VARIATIONS OF STAR COLORING ON GRAPHS

ÇİZGELERDE YILDIZ RENKLENDİRME VARYASYONLARI

ALAİTTİN KIRTIŞOĞLU

ASSOC. PROF. DR. SELMA ALTINOK BHUPAL
Supervisor

Submitted to
Graduate School of Science and Engineering of Hacettepe University
as a Partial Fulfillment to the Requirements
for the Award of the Degree of Master of Science
in Mathematics

2021

ABSTRACT

VARIATIONS OF STAR COLORING ON GRAPHS

Alaittin KIRTIŞOĞLU
Master of Science, Mathematics Department
Supervisor: Assoc. Prof. Dr. Selma ALTINOK BHUPAL
Co-Supervisor: Assoc. Prof. Dr. Lale ÖZKAHYA
June 2021, 28 pages

This thesis is constructed on a variety of coloring types in five chapters. Following the Introduction chapter, elementary definitions and methods used throughout the work are presented in Chapter 2.

Chapter 3 presents some results on acyclic and star colorings that forbid bicolored copies of cycles and paths on four vertices, respectively. Non-repetitive and k-distance colorings are closely related to the star coloring, and these colorings are also presented here to provide a perspective on the star coloring.

P_k-coloring is a proper coloring with no bicolored paths with k vertices. Chapter 4 is devoted to products of graphs, in particular, cylinder, 2-dimensional grid, and 2-dimensional tori that are the variations of products of paths and cycles. We find exact values of P_k-chromatic numbers of these graph families for $k = 5, 6$.

The probabilistic method is a fundamental tool to show that the desired object exists with a positive probability under random construction. In Chapter 5, we provide general bounds on the P_k-coloring. Moreover, we obtain similar bounds considering colorings with no bicolored cycles.

Keywords: Graphs, Star coloring, Acyclic Coloring

ÖZET

ÇİZGELERDE YILDIZ RENKLENDİRME VARYASYONLARI

Alaittin KIRTIŞOĞLU

Yüksek Lisans, Matematik

Danışman: Doç. Dr. Selma ALTINOK BHUPAL

Eş Danışman: Doç. Dr. Lale ÖZKAHYA

Haziran 2021, 28 sayfa

Bu tez, beş bölümden oluşup çeşitli boyama türlerine odaklanmaktadır. Giriş bölümünden sonra, ikinci bölümde çalışma boyunca kullanılan temel tanımlar ve metodlar tanıtılır.
Üçüncü bölümde, sırasıyla iki renkli döngüleri ve yolları yasaklayan döngüsüz ve yıldız boyama üzerine literatürdeki bazı çalışmalar sunulmaktadır. Tekrar etmeyen ve k-mesafeli boyamalar, yıldız boyamayla yakında ilişkili olduğu için, yıldız boyamaya bir bakış açısı kazandırmak adına bu bölümde tartışılır.
Bir G çizgesinde P_k-boyama, komşu köşelerin farklı renklere sahip olduğu bir boyamadır ve çizgedeki k köşeye sahip yolların iki renkli olmasını yasaklar. Dördüncü bölüm, yolların ve döngülerin çarpımları olan silindir, 2 boyutlu kafes ve tori gibi çizge çarpımlarına ayrılmıştır.
Bu bölümde, bu çizge ailelerinin P_k-kromatik sayıları $k = 5, 6$ için tam olarak belirlenir.
Olasılıksal yöntem, istenen bir objeyi rastgele inşa ederek, objenin varlığının pozitif olasılığa sahip olduğunu göstermek için kullanılan temel bir araçtır. Beşinci bölümde, herhangi bir çizgenin P_k-kromatik sayısına yönelik genelleştirilmiş sınırlar bulunmuştur. Benzeri sınırlar, iki renkli bazı döngüleri içermeyen çizgeler için elde edilmiştir.

Anahtar Kelimeler: Çizge Teorisi, Döngüsüz Boyama, Yıldız Boyama

ACKNOWLEDGEMENTS

I thank professors, who are serving in my thesis committee, for their valuable contribution. I thank professors, Rıza Ertürk, Uğur Gül, Ayşe Çiğdem Özcan, and Sinem Onaran, whom I took lessons from, for teaching and guiding me. I would like to express many thanks to my advisor Selma Altınok Bhupal for her patience and help throughout my education.
I thank my advisor Lale Ozkahya for her great supports on this thesis and her extraordinary contribution to my life. We worked on many problems together and I learned a lot from her.
I am grateful to my friends Haydar Jafaar, Erdem Şafak Öztürk, and Fikret Sarı for discussions on mathematics and their belief in me. I sincerely thank Enfal Sartaş, Sadık Eyidoğan, and Ceren Aydın for not leaving me alone in these years and for their warm friendship. Thank you to my family for their love and support.

Alaittin KIRTIŞOĞLU

June 2021, Ankara

CONTENTS

FIGURES

SYMBOLS

$E(G)$	set of edges of a graph G
$V(G)$	set of vertices of a graph G
P_k	path on k vertices
C_k	cycle on k vertices
$d(v)$	degree of v
$\Delta(G)$	maximum degree of a graph G
$G[X]$	subgraph induced by a vertex set X in a graph G
$d(u,v)$	distance between u and v
$ex(n,H)$	Turan number for the graph H
K_n	complete graph on n vertices
$K_{n,m}$	complete bipartite graph with parts having n and m vertices
$G\square H$	cartesian product of graphs G and H
$G(n_1,...,n_d)$	d-dimensional grid with n_i vertices in each coordinate i
Q_d	d-dimensional hypercube
$TG(n_1,...,n_d)$	d-dimensional tori with n_i vertices in each coordinate i
$\pi(G)$	Thue chromatic number of a graph G
$\chi(G)$	chromatic number of a graph G
$a(G)$	acyclic chromatic number of a graph G
$\chi_S(G)$	star chromatic number of a graph G
$s_k(G)$	P_k-chromatic number of a graph G
$a_k(G)$	C_k-chromatic number of a graph G
$\chi_k(G)$	k-distance chromatic number of a graph G

1 INTRODUCTION

The main goal in graph coloring is to color the vertices or edges with the minimum number of colors under certain conditions. However, it can be very difficult to solve problems for some graph families. The coloring of graphs has its roots in the four-coloring problem. This problem asks if it is possible to color any planar map divided into contiguous regions using four colors with different colored neighbors for each region. This problem could not be solved for more than 100 years. After many false proofs and counterexamples, a computer-assisted proof was presented in several hundred pages of an article [3].

A coloring is called *proper coloring* if no two neighboring vertices have the same color. Many variations of the proper coloring were defined after the four-color problem. For instance, the proper coloring without containing any bicolored copy of a fixed family of subgraphs is another well-studied problem. More coloring problems can be found in [4]. In this thesis, we will focus on the star and acyclic colorings, where bicolored copies of paths on 4 vertices and cycles are not allowed, respectively. In addition, we introduce generalizations of these colorings and provide some results.

Star and acyclic colorings are defined in 1973 by Grünbaum [5]. He proves that it is possible to obtain an acyclic coloring of every planar graph using nine colors, and conjectures that five is enough. Finding a chromatic number for a given family of graphs may not be always computationally fast. In fact, researchers often try to find a bound on the chromatic number. The following bounds given by Alon et al. in [6] are the best available asymptotic bounds for the acyclic chromatic number. This bound holds for any graph with $\Delta(G) = d$.

$$\Omega\left(\frac{d^{\frac{4}{3}}}{(logd)^{\frac{4}{3}}}\right) = a(G) = O(d^{\frac{4}{3}})$$

There have been several improvements in the constant factor of the upper bound, which we mention in this thesis. Similar results are obtained by Fertin, Raspaud, and Reed in [7], showing $\chi_s(G) \leq \lceil 20d^{3/2} \rceil$. We will provide a generalization of these results in Chapter 5

As well as the general bound, some particular graph families have been also studied. Especially, products of regular graphs take a wide portion in the literature. For example, various bounds on the star chromatic numbers of hypercube, grid, tori, cycles, and complete bipartite graphs are shown in [7]. More recent results on the acyclic coloring of grid and tori can be found in [8] and [9]. Similarly, grid and hypercube are studied in [10]. Moreover, [11], [12], and [13] investigate the acyclic chromatic number for products of trees, products of cycles and Hamming graphs. Finding the exact values of all these chromatic numbers has been a longstanding problem.

Chapter 2 includes a quick review of essential definitions and theorems used throughout the work. In Chapter 3, we describe coloring types and furthermore discuss the literature. In Chapter 4, we present the acyclic and star colorings of cartesian products of graphs in the literature and work on generalizing these to P_k-coloring for $k = 5, 6$. Finally, in Chapter 5 we present lower bounds on P_k and C_k-chromatic numbers and generalize these results showing some bounds for all graphs.

2 BACKGROUND

In this chapter, we review elementary definitions, coloring in graphs, counting, and probabilistic methods in combinatorics, which we need throughout the thesis. In general, we use the notations and terminology given by West [14].

2.1 Graph Terminology

The use of graphs is introduced by L. Euler in the 18th century. He finds a method to solve the famous Königsberg bridge problem that asks if there exists a walk crossing each of the seven bridges of Königsberg (now Kaliningrad, Russia) once and only once [15]. The left image in Figure 2.1 shows us islands that are marked with letters and seven bridges connecting islands. On the right image, the graph is drawn, in which islands are shown as vertices and bridges as edges connecting these vertices to make the problem easier and save the shape from unnecessary components.

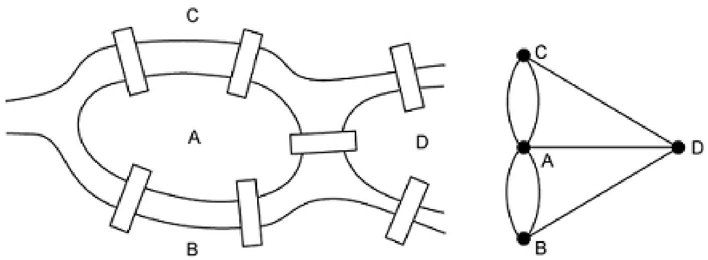

FIGURE 2.1: Königsberg problem [1].

For a graph G, $V(G)$ and $E(G)$ are used to denote vertex and edge sets respectively.

If uv is an edge, then u and v are called *adjacent* vertices or *neighbors*. The *degree* of a vertex v indicates the number of neighbors of v and the maximum of all degrees in a graph is called the *maximum degree* denoted by $\Delta(G)$.

A graph H is a *subgraph* of a graph G if all vertices and edges of H are contained in G. A subgraph H of G is called *induced* $G[X]$ if $E(H)$ contains all the edges of G with both endpoints in $V(H)$.

A *path* on k vertices, denoted by P_k, is a graph in which the vertices can be ordered such that the edges are consecutive pairs of vertices. Similarly, a *cycle* on k vertices, denoted by C_k

is a graph defined similarly, where the edge set is $E = \{v_i v_{i+1} : 1 \leq i \leq k-1\} \cup \{v_1 v_k\}$. Observe that deleting an edge from C_k produces P_k. The *distance* between vertices u, v, denoted by $d(u, v)$, is the number of edges in the shortest path connecting u and v. A *tree* T is an acyclic graph, in which every pair of vertices is connected by some path.

The *cartesian product* of two graphs $G = (V, E)$ and $G' = (V', E')$ is denoted by $G \Box G'$ and its vertex set is $V \times V'$. For any vertices $x, y \in V$ and $x', y' \in V'$, there is an edge between (x, y) and (x', y') in $G \Box G'$ if and only if either $x = y$ and $x'y' \in E'$ or $x' = y'$ and $xy \in E$.

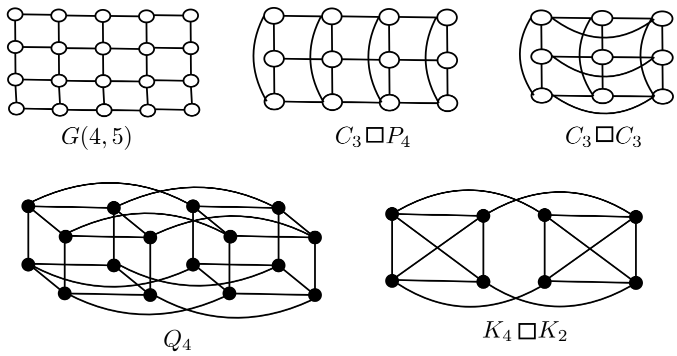

FIGURE 2.2: Cartesian products of graphs

A d*-dimensional grid* is $P_{n_1} \Box P_{n_2} \Box ... \Box P_{n_d}$ of paths on $n_1, n_2, ..., n_d$ vertices. It is denoted by $G(n_1, n_2, ..., n_d)$. In general, $G(n_1, n_2, ..., n_d)$ is also known as a graph whose vertices are of the form $v = (v_1, v_2, ..., v_d)$ such that $1 \leq v_i \leq n_i$ for each $1 \leq i \leq d$. The edges consist of pairs that have exactly one coordinate different. Figure 2.2 shows 2-dimensional grid $G(4, 5)$ which is also called lattice. A d*-dimensional hypercube* Q_d is $G(n_1, n_2, ..., n_d)$ such that $n_i = 2$ for all $1 \leq i \leq d$. A *cylinder* is $P_n \Box C_m$ of a path on n vertices and cycle on m vertices, which contains 2-dimensional grid $G(n, m) = P_n \Box P_m$ as a subgraph. Finally, a d*-dimensional tori* $TG(n_1, n_2, ..., n_d)$ is $C_{n_1} \Box C_{n_2} \Box ... \Box C_{n_d}$ with $n_i \geq 3$ for all i. Figure 2.2 provides some examples of grid, hypercube, cylinder and tori.

2.2 Coloring of Graphs

The roots of graph coloring are based on the four-coloring problem which asks whether one can color any planar map using four colors with no two neighbors receiving the same color. After more than 100 years of effort, Appel and Haken [3] proved that it is possible. In Figure 2.3, we see an example of maps colored by using four colors and the graph representation of the map.

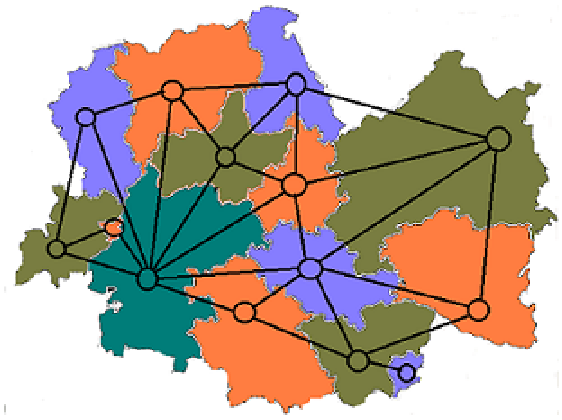

FIGURE 2.3: Four-coloring of Germany map and its graph [2]

To state differently, *vertex coloring* of a graph G is a function $f : V(G) \mapsto \mathbb{Z}$. It is called *proper* if $f(u) \neq f(v)$ for neighboring pairs u, v. The smallest number of colors achieving that for a graph G is called the *chromatic number* of G, $\chi(G)$. Hence, G is called k-*colorable* for any $k \geq \chi(G)$.

In addition to proper coloring, avoiding bicolored copies of a given subgraph in a graph is also widely studied. The following colorings are well-known examples for avoiding bicolored particular subgraphs.

Definition 2.1. An *acyclic coloring* of a graph G is a proper coloring in such a way that any cycle in G has at least three colors, with *acyclic chromatic number*, $a(G)$, being the minimum number colors needed.

Definition 2.2. A *star coloring* of a graph G is a proper coloring with no bicolored P_4, with *star chromatic number*, $\chi_s(G)$, being the minimum number colors needed.

By these, we have

$$\chi(G) \leq a(G) \leq \chi_S(G).$$

In this thesis, we are only interested in acyclic coloring, star coloring, and the related variants defined in Chapter 3.

2.3 Counting on Graphs

The product and sum rules are very commonly used in counting objects, including graphs. In this section, we first explain these rules and the pigeonhole principle. Then, we give a basic upper bound on the numbers of P_k including a fixed vertex in a graph, which we use in Chapter 5.

The *pigeonhole principle* says that if there are at least $r + 1$ objects, then we cannot place them into smaller number of boxes and have at least one object in all the boxes. For instance, in a group of more than twelve people, there are at least two people who were born in the same month according to the pigeonhole principle. We use this simple principle in Theorem 4.6 when we color vertices of $P_3 \square P_3$ with three colors.

The *rule of sum* helps to count the objects in m disjoint sets/cases $S_1, ..., S_m$ stating that $| \cup S_i| = \sum |S_i|$. The *rule of product* says that if there are m options at a point and n_i possibilities for each option i, then the number of possible ways is $n_1 n_2 ... n_m$.

One application of the sum and product rules, also used in this thesis, is the proof for showing that the number of P_k's containing a vertex v is at most $\lceil \frac{k}{2} \rceil d^{k-1}$, where d is the maximum number of neighbors of v. To see this, we count how many P_k's are possible to include v in G. We determine unsymmetric positions of v on P_k. Otherwise, we count some paths twice. Recall that P_k is a path consisting of ordered vertices $v_1, v_2, ..., v_k$ and the edges $v_i v_{i+1}$ for all $1 \leq i \leq k-1$. Since vertices in the first half of $v_1, v_2, ..., v_k$ are symmetric to the vertices in the second half, there are at most $\lceil \frac{k}{2} \rceil$ different positions for v (k may be odd). For any position of v, we can choose an adjacent vertex of v in d ways, and so an adjacent vertex of this new vertex in $d-1$ ways (one of them is already v). Since there are exactly $k-1$ vertices on P_k except v, the maximum number of P_k's for each position is $d(d-1)^{k-2} \leq d^{k-1}$ by the rule of product. Hence, the total number of P_k's through v is at most $\lceil \frac{k}{2} \rceil d^{k-1}$ by the rule of sum.

2.4 The Probabilistic Method

It is possible to show that an object exists satisfying a given property, without producing a specific example. To do that, a random object is constructed in probability space, and then it is shown that the probability of such an object is non-zero. Thus, we conclude that the desired object must exist in one of the random instances.

The *probability function (or distribution)* is defined on a set S as $Pr : S \mapsto [0, 1]$, where $\sum_{x \in S} Pr(x) = 1$. Mostly, one calls S the *sample space*. An *event* A is a subset of the sample space. In this work, we only consider the *uniform distribution* on S, that is

$$Pr(x) = \frac{1}{|S|} \quad \text{for all} \quad x \in S$$

In a probability space, we call events A and B *independent* if and only if $Pr(A \cap B) = Pr(A)Pr(B)$. The *conditional probability of A given B* is calculated using the same formula, only replacing $Pr(A)$ with $Pr(A|B)$, assuming $Pr(B) \neq 0$.

In combinatorics, one may wonder about the existence of a graph satisfying some properties. However, one may not be able to construct such graphs so easily, because there are exponentially many possibilities to check. Instead of constructing an object directly, one randomizes over all the possible configurations and then shows that the probability of the randomized object is non-zero. Consequently, the main idea behind the method is the random construction of an object.

To guarantee that the "good" event happens, one investigates what is needed to satisfy $Pr(f(x) \geq t) > 0$. For that, one needs to determine the "bad" events $A_1, A_2, ..., A_n$ that prevent the realization of this. Thus, the above condition is rewritten as

$$Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0. \tag{1}$$

This simple fact means that the object with desired "good" properties exists.

3 STAR COLORING AND ITS VARIATIONS

The acyclic coloring in Definition 2.1 was introduced in 1973 by Grünbaum [5]. Past 40 years, acyclic coloring has been studied on planar graphs with large girth [16], cographs [17], subcubic graphs [18], graphs of maximum degree 5 and 6 [19, 20], including its NP-completeness [21].

For any graph G, with $|V(G)| = n, |E(G)| = m$, $a(G) \geq \frac{2n+1-\sqrt{\Delta}}{2}$, where $\Delta = 4n(n-1) - 8m + 1$ [10]. This result provides an optimal lower bound for some graph families. For instance, it implies that $a(T) \geq 2$, $a(C_n) \geq 3$, and $a(K_n) \geq n$, since $m = n - 1$ in a tree, $m = n$ for cycles and $m = \frac{n(n-1)}{2}$ for complete graphs. Moreover, it can be calculated that $a(G) \geq 2 + \lfloor m/n \rfloor$ is a slight approximation of the lower bound. Applying this approximation, Fertin et al. obtain the lower bound below.

$$2 + \left\lfloor d - \sum_{i=1}^{d} \frac{1}{n_i} \right\rfloor \leq a(G(n_1, ..., n_d)) \leq d + 1. \tag{2}$$

In particular, $a(G(n_1, ..., n_d)) = d + 1$ when $\sum_{i=1}^{d} \frac{1}{n_i} \leq 1$.
These results have implications on Q_d using $n_i = 2$ for each i. Moreover, $a(G(n_1, ..., n_d)) = d + 1$, when $n_i \geq d$ for all i. So, the first question that comes to our mind is whether $a(Q_d)$ is near the lower bound, since each n_i is equal to the smallest value 2 for Q_d. Jamisson, and Matthews [12] support our doubts proving that $a(Q_d) = \frac{d+4}{2}$, if $\frac{d+4}{2}$ is a Fermat prime of the form $F_n = 2^{2^n} + 1$. Any coloring of Q_3 with three colors contains a bicolored cycle, and we show such an example at the right side in Figure 4.1.

Another type of coloring that Grünbaum [5] introduced together with acyclic coloring is star coloring. Star coloring is studied on locally planar graphs [22], bipartite planar graphs [23], graphs with girth at least five [24], sparse graphs [25] and subcubic graphs [26], and on many more graph families, including its NP-completeness [27].

In general, acyclic and star colorings are closely related and $a(G) \leq \chi_S(G)$, hence the lower bounds for $a(G)$ also apply to $\chi_S(G)$. Albertson et al. [27] prove the following result.

$$\chi_S(G) \leq a(G)(2a(G) - 1)$$

3.1 Non-repetitive Coloring

A finite sequence of symbols $x = x_1x_2...x_n$ is called *repetitive* if it contains the same consecutive subsequences. Otherwise, we say that the word x is *non-repetitive*. For instance, 2312315 is a repetitive sequence, while $subsetsub$ is non-repetitive. The non-repetitive sequence has its origin about 100 years ago. In 1906, Axel Thue proves the existence of arbitrarily long non-repetitive sequences with three letters [28] (see for English [29]). Although there are many applications of the non-repetitive sequence ranging from group theory to number theory, we are only interested in the graph-theoretical perspective in this section.

A coloring of a graph G is called *non-repetitive*, if the sequence of colors on any path in G is non-repetitive. The *Thue chromatic number* of G, denoted by $\pi(G)$, is the minimum number of colors needed for a non-repetitive coloring of G.

A sequence of any proper coloring on P_3 cannot contain a repetitive sequence, and so $\pi(P_3) = 2$. This result implies $\pi(C_n) \leq 4$, $n \geq 3$. Because, adding an edge to P_n produces C_n, and the number of colors increases at most one. Moreover, the Thue chromatic number of cycles is detailed by Currie [30].

Because any color sequence of a bicolored P_4 is a repetitive sequence such as 1212, nonrepetitive property yields a star coloring. However, results on $\pi(P_n)$ and $\pi(C_n)$ reveal that determining $\pi(G)$ is nontrivial even for paths and cycles. Brešar et al. [31] show that $\pi(T) \leq 4$ assigning a coloring function to vertices of any tree T.

Non-repetitive coloring is a topic that is studied on many graph families with different variants. We only present related results with star and acyclic colorings in order to compare them. Grytczuk [32] shows $\pi(G) \leq 16\Delta(G)^2$ for any graph G. Similarly, Alon et al. [33] find $O(d^2)$ for the edge coloring version of this problem.

3.2 k-Distance Coloring

A *k-distance coloring* is a coloring of the vertex set of a graph such that if any pair of vertices with distance at most k receive different colors. The *k-distance chromatic number* of a graph G, denoted by $\chi_k(G)$, is the smallest number of colors, r, needed for a k-distance coloring of G. Besides, any 2-distance coloring of a graph is also a star coloring, since $\chi_2(P_4) = 3$. Therefore, we have

$$\chi(G) = \chi_1(G) \leq a(G) \leq \chi_S(G) \leq \chi_2(G) = \chi(G^2)$$

for any graph G. Here, G^2 is the *square of* G, which has the same vertex set as G and has all edges of G in its edge set, only with additional edges between pairs of vertices with some common neighbor.

The 2-distance coloring of graphs is introduced in 1969 by F. Kramer, and H. Kramer [34], [35]. In these years, Wegner [36] claims that $\chi_2(G) \leq \lceil \frac{3d}{2} \rceil + 1$, if G is a planar graph with maximum degree $d \geq 8$. It is shown by Havet et al. in [37] that this claim is asymptotically true. The best upper bound today is $\lceil \frac{5d}{3} \rceil + 78$ [38], which is still bigger than the Wegner's claim.

The 2-distance coloring and related problems are widely studied on particular graph families. For any d-dimensional hypercube, Wan [39] shows that $2d \leq \chi_2(Q_d) \leq 2^{\lceil \log_2(d+1) \rceil + 1}$. Fertin et al. [10] determine the exact value for grids as $\chi_2(G(n_1, n_2, ..., n_d)) = 2d + 1$. For the general bound on χ_2, Alon, and Mohar prove the following.

Theorem 3.1. *[40] Let G be a graph with maximum degree d and girth g.*

- *If $g \leq 6$, there exists a function $\epsilon(d)$ that tends to 0 as d tends to infinity such that $(1 - \epsilon(d))d^2 \leq \chi_2(G) \leq d^2 + 1$*
- *If $d \geq 2$ and $g \geq 7$, $\chi_2(G) = \Theta(\frac{d^2}{logd})$.*

In summary, if G does not contain "small" cycles, we have the tight bound $\chi_2(G) = \Theta(\frac{d^2}{logd})$ We see in Chapter 5 that best known upper bounds on the acyclic and star chromatic numbers are slightly better than $O(\frac{d^2}{logd})$.

4 P_k-COLORING OF GRAPHS

The P_k*-coloring* of a simple graph G, where $k \geq 4$, is a proper vertex coloring of G such that there is no bicolored copy of P_k in G, and the minimum number of colors needed for a P_k-coloring of G is called the P_k*-chromatic number* of G, denoted by $s_k(G)$.

A special case of this coloring is the star coloring, when $k = 4$, introduced by Grünbaum [5]. Hence, $\chi_s(G) = s_4(G)$ and all of the bounds on $s_k(G)$ in Chapter 4 and 5 can be applied to the star chromatic number using $k = 4$.

4.1 Related Work

In this section, we discuss the 2-distance, acyclic, and star chromatic numbers of the grid, cylinder, and tori in small dimensions. Our aim is to provide an idea for Section 4.2 and 4.3 including our results on the P_5 and P_6-chromatic numbers of these graphs.

Since paths and cycles are almost the same graphs, one may expect that $\chi_2(C_m \Box C_n)$ and $\chi_2(P_m \Box P_n) = 5$ have the same value. However, Sopena and Wu [41] assert a surprising result in Theorem 4.1, which shows that 2-distance coloring makes a big difference between 2-dimensional grid and tori.

Theorem 4.1. *[41] If* $m, n \geq 3$ *Then,*

$$\chi_2(C_m \Box C_n) = \begin{cases} 7 & \textit{if } (m,n) = (4,4),(3,5) \\ 9 & \textit{if } (m,n) = (3,3) \\ 9 & otherwise. \end{cases}$$

For the acyclic coloring, Theorem 4.2 gives the exact values of the acyclic chromatic numbers of cylinder and tori.

Theorem 4.2. *[13] For all* $m \geq 2$, $a(P_m \Box C_4) = 4$, *and* $a(P_m \Box C_n) = 3$ *where* $n \neq 4$. *Moreover,* $a(C_3 \Box C_3) = 5$ *and* $a(C_m \Box C_n) = 4$ *when* $(m,n) \neq (3,3)$.

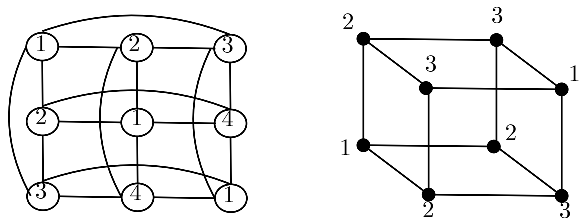

FIGURE 4.1: Colorings of $C_3\square C_3$ and Q_3

By the particular case of (2), we have $a(P_m\square P_n) = d + 1 = 3$ for all $m, n \geq 2$. This result provides a lower bound for $P_m\square C_n$ and $C_m\square C_n$, which contain $P_m\square P_n$ as a subgraph. Therefore, showing an acyclic coloring with three colors is enough to prove that $a(P_m\square C_n) = 3$ where $n \neq 4$ in Theorem 4.2. For the other results, Jamisson and Matthews first provide a lower bound proving that there is no acyclic coloring of these graphs with three colors. Then, they show an acyclic coloring for the upper bounds using some copying techniques similar to the one we use in Chapter 4. Figure 4.1 shows a coloring of $C_3\square C_3$ with four colors including bicolored cycles.

If G is a product of the trees, $T_1, T_2, ..., T_d$, then $\lceil \frac{d+3}{2} \rceil \leq a(G) \leq d + 1$ [11]. Moreover, they show $a(T_1\square T_2) = 3$ and $a(T_1\square T_2\square T_3) = 4$, where each tree has at least two vertices. The lower bound holds because any product of trees contains a product of T_2's that is the hypercube. For the upper bound, they assign an acyclic coloring function on the vertex set.

The star coloring is also studied on the grid, cylinder, and tori in small dimensions. Theorem 4.3 gives the chromatic number of products of two paths.

Theorem 4.3. *[7]* $\chi_S(G(2, 2)) = 3$*,* $\chi_S(G(2, m)) = \chi_S(G(3, m)) = 4$ *for* $m \geq 4$*, and* $\chi_S(G(m, n)) = 5$ *for* $m, n \geq 4$.

Han et al. [9] work on star colorings of $P_n\square P_m$ and $C_n\square C_m$. The following theorem says that $\chi_S(P_n\square C_m)$ and $\chi_S(C_n\square C_m)$ are equal to $\chi_S(P_n\square P_m)$ except in finitely many cases.

Theorem 4.4. *[9] If* $m \geq 3$ *is even,* $\chi_S(P_3\square C_m) = 4$*. Otherwise,* $\chi_S(P_n\square C_m) = 5$ *for* $n, m \geq 3$*. For 2-dimensional tori,* $\chi_S(C_n\square C_m) = 5$*, where* $n, m \geq 30$.

In Figure 4.2, we show a coloring of $C_3\square C_5$, which contains a few bicolored P_4's.

For $m, n < 30$, Akbari et al. [8] determine the value of $\chi_S(C_n\square C_m)$ in Theorem 4.5 using similar techniques.

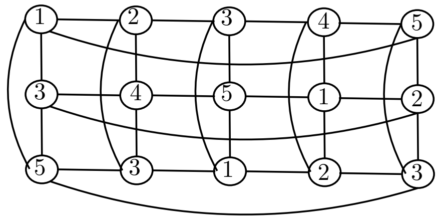

FIGURE 4.2: A coloring of $C_3 \square C_5$

Theorem 4.5. *[8]* $\chi_S(C_3 \square C_3) = \chi_S(C_3 \square C_5) = 6$. *If* $m, n \geq 3$ *and* $m, n \notin \{(3,3), (3,5)\}$, *then* $\chi_S(C_m \square C_n) = 5$.

Furthermore, Han et al. [9] show that $\chi_S(P_i \square P_j \square P_k) = 6$ for $i, j, k \geq 4$, and $\chi_S(C_3 \square C_3 \square C_k) = 7$ for $k \geq 3$, and $\chi_S(C_{4i} \square C_{4j} \square C_{4k} \square C_{4l}) \leq 9$ for $i, j, k, l \geq 1$.

4.2 P_5-coloring of Graphs

Theorem 4.3 and Theorem 4.5 show that four colors are enough for a star coloring of the cylinder, 2-dimensional grid, and tori. In this section, we ask whether four colors are enough for the P_k-coloring of these graphs.

Theorem 4.6.

$$s_5(P_3 \square P_3) = s_5(C_3 \square C_3) = s_5(C_3 \square C_4) = s_5(C_4 \square C_4) = 4.$$

Proof. We start by showing that $s_5(P_3 \square P_3) \geq 4$. Assume that there is a coloring of $P_3 \square P_3$ using three colors. Note that each color appears at most 3 times in consecutive columns. If a color, say a, appears 3 times, then a color, say c, appears exactly once on these consecutive columns. In this case, the vertices colored a and b contain a bicolored P_5. Hence, each color is used exactly twice and all colors appear in any consecutive columns.

Suppose that a is used twice in a column. Then, in a consecutive column, either b or c is used twice, which is impossible in a proper coloring using $\{a, b, c\}$ only. Thus, each column has colors a, b, c exactly once. According to this property, if the vertex at the center of $P_3 \square P_3$ has, say color a, then some pair of vertices at opposing corners have color a as well. When the remaining vertices are colored, there is always a bicolored P_5, thus $s_5(P_3 \square P_3) \geq 4$.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array} \tag{3}$$

Since $C_3 \square C_3$, $C_3 \square C_4$ and $C_4 \square C_4$ contain $P_3 \square P_3$ as a subgraph, 3 colors are not enough. Such a coloring can be obtained as in (3) by taking the first three or four rows/columns depending on the change in the grid dimension. □

Theorem 4.7 follows from Theorem 4.6.

Theorem 4.7. $s_5(G(n,m)) = 4$ *for all* $n, m \geq 3$.

Proof. Note that $4 = s_5(G(3,3)) \leq s_5(G(n,m))$ for all $m, n \geq 3$. Since there exists some integer k for which $3k \geq n, m$ and $G(n,m)$ is a subgraph of $G(3k, 3k)$, $s_5(G(n,m)) \leq s_5(G(3k, 3k))$ for some k. Hence, we show that $s_5(G(3k, 3k)) = 4$. In Theorem 4.6, a P_5-coloring of $C_3 \square C_3$ is given by the upper left corner of the coloring in (3) by using 4 colors. By repeating this coloring of $C_3 \square C_3$ k times in $3k$ rows, we obtain a coloring of $G(3k, 3)$. Then repeating this colored $G(3k, 3)$ k times in $3k$ columns, we obtain a P_5-coloring of $G(3k, 3k)$ using 4 colors. There exists no bicolored P_5 in this coloring. □

In the following, we generalize the previous cases by making use of the well-known result below.

Theorem 4.8 (Sylvester, [42]). *If* $r, s > 1$ *are relatively prime integers, then there exist* $\alpha, \beta \in \mathbb{N}$ *such that* $t = \alpha r + \beta s$ *for all* $t \geq (r-1)(s-1)$.

Theorem 4.9. *Let* $p, q \geq 3$ *and* $p, q \neq 5$. *Then* $s_5(C_p \square C_q) = 4$.

Proof. By Theorem 4.6, one observes that 4 colors are needed. By Theorem 4.8, p and q are linear combinations of 3 and 4 using nonnegative coefficients. By using this, we are able to tile the $p \times q$-grid of $C_p \square C_q$ using these blocks of 3×3, 3×4, 4×3, and 4×4 grids. Recall that the coloring pattern in (3) also provides a P_5-coloring of smaller grids listed above by using the upper left portion for the required size. Therefore, using these coloring patterns on the smaller blocks of the tiling yields a P_5-coloring of $C_p \square C_q$.

$\square$

Corollary 4.10. *Let* $i, j \geq 3$ *and* $i, j \neq 5$. *Then,* $s_5(P_i \square C_j) = 4$.

Proof. Since $P_i \square P_j$ is a subgraph of $P_i \square C_j$, Theorem 4.7 gives the lower bound. By Theorem 4.9, we have equality. $\square$

4.3 P_6-coloring of Graphs

In addition to P_5-coloring, we also investigate P_6-coloring of products of paths and cycles. We already have $s_6(P_n \square P_m) \leq s_5(P_n \square P_m)$ and $s_6(C_n \square C_m) \leq s_5(C_n \square C_m)$ by the definition of P_k-coloring. Moreover, our main purpose in this section is to show that $s_6(P_n \square P_m) = s_6(C_n \square C_m) = s_5(C_n \square C_m) = s_5(P_n \square P_m)$ for $m, n \geq 3$ and $m, n \neq 5$.

Theorem 4.11. $s_6(G(4, 4)) = 4$.

Proof. Since $s_6(G(4, 4)) \leq s_5(G(4, 4)) = 4$, we prove $s_6(G(4, 4)) \geq 4$. Assume that f is a coloring of $G(4, 4)$ using the colors $\{1, 2, 3\}$ only. We consider possible colorings on the C_4 at the center of the grid, call it C.

Case 1: C is bicolored. Assume that C has only two colors, 1 and 2. Then, either x or y shown in Figure 4.3 has color 3. Assume that $f(x) = 3$. This implies $f(y) = 2$. To avoid a bicolored P_6, we have $f(q) = 3$. This implies that $f(w) = 2$, and therefore $f(z) = 3$ so that $V(C) \cup \{z, w\}$ is not bicolored. However, this yields a bicolored P_7 as seen in Figure 4.3.

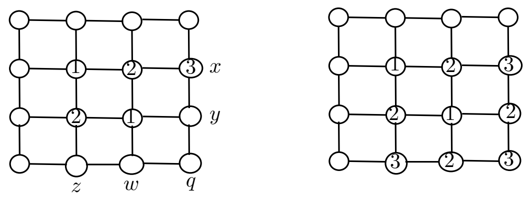

FIGURE 4.3: Possible colorings in Case 1.

Case 2: C has all three colors. We assume that the repeating color on C is 1.

Case 2.a: Color 1 is also used on the pair of vertices in opposing corners as in Figure 4.4a. Note that x and y cannot have the same color, otherwise there is a bicolored P_6. Same holds for w and z. Hence, both 2 and 3 appear as colors on the pairs $\{x, y\}$ and $\{w, z\}$, yielding a bicolored P_6.

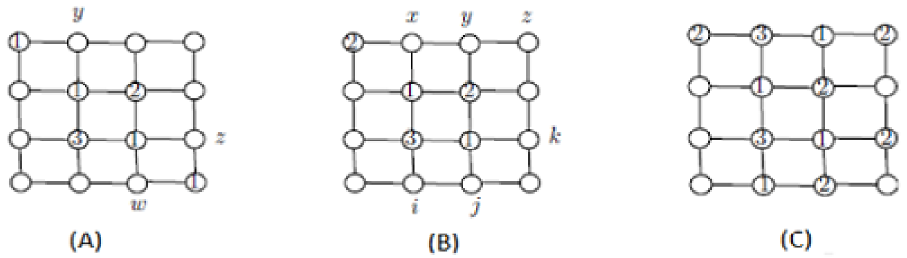

FIGURE 4.4: Possible olorings in Case 2.

Case 2.b: Color 1 is not used on both of the vertices in opposing corners as in Figure 4.4a. Assume that one of the vertices at the corners is colored 2 as in Figure 4.4b. This case is also symmetric to the case when this color is 3. This implies that $f(x) = 3$ and $f(y) = 1$ yielding a bicolored P_5. To avoid a bicolored (with colors 1 and 3) P_6, it is necessary that $f(j) = f(k) = f(z) = 2$. However, this produces a bicolored P_6 seen in Figure 4.4c. □

Corollary 4.12. $s_6(G(n, m)) = 4$ *for all* $n, m \geq 4$.

Proof. By Theorem 4.7, we have $s_6(G(n, m)) \leq s_5(G(n, m)) = 4$. And, by Theorem 4.11, we have equality. □

Corollary 4.13. $s_6(C_m \square C_n) = 4$ *for all* $m, n \geq 4$ *and* $m, n \neq 5$.

Proof. By the definition of P_k-coloring and Theorem 4.9, $s_6(C_m \square C_n) \leq s_5(C_m \square C_n) = 4$ for all $m, n \geq 3$ and $m, n \neq 5$. Since $G(4, 4)$ is a subgraph of $C_m \square C_n$ for all $m, n \geq 4$ and by Corollary 4.12, $s_6(C_m \square C_n) \geq s_6(G(4, 4)) = 4$. □

5 GENERAL BOUNDS ON P_k AND C_k-COLORINGS OF GRAPHS

5.1 Lower Bounds

For the P_k-coloring, in 2020, Hou, and Zhu [43] find a lower bound on $s_k(G)$ depending on the maximum degree d in G. For all ≥ 5, they prove that

$$\Omega\left(\frac{d^{\frac{k-1}{k-2}}}{(logd)^{\frac{1}{k-2}}}\right) = s_k(G).$$

Below, we present bounds using a result of Erdős, and Gallai.

Theorem 5.1. *[44] For any graph G,*

1. *if* $|E(G)| > \frac{1}{2}(k-2)|V(G)|$*, then G contains* P_k *as a subgraph,*
2. *if* $|E(G)| > \frac{1}{2}(k-1)(|V(G)|-1)$*, then a member of* $\mathcal{C}_k$ *is a subgraph,*

for any P_k *with* $k \geq 2$*, and for any* $\mathcal{C}_k$ *with* $k \geq 3$.

As also observed in [7], the subgraphs induced by any two color classes do not contain P_k yielding the following observation for any graph G on n verties and m edges.

$$s_k(G) \geq \frac{2m}{n(k-2)} + 1,$$

for any $k \geq 3$. To see this, let $s_k(G) = x$ and below, let $E_{i,j}$ be the set of edges induced by the color classes $V_i \cup V_j$ as defined as in [7]. By the observation above, we obtain that

$$|E(G)| = \sum_{(i,j)} |E_{i,j}| \leq \sum_{(i,j)} \frac{1}{2}(k-2)(|V_i| + |V_j|) \leq \frac{n}{2}(k-2)(x-1),$$

where the first inequality follows from Theorem 5.1.

The C_k*-coloring* ($k \geq 3$) is a proper vertex coloring of G without any bicolored copy of members from family $\mathcal{C}_k = \{C_i : i \geq k\}$. The minimum number of colors needed in a C_k-coloring of a graph G is written as $a_k(G)$. If a graph does not contain a bicolored P_k, then it does not contain any bicolored cycle from the family $\mathcal{C}_k = \{C_i : i \geq k\}$. Hence, inequalities given below hold for all $k \geq 3$.

$$a_{k+1}(G) \leq a_k(G) \leq s_k(G) \quad \text{and} \quad s_{k+1}(G) \leq s_k(G)$$

As before, we have a lower bound for C_k-coloring as $a_k(G) \geq \frac{1}{2}(2n + 1 - \sqrt{\Delta})$, for any $k \geq 3$, where $\Delta = 4n(n-1) - \frac{16m}{k-1} + 1$. To see this, let $a_k(G) = x$ and consider a C_k-coloring of G. Similarly, by Theorem 5.1, we have

$$|E(G)| \leq \sum_{(i,j)} \frac{1}{2}(k-1)(|V_i| + |V_j| - 1) = (k-1)[2n(x-1) - x(x-1)],$$

which gives $0 \geq x^2 - (2n+1)x + (2n + \frac{4m}{k-1})$. Let $\Delta = 4n^2 - 4n - \frac{16m}{k-1} + 1$. We note that $\Delta \geq 1$, since $k \geq 3$ and $m \leq \frac{n(n-1)}{2}$. Thus, we have $x \geq \frac{1}{2}(2n + 1 - \sqrt{\Delta})$.

5.2 Lovasz Local Lemma

The *Lovasz Local Lemma (LLL)* is introduced in 1973 (published in 1975) by Lovasz and Erdős [45]. It is a fundamental tool of probabilistic combinatorics to show that the desired object exists with a positive probability under random construction. To show the existence, LLL uses ”bad” events such that the intersection of complements of these events gives the desired object. There are many new versions and improvements for LLL. The first version is divided into two cases called symmetric and general. In 1991, Beck [46] proves that there exists an algorithmic version of LLL to compute that none of the bad events occur. Moser and Tardos [47] give a polynomial-time algorithm, which earn them the Gödel Prize in 2020. In this thesis, we only focus on the general case.

As we mention in Section 2.4, avoiding bad events $A_1, A_2, ..., A_n$ with positive probability is to show that

$$Pr\left[\bigcap_{i=1}^{n} \overline{A_i}\right] > 0. \tag{4}$$

An event A_i is *mutually independent* of a set of events $\{B_i \mid i = 1, 2..., n\}$ if for any subset $\mathcal{B}$ of events or their complements contained in $\{B_i\}$, we have $Pr(A_i \mid \mathcal{B}) = Pr(A_i)$. Thus, if $\mathcal{A} = \{A_i \mid i = 1, 2..., n\}$ is a set of mutually independent events and $0 < Pr(A_i) < 1$ for all i, then

$$Pr\left[\bigcap_{i=1}^{n} \overline{A_i}\right] = \prod_{i=1}^{k} Pr(\overline{A_i}) > 0, \tag{5}$$

yielding that none of the bad events occur. Lovasz local lemma allows that some of A_i's could be dependent. To indicate the dependence between these events, a dependency graph is constructed as follows.

For a collection of events $\mathcal{A} = \{A_1, A_2, ..., A_n\}$, the *dependency graph* is defined as a graph, in which the vertex set is $\mathcal{A}$ and the edge set is the set of pairs of $\{A_i, A_j\}$ that are *not* mutually independent.

Theorem 5.2. *[45] Let $H = (V, E)$ be a dependency graph for $A_1, A_2, ..., A_n$ and suppose there are real numbers $y_1, y_2, ..., y_n$ such that $0 \leq y_i \leq 1$ and*

$$Pr(A_i) \leq y_i \prod_{(i,j) \in E} (1 - y_j) \qquad \textit{for all } 1 \leq i \leq n. \tag{6}$$

Then $Pr(\bigwedge_{i=1}^{n} \bar{A}_i) \geq \prod_{i=1}^{n}(1 - y_i)$.

To use this tool, the desired object is constructed randomly. Then, bad events are defined, which prevent the realization of the object. After counting the maximum number of events, one constructs a dependency graph and finds the degrees of vertices. The proof of Theorem 5.3 is an example of this process.

5.3 Upper Bounds

In 1976, Erdős [48] conjectures for a graph G with maximum degree d that $a(G) = o(d^2)$ as d tends to infinity. Alon, McDiarmid, and Reed [6] confirm this conjecture, showing $a(G) \leq \lceil 50d^{\frac{4}{3}} \rceil$ and the following lower bound.

$$\Omega\left(\frac{d^{\frac{4}{3}}}{(logd)^{\frac{1}{3}}}\right) = a(G) = O(d^{\frac{4}{3}}).$$

Although those are the best available asymptotic bounds, there are some improvements in the constant factor of the upper bound. Ndreca, Procacci, and Scoppola [49] reduce this upper bound to $\lceil 6.59d^{4/3} + 3.3d \rceil$. Furthermore Sereni, and Volec [50] lowers it to $2.835d^{3/2} + d$, by using the entropy compression method. Recently Gonçalves et al. [51] improves it to $\frac{3}{2}d^{\frac{4}{3}} + O(d)$ for all $d \geq 24$.

In particular, Alon, McDiarmid, and Reed provide the result $a(G) \leq \lceil 32\sqrt{\gamma}d \rceil$, where G has no copy of the complete bipartite graph $K_{2,\gamma+1}$, for $\gamma \geq 1$. Gonçalves et al. [51] reduce this upper bound to $\lceil (1 + \sqrt{2\gamma + 4})d \rceil$.

Fertin, Godard, and Raspaud [7] show that $\chi_s(G) \leq \lceil 20d^{3/2} \rceil$. This upper bound is improved in the constant factor by Ndreca, Procacci, and Scoppola [49] to $\chi_S(G) \leq \lceil 4.34d^{3/2} + 1.5d \rceil$.

Alon, McDiarmid, and Reed [6] claim that $s_k(G) = O(d^{\frac{k-1}{k-2}})$ for all $k \geq 4$, introducing the P_k-coloring of graphs. After about a quarter century, Esperet, and Parreau [52] confirm this claim for even values of $k \geq 4$ and improve the bound in Fertin et al. [7] for $k = 4$. In 2020, Hou, and Zhu [43] find a slightly better upper bound on $s_k(G)$, for all $k \geq 5$, showing that

$$s_k(G) \leq \left(1 + \left\lceil \frac{k}{2} \right\rceil^{\frac{1}{k-3}}\right) d^{\frac{k-1}{k-2}} + d + 1.$$

Furthermore, this result independently improves our current work presented in Theorem 5.3. Their method is based on an algorithmic approach that is slightly different than ours.

Theorem 5.3. *For any graph G, $s_k(G) \leq \lceil 6\sqrt{10}d^{\frac{k-1}{k-2}} \rceil$ for any $k \geq 4$ and $d = \Delta(G) \geq 2$.*

Proof. Assume that $x = \lceil ad^{\frac{k-1}{k-2}} \rceil$ and $a = 6\sqrt{10}$ and the vertex set of G is colored uniformly at random by $f : V \mapsto \{1, 2, ..., x\}$. We aim to show that f does not produce a bicolored P_k with positive probability.

Below are the types of probabilistic events that are not allowed:

- I ($A_{u,v}$): $f(u) = f(v)$ for $uv \in E(G)$.
- II (A_P): The path P, a copy of P_k, is colored properly with two colors.

By definition of our coloring, none of these events are allowed to occur. We introduce a dependency graph H, where the events of above become the vertices. For two vertices A_1 and A_2 to be adjacent in H, the subgraphs corresponding to these events should have common vertices in G. The dependency graph of the events is called H, where the vertices are the union of the events.

Observation 1. For all $v \in V(G)$, at most

- d pairs $\{u, v\}$ are associated with an event of Type I, and
- $\frac{k+1}{2}d^{k-1}$ copies of P_k containing v, are associated with an event of Type II.

Proof. The first claim is true because $\Delta(G) = d$. To see the second observation, let us label the vertices of a P_k containing v as $x_1, x_2, ..., x_k$. The maximum number of P_k's with $x_i = v$ is d^{k-1}. Considering that $1 \le i \le k$, there are at most $\lceil \frac{k}{2} \rceil d^{k-1}$ copies of P_k containing v considering the symmetric positions on the path. □

Lemma 5.4. *The maximum possible number of neighbors of type j for a type i vertices:*

	I	II
I	$2d$	$(k+1)d^{k-1}$
II	kd	$\frac{k}{2}(k+1)d^{k-1}$

Proof. Consider a vertex $A_{u,v}$ in H for the first row. We have $2d$ since this vertex may be adjacent to events $A_{u,z}$ and $A_{v,x}$ for some $x, z \in V(G)$. Similarly, $A_{u,v}$ may be adjacent to events A_P, where P is a P_k containing u or v. There are at most $(k+1)d^{k-1}$ such events. For the second row, a path P that is a copy of P_k may have kd events intersecting it. Similarly, there may be at most $(k+1)d^{k-1}/2$ other P_k's containing some particular vertex of P. □

Observation 2. The probabilities of the events of type I and II are respectively

- $Pr(A_{u,v}) = \frac{1}{x}$, and
- $Pr(A_P) = \frac{1}{x^{k-2}}$.

To apply Theorem 5.2, we choose the weights y_i as below:

$$y_1 = \frac{1}{3d}, \qquad y_2 = \frac{1}{2(k+1)d^{k-1}}.$$

Below are the conditions that are to be satisfied for (6) to hold.

$$\frac{1}{x} \leq \frac{1}{3d}\left(1 - \frac{1}{3d}\right)^{2d}\left(1 - \frac{1}{2(k+1)d^{k-1}}\right)^{(k+1)d^{k-1}} \tag{7}$$

$$\frac{1}{x^{k-2}} \leq \frac{1}{2(k+1)d^{k-1}}\left(1 - \frac{1}{3d}\right)^{kd}\left(1 - \frac{1}{2(k+1)d^{k-1}}\right)^{\frac{k}{2}(k+1)d^{k-1}} \tag{8}$$

Since

$$(1+x)^n \geq 1 + nx \text{ for } x \geq -1 \text{ and any nonnegative integer } n, \tag{9}$$

it is sufficient to verify the following to satisfy (7), and we observe that it holds when $a = 6\sqrt{10} \geq 18$ and $k \geq 4$.

$$\frac{1}{ad^{\frac{k-1}{k-2}}} \leq \frac{1}{3d}\left(1 - \frac{2d}{3d}\right)\left(1 - \frac{(k+1)d^{k-1}}{2(k+1)d^{k-1}}\right) = \frac{1}{18d}$$

We can rewrite (8) as below.

$$\frac{1}{a} \leq \left(\frac{1}{2(k+1)}\right)^{\frac{1}{k-2}}\left(1 - \frac{1}{3d}\right)^{\frac{kd}{k-2}}\left(1 - \frac{1}{2(k+1)d^{k-1}}\right)^{\frac{k(k+1)d^{k-1}}{2(k-2)}}$$

By (9), it is sufficient to verify the following to satisfy (8). We omit the use of ceiling for simplicity.

$$\frac{1}{a} \leq \left(\frac{1}{2(k+1)}\right)^{1/k-2}\left(1 - \frac{k}{3(k-2)}\right)\left(1 - \frac{k}{4(k-2)}\right)$$

Since all factors on the right are decreasing for $k \geq 4$, (8) is verified. $\square$

REFERENCES

[1] Pawel Boguslawski. *Modelling and analysing 3d building interiors with the dual half-edge data structure*. Ph.D. thesis, University of Glamorgan Pontypridd, Wales, UK, **2011**.

[2] Thomas Wright Sulcer. Germany with provinces and circles with lines. `https://en.citizendium.org/wiki/File:Germany_Map_2.png`, **2010-04-19**.

[3] Kenneth I Appel and Wolfgang Haken. *Every planar map is four colorable*, volume 98. American Mathematical Soc., **1989**.

[4] Tommy R Jensen and Bjarne Toft. *Graph coloring problems*, volume 39. John Wiley & Sons, **2011**.

[5] Branko Grünbaum. Acyclic colorings of planar graphs. *Israel journal of mathematics*, 14(4):390–408, **1973**.

[6] Noga Alon, Colin Mcdiarmid, and Bruce Reed. Acyclic coloring of graphs. *Random Structures & Algorithms*, 2(3):277–288, **1991**.

[7] Guillaume Fertin, André Raspaud, and Bruce Reed. Star coloring of graphs. *Journal of Graph Theory*, 47(3):163–182, **2004**.

[8] S Akbari, M Chavooshi, M Ghanbari, and S Taghian. Star coloring of the cartesian product of cycles. *arXiv preprint arXiv:1906.06561*, **2019**.

[9] Tianyong Han, Zehui Shao, Enqiang Zhu, Zepeng Li, and Fei Deng. Star coloring of cartesian product of paths and cycles. *Ars Comb.*, 124:65–84, **2016**.

[10] Guillaume Fertin, Emmanuel Godard, and André Raspaud. Acyclic and k-distance coloring of the grid. *Information Processing Letters*, 87(1):51–58, **2003**.

[11] Robert E Jamison, Gretchen L Matthews, and John Villalpando. Acyclic colorings of products of trees. *Information Processing Letters*, 99(1):7–12, **2006**.

[12] Robert E Jamison and Gretchen L Matthews. On the acyclic chromatic number of hamming graphs. *Graphs and Combinatorics*, 24(4):349–360, **2008**.

[13] Robert E Jamison and Gretchen L Matthews. Acyclic colorings of products of cycles. *Bulletin of the Institute of Combinatorics and its Applications*, 54:59–76, **2008**.

[14] Douglas Brent West et al. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, **2001**.

[15] Leonhard Euler. Solutio problematis ad geometriam situs pertinentis. *Commentarii academiae scientiarum Petropolitanae*, pages 128–140, **1741**.

[16] Oleg V Borodin, Alexandr V Kostochka, and Douglas R Woodall. Acyclic colourings of planar graphs with large girth. *Journal of the London Mathematical Society*, 60(2):344–352, **1999**.

[17] Andrew Lyons. Acyclic and star colorings of cographs. *Discrete applied mathematics*, 159(16):1842–1850, **2011**.

[18] San Skulrattanakulchai. Acyclic colorings of subcubic graphs. *Information processing letters*, 92(4):161–167, **2004**.

[19] Guillaume Fertin and André Raspaud. Acyclic coloring of graphs of maximum degree δ. *DMTCS Proceedings*, (1), **2006**.

[20] Juan Wang and Lianying Miao. Acyclic coloring of graphs with maximum degree at most six. *Discrete Mathematics*, 342(11):3025–3033, **2019**.

[21] Alexandr V Kostochka. Upper bounds of chromatic functions of graphs. In *Doct. Thesis*. Novosibirsk, **1978**.

[22] Ken-ichi Kawarabayashi and Bojan Mohar. Star coloring and acyclic coloring of locally planar graphs. *SIAM Journal on Discrete Mathematics*, 24(1):56–71, **2010**.

[23] Hal A Kierstead, André Kündgen, and Craig Timmons. Star coloring bipartite planar graphs. *Journal of Graph Theory*, 60(1):1–10, **2009**.

[24] MA Shalu and TP Sandhya. Star coloring of graphs with girth at least five. *Graphs and Combinatorics*, 32(5):2121–2134, **2016**.

[25] Yuehua Bu, Daniel W Cranston, Mickaël Montassier, André Raspaud, and Weifan Wang. Star coloring of sparse graphs. *Journal of Graph Theory*, 62(3):201–219, **2009**.

[26] Min Chen, André Raspaud, and Weifan Wang. 6-star-coloring of subcubic graphs. *Journal of Graph Theory*, 72(2):128–145, **2013**.

[27] Michael O Albertson, Glenn G Chappell, Hal A Kierstead, André Kündgen, and Radhika Ramamurthi. Coloring with no 2-colored p_4's. *the electronic journal of combinatorics*, pages R26–R26, **2004**.

[28] A Thue. Uber unendliche zeichenreichen. norske vid selsk. skr. i. mat. nat. kl. christian, 7: 1–22, 1906. *English translation:[9].*

[29] Jean Berstel. *Axel Thue's papers on repetitions in words: a translation*, volume 20. Départements de mathématiques et d'informatique, Université du Québec à Montréal, **1995**.

[30] James D Currie. There are ternary circular square-free words of length n for n 18. **2002**.

[31] Boštjan Brešar, Jaroslaw Grytczuk, Sandi Klavžar, Staszek Niwczyk, and Iztok Peterin. Nonrepetitive colorings of trees. *Discrete Mathematics*, 307(2):163–172, **2007**.

[32] Jaroslaw Grytczuk. Nonrepetitive colorings of graphs a survey. *International journal of mathematics and mathematical sciences*, 2007, **2007**.

[33] Noga Alon, Jarosław Grytczuk, Mariusz Hałuszczak, and Oliver Riordan. Nonrepetitive colorings of graphs. *Random Structures & Algorithms*, 21(3-4):336–346, **2002**.

[34] Florica Kramer and Horst Kramer. Ein färbungsproblem der knotenpunkte eines graphen bezüglich der distanz p. *Rev. Roumaine Math. Pures Appl*, 14(2):1031–1038, **1969**.

[35] Florica Kramer and Horst Kramer. Un probleme de coloration des sommets d'un graphe. *CR Acad. Sci. Paris A*, 268(7):46–48, **1969**.

[36] Gerd Wegner. Graphs with given diameter and a coloring problem. **1977**.

[37] Frédéric Havet, Jan Van Den Heuvel, Colin McDiarmid, and Bruce Reed. List colouring squares of planar graphs. *Electronic Notes in Discrete Mathematics*, 29:515–519, **2007**.

[38] Michael Molloy and Mohammad R Salavatipour. A bound on the chromatic number of the square of a planar graph. *Journal of Combinatorial Theory, Series B*, 94(2):189–213, **2005**.

[39] Peng-Jun Wan. Near-optimal conflict-free channel set assignments for an optical cluster-based hypercube network. *Journal of Combinatorial Optimization*, 1(2):179–186, **1997**.

[40] Noga Alon and Bojan Mohar. The chromatic number of graph powers. *Combinatorics Probability and Computing*, 11(1):1–10, **2002**.

[41] Éric Sopena and Jiaojiao Wu. Coloring the square of the cartesian product of two cycles. *Discrete mathematics*, 310(17-18):2327–2333, **2010**.

[42] James J Sylvester et al. Mathematical questions with their solutions. *Educational times*, 41(21):171–178, **1884**.

[43] Jianfeng Hou and Hongguo Zhu. Coloring graphs without bichromatic cycles or paths. *Bulletin of the Malaysian Mathematical Sciences Society*, pages 1–13, **2020**.

[44] Paul Erdős and Tibor Gallai. On maximal paths and circuits of graphs. *Acta Mathematica Academiae Scientiarum Hungarica*, 10(3-4):337–356, **1959**.

[45] Paul Erdős and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *in Infinite and Finite Sets (A. Hajnal et al., eds)*. North-Holland, Amsterdam, **1975**.

[46] József Beck. An algorithmic approach to the lovász local lemma. i. *Random Structures & Algorithms*, 2(4):343–365, **1991**.

[47] Robin A Moser and Gábor Tardos. A constructive proof of the general lovász local lemma. *Journal of the ACM (JACM)*, 57(2):1–15, **2010**.

[48] MO Albertson and DM Berman. The acyclic chromatic number. *Congr. Numer*, 17:51–69, **1976**.

[49] Sokol Ndreca, Aldo Procacci, and Benedetto Scoppola. Improved bounds on coloring of graphs. *European Journal of Combinatorics*, 33(4):592–609, **2012**.

[50] Jean-Sébastien Sereni and Jan Volec. A note on acyclic vertex-colorings. *arXiv preprint arXiv:1312.5600*, **2013**.

[51] Daniel Gonçalves, Mickaël Montassier, and Alexandre Pinlou. Entropy compression method applied to graph colorings. *arXiv preprint arXiv:1406.4380*, **2014**.

[52] Louis Esperet and Aline Parreau. Acyclic edge-coloring using entropy compression. *European Journal of Combinatorics*, 34(6):1019–1027, **2013**.