct is a family of morphisms $\{u_i : A_i \longrightarrow A\}_{i\in I}$, called *injections*, such that for each family of morphisms $\{\alpha_i : A_i \longrightarrow A'\}_{i\in I}$ we have a unique morphism $\alpha : A \longrightarrow A'$ with $\alpha u_i = \alpha_i$ for all $i \in I$. The object A will be denoted by $\bigoplus_{i\in I} A_i$.

Proposition 2.5.6. *[9, 3.5.1] Let $\mathcal{C}$ be a category with an initial object I and let the coproduct object $A \amalg B$ for each pair of objects A and B exists in $\mathcal{C}$. Then*

(i) *$\mathcal{C}$ has finite coproducts.*

(ii) *There exists a bifunctor* $\amalg : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ *such that* $\amalg(A, B) = A \amalg B$.

(iii) *For any three objects* A, B *and* C *of* $\mathcal{C}$*, we have the isomorphism*

$$(A \amalg B) \amalg C \cong A \amalg (B \amalg C).$$

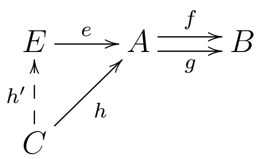
More generally, let $\mathcal{C}$ be a category with a zero object. Then for any pair A, B of objects there exists a zero-morphism $0 : A \longrightarrow B$. Additionally if finite products and finite coproducts exist in $\mathcal{C}$, then there exists a canonical morphism

$$f : A_1 \amalg A_2 \amalg ... \amalg A_n \longrightarrow A_1 \times A_2 \times ... \times A_n$$

of the coproduct to the product.
In R-Mod and Ab, finite products and finite coproducts of a family of objects $\{A_i\}$, where $i = 1, \ldots, n$, are isomorphic to each other and called the *direct sum*, denoted by $\bigoplus_{i=1}^{n} A_i$.

2.6 Kernels and Cokernels

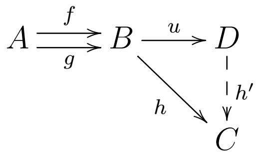
Definition 2.6.1. [9] Let $\mathcal{C}$ be a category and let $f, g : A \longrightarrow B$ be a pair of morphisms in $\mathcal{C}$. The *equalizer* of f and g is then defined as a pair $(E, e : E \longrightarrow A)$ such that $fe = ge$ and the following property is satisfied: For any other morphism $h : C \longrightarrow A$ satisfying $fh = gh$, there exists a unique morphism $h' : C \longrightarrow E$ such that $h = eh'$, i.e. the following diagram commutes.



Proposition 2.6.2. *[9, 3.4] Let* $\mathcal{C}$ *be a category. If* $e : C \longrightarrow A$ *is an equalizer for a pair of morphisms* $f, g : A \longrightarrow B$*, then* e *is monic.*

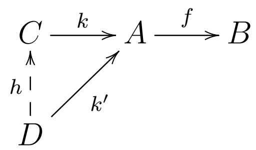
Definition 2.6.3. [9] Let $\mathcal{C}$ be a category and let $f, g : A \longrightarrow B$ be a pair of morphisms in $\mathcal{C}$. The *coequalizer* of f and g is then defined as a pair $(D, u : B \longrightarrow D)$ such that $uf = ug$ holds, and the following property is satisfied: For any other morphism $h : B \longrightarrow C$

satisfying $uf = ug$, there exists a unique morphism $h' : D \longrightarrow C$ such that $h = h'u$ holds, i.e., following diagram commutes.

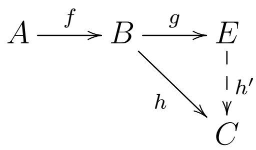


Proposition 2.6.4. *[9, 3.4] Let $\mathcal{C}$ be a category. If $u : A \longrightarrow D$ is a coequalizer for a pair of morphisms $f, g : A \longrightarrow B$, then e is an epimorphism.*

Definition 2.6.5. [9] Let $\mathcal{C}$ be a category with a zero object Z. The *kernel* of a morphism $f : A \longrightarrow B$ is defined as equalizer of the morphisms $f, 0 : A \longrightarrow B$. To put more directly, kernel of f is a morphism $k : C \longrightarrow A$ such that $fk = 0$ holds and the following property is satisfied: For any other morphism $k' : D \longrightarrow A$ with $fk' = 0$, there exists a morphism $h : D \longrightarrow C$ such that $k' = kh$. Visually, the following diagram commutes.



Definition 2.6.6. We define the *cokernel* of a morphism $f : A \longrightarrow B$ as the coequalizer of morphisms $f, 0 : A \longrightarrow B$, i.e., a morphism $g : B \longrightarrow C$ such that $gf = 0$ and for any other morphism $h : B \longrightarrow D$ satisfying $hf = 0$, there exists a unique morphism $h' : E \longrightarrow C$ such that $h = h'g$.



Example 2.6.1. *In Grp, the group I with exactly one element is a zero-object and for any two groups G, H the zero-morphism $0 : G \longrightarrow H$ is the homomorphism sending each element of G to the identity element of H. Then the kernel of an arbitrary morphism $f : G \longrightarrow H$ is the inclusion map $i : \ker(f) \longrightarrow G$ of the usual kernel $\ker(f)$ of f, defined by $i(g) = g$.*

Example 2.6.2. *In any Ab-category $\mathcal{C}$, all equalizers are kernels since for any pair of parallel morphisms $f, g : A \longrightarrow B$ with equalizer e, abelian group structure on $\mathrm{Hom}(A, B)$ gives*

$$fe = ge \Rightarrow (f - g)e = 0,$$

implying e is a kernel of $f - g$.

2.7 Exact sequences

Definition 2.7.1. [10, 1.15] A category $\mathcal{C}$ is called *normal* (*conormal*) if every monomorphism (epimorphism) $f : A \longrightarrow B$ in $\mathcal{C}$ is a kernel (cokernel) of some morphism. A normal and conormal category $\mathcal{C}$ is called *exact* if every morphism $f : A \longrightarrow B$ can be written as a composition $f = vq$ where $q : A \longrightarrow I$ is an epimorphism and $v : I \longrightarrow B$ is a monomorphism.

Furthermore, if this factorization exists, then $q = \ker(\mathrm{coker}(f))$ and $v = \mathrm{coker}(\ker(f))$ and image and coimage of f are defined respectively by $\mathrm{Im}(f) = v$ and $\mathrm{Coim}(f) = q$.

Definition 2.7.2. A composable pair of morphisms

$$A \xrightarrow{f} I \xrightarrow{g} B$$

is said to be *exact* at B if $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Definition 2.7.3. An *exact sequence* is a sequence

$$\dots \longrightarrow A_{-1} \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots$$

of objects A_i such that (f_{i-1}, f_i) is exact at A_i for all i.

Proposition 2.7.4. *[10, Proposition 15.1] Let $\mathcal{C}$ be an exact category. Then following properties hold.*

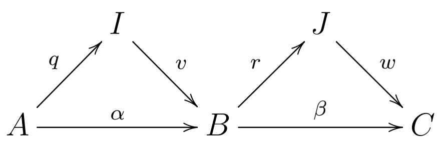
(i) $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ *is exact if and only if* $A \xleftarrow[\alpha]{} B \xleftarrow[\beta]{} C$ *is exact in* $\mathcal{C}^{op}$

(ii) $0 \longrightarrow A \xrightarrow{\alpha} B$ *is exact if and only if* α *is a monomorphism.*

(iii) $A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if* α *is an epimorphism.*

(iv $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if* α *is an isomorphism.*

Proof. (i) Since $\mathcal{A}$ is exact, factorizations $\alpha = vq$ and $\beta = wr$ where $\mathrm{Im}(\alpha) = v : A \longrightarrow I$ and $\mathrm{Im}(\beta) = w : J \longrightarrow C$ exist. Then $r = \mathrm{Coim}(\beta)$.



If $A \longrightarrow B \longrightarrow C$ is exact, then $v = \ker(\beta)$ and hence $v = \ker(r)$. Therefore $r = \mathrm{coker}(v)$ and hence $r = \mathrm{coker}(\alpha)$. We then have in dual category that $\ker(\alpha) = v = \mathrm{Im}(\beta)$, and $A \xleftarrow[\alpha]{} B \xleftarrow[\beta]{} C$ is exact.
(ii) Let α be a monomorphism. Then $\mathrm{Ker}(\alpha) = 0$ and $0 \longrightarrow A \xrightarrow{\alpha} B$ is exact.
Now let $0 \longrightarrow A \xrightarrow{\alpha} B$ be exact and let $A \xrightarrow{q} I \xrightarrow{v} B$ be a factorization of α where v is a monomorphism and q is an epimorphism. Since $q = \mathrm{coker}(\ker(\alpha))$ and $\ker(\alpha) = 0$, q must be an isomorphism. Hence $\alpha = vq$ is a monomorphism.
(iii) Follows from (i) and (ii).
(iv) Any normal category is *balanced*, i.e., every morphism that is both epic and monic is an isomorphism. Hence (iv) follows from (ii) and (iii). □

Definition 2.7.5. [10, Chapter I, p.19] Let $\mathcal{C}$ be an exact category. A sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is said to be *short exact sequence* if and only if α is a monomorphism, β is an epimorphism and $\alpha = \ker(\beta)$ equivalently $\beta = \mathrm{coker}(\alpha)$.

Definition 2.7.6. [10, Chapter I, p.32] Let $\mathcal{A}$ be an exact category. Then a short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

splits if β is a retraction.

2.8 Pullbacks and Pushouts

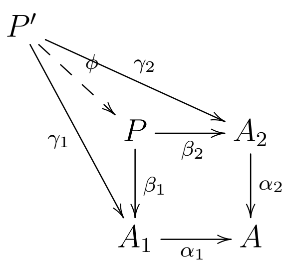
Definition 2.8.1. Given two morphisms $\alpha_1 : A_1 \longrightarrow A$ and $\alpha_2 : A_2 \longrightarrow A$ with common codomain, a *pullback* is a commutative square

$$\begin{array}{ccc} P & \xrightarrow{\beta_2} & A_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ A_1 & \xrightarrow[\alpha_1]{} & A \end{array}$$

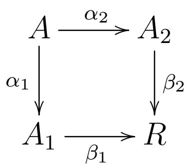
such that for any other commutative square

$$\begin{array}{ccc} P' & \xrightarrow{\gamma_2} & A_2 \\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ A_1 & \xrightarrow[\alpha_1]{} & A \end{array}$$

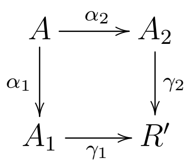
there exists a unique morphism $\phi : P' \longrightarrow P$ such that $\gamma_1 = \beta_1\phi$ and $\gamma_2 = \beta_2\phi$.



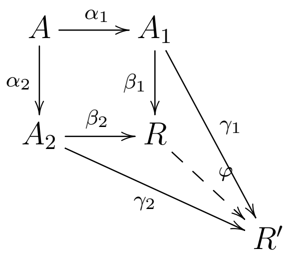
Definition 2.8.2. Given two morphisms $\alpha_1 : A \longrightarrow A_1$ and $\alpha_2 : A \longrightarrow A_2$ with common domain, a *pushout* is a commutative square



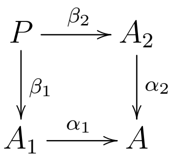
such that for any other commutative square



there exists a unique morphism $\varphi : R \longrightarrow R'$ such that $\gamma_1 = \varphi\beta_1$ and $\gamma_2 = \varphi\beta_2$.



Proposition 2.8.3. *[10, Proposition 7.1] In the pullback diagram*



if α_1 is a monomorphism, then β_2 is also a monomorphism.

Proof. Suppose that $\beta_2 f = \beta_2 g$. Then $\alpha_1\beta_1 f = \alpha_2\beta_2 f = \alpha_2\beta_2 g = \alpha_1\beta_1 g$, and we have $\beta_1 f = \beta_1 g$ since α_1 is a monomorphism. Uniqueness of factorizations through the pullback then gives $f = g$. □

Proposition 2.8.4. *Given pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha_2} & A_2 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_2} \\ A_1 & \xrightarrow{\beta_1} & R \end{array}$$

if α_1 is an epimorphism, then β_2 is also an epimorphism.

Proof. Let $f\beta_2 = g\beta_2$. Then $f\beta_1\alpha_1 = f\beta_2\alpha_2 = g\beta_2\alpha_2 = g\beta_1\alpha_1$, and we have $f\beta_1 = g\beta_1$ since α_1 is an epimorphism. Hence by the uniqueness of factorizations through pushout we get $f = g$. □

Proposition 2.8.5. *[10, Proposition 7.2] If each of the squares in the diagram*

$$\begin{array}{ccccc} P & \longrightarrow & Q & \longrightarrow & B' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & I & \longrightarrow & B \end{array}$$

is a pullback and $B' \longrightarrow B$ is a monomorphism then the outer rectangle is also a pullback.

Proof. Let X be an object and let $X \longrightarrow B'$ and $X \longrightarrow A$ be two morphisms such that $X \longrightarrow B' \longrightarrow B = X \longrightarrow A \longrightarrow I \longrightarrow B$. Then, since right-hand square is a pullback, we have a morphism $X \longrightarrow Q$ such that $X \longrightarrow B' = X \longrightarrow Q \longrightarrow B'$ and $X \longrightarrow I = X \longrightarrow Q \longrightarrow I$. Now since the outer rectangle is commutative by our assumption, it follows that $X \longrightarrow Q \longrightarrow I = X \longrightarrow A \longrightarrow I$. Hence using the fact that left-hand square is a pullback, we have a morphism $X \longrightarrow P$ with $X \longrightarrow Q = X \longrightarrow P \longrightarrow Q$ and $X \longrightarrow A = X \longrightarrow P \longrightarrow A$. Then we see that $P \longrightarrow A$ is a monomorphism by Proposition 2.8.3, and therefore the morphism $X \longrightarrow P$ is unique. □

Proposition 2.8.6. *If each square in the diagram*

$$\begin{array}{ccccc} A & \longrightarrow & I & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & I & \longrightarrow & Q \end{array}$$

is a pushout and $A \longrightarrow A'$ is an epimorphism then the outer rectangle is a pushout.

Proof. Let X be an object with two morphisms $B \longrightarrow X$ and $A' \longrightarrow X$ such that $A \longrightarrow B \longrightarrow X = A \longrightarrow A' \longrightarrow X$. Since the right-hand square is a pushout, there exists a morphism $Q \longrightarrow X$ such that $B \longrightarrow X = B \longrightarrow Q \longrightarrow X$ and $P \longrightarrow X = P \longrightarrow Q \longrightarrow X$. Left-hand square is also a pushout, so we have a morphism $P \longrightarrow X$ such that $I \longrightarrow\longrightarrow B \longrightarrow X = I \longrightarrow P \longrightarrow X$ and $A' \longrightarrow P \longrightarrow X = A' \longrightarrow X$. Hence the morphism $Q \longrightarrow X$ satisfies the required conditions, and by Proposition 2.8.4 we see that $B \longrightarrow Q$ is an epimorphism and uniqueness follows. $\square$

Proposition 2.8.7. *[10, Proposition 13.1] Consider a commutative diagram*

$$\begin{array}{ccccc} K & \xrightarrow{\gamma} & P & \xrightarrow{\beta_2} & A_2 \\ \| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \alpha_2} \\ K & \xrightarrow{u} & A_1 & \xrightarrow{\alpha_1} & A \end{array}$$

where the right-hand square is a pullback, $u = \ker(\alpha_1)$ and γ is the morphism into the pullback induced by two morphisms $u : K \longrightarrow A_1$ and $0 : K \longrightarrow A_2$. Then $\gamma = \ker(\beta_2)$.

Proof. Since $u = \beta_1\gamma$ and u is a monomorphism γ must be a monomorphism. Also $\beta_2\gamma = 0$ by the construction of γ. Let $v : X \longrightarrow P$ such that $\beta_2 v = 0$. Then $0 = \alpha_2\beta_2 v = \alpha_1\beta_1 v$ and since u is the kernel of α_1, we have a morphism $w : X \longrightarrow K$ such that $vw = \beta_1 v$. Hence $\gamma w = v$, since each of these homomorphisms give the same thing when composed with β_1 and β_2. This proves that $\gamma = \ker(\beta_2)$. $\square$

Proposition 2.8.8. *[10, Proposition 13.2] Consider the following diagram*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

where $B' \longrightarrow B$ is the kernel of some morphism $B \longrightarrow B''$. Then the diagram can be extended to a pullback if and only if $A' \longrightarrow A$ is the kernel of the composition $A \longrightarrow B \longrightarrow B''$.

Proof. ($\Rightarrow$) Suppose there is a morphism $A' \longrightarrow B'$ such that $A' \longrightarrow A \longrightarrow B = A' \longrightarrow B' \longrightarrow B$. If there is another A'' and there are morphisms $A'' \longrightarrow A$ and $A'' \longrightarrow B'$ such that $A'' \longrightarrow A \longrightarrow B = A'' \longrightarrow B' \longrightarrow B$, then there is a unique morphism $A'' \longrightarrow A'$ such that $A'' \longrightarrow A' \longrightarrow B' = A'' \longrightarrow B'$ and $A'' \longrightarrow A' \longrightarrow A = X \longrightarrow A'' \longrightarrow A$. Since u is the kernel of $B \longrightarrow B''$, $B' \longrightarrow B \longrightarrow B''$ is zero. $A' \longrightarrow A \longrightarrow B \longrightarrow B' = A' \longrightarrow B' \longrightarrow B \longrightarrow B''$ is zero and $A'' \longrightarrow A \longrightarrow B \longrightarrow B'' = A'' \longrightarrow B' \longrightarrow B \longrightarrow B''$ is zero. So there is a unique morphism $A'' \longrightarrow A$ such that $A'' \longrightarrow A' \longrightarrow A = A'' \longrightarrow A$. Hence $A' \longrightarrow A$ is the kernel of $A \longrightarrow B \longrightarrow B''$.
($\Leftarrow$) Suppose that $A' \longrightarrow A$ is the kernel $A \longrightarrow B \longrightarrow B''$. Then $A' \longrightarrow A \longrightarrow B \longrightarrow B''$ is zero. Since $B' \longrightarrow B$ is the kernel of $B \longrightarrow B''$, there is a unique morphism $A' \longrightarrow B'$ such that $A' \longrightarrow B' \longrightarrow B = A' \longrightarrow A \longrightarrow B$. Suppose that $X \longrightarrow A \longrightarrow B = X \longrightarrow B' \longrightarrow B$. Then $X \longrightarrow A \longrightarrow B \longrightarrow B'$ is zero. Hence there is a unique morphism $X \longrightarrow A'$ such that $X \longrightarrow A' \longrightarrow A = X \longrightarrow A$. Then also $X \longrightarrow A' \longrightarrow B' \longrightarrow B = X \longrightarrow A' \longrightarrow A \longrightarrow B = X \longrightarrow B' \longrightarrow B$ and since $B' \longrightarrow B$ is a monomorphism $X \longrightarrow A' \longrightarrow B' = X \longrightarrow B'$. So the diagram can be extended to a pullback. □

Proposition 2.8.9. *Consider the diagram*

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \\ B' & \longrightarrow & C' \end{array}$$

where $B \longrightarrow C'$ is the cokernel of some morphism $A \longrightarrow G$. then the diagram can be extended to a pushout if and only if $B' \longrightarrow C'$ is the cokernel of the composition $A \longrightarrow B \longrightarrow B'$.

Proof. ($\Rightarrow$) Suppose there is a morphism $C \longrightarrow C'$ such that $B \longrightarrow C \longrightarrow C' = B \longrightarrow B' \longrightarrow C'$. If there is another C'' and there are morphisms $C \longrightarrow C''$ and $B' \longrightarrow C''$

such that $B \longrightarrow C \longrightarrow C'' = B \longrightarrow B' \longrightarrow C''$, then there is a unique morphism $C' \longrightarrow C''$ such that $C \longrightarrow C' \longrightarrow C'' = C \longrightarrow C''$. Since u is the cokernel of $A \longrightarrow B$, $A \longrightarrow B \longrightarrow C$ is zero. $A \longrightarrow B \longrightarrow B' \longrightarrow C' = A \longrightarrow B \longrightarrow C \longrightarrow C'$ is zero and $A \longrightarrow B \longrightarrow B' \longrightarrow C'' = A \longrightarrow B \longrightarrow C \longrightarrow C''$ is zero. So there is a unique $C' \longrightarrow C''$ such that $B' \longrightarrow C' \longrightarrow C'' = B' \longrightarrow C''$. Hence $B' \longrightarrow C'$ is the cokernel of $A \longrightarrow B \longrightarrow B'$.

($\Leftarrow$) Suppose that $B' \longrightarrow C'$ is the cokernel $A \longrightarrow B \longrightarrow B'$. Then $A \longrightarrow B \longrightarrow B' \longrightarrow C'$ is zero. Since $B \longrightarrow C$ is the kernel of $A \longrightarrow B$, there is a unique morphism $C \longrightarrow C'$ such that $B \longrightarrow C \longrightarrow C' = B \longrightarrow B' \longrightarrow C'$. Suppose that $B \longrightarrow C \longrightarrow C'' = B \longrightarrow B' \longrightarrow C''$. Then $B \longrightarrow B' \longrightarrow C' \longrightarrow C''$ is zero. Hence there is a unique morphism $C \longrightarrow C''$ such that $C \longrightarrow C' \longrightarrow C'' = C \longrightarrow C''$. Then also $B \longrightarrow B' \longrightarrow C' \longrightarrow C'' = B \longrightarrow C \longrightarrow C' \longrightarrow C'' = B \longrightarrow B' \longrightarrow C''$ and since $B' \longrightarrow C'$ is an epimorphism $B' \longrightarrow C' \longrightarrow C'' = B' \longrightarrow C''$. So the diagram can be extended to a pushout. □

2.9 Grothendieck categories

Definition 2.9.1. [9] A category $\mathcal{C}$ is said to be an *abelian category* if it is a preadditive category which satisfies the following properties.

(1) $\mathcal{C}$ has a zero object,

(2) $\mathcal{C}$ has biproducts for any pair of objects A, B of $\mathcal{C}$,

(3) Every morphism in $\mathcal{C}$ has a kernel and a cokernel,

(4) Every monomorphism in $\mathcal{C}$ is a kernel and every epimorphism in $\mathcal{C}$ is a cokernel.

Example 2.9.1. *The category of abelian groups Ab is an abelian category.*

Example 2.9.2. *Let R be a ring. The category of left R-modules R-Mod is an abelian category.*

(1) *It is clear that the trivial* 0*-module is a zero object in* R*-Mod.*

(2) *Products in* R*-Mod coincides with cartesian products, given with componentwise addition and* R*-action. Coproducts in* R*-Mod is given by direct sums, which is the submodule of cartesian product consisting of tuples of elements such that only finitely many are non-zero. Then products and coproducts of finitely many objects clearly coincide, hence* R*-Mod has finite biproducts.*

(3) *Kernel of a morphism* $f : M \longrightarrow N$ *in* R*-Mod is given by the canonical injection map* $i : \operatorname{Ker}(f) \longrightarrow M$ *from* $\operatorname{Ker}(f) = f^{-1}(0)$ *to* M*. Dually, cokernel of* $f : M \longrightarrow N$ *is given by the canonical projection map* $p : N \longrightarrow N/\operatorname{Im}(f)$ *from* N *to the quotient abelian group* $N/\operatorname{Im}(f)$*.*

(4) *Lastly, let* $f : M \longrightarrow N$ *be a monomorphims and let* $g : N \longrightarrow N/M$ *be its cokernel. Then, clearly,* $g \circ f = 0$ *and if* $h : L \longrightarrow N$ *is another morphism satisfying* $g \circ h = 0$*, then* $\operatorname{Im}(h) \subseteq M$*, hence we can find a morphism* $h' : L \longrightarrow M$ *such that* $f \circ h' = h$*, giving that* f *is the kernel of* g*.*

Remark 2.9.2. For every morphism $f : M \longrightarrow N$ in an abelian category $\mathcal{A}$ we have the following notation and analysis:

$$\begin{array}{ccccccc} \operatorname{Ker}(f) & \xrightarrow{\ker(f)} & M & \xrightarrow{f} & N & \xrightarrow{\operatorname{coker}(f)} & \operatorname{Coker}(f) \\ & & \downarrow{\scriptstyle \operatorname{coim}(f)} & & \uparrow{\scriptstyle \operatorname{im}(f)} & & \\ & & \operatorname{Coim}(f) & \xrightarrow[\bar{f}]{} & \operatorname{Im}(f) & & \end{array}$$

where $\bar{f}$ is an isomorphism.

Definition 2.9.3. Let $\mathcal{A}$ be an abelian category. Two exact sequences $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and $0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$ are said to be *isomorphic* if there exists a triple (φ, γ, ψ) of isomorphisms such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

Additionally we say that two short exact sequences are *equivalent* if $A' = A$ and $C' = C$ with $\varphi = 1_A$ and $\psi = 1_C$.

Lemma 2.9.4. *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be a short exact sequence in an abelian category* $\mathcal{A}$*. Then the following conditions are equivalent.*

(i) f *is a section.*

(ii) g *is a retraction.*

(iii) The sequence is equivalent to the sequence

$$0 \longrightarrow A \xrightarrow{i_A} A \oplus C \xrightarrow{p_C} C \longrightarrow 0$$

given by the direct sum and canonical injection and projection morphisms $i_A : A \longrightarrow A \oplus C$ *and* $p_C : A \oplus C \longrightarrow C$*.*

Proof. $(i) \Rightarrow (iii)$ Let f be a section. Then there exists a $h : B \longrightarrow A$ such that $hf = 1_A$. Define $P = fh$. Then $P^2 = P$. So every element $b \in B$ can be written as $b = (b - P(b)) + P(b)$ where $b - P(b) \in \mathrm{Ker}(h)$ and $P(b) \in \mathrm{Im}(f)$. Additionally, this decomposition is unique since if $b \in \mathrm{Im}(f)$ with $f(a) = b$ for some $a \in A$ and $b \in \mathrm{Ker}(h)$ in the same time, $0 = h(b) = h(f(a)) = a$. Hence $B \simeq \mathrm{Im}(f) \oplus \mathrm{Ker}(h)$ is a direct sum and $f : A \longrightarrow B$ is the canonical inclusion of $\mathrm{Im}(f)$. Since the sequence is exact, we have $\mathrm{Ker}(h) \simeq \mathrm{Im}(g)$ and $B \simeq A \oplus C$ with the canonical inclusion and projection.

$(ii) \Rightarrow (iii)$ This part can be proved dually by $(i) \Rightarrow (iii)$.

$(iii) \Rightarrow (i)$ Let $\varphi : A \oplus C \longrightarrow B$ be an isomorphism such that the following diagram with canonical injections and projections commutes.

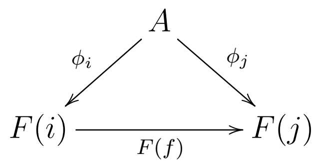
$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow 1_A & & \downarrow \varphi & & \downarrow 1_C & & \\
0 & \longrightarrow & A & \xrightarrow{i_A} & A \oplus C & \xrightarrow{p_C} & C & \longrightarrow & 0
\end{array}$$

Define $h : B \longrightarrow A$ by $h = p_A \varphi^{-1}$. Then $hf = 1_A$.

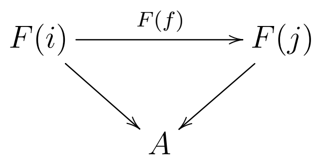
$(iii) \Rightarrow (ii)$ Clear by $(iii) \Rightarrow (i)$. $\square$

Definition 2.9.5. A short exact sequence in an abelian category satisfying either of the equivalent conditions in Lemma 2.9.4 is called a *split* short exact sequence.

Definition 2.9.6. Let $\mathcal{A}$ be a category, A an object of $\mathcal{A}$. For an arbitrary category $\mathcal{J}$, let $F : \mathcal{J} \longrightarrow \mathcal{A}$ be a functor. A family of morphisms $\phi_i : A \longrightarrow F(i), i \in \mathcal{J}$, such that for each morphism $f : i \longrightarrow j$ in $\mathcal{J}$ the following diagram commutes



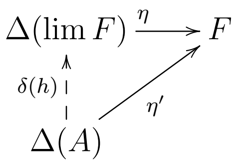
is called a *cone* with A over F and it is denoted by $Cone(A|F)$. Dually, a collection of morphisms $\varphi_i : F(i) \longrightarrow A, i \in \mathcal{J}$ such that for every morphism $f : i \longrightarrow j$ in $\mathcal{J}$ the following diagram commutes



is called a *cone* with F *over* A and it is denoted by $Cone(F|A)$.

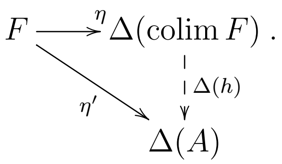
Definition 2.9.7. Let $\mathcal{A}$ and $\mathcal{J}$ be two categories. A functor $\Delta : \mathcal{A} \longrightarrow Fun(\mathcal{J}, \mathcal{A})$ which sends an object A of $\mathcal{A}$ to the functor $\Delta(A) : \mathcal{J} \longrightarrow \mathcal{A}$ which maps every object $i \in \mathcal{J}$ to A and every morphism in $\mathcal{J}$ is to the identity morphism 1_A of A. Functor Δ is called *diagonal functor*.

Definition 2.9.8. Let $F : \mathcal{J} \longrightarrow \mathcal{A}$ be a functor. The *limit* of F is an object $\lim F$ equipped with a natural transformation $\eta : \Delta(\lim F) \longrightarrow F$ such that $(\lim F, u)$ is universal from Δ to $\lim F$. This means that for any other pair $(A, \eta' : \Delta(A) \longrightarrow F)$ with η' a natural transformation and $A \in \mathcal{A}$, there exists a unique morphism $h : A \longrightarrow \lim F$ in $\mathcal{A}$ such that the following diagram is commutative.



The morphism $\eta : \Delta(\lim F) \longrightarrow F$ forms a cone with $\lim F$ over F via family of morphisms $\eta_i : \lim F \longrightarrow F(i)$ for all $i \in I$.

Definition 2.9.9. Let $F : \mathcal{J} \longrightarrow \mathcal{A}$ be a functor. The *colimit* of F is an object $\operatorname{colim} F$ equipped with a natural transformation $\eta : F \longrightarrow \Delta(\operatorname{colim} F)$ such that $(\operatorname{colim} F, \eta : F \longrightarrow \Delta(\operatorname{colim} F))$ is universal from F to Δ. This means that for any pair $(A, \eta' : F \longrightarrow \Delta(A))$ with η' a natural transformation where $A \in \mathcal{A}$, there exists a unique morphism $h : \operatorname{colim} F \longrightarrow A$ in $\mathcal{A}$ such that the following diagram is commutative.



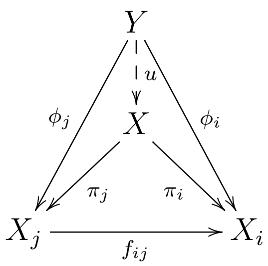
Definition 2.9.10. Let $(I, \leq)$ be a poset, let $(A_i)_{i \in I}$ be a family of objects and $f_{ij} : A_j \longrightarrow A_i$ for all $i \leq j$ be a family of morphisms. Then $((A_i)_{i \in I}, (f_{ij})_{i \leq j})$ is called an *inverse system* of objects and morphisms on I if the following properties hold.

(1) f_{ii} is the identity on A_{ii}.

(2) $f_{ik} = f_{ij} = f_{jk}$ for all $i \leq j \leq k$.

Definition 2.9.11. Let (X_i, f_{ij}) be an inverse system of objects and morphisms in a category $\mathcal{A}$. The *inverse limit* of this system is an object X, which is denoted by $\lim_{\Leftarrow} X_i$ in $\mathcal{A}$ together with morphisms $\pi_i : X \longrightarrow X_i$ satisfying $\pi_i = f_{ij}\pi_j$ for all $i \leq j$. The pair (X, π_i) must be universal in the sense that for any other such pair (Y, ϕ_i) there exists a unique morphism

$u : Y \longrightarrow X$ making the following diagram commute.



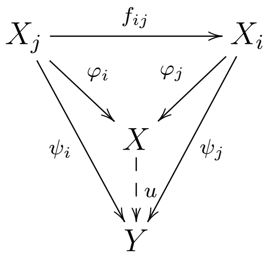
Definition 2.9.12. Let $\{A_i | i \in I\}$ be a collection of objects in a category $\mathcal{A}$ indexed by directed set $(I, \leq)$ and $f_{ij} : A_i \longrightarrow A_j$ be a morphism for all $i \leq j$ with the following properties.

(1) f_{ij} is the identity on I.

(2) $f_{ik} = f_{ij} = f_{jk}$ for all $i \leq j \leq k$.

Then the pair $((A_i), (f_{ij}))$ is called a direct system over I.

Definition 2.9.13. Let (X_i, f_{ij}) be an direct system of objects and morphisms in a category $\mathcal{A}$. The *direct limit* of this system is an object X, which is denoted by $\lim_{\Rightarrow} X_i$ in $\mathcal{A}$ together with morphisms $\varphi_i : X \longrightarrow X_i$ satisfying $\varphi_i = f_{ij}\pi_j$ for all $i \leq j$. The pair (X, φ_i) must be universal in the sense that for any other such pair (Y, ψ_i) there exists a unique morphism $v : X \longrightarrow Y$ making the following diagram commute.



Definition 2.9.14. A family of objects $\{U_i\}_{\{}i \in I\}$ is called a *family of generators* for a category $\mathcal{A}$ if for every pair of distinct morphisms $\alpha, \beta : A \longrightarrow B$ there is a morphism $u : U_i \longrightarrow A$ for some i such that $\alpha u \neq \beta u$. An object U in $\mathcal{A}$ is called a generator $\mathcal{A}$ if $\{U\}$ is a family of generators.

Definition 2.9.15. [12] A category $\mathcal{A}$ is called a *Grothendieck category* if

(1) $\mathcal{A}$ is an abelian category.

(2) Every (possibly infinite) family of objects in $\mathcal{A}$ has a coproduct.

(3) Direct limits are exact in $\mathcal{A}$.

(4) $\mathcal{A}$ has a generator, i.e. there is an object U of $\mathcal{A}$ for every object X of $\mathcal{A}$ there exists an epimorphism $U^{(I)} \longrightarrow X$, where $U^{(I)}$ denotes a coproduct copies of U.

Example 2.9.3. *[12] Let R be a ring. Then the category of right R-modules Mod-R is an abelian category.*

Definition 2.9.16. [13] Let C be a coalgebra over a field. The category of right comodules over the coalgebra C is denoted by $\mathcal{M}^{\mathcal{C}}$. The objects are all right $\mathcal{C}-$comodules and the morphisms between two objects are the morphisms of comodules. We will also denote the morphisms in $\mathcal{C}^{\mathcal{C}}$ from M to N by $\mathcal{C}om_{\mathcal{C}}(M, N)$. Similarly, the category of left $\mathcal{C}-$comodules will be denoted ${}^{\mathcal{C}}\mathcal{M}$.

Example 2.9.4. *[13, Corollary 2.2.8] The category $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category.*

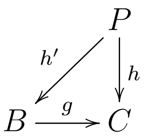
Definition 2.9.17. [14] An object M of a category $\mathcal{A}$ is said to be *finitely generated* if whenever $M = \Sigma M_i$ for a family $(M_i)_I$ of subobjects of M, there is an $i \in I$ such that $M = M_i$.

Definition 2.9.18. [14] An object M of a category $\mathcal{A}$ is said to be *finitely presented* if it is finitely generated and every epimorphism $L \longrightarrow M$ when L is finitely generated has a finitely generated kernel.

Definition 2.9.19. [14] A category $\mathcal{A}$ is said to be *locally finitely generated (presented)* if it has a family of finitely generated (presented) generators.

2.10 Projective objects

Definition 2.10.1. Let $\mathcal{A}$ be a category. An object P of $\mathcal{A}$ is said to be *projective* if every morphism $h : P \longrightarrow C$ whose domain is P factors through every epimorphism $g : B \longrightarrow C$ as $h = gh'$ for some $h' : P \longrightarrow B$.



Definition 2.10.2. A category $\mathcal{A}$ is said to have *enough projectives* if for each object A in $\mathcal{A}$ there is an epimorphism $f : P \longrightarrow A$ with P projective.

Proposition 2.10.3. *[10, Proposition 14.2] Let P be a projective object in a category $\mathcal{A}$. Then every morphism $f : A \longrightarrow P$ is a retraction. Conversely if every epimorphism $g : A \longrightarrow P$ is a retraction and if $\mathcal{A}$ either has enough projectives or is abelian, then P is projective.*

Proposition 2.10.4. *Every short exact sequence ending with a projective object P, i.e., of the form*

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

splits.

Proof. Clear by Proposition 2.10.3. □

Proposition 2.10.5. *[10, Proposition 14.3] If $P = \bigoplus_i P_i$ and P_i is projective for each i, then P is projective. Converse is true in categories with zero objects.*

2.11 Purity

Definition 2.11.1. [14, Definition, p.353] A short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in a Grothendieck category $\mathcal{A}$ is said to be *pure* if every finitely presented object is relatively projective to it. In this case L is a pure subobject of M.

Definition 2.11.2. [14, Definition, p.354] Also an object M of a Grothendieck category $\mathcal{A}$ is said to be *flat* if every short exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

is pure.

Definition 2.11.3. [15, p. 160] Let $\mathcal{E}$ be a class of short exact sequences of objects of an abelian category $\mathcal{A}$ such that every sequence isomorphic to a sequence in $\mathcal{E}$ is also in $\mathcal{E}$. Denote by $\mathcal{E}_e$ and $\mathcal{E}_m$ the corresponding classes of epimorphisms and monomorphisms, respectively. $\mathcal{E}$ is then called a proper class if it has the following properties.

(1) Every split short exact sequence is in $\mathcal{E}$.

(2) If $\alpha, \beta \in \mathcal{E}_m$ then $\beta\alpha \in \mathcal{E}_m$ if defined.

(3) If $\alpha, \beta \in \mathcal{E}_e$ then $\beta\alpha \in \mathcal{E}_e$ if defined.

(4) If $\beta\alpha \in \mathcal{E}_m$ then $\alpha \in \mathcal{E}_m$ if defined.

(5) If $\beta\alpha \in \mathcal{E}_e$ then $\beta \in \mathcal{E}_e$ if defined.

Lemma 2.11.4. *[14, Lemma 6 (i)] The class of pure short exact sequences $\mathcal{P}ure$ in a Grothendieck category $\mathcal{A}$ is a proper class.*

Definition 2.11.5. [16, p.313] A Grothendieck category $\mathcal{A}$ is said to be *regular* if every object M of $\mathcal{A}$ is regular in the sense that every short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is pure in $\mathcal{A}$.

Theorem 2.11.6. *[14, Theorem 4] Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following statements are equivalent.*

(i) $\mathcal{A}$ is regular.

(ii) All objects are flat.

(iii) All short exact sequences are pure.

(iv) All finitely presented objects are projective.

2.12 Pure-projective objects

Definition 2.12.1. An object M of a Grothendieck category $\mathcal{A}$ is called *pure-projective* if it is projective relative to all pure exact sequences.

Proposition 2.12.2. *[17, Proposition 4.1] Let $M = \bigoplus M_i$ in a Grothendieck category $\mathcal{A}$. Then M is pure-projective if and only if M_i is pure-projective for every i.*

Proposition 2.12.3. *Let $\mathcal{A}$ be a Grothendieck category. Then the followings are equivalent for an object N of $\mathcal{A}$.*

(i) N is pure-projective.

(ii) Every pure exact sequence ending with N splits.

Proof. $(i) \Rightarrow (ii)$ Let

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\alpha} N \longrightarrow 0$$

be a pure short exact sequence in $\mathcal{A}$. Since N is pure-projective, for every morphism $f : N \longrightarrow N$ there exists a morphism $g : N \longrightarrow M$ such that $\alpha g = f$. Choosing $f = 1_N$, we get $\alpha g = 1_N$. So the sequence splits.
$(ii) \Rightarrow (i)$ Let

$$0 \longrightarrow K \xrightarrow{k'} P \xrightarrow{\alpha} M \longrightarrow 0$$

be a pure exact sequence and $f : N \longrightarrow M$ a morphism. Forming a pullback, we have the following commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{k} & G & \xrightarrow{h} & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & K & \xrightarrow{k'} & P & \xrightarrow{\alpha} & M & \longrightarrow & 0 \end{array}$$

We know that $gk = k'$ is in $\mathcal{P}ure_M$. Since the class of pure exact sequences $\mathcal{P}ure$ is a proper class by [14, Lemma 6 (i)], k is also in $\mathcal{P}ure_M$. So the first row of the diagram is a pure exact sequence. Then

$$0 \longrightarrow K \xrightarrow{k} G \xrightarrow{h} N \longrightarrow 0$$

splits by assumption. So there exists a morphism $e : N \longrightarrow G$ such that $he = 1_N$. Now put $\gamma = ge$. Then we have $\alpha\gamma = \alpha ge = fhe = f$. Thus M is pure-projective. $\square$

Definition 2.12.4. The category $\mathcal{A}$ is said to have *enough pure-projectives* if for every object M of $\mathcal{A}$ there is a pure epimorphism $f : P \longrightarrow M$ with P pure-projective that is we have a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

for every object M of $\mathcal{A}$.

Lemma 2.12.5. *[14, Lemma 6 (ii)] Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then $\mathcal{A}$ has enough pure-projectives.*

Theorem 2.12.6. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the followings are equivalent.*

(i) $\mathcal{A}$ is regular.

(ii) Every pure-projective object is projective.

(iii) Every pure-projective object is flat.

Proof. $(i) \Rightarrow (ii)$ Since $\mathcal{A}$ is regular, every short exact sequence in $\mathcal{A}$ is pure by [14, Theorem 4].

$(ii) \Rightarrow (iii)$ Since every projective object is flat by [14, Lemma 7 (i)], it is clear.

$(iii) \Rightarrow (i)$ Let M be an object. Since $\mathcal{A}$ has enough pure-projective objects by [14, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. Now P is flat by assumption and therefore M is flat by [18, Proposition 2.2 (c) (ii)]. Thus $\mathcal{A}$ is regular by [14, Theorem 4]. $\square$

3. DIRECT PROJECTIVE OBJECTS IN ABELIAN CATEGORIES

3.1 Direct-projective objects

Definition 3.1.1. Let A be an abelian category. An object M of $\mathcal{A}$ is said to be *direct-projective* if every subobject N of M with M/N isomorphic to a direct summand of M is a direct summand of M.

Proposition 3.1.2. *Let $\mathcal{A}$ be an abelian category. Then the following conditions are equivalent for an object M of $\mathcal{A}$.*

(i) Given any direct summand A of M with the projection map $p : M \longrightarrow A$ and any epimorphism $f : M \longrightarrow A$ there exists an endomorphism g of M such that $fg = p$.

(ii) M is direct-projective.

(iii) Any epimorphism $f : M \longrightarrow A$ with A a direct summand of M splits.

Proof. $(i) \Rightarrow (ii)$ Let A be a direct summand of M and N be a subobject of M with M/N isomorphic to A. Let $f : M/N \longrightarrow A$ be that isomorphism. There is an endomorphism $g : M \longrightarrow M$ such that $fg = \pi$,where $\pi : M \longrightarrow A$ is the canonical projection, by assumption. Let $h = gi$ where $i : A \longrightarrow M$ is the inclusion map. Then $fh = fgi = \pi i = 1_A$. So f splits. Thus N is a direct summand of M.
$(ii) \Rightarrow (iii)$ Let A be a direct summand of M and $f : M \longrightarrow N$ be an epimorphism. Then since $M/\operatorname{Ker}(f) \cong N$, $\operatorname{Ker}(f)$ is a direct summand of M by assumption. Therefore f splits.
$(iii) \Rightarrow (i)$ Let A be a direct summand of M, $\pi : M \longrightarrow A$ be the canonical projection map and $f : M \longrightarrow A$ be an epimorphism. Since f splits by assumption, there exists a morphism $h : A \longrightarrow M$ such that $fh = 1_A$. Define $g : M \longrightarrow M$ by $g = h\pi$. Then $fg = f(h\pi) = (fh)\pi = 1_A\pi = \pi$. □

Lemma 3.1.3. *Let $\mathcal{A}$ be an abelian category. If $M \oplus N$ is direct-projective, then an exact sequence*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{g} N \longrightarrow 0$$

of objects and morphisms of $\mathcal{A}$ splits.

Proof. Let $p_1 : M \oplus N \longrightarrow M$ and $p_2 : M \oplus N \longrightarrow N$ be canonical projection maps. Since $M \oplus N$ is direct-projective and $gp_1 : M \oplus N \longrightarrow N$ is an epimorphism, there exists an endomorphism h of $M \oplus N$ such that $gp_1h = p_2$ by Proposition 3.1.2. Then choosing $f = p_1hi_2$ where $i_2 : N \longrightarrow M \oplus N$ is the inclusion map, we have $gf = 1_N$. So g splits. □

Theorem 3.1.4. *Let $\mathcal{A}$ be an abelian category with enough projective objects, M be an object of $\mathcal{A}$ and $\nu : K \longrightarrow M$ be an epimorphism from a projective object K of $\mathcal{A}$ to M. Then M is projective if and only if $K \oplus M$ is direct-projective.*

Proof. ($\Rightarrow$) Let M be projective. Then $K \oplus M$ is projective by Proposition 2.10.5 and since projective objects are direct-projective, $K \oplus M$ is direct-projective.
($\Leftarrow$) Let $K \oplus M$ be direct-projective. Then the canonical short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\nu) \longrightarrow K \xrightarrow{\nu} M \longrightarrow 0$$

splits by Lemma 3.1.3. Hence M is projective. □

Corollary 3.1.5. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then the class of direct-projective objects of $\mathcal{A}$ need not be closed under factor objects.*

Proof. Let M be an object in $\mathcal{A}$ which is not direct-projective. Since $\mathcal{A}$ has enough projective objects, there is an exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$$

with N projective. Since N is projective, it is direct-projective and $N/K \cong M$. □

Corollary 3.1.6. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then the class of direct-projective objects of $\mathcal{A}$ need not be closed under taking finite coproducts.*

Proof. Let N be a direct-projective object in $\mathcal{A}$ which is not projective. Since $\mathcal{A}$ has enough projective objects, there is an exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

with M projective. Now $M \oplus N$ is not direct-projective since N is not projective by Theorem 3.1.4. □

Corollary 3.1.7. *Let $\mathcal{A}$ be an abelian category with enough projectives. Then every direct-projective object is projective if and only if the coproduct of two direct-projective objects of $\mathcal{A}$ is direct-projective.*

Proof. ($\Rightarrow$) It is clear by Proposition 2.10.5.
($\Leftarrow$) Let M be a direct-projective object in $\mathcal{A}$. Since $\mathcal{A}$ has enough projectives, there exists a projective object P of $\mathcal{A}$ and an epimorphism $\alpha : P \longrightarrow M$. Projective objects are direct-projective, so $P \oplus M$ is direct-projective by assumption. Then M is projective by Corollary 3.1.4. □

Proposition 3.1.8. *Direct summands of direct-projective objects in an abelian category $\mathcal{A}$ are direct-projective.*

Proof. Let M be direct-projective, N be a direct summand of M, S be a direct summand of N and let $\alpha : N \longrightarrow S$ be an epimorphism. Since S is a direct summand of N and N is a direct summand of M, S is a direct summand of M. Now αp_{MN} is an epimorphism where $p_{MN} : M \longrightarrow N$ is the projection map. Since M is direct-projective, there exists an endomorphism ψ of M such that $(\alpha p_{MN})\psi = p_{MS}$ where $p_{MS} : M \longrightarrow S$ is the projection map. Then choosing $\beta = p_{MN}\psi p_{NM} : N \longrightarrow N$, $\alpha\beta = p_{NS}$. □

Definition 3.1.9. An abelian category $\mathcal{A}$ is said to be *hereditary* if and only if every subobject of a projective object is projective if and only if every quotient object of an injective object is injective. $\mathcal{A}$ is said to be *semihereditary* if every finitely generated subobject of a projective object is projective and *cosemihereditary* if every finitely cogenerated quotient object of an injective object is injective.

Theorem 3.1.10. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is (semi)hereditary.*

(ii) *Every (finitely generated) subobject of a projective object is direct-projective.*

Proof. $(i) \Rightarrow (ii)$ Clear.
$(ii) \Rightarrow (i)$ Let N be a subobject of a projective object P. Since $\mathcal{A}$ has enough projectives, there is an epimorphism $f : P_1 \longrightarrow N$ with P_1 projective. Now $P_1 \oplus N$ is a subobject of the projective object $P_1 \oplus P$ and therefore direct-projective by assumption. Then N is projective by Theorem 3.1.4. □

Definition 3.1.11. Let M and N be objects of an abelian category $\mathcal{A}$. M is said to be *N-projective* if given any epimorphism from N to an object L of $\mathcal{A}$, any homomorphism from M to L can be lifted to a homomorphism from M to N.

Definition 3.1.12. An object M of an abelian category $\mathcal{A}$ is called a *quasi-projective* object if it is M-projective.

Proposition 3.1.13. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(i) $\mathcal{A}$ is regular.

(ii) Every pure-projective object of $\mathcal{A}$ is a quasi-projective.

Proof. $(i) \Rightarrow (ii)$ Since $\mathcal{A}$ is regular, every pure-projective object of $\mathcal{A}$ is projective by Theorem 2.12.6. Since every projective object is quasi-projective, every pure-projective object of $\mathcal{A}$ is quasi-projective.

$(ii) \Rightarrow (i)$ Let M be a pure-projective object of $\mathcal{A}$. Since $\mathcal{A}$ has enough projective objects by [15, Lemma 6 (ii)], there is a projective object P and an epimorphism $f : P \longrightarrow M$. Since P is projective, it is pure-projective. So $P \oplus M$ is pure-projective. Now $P \oplus M$ is quasi-projective by assumption. Since every quasi-projective object is direct-projective, $P \oplus M$ is direct-projective. We have an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective and $P \oplus M$ is direct-projective. Thus M is projective by Theorem 3.1.4. Hence $\mathcal{A}$ is regular by Theorem 2.12.6. □

3.2 Classes all of whose objects are direct-projective

Definition 3.2.1. An abelian category $\mathcal{A}$ is called a *spectral* category if every short exact sequence in $\mathcal{A}$ splits.

Definition 3.2.2. An epimorphism $f : P \longrightarrow M$ in an abelian category $\mathcal{A}$ is called a *projective cover* of M if P is a projective object and $\operatorname{Ker}(f) \ll P$, that is for every subobject U of P with $\operatorname{Ker}(f) + U = P$ implies that $U = P$.

Definition 3.2.3. An abelian category $\mathcal{A}$ is said to be *perfect* if every object of $\mathcal{A}$ has a projective cover.

Theorem 3.2.4. *[8, Theorem 3.5] Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}$ is perfect if and only if it has enough projectives and every projective object of $\mathcal{A}$ is lifting.*

Theorem 3.2.5. *Let $\mathcal{A}$ be an abelian category. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is spectral.*

(ii) *$\mathcal{A}$ is perfect and every object of $\mathcal{A}$ is direct-projective.*

(iii) *$\mathcal{A}$ is perfect and every factor object of a direct-projective object is direct-projective.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ is clear.
$(iii) \Rightarrow (ii)$ Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough projective objects by Theorem 3.2.4, there exists an epimorphism $f : P \longrightarrow M$ with P projective. Therefore M is direct-projective being a quotient object of a direct-projective object P.
$(ii) \Rightarrow (i)$ Let M be an object of $\mathcal{A}$. Since there are enough projective objects in $\mathcal{A}$ by Theorem 3.2.4, there is a short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. Since every object of $\mathcal{A}$ is direct-projective, $P \oplus M$ is direct-projective. Then the sequence splits by Corollary 3.1.3. Therefore $\mathcal{A}$ is spectral. □

Definition 3.2.6. An object M in a category $\mathcal{A}$ with a zero object is *simple* if there are precisely two quotient objects of M, namely 0 and M.

Definition 3.2.7. A Grothendieck category $\mathcal{A}$ is said to be semisimple if each object of $\mathcal{A}$ is a coproduct of simple objects.

Proposition 3.2.8. *[12, Proposition 6.7 Chapter V] A locally finitely generated Grothendieck category is semisimple if and only if it is spectral.*

Theorem 3.2.9. *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is semisimple.*

(ii) *$\mathcal{A}$ has enough projectives and every object of $\mathcal{A}$ is direct-projective.*

(iii) *$\mathcal{A}$ has enough projectives and the coproduct of two direct-projective objects is direct-projective.*

Proof.

$(i) \Rightarrow (ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (i)$ Let S be a simple object in $\mathcal{A}$. Since $\mathcal{A}$ has enough projective objects, there exists an epimorphism $f : P \longrightarrow S$ with P projective. S is clearly quasi-projective and therefore direct-projective. So $P \oplus S$ is direct-projective by assumption. Thus S is projective by Theorem 3.1.4. Then $\mathcal{A}$ is semisimple by [16, 20.7] whose proof works for locally finitely generated Grothendieck categories. $\square$

4. PURE DIRECT PROJECTIVE OBJECTS IN GROTHENDIECK CATEGORIES

4.1 Pure-direct-projective objects

Definition 4.1.1. Let $\mathcal{A}$ be a Grothendieck category. An object M of $\mathcal{A}$ is said to be *pure-direct-projective* if every pure subobject K with M/K isomorphic to a direct summand of M is a direct summand.

Proposition 4.1.2. *[19, Proposition 3.3] Let $\mathcal{A}$ be a Grothendieck category. Then the following conditions are equivalent for an object M of $\mathcal{A}$.*

(i) Given a direct summand N of M with the projection $p : M \longrightarrow N$ and any epimorphism $f : M \longrightarrow N$ with $\operatorname{Ker}(f)$ pure in M there exists an endomorphism $g : M \longrightarrow M$ such that $fg = p$.

(ii) M is pure-direct-projective.

(iii) Any epimorphism $f : M \longrightarrow N$ with N a direct summand of M and $\operatorname{Ker}(f)$ pure in M splits.

Proof. $(i) \Rightarrow (ii)$ Let K be a pure subobject of M such that M/K is isomorphic to a direct summand N of M. Let $f : N \longrightarrow M/K$ be that isomorphism. By assumption there exists an endomorphism g of M such that $fg = p$, where $p : M \longrightarrow N$ is the projection map. Define $h : M \longrightarrow M$ by $h = gi$, $i : N \longrightarrow M$ being the inclusion map. Then $fh = f(gi) = (fg)i = i$ holds, so f splits. Thus K is a direct summand of M.
$(ii) \Rightarrow (iii)$ Let N be a direct summand of M and $f : M \longrightarrow N$ be an epimorphism with $\operatorname{Ker}(f)$ pure in M. Then, since $M/\operatorname{Ker}(f) \cong N$, $\operatorname{Ker}(f)$ is a direct summand of M and f splits.
$(iii) \Rightarrow (i)$ Let N be a direct summand of M, $p : M \longrightarrow N$ be the canonical projection map and $f : M \longrightarrow N$ be an epimorphism with $\operatorname{Ker}(f)$ pure in M. Since f splits by assumption,

there exists a morphism $h : N \longrightarrow M$ such that $fh = 1_N$. Define $g : M \longrightarrow M$ by $g = hp$, where $p : M \longrightarrow N$ is the canonical projection map. Then $fg = f(hp) = (fh)p = 1_N p = p$. □

Lemma 4.1.3. *Let $\mathcal{A}$ be a Grothendieck category. If $M \oplus N$ is pure-direct-projective, then a pure exact sequence*

$$0 \longrightarrow K \longrightarrow M \xrightarrow{g} N \longrightarrow 0$$

of objects and morphisms of $\mathcal{A}$ splits.

Proof. Suppose that $M \oplus N$ is pure-direct-projective in $\mathcal{A}$. Let $p_1 : M \oplus N \longrightarrow M$ and $p_2 : M \oplus N \longrightarrow N$ be the canonical projections. Since $M \oplus N$ is pure-direct-projective, there exists an endomorphism h of $M \oplus N$ such that $gp_1h = p_2$ by Proposition 4.1.2. Define $f : N \longrightarrow M$ by $f = p_1hi_2$ where $i : N \longrightarrow M \oplus N$ is the inclusion map. Then $gf = g(p_1hi_2) = p_2i_2 = 1_N$. Thus the sequence splits. □

Theorem 4.1.4. *Let $\mathcal{A}$ be a Grothendieck category and*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a pure exact sequence in $\mathcal{A}$ with M pure-projective. $M \oplus N$ is pure-direct-projective if and only if N is pure-projective.

Proof. ($\Rightarrow$) Suppose $M \oplus N$ is pure-direct-projective. Then the sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

splits by Lemma 4.1.3. Then N is also pure-projective by Proposition 2.12.2.
($\Leftarrow$) Suppose N is pure-projective. Then $M \oplus N$ is pure-projective by Proposition 2.12.2 and therefore $M \oplus N$ is pure-direct-projective since pure-projective objects are pure-direct-projective. □

Corollary 4.1.5. *The class of pure-direct-projective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ need not be closed under pure factors.*

Proof. Let M be an object in $\mathcal{A}$ which is not pure-direct-projective. Since $\mathcal{A}$ has enough pure-projective objects by [14, Lemma 6(ii)], there is a pure exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$$

with N pure-projective. Since N is pure-projective, it is pure-direct-projective and $N/K \cong M$. □

Corollary 4.1.6. *The class of pure-direct-projective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ need not be closed under finite coproducts.*

Proof. Let N be a pure-direct-projective object in $\mathcal{A}$ that is not pure-projective. Since $\mathcal{A}$ has enough pure-projectives by [14, Lemma 6(ii)], there is a pure exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

with M pure-projective. Then $M \oplus N$ is not pure-direct-projective since N is not pure-projective by Theorem 4.1.4. □

Corollary 4.1.7. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then every pure-direct-projective object of $\mathcal{A}$ is pure-projective if and only if the coproduct of any two pure-direct-projective objects of $\mathcal{A}$ is pure-direct-projective.*

Proof. ($\Rightarrow$) Clear by [17, Proposition 4.1].
($\Leftarrow$) Let M be a pure-direct-projective object of $\mathcal{A}$. Since $\mathcal{A}$ has enough pure-projectives by [14, Lemma 6(ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. Since P is pure-projective, P is pure-direct-projective, and therefore $P \oplus M$ is pure-direct-projective by assumption . So M is pure-projective by Theorem 4.1.4. □

Proposition 4.1.8. *Direct summands of pure-direct-projective objects of a Grothendieck category $\mathcal{A}$ are pure-direct-projective.*

Proof. Let M be a pure-direct-projective object of $\mathcal{A}$ and N be a direct summand of M and $\pi' : M \longrightarrow N$ be the projection map. Let K be a pure subobject of N and $\pi : N \longrightarrow K$ be the projection map. Let $f : N \longrightarrow K$ be an epimorphism with $\mathrm{Ker}(f)$ pure in N and $f' : M \longrightarrow N$ be an epimorphism. Then $ff' : M \longrightarrow K$ is also an epimorphism and since $\mathrm{Ker}(f)$ is a pure subobject of N and N is a pure subobject of M, $\mathrm{Ker}(f)$ is a pure subobject of M by [14, Lemma 6 (i)]. Since M is pure-direct-projective, there is an endomorphism $g : M \longrightarrow M$ such that $ff'g = \pi\pi'$. Let $i : K \longrightarrow N$ and $i' : N \longrightarrow M$ be inclusion maps. Put $h = f'gi'$. Then $fh = ff'gi' = \pi\pi'i' = \pi 1_N = \pi$. Thus N is pure-direct-projective by Proposition 4.1.2. □

Theorem 4.1.9. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(i) $\mathcal{A}$ is regular.

(ii) Every pure-direct-projective object is flat.

Proof. $(i) \Rightarrow (ii)$ Clear by [14, Theorem 4].
$(ii) \Rightarrow (i)$ Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough pure-projective objects by [14, Lemma 6 (ii)], we have a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. So P is flat by assumption. Then M is flat by [18, Proposition 2.2. (c) (i)]. □

Theorem 4.1.10. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Assume $\mathcal{A}$ has enough projectives. Then the following statements are equivalent.*

(i) *$\mathcal{A}$ is regular.*

(ii) *Every pure-direct-projective object of $\mathcal{A}$ is direct-projective.*

Proof. $(i) \Rightarrow (ii)$ Clear since every short exact sequence is pure exact in a regular category by [14, Theorem 4].
$(ii) \Rightarrow (i)$ Let M be a pure-projective object in $\mathcal{A}$. Since $\mathcal{A}$ has enough projectives, there is a projective object P and an epimorphism $f : P \longrightarrow M$. Since P is projective, it is pure-projective and so $M \oplus P$ is pure-projective by [17, Proposition 4.1]. Therefore $M \oplus P$ is pure-direct-projective. Now $M \oplus P$ is direct-projective by assumption, thus M is projective by Theorem 3.1.4. Now it $\mathcal{A}$ is regular by Theorem 2.12.6. □

Proposition 4.1.11. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(i) $\mathcal{A}$ is regular and the coproduct of two pure-direct-projective objects is pure-direct-projective.

(ii) Every pure-direct-projective object of $\mathcal{A}$ is projective.

Proof. $(i) \Rightarrow (ii)$ Let M be a pure-direct-projective object of $\mathcal{A}$. Since every locally finitely presented Grothendieck category $\mathcal{A}$ has enough pure-projective objects by [14, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. Now $M \oplus P$ is pure-direct-projective by assumption. Then M is pure-projective by Theorem 3.1.4 and therefore M is projective by Theorem 2.12.6.
$(ii) \Rightarrow (i)$ Let M be a pure-direct-projective object of $\mathcal{A}$. Then M is projective by assumption and therefore it is flat by [14, Lemma 7 (i)]. So $\mathcal{A}$ is regular by Theorem 4.1.9. Since the coproduct of two projective objects is projective, the coproduct of two pure-direct-projective objects is pure-direct-projective. □

Corollary 4.1.12. *Let $\mathcal{A}$ be a locally finitely presented regular Grothendieck category. If the coproduct of two pure-direct-projective objects is pure-direct-projective, then every pure-direct-projective object of $\mathcal{A}$ is quasi-projective.*

Proposition 4.1.13. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projective objects. If every pure-direct-projective object is quasi-projective, then $\mathcal{A}$ is regular.*

Proof. Let M be a finitely presented object of $\mathcal{A}$. So M is pure-projective. Since there are enough projective objects in $\mathcal{A}$, there is an epimorphism $f : P \longrightarrow M$ with P projective. Now $P \oplus M$ is pure-projective and therefore it is pure-direct-projective. So $P \oplus M$ is quasi-projective by assumption. Since every quasi-projective object is direct-projective, $P \oplus M$ is direct-projective. Then M is projective by Theorem 3.1.4. Hence $\mathcal{A}$ is regular by [15, Theorem 4]. □

Proposition 4.1.14. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category and M a flat object of $\mathcal{A}$. Then M is pure-direct-projective if and only if its direct-projective.*

Proof. ($\Rightarrow$) Let M be a pure-direct-projective object of $\mathcal{A}$, N be a direct summand of M and $f : M \longrightarrow N$ be an epimorphism. Since the class of flat objects is closed under coproducts by [18, Proposition 2.3 (a)], N is also flat. Therefore f is a pure epimorphism. Since M is pure-direct-projective, f splits. Then M is direct-projective.
($\Leftarrow$) Clear. □

Definition 4.1.15. Let $\mathcal{A}$ be a Grothendieck category. $\mathcal{A}$ is said to be pure-hereditary if every epimorphic image of an injective object is pure-injective.

Definition 4.1.16. An object M of a Grothendieck category $\mathcal{A}$ is said to be *cotorsion* if $Ext^1_{\mathcal{A}}(M, N) = 0$ for any flat object N of $\mathcal{A}$.

Definition 4.1.17. A class $\mathcal{C}$ of objects of a $\mathcal{A}$ is said to be closed under extension if N, $M/N \in \mathcal{C}$ implies $M \in \mathcal{C}$.

Definition 4.1.18. Let $\mathcal{A}$ be a Grothendieck category and M be an object of $\mathcal{A}$. M is called a *pure-injective* object if it is relatively injective to every pure short exact sequence in $\mathcal{A}$.

Proposition 4.1.19. *Let $\mathcal{A}$ be a Grothendieck category with enough projectives. Suppose that the class of pure-injective objects in $\mathcal{A}$ is closed under extension. Then the following conditions are equivalent.*

(i) $\mathcal{A}$ is pure hereditary.

(iii) Every pure subobject of any projective object is projective.

(iv) Every flat object is of projective-dimension at most 1.

Proof. $(i) \Rightarrow (ii)$ Let M be a projective object of $\mathcal{A}$ and P be a pure subobject of M. Let $\beta : I \longrightarrow L$ be an epimorphism with I injective and let $f : P \longrightarrow L$ be a morphism from P to L. Since $\mathcal{A}$ is pure hereditary, L is pure-injective. So there exists a morphism $g : M \longrightarrow L$ such that $gh = f$. Since M is projective, there exists a morphism $\alpha : M \longrightarrow I$ such $\beta\alpha = g$. Put $\gamma = \alpha h : P \longrightarrow I$. Then gives $\beta\gamma = \beta(\alpha h) = gh = f$. Hence P is projective.
$(ii) \Rightarrow (iii)$ Let M be a flat object. Since $\mathcal{A}$ has enough projectives, there exists an epimorphism $f : P \longrightarrow M$ with P projective. Then we have the short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

which is pure since M is flat. So K is pure in P and therefore K is projective by assumption. Thus projective dimension of M is at most 1.
$(iii) \Rightarrow (i)$ Let I be an injective object of $\mathcal{A}$ and N be a subobject of I. Then we have a short exact sequence

$$0 \longrightarrow N \longrightarrow I \longrightarrow I/N \longrightarrow 0.$$

Let M be a flat object of $\mathcal{A}$. Since projective dimension of M is at most 1 by assumption, $Ext^1_{\mathcal{A}}(M, I/N) = 0$. So I/N is a cotorsion object of $\mathcal{A}$. Therefore I/N is pure-injective by [20, Theorem 3.5.1] whose proof works in locally finitely presented Grothendieck categories. So $\mathcal{A}$ is pure-hereditary. $\square$

Theorem 4.1.20. *Let $\mathcal{A}$ be a locally finitely presented regular Grothendieck category with enough projective objects. If the class of pure-injective objects is closed under extensions then the following conditions are equivalent.*

(i) $\mathcal{A}$ is pure hereditary.

(ii) Every subobject of any projective object of $\mathcal{A}$ is pure-direct-projective.

(iii) Every subobject of any pure-projective object of $\mathcal{A}$ is pure-direct-projective.

Proof. $(i) \Rightarrow (ii)$ Clear by Proposition 4.1.19.
$(ii) \Rightarrow (iii)$ Since $\mathcal{A}$ is regular by assumption, pure-projective objects are projective.
$(iii) \Rightarrow (i)$ By assumption every subobject of a projective object is pure-direct-projective and therefore direct-projective since $\mathcal{A}$ is regular. So $\mathcal{A}$ is hereditary by Theorem 3.1.10 and therefore it is pure-hereditary. □

Corollary 4.1.21. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the class of pure-direct-projective objects of $\mathcal{A}$ need not be closed under subobjects.*

4.2 Classes all of whose objects are pure-direct-projective

Recall that an abelian category $\mathcal{A}$ is called a *spectral category* if every short exact sequence in $\mathcal{A}$ splits. We can give the following immediate result without proof.

Proposition 4.2.1. *Let $\mathcal{A}$ be a spectral Grothendieck category. Then the following conditions are equivalent.*

(i) Every object in $\mathcal{A}$ is pure-direct-projective.

(ii) Every pure-direct-projective object of $\mathcal{A}$ is projective.

(iii) Every pure-direct-projective object of $\mathcal{A}$ is direct-projective.

Definition 4.2.2. [21, Abstract] A Grothendieck category $\mathcal{A}$ is said to be *pure-semisimple if it is locally finitely presented and each of its objects is pure-projective.*

Remark 4.2.3. If a Grothendieck category $\mathcal{A}$ is pure-semisimple, then every object is pure-injective by [22, Theorem 2].

Remark 4.2.4. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. $\mathcal{A}$ is called *pure-semisimple* if it has pure global dimension zero, which means that each of its objects is a direct summand of a coproduct of finitely presented objects (see [23]). $\mathcal{A}$ is pure-semisimple if and only if it satisfies the pure noetherian property a coproduct of any family of pure-injective objects in $\mathcal{A}$ is pure-injective (see [21, Theorem 1.9]).

Lemma 4.2.5. *Let $\mathcal{A}$ be a finitely presented Grothendieck category. Then the following conditions are equivalent.*

(i) $\mathcal{A}$ is pure-semisimple.

(ii) Every pure exact sequence in $\mathcal{A}$ splits.

Proof. $(i) \Rightarrow (ii)$ By definition every object in $\mathcal{A}$ is pure-projective. Since any pure exact sequence ending with pure-projective object splits by Lemma 2.12.3, every pure-exact sequence splits.

$(ii) \Rightarrow (i)$ Suppose that every pure exact sequence in $\mathcal{A}$ splits. Let M be an object of $\mathcal{A}$. We want to show that M is pure-projective. Since every locally finitely presented Grothendieck category has enough pure-projective objects by [14, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. This sequence splits by assumption and therefore M is a direct summand of the pure-projective object P. Then M is pure-projective by [17, Proposition 4.1]. This means that $\mathcal{A}$ is pure-semisimple. □

Theorem 4.2.6. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(i) $\mathcal{A}$ is pure-semisimple.

(ii) Every object of $\mathcal{A}$ is pure-projective.

(iii) Every object of $\mathcal{A}$ is pure-direct-projective.

(iv) Every pure quotient of a pure-direct-projective object is pure-direct-projective.

Proof. $(i) \Rightarrow (ii)$ Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough pure-projective objects by [14, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$$

with N pure-projective. Since $\mathcal{A}$ is pure-semisimple, $0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$ splits by Lemma 4.2.5. So M is pure-projective [17, Proposition 4.1].

$(ii) \Rightarrow (iii)$ Clear.

$(iii) \Rightarrow (iv)$ Clear.

$(iv) \Rightarrow (iii)$ Since $\mathcal{A}$ has enough pure-projective objects by [14, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

with M pure-projective for every object N of $\mathcal{A}$. Since $M/K \cong N$, N is pure-direct-projective.

$(iii) \Rightarrow (i)$ Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough pure-projective objects by [14, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$$

with N pure-projective. Since $N \oplus M$ is pure-direct-projective by assumption, M is pure-projective by Theorem 3.1.4. □

5. APPLICATIONS

In this chapter we give applications some of our results to module and comodule categories.

5.1 Modules and Comodules

Remark 5.1.1. Let R be a unitary ring and Mod(R) be the category of right R-modules. Mod(R) is a locally finitely generated Grothendieck category with enough injectives and enough projectives. Mod(R) is hereditary if and only if the ring R is right hereditary.

Then we have the following corollary of Theorem 3.1.10 for module categories.

Corollary 5.1.2. *[24, Theorem 4] Let R be a unitary ring. Then the following conditions are equivalent.*

(1) R is right hereditary.

(2) Every submodule of a projective right R-module is direct-projective.

Remark 5.1.3. Let C be a coalgebra over a field and $\mathcal{M}^C$ be the category of right C-comodules. $\mathcal{M}^C$ is a locally finitely generated Grothendieck category. Then it has enough injectives. The category $\mathcal{M}^C$ is hereditary if and only if C is a (left and right) hereditary coalgebra (see [25]).

Then we have the following corollary of Theorem 3.1.10 for comodule categories.

Corollary 5.1.4. *Let C be a coalgebra over a field. Then the following conditions are equivalent.*

(1) C is hereditary.

(2) Every subcomodule of a projective right C-comodule is direct-projective.

Remark 5.1.5. Let C be a coalgebra over a field. Then the category $\mathcal{M}^C$ of right C-comodules is spectral if and only if $\mathcal{M}^C$ is semisimple if and only if C is cosemisimple.

Now we have the following result of Theorem 3.2.5 for comodule categories.

Corollary 5.1.6. *Let C be a coalgebra over a field. Then the following conditions are equivalent.*

(1) C is cosemisimple.

(2) C is right semiperfect and every right C-comodule is direct-projective.

(3) C is right semiperfect and every factor comodule of a direct-projective right C-comodule is direct-projective.

Proof. C has enough projectives if and only if C is right semiperfect (see [13, Theorem 3.2.3]). Since every cosemisimple coalgebra is right semiperfect, C has enough projectives. C is a cosemisimple coalgebra if and only if every right C-comodule is projective in the category $\mathcal{M}^C$ (see [13, Theorem 3.1.5]). □

We have the following corollary of Theorem 4.1.10 for module categories.

Corollary 5.1.7. *[6, Proposition 2.10] Let R be a unitary ring. Then the following conditions are equivalent.*

(1) R is a von Neumann regular ring.

(2) Every pure-projective right R-module is projective.

(3) Every pure-direct-projective right R-module is direct-projective.

Also we have the following corollary of Theorem 4.1.10 for comodule categories.

Corollary 5.1.8. *Let C be a semiperfect coalgebra over a field. Then the following statements are equivalent.*

(1) C is cosemisimple.

(2) *Every pure-projective right C-comodule is projective.*

(3) *Every pure-direct-projective right C-comodule is direct-projective.*

Proof. C has enough projectives if and only if C is right semiperfect (see [13, Theorem 3.2.3]). Since every cosemisimple coalgebra is right semiperfect, C has enough projectives. C is cosemisimple if and only if every right C-comodule is injective if and only if every right C-comodule is projective by [13, Theorem 3.1.5]. If a coalgebra C over a field is cosemisimple, then the category of right C-comodules $\mathcal{M}^C$ is regular. Conversely, if $\mathcal{M}^C$ is regular, then every right C-comodule K is FP-injective, that is every short exact sequence of the form $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ is pure (see [14]). The category of right C-comodules $\mathcal{M}^C$ coincides with the category $\sigma[{}^*_C\mathcal{C}]$ of submodules of C-generated left C^*-modules (see [14, Section 2.5]). Since $\mathcal{M}^C$ is locally noetherian, every FP-injective right C-comodule is injective by [16, 35.7]. Therefore, every right C-comodule is injective. Hence, C is cosemisimple by [14, Theorem 3.1.5]. $\square$

6. CONCLUSION

In this thesis we study direct-projective objects in abelian categories and pure-direct-projective objects in Grothendieck categories. Also we give applications some of our results to module and comodule categories. Our results given in Chapter 3. and Chapter 4. are new in abelian categories and Grothendieck categories respectively. We can list some of the most important results as follows:

6.1 Direct-projective objects

Theorem 6.1.1. *Let $\mathcal{A}$ be an abelian category with enough projective objects, M be an object of $\mathcal{A}$ and $\nu : K \longrightarrow M$ be an epimorphism from a projective object K of $\mathcal{A}$ to M. Then M is projective if and only if $K \oplus M$ is direct-projective.*

Theorem 6.1.2. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is (semi)hereditary.*

(ii) *Every (finitely generated) subobject of a projective object is direct-projective.*

Proposition 6.1.3. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(i) $\mathcal{A}$ is regular.

(ii) Every pure-projective object of $\mathcal{A}$ is a quasi-projective.

Theorem 6.1.4. *Let $\mathcal{A}$ be an abelian category. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is spectral.*

(ii) *$\mathcal{A}$ is perfect and every object of $\mathcal{A}$ is direct-projective.*

(iii) *$\mathcal{A}$ is perfect and every quotient object of a direct-projective object is direct-projective.*

Theorem 6.1.5. *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is semisimple.*

(ii) *$\mathcal{A}$ has enough projectives and every object of $\mathcal{A}$ is direct-projective.*

(iii) *$\mathcal{A}$ has enough projectives and the coproduct of two direct-projective objects is direct-projective.*

6.2 Pure-direct-projective objects

Theorem 6.2.1. *Let $\mathcal{A}$ be a Grothendieck category and*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a pure exact sequence in $\mathcal{A}$ with M pure-projective. $M \oplus N$ is pure-direct-projective if and only if N is pure-projective.

Theorem 6.2.2. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(i) $\mathcal{A}$ is regular.

(ii) Every pure-direct-projective object is flat.

Theorem 6.2.3. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Assume $\mathcal{A}$ has enough projectives. Then the following statements are equivalent.*

(i) *$\mathcal{A}$ is regular.*

(ii) *Every pure-direct-projective object of $\mathcal{A}$ is direct-projective.*

Proposition 6.2.4. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is regular and the coproduct of two pure-direct-projective objects is pure-direct-projective.*

(ii) *Every pure-direct-projective object of $\mathcal{A}$ is projective.*

Theorem 6.2.5. *Let $\mathcal{A}$ be a locally finitely presented regular Grothendieck category with enough projective objects. If the class of pure-injective objects is closed under extensions then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is pure hereditary.*

(ii) *Every subobject of any projective object of $\mathcal{A}$ is pure-direct-projective.*

(iii) *Every subobject of any pure-projective object of $\mathcal{A}$ is pure-direct-projective.*

Proposition 6.2.6. *Let $\mathcal{A}$ be a spectral Grothendieck category. Then the following conditions are equivalent.*

(i) *Every object in $\mathcal{A}$ is pure-direct-projective.*

(ii) *Every pure-direct-projective object of $\mathcal{A}$ is projective.*

(iii) *Every pure-direct-projective object of $\mathcal{A}$ is direct-projective.*

Theorem 6.2.7. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is pure-semisimple.*

(ii) *Every object of $\mathcal{A}$ is pure-projective.*

(iii) *Every object of $\mathcal{A}$ is pure-direct-projective.*

(iv) *Every pure quotient of a pure-direct-projective object is pure-direct-projective.*

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