THE INJECTIVE PROFILE OF A RING AND ITS EFFECT ON THE STRUCTURE OF RINGS

HALKALARIN İNJEKTİF PROFİLİ VE PROFİLİN HALKA YAPISI ÜZERİNDEKİ ETKİSİ

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ABSTRACT

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This thesis is partly based on recent developments on the subject of what is known as "injectivity domains" in the theory of modules over rings with identity. The subject was suggested as a measurement of how far a module is away from being injective and has gained increased interest over the last few years from people studying rings towards homological properties. The aim of this thesis is to present significant achievements with a unifying approach. Our thesis is primarily concerned with the investigation of a particular class of rings, called rings with no middle class, which is defined by means of injectivity domains.

This thesis consists of four chapter. The first chapter contains motivation and historical background of the subject of this thesis. In the second chapter, we give some necessary background material and classifications of some rings by their homological properties to better understand next chapters. In the third chapter, we introduce the notion of injectivity domains and that of poor modules, defined in terms of injectivity domains. The last chapter is concerned with the rings without a middle class. We give a number of properties of these rings and characterize them with respect to hereditary pretorsion classes. We also explore decomposability of rings with no middle class and obtain, in an incisive way, that they can decompose into the product of an indecomposable ring and a semisimple Artinian ring. Finally, we investigate commutative rings without a middle class.

Keywords: Injective module, Injectivity domain, Poor module, preradical, V-ring, QI-ring, PCI-ring, SI-ring, Middle class.

ÖZET

HALKALARIN İNJEKTİF PROFİLİ VE PROFİLİN HALKA YAPISI ÜZERİNDEKİ ETKİSİ

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Bu tez, birimli halkalar üzerindeki modüller teorisinde "injektiflik bölgeleri" olarak bilinen konudaki son gelişmelere dayanmaktadır. Bu konu, bir modülün injektiflikten ne kadar uzak olduğunun bir ölçütü olarak ortaya atılmış ve halkaların homolojik özellikleri üzerinde çalışan insanlar tarafından son birkaç yıldır artan bir ilgi görmüştür. Bu tezin amacı bazı önemli kazanımları birleştirici bir yaklaşımla sunmaktır. Tezimiz öncelikli olarak injektiflik bölgeleri vasıtasıyla tanımlanan ve özel bir sınıf olan orta sınıfsız halkaların incelenmesi ile ilgilidir.

Dört bölümden oluşan bu tezin ilk bölümünde, ilgilenilen konunun tarihsel geçmişi ve motivasyonu hakkında bilgiler verilmiştir. İkinci bölümde, diğer bölümlerin daha iyi anlaşılabilmesi için bazı gerekli bilgiler ve halkaların üzerindeki homolojik özellikler aracılığıyla sınıflandırılması verilmiştir. Üçüncü bölümde, injektiflik bölgeleri ve bu kavram üzerinden tanımlanan yoksul modül kavramı ele alınmıştır. Son bölüm ise orta sınıfsız halkalar üzerinedir. Bu tür halkalarla ilgili özellikler verilmiş ve kalıtsal önburulma sınıfları ile ilişkili olarak karakterize edilmiştir. Ayrıca, orta sınıfsız halkaların indirgenemez ve yarı-basit Artin halkaların direkt çarpımı olarak ayrıştırılması üzerine çalışılmıştır. Son olarak, orta sınıfsız değişmeli halkalar araştırılmıştır.

Anahtar Kelimeler: İnjektif modül, İnjektiflik bölgelesi, Yoksul modül, Önradikal, V-halka, QI-halka, PCI-halka, SI-halka, Orta sınıf.

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Nergiz YUCA

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Contents

LIST OF SYMBOLS

Symbols

R	An associative ring with identity
$\mathrm{J}(R)$	The Jacobson radical of a ring R
$\mathrm{Soc}(M)$	The socle of an R-module M
$\mathrm{Z}(M)$	The singular submodule of an R-module M
$\mathrm{ann}(m)$	The annihilator of $m \in M$ in R
$\mathrm{rad}(M)$	The Jacobson radical of an R-module M
$N \leq M$	A submodule N of an R-module M
$N \leq_e M$	An essential submodule N of an R-module M
$N \ll M$	A small submodule N of an R-module
$N \subseteq_c M$	The complement of N to M
$M^{(I)}$	$\bigoplus_{i \in I} M_i$ with $M_i \cong M$ for each $i \in I$
M^I	$\prod_{i \in I} M_i$ with $M_i \cong M$ for each $i \in I$
$\mathrm{Im}\, f$	The image of an R-homomorphism f
$\ker f$	The kernel of an R-homomorphism f
$\mathrm{E}(M)$	Injective envelope (hull) of an R-module M
$\mathrm{Hom}_R(M, N)$	The group of all module homomorphism from M to N
$\mathrm{Tr}_R(N, M)$	The trace of N in M
$\mathrm{End}_R(M)$	The endomorphism ring of an R-module
$\mathrm{Mod}\text{-}\mathrm{R}$	The category of right R-modules
$\mathrm{SSMod}\text{-}\mathrm{R}$	The category of semisimple right R-modules
$\mathrm{Sing}\text{-}\mathrm{R}$	The category of singular right R-modules
$\sigma[M]$	Subcategory of $\mathrm{Mod}\text{-}\mathrm{R}$ subgenerated by M
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{Z}_n$	The cyclic group $\mathbb{Z}/n\mathbb{Z}$

Chapter 1

INTRODUCTION

Our thesis is a study that will be carried out within the scope of the research program that has been actively ongoing for more than half a century under the title of "characterization of rings through the homological properties provided by the modules over them", which is within the scope of the algebraic theory known as "classification of rings". An important subject that has been studied for a long time since the second half of the 20th century is the question of how certain types of modules of a ring being injective affects the ring structure. In this direction, many new ring types have been discovered and their relationships with each other have been revealed. In our study, we will focus on a brand new ring type introduced in 2010, and the relationship of these new rings with some of the previously defined ring types will be presented. We will begin our study with the "injectivity domains", introduced as a measure of how far the modules are from being injectivity, and the concept of "poor modules" defined with the help of this concept. Then, we will address the structural properties and various characterizations of the rings over which every module is either injective or poor. In addition, we will give many examples of different character to these new rings.

In our thesis, we aim to discuss some new concepts and techniques which we believe will contribute to the solution of the conjecture known as the “Boyle Conjecture” in the literature that has not been resolved since the day it was brought forward. In this context, we aim to constitute a theory within the scope of our thesis which would involve the researches on the rings without a middle class carried out in the last ten years and scattered in the literature. This thesis will also be an important source for understanding the position of the QI-rings, because the rings without middle class are closely related to the QI-rings in terms of each non-semisimple quasi-injective module being injective. Another aim of our thesis research is to be an important reference as part of the theory of classification of rings by their homological properties.

According to a result obtained by Barbara Osofsky in 1964, the fact that all the cyclic modules over a unital and associative ring are injective is equivalent to the ring being a semisimple Artinian ring. This result is considered as one of the most important and fundamental results provided for determining a ring with the homological properties (e.g., injectivity, projectivity, flatness, etc.) of the modules over it. Since then, significant steps were taken to classify a ring in terms of the structure of the modules over it. One part of the research program, extending from the second half of the twentieth century to the present day and continues actively in this direction, is carried out when the certain types of modules over a ring are injective. For example, we can say that there are ring classes such as right V-rings (rings over which every simple right R-module is an injective module), right PCI-rings (rings over which every proper cyclic right R-module is an injective module), right QI-rings (rings over which every quasi-injective right R-module is an injective module), right SI-rings (rings over which every singular right R-module is an injective module).

Many important steps have been taken to uncover the relations between these ring classes so far. For example, in 1969, Ann Boyle showed that Noether hereditary V-rings are the QI-rings, and threw out a conjecture (known as Boyle's Conjecture) that "every right QI-ring is right hereditary". Some new methods and concepts that seem to provide the opportunity to develop different approaches to the solution of Boyle's Conjecture, which remains unsolved until today, have been introduced in the last decade in several studies. The most important of these is the concept of "ring without middle class", defined by the injectivity domains. First introduced by Alahmadi, Alkan and Lopez-Permouth in 2010 [1], this ring class was later studied in all aspects and its relationship with some other ring classes was investigated. It is currently being investigated whether these rings have to satisfy the ascending chain condition of right ideals. Rings without middle class are defined using the concept of poor modules.

In our thesis, contemporary theory of the previously mentioned subjects which are still being developed will be compiled and presented as an integrated source of reference. Since the rings without middle class, which is subject to our thesis, is an attractive subject in terms of their interesting properties and their close relationship with other ring types, we think that the subjects to be compiled in our thesis will be a good source for all researchers studying related subjects.

Chapter 2

PRELIMINARIES

2.1 Modules, Submodules and Homomorphisms

Definition 2.1.1. Let R be a ring with unity. A right R-module is an additive abelian group M together with a mapping $. : M \times R \longrightarrow M$, called scalar multiplication, which satisfies the following axioms for all $r, s \in R$ and $m, n \in M$.

(i) $m \cdot 1 = m$

(ii) $m \cdot (r \cdot s) = (m \cdot r) \cdot s$

(iii) $m \cdot (r + s) = m \cdot r + m \cdot s$

(iv) $(m + n) \cdot r = m \cdot r + n \cdot r$

We can define a left R-module similarly. In this thesis, we will always use the right R-modules.

Definition 2.1.2. A submodule N of an R-module M is a subgroup of $(M, +)$ that is closed under taking scalar multiplication, i.e, $rn \in N$ for all $r \in R$ and $n \in N$. If N is a submodule of M, then we write $N \leq M$.

Definition 2.1.3. A nonzero R-module S is said to be a simple module if 0 and S are the only submodules of S.

Definition 2.1.4. Let M and N be two right R-modules. A homomorphism ψ: $M \longrightarrow N$ is a right R-homomorphism if for all $a \in R$ and $x, y \in M$ we have

(i) $\psi(x + y) = \psi(x) + \psi(y)$,

(ii) $\psi(xa) = \psi(x)a$.

ψ is called a monomorphism if $\ker \psi = 0$ (in which case ψ is an injection), ψ is an epimorphism if $\operatorname{Im} \psi = N$ (in which case ψ is a surjection) and ψ is called an isomorphism when ψ is both a monomorphism and an epimorphism. If $M = N$, then ψ is called an endomorphism. All right R-module homomorphisms from M to N form an additive group, denoted $\operatorname{Hom}_R(M, N)$, with the following addition: If $f, g \in \operatorname{Hom}_R(M, N)$, then $(f + g)(x) = f(x) + g(x)$, for all $x \in M$. Also the endomorphisms of R-module M is denoted $\operatorname{End}_R(M)$.

Definition 2.1.5. Let M be an R-module. A submodule U of M is said to be **fully invariant** provided $f(U) \subseteq U$ for every $f \in \operatorname{End}_R(M)$.

Definition 2.1.6. A sequence of homomorphisms $\cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \cdots$ is exact if for each successive pair f_n, f_{n+1}, we have $\operatorname{Im} f_n = \ker f_{n+1}$.

Proposition 2.1.7. *[2, Proposition 3.12] Given modules X and Y and a homomorphism $\psi : X \longrightarrow Y$, the sequence*

(i) $0 \longrightarrow X \xrightarrow{\psi} Y$ *is exact if and only if* ψ *is a monomorphism;*

(ii) $X \xrightarrow{\psi} Y \longrightarrow 0$ *is exact if and only if* ψ *is an epimorphism;*

(iii) $0 \longrightarrow X \xrightarrow{\psi} Y \longrightarrow 0$ *is exact if and only if* ψ *is an isomorphism.*

Definition 2.1.8. We say that a class of right R-modules $\mathscr{C}$ is closed under taking extensions if whenever $0 \to X' \to X \to X'' \to 0$ is an exact sequence of right R-modules with X' and X'' in $\mathscr{C}$, then X is also in $\mathscr{C}$.

2.2 Quotient Modules and Isomorphism Theorems

Definition 2.2.1. Let M be an R-module and A be a submodule of M. Then we can define a module structure on the additive group M/A with the scalar multiplication

$$M/A \times R \longrightarrow M/A$$
$$(m + A, r) \longrightarrow mr + A,$$

i.e , $(m + A)r = mr + A$. The module M/A is called the quotient of M modulo A.

The canonical map $\pi : M \longrightarrow M/A$, defined by $\pi(m) = m + A$, $\forall m \in M$ is an R-epimorphism with $\ker \pi = A$.

Proposition 2.2.2. *[2, Corollary 3.7] Let M and A be right R-modules.*

(i) If $f : M \longrightarrow A$ is an epimorphism with $\ker f = B$, then there is a unique isomorphism $\eta : M/B \longrightarrow A$ such that $\eta(m + B) = f(m)$ for all $m \in M$.

(ii) If $B \leq C \leq M$, then $M/C \cong (M/B)/(C/B)$.

(iii) If $D \leq M$ and $B \leq M$, then $(D + B)/B \cong D/(D \cap B)$.

Definition 2.2.3. Let M be a right R-module and let A be a nonempty subset of R. Then the right annihilator of A in R is the set $\operatorname{ann}_R(A) = \{r \in R : a.r = 0 \text{ for all } a \in A\}$. Moreover, if the nonzero subset A has exactly one element $a \in R$, then we write $\operatorname{ann}_R(a)$ instead of $\operatorname{ann}_R(A)$.

Note that $\operatorname{ann}_R(A)$ is a right ideal of R. Also note that if A is a left ideal of R, then $\operatorname{ann}_M(A)$ is a submodule of M.

Definition 2.2.4. A right R-module M is said to be cyclic if it is generated by one element, i.e., $M = mR = \{mr : r \in R\}$ for some $m \in M$. If cyclic right R-module M is not isomorphic to R, then we say that it is proper.

Lemma 2.2.5. *Let M be a cyclic right R-module generated by m. Then there exists an isomorphism such that $M \cong R/\operatorname{ann}_R(m)$. Therefore, every cyclic R-module M is a factor of R.*

2.3 Direct Products and Direct Sums

Let $\{A_i\}_{i \in I}$ be a family of right R-modules. Their cartesian product

$$\prod_I A_i = \{(x_i)_I : x_i \in (A_i)\}$$

is a right R-module under addition and scalar multiplication defined in the usual way, as follows:

$$(a_i)_I + (y_i)_I = (a_i + b_i)_I$$

$$(a_i)_I \cdot r = (a_i \cdot r)_I$$

Then this module is called the direct product of the modules A_i. The direct sum of the modules A_i is defined as the subset

$$\bigoplus_I A_i = \{(a_i)_I : a_i \in \prod_I A_i \text{ with } a_i = 0 \text{ for almost } i \in I\}.$$

The direct product of the modules A_i contains the direct sum of modules A_i as a submodule and if I is a finite set, then, obviously,

$$\bigoplus_I A_i = \prod_I A_i.$$

There are natural homomorphism, $\prod_I A_i \longrightarrow A_j$ (for all $j \in I$), called a projection map onto A_j, and $A_i \longrightarrow \prod_I A_j$, called an injection map.

Lemma 2.3.1. *[2, Lemma 5.1] Let $\psi : M \longrightarrow A$ and $\psi' : A \longrightarrow M$ be R-homomorphisms with*

$$\psi\psi' = 1_A.$$

Then ψ is an epimorphism, ψ' is a monomorphism and we have $M = \ker\psi \bigoplus \operatorname{Im}\psi'$. In this case ψ is called a split epimorphism and ψ' is called a split monomorphism.

Lemma 2.3.2. *Let C and D be right R-modules such that $R_R = C \oplus D$ and let $\operatorname{Hom}_R(C, D) = 0$. Then C is a two-sided ideal.*

Proof. Let us consider the following equations:

$$RC = C^2 + DC$$

We complete proof by showing $DC = 0$. Let $d \in D$. Define a mapping $\psi_d : C \to D$ with $\psi_d(c) = dc$. Then we have $\psi_d \in \operatorname{Hom}_R(C, D) = 0$ and $dC = \operatorname{Im}\psi_d = 0$. It means that for all $d \in D$, $dC = 0$. □

Proposition 2.3.3. *Let M be a right R-module, then the following statements are equivalent:*

(i) M is semisimple.

(ii) $M = \bigoplus_I M_i$, where M_i is a simple submodule of M for each $i \in I$.

(iii) $M = \sum_I M_i$, where M_i is a simple submodule of M for each $i \in I$.

(iv) Every submodule of M is a direct summand of M, i.e., for some submodules L, N of M, we have $M = N \oplus L$.

Definition 2.3.4. An R-module M is called semisimple if any of the equivalent conditions in Proposition 2.3.3 holds. We call the ring R semisimple in case R_R is semisimple.

Proposition 2.3.5. *[3, Proposition 1.17] The class of all semisimple right R-modules is closed under taking submodules, factor modules, and direct sums.*

2.4 The Socle and The Radical

Definition 2.4.1. Let M be a right R-module, the socle of M is the sum of all simple submodules of M. We write $\mathrm{Soc}(M)$ to denote the socle of M.

By definition it is clear that the socle of M is the largest semisimple submodule of M and note that M is semisimple if and only if $\mathrm{Soc}(M) = M$.

Definition 2.4.2. Let M be a right R-module. The radical of M is defined as the intersection of all maximal submodules of M. We denote the radical of M by $\mathrm{rad}(M)$.

Proposition 2.4.3. *[2, Proposition 9.14, 9.8] Let M and A be right R-modules and let $f : M \longrightarrow A$ be an R-homomorphism. Then $f(\mathrm{rad}(M)) \leq \mathrm{rad}(A)$ and $f(\mathrm{Soc}(M)) \leq \mathrm{Soc}(A)$.*

Definition 2.4.4. Let M be a right R-module. Then the top of M is defined as $M/\operatorname{rad} M$, denoted $\operatorname{top} M$.

Lemma 2.4.5. *Let M and N be right R-modules. If $M \cong N$, then $\operatorname{top} M \cong \operatorname{top} N$.*

As we shall see later in Lemma 2.12.7, the converse of Lemma 2.4.5 is also true for finitely generated projective modules.

2.5 Essential and Small Submodules

Definition 2.5.1. Let A be a nonzero submodule of the R-module M. Then we say that A is essential in M or M is an essential extension of A if for each submodule B of M, $A \cap B = 0$ implies $B = 0$. In this case, we denote $A \leq_e M$. Dually, a submodule S of M is called small or superfluous in M if, for any submodule L of M, $L + S = M$ implies $L = M$. In this case, we write $S \ll M$.

Proposition 2.5.2. *[2, Proposition 5.16] Let M be a right R-module with submodules $A \leq B \leq M$ and $C \leq M$. Then,* *(i)* $A \leq_e M$ *if and only if* $A \leq_e B$ *and* $B \leq_e M$*;*

(ii) $C \cap A \leq_e M$ *if and only if* $C \leq_e M$ *and* $A \leq_e M$*.*

Definition 2.5.3. Let M be a right R-module and $N \leq M$. If a submodule C of M is maximal with respect to condition $C \cap N = 0$, which means that $C \subseteq C'$ and $C' \cap N = 0$ implies $C = C'$, then C is called a complement to N (in M), denoted $C \subseteq_c N$. If C has no proper essential extensions in M, we say that C is essentially closed in M.

Proposition 2.5.4. *[2, Proposition 5.21] Let M be a right R-module and $A' \leq M$. If A' is a complement to A, then*

(i) $A \bigoplus A' \leq_e M$*;*

(ii) $(A \bigoplus A')/A' \leq_e M/A'$*.*

Proposition 2.5.5. *[4, Proposition 6.22] Assume that $C \subseteq_c M$ and that T is a submodule of M such that $C \cap T = 0$. Then C is a complement to T if and only if $C \oplus T \leq_e M$.*

Proposition 2.5.6. *[4, Proposition 6.32] Let M be a right R-module and $C \leq M$. Then the following statements are equivalent:*

(i) $C \subseteq_c M$*.*

(ii) C *is essentially closed in* M*.*

Proposition 2.5.7. *[2, Proposition 5.17] Let M be a right R-module with submodules $A \leq B \leq M$ and $C \leq M$. Then,*

(i) $B \ll M$ *if and only if* $A \ll M$ *and* $B/A \ll M/A$*;*

(ii) $C + A \ll M$ *if and only if* $C \ll M$ *and* $A \ll M$*.*

Proposition 2.5.8. *[5, Proposition 3.16] Let A be a submodule of the right R-module M. Then* $\mathrm{Soc}(A)$ *is the submodule of* $\mathrm{Soc}(M)$*. Moreover, if A is essential in M, then* $\mathrm{Soc}(A) = \mathrm{Soc}(M)$.

Proof. The first claim is obvious. Let $A \leq_e M$. Now consider a simple submodule B of M. It is enough to show that B is also a submodule of A. Since A is essential in M, we obtain $A \cap B \neq 0$, and so $B \subseteq A$. Hence, $Soc(A) = Soc(M)$. □

Proposition 2.5.9. *[5, Proposition 3.17] Let M be a right R-module. Then the following statements are equivalent:*

(i) Any nonzero submodule of M contains a simple submodule;

(ii) $\mathrm{Soc}(M)$ is essential in M.

Proof. We assume that (i) holds. If A is a nonzero submodule of M, then A contains a simple submodule, say U. Then we obtain $0 \neq U \subseteq \mathrm{Soc}(M) \cap A \neq 0$. This shows that $\mathrm{Soc}(M)$ is essential in M. Conversely, let $\mathrm{Soc}(M)$ is essential in M. If A is a nonzero submodule of M, then $A \cap \mathrm{Soc}(M) \neq 0$. Since $A \cap \mathrm{Soc}(M)$ is the submodule of semisimple module $\mathrm{Soc}(M)$, then $A \cap \mathrm{Soc}(M)$ is also semisimple. Hence, $A \cap \mathrm{Soc}(M)$ contains a simple submodule. This shows that A also contains a simple submodule. □

The following corollary provides a useful characterizations of the socle and radical.

Corollary 2.5.10. *For a right R-module M, the following statements satisfy.*

(i) $\mathrm{Soc}(M) = \bigcap\{K \leq M : K$ *is essential in* ***M*** $\}$

(ii) $\mathrm{rad}(M) = \sum\{K \leq M : K$ *is small in* ***M*** $\}$

Corollary 2.5.11. *[6, Corollary 10] Let M be a right R-module. Then M is semisimple if and only if it contains no proper essential submodules.*

2.6 Noetherian and Artinian Modules

Definition 2.6.1. A right R-module N is called Artinian if for every descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_i \supseteq N_{i+1} \supseteq \cdots$ of submodules of N, there exists an $n \in \mathbb{N}$ such that $N_{n+i} = N_n$ $(i = 1, 2, \ldots)$.

A right R-module M is called Noetherian if for every ascending chain $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq M_{i+1} \subseteq \cdots$ of submodules of M, there exists an $n \in \mathbb{N}$ such that $M_{n+i} = M_n$ $(i = 1, 2, \ldots)$.

Proposition 2.6.2. *[7, Proposition 2.4.3] The following statements are equivalent for a right R-module M:*

(i) M is Noetherian;

(ii) *any submodule of M is finitely generated;*

(iii) *any nonempty set of submodules of M has a maximal member.*

Proposition 2.6.3. *[7, Proposition 2.4.4] The following statements are equivalent for a right R-module M:*

(i) *M is Artinian;*

(ii) *any nonempty set of submodules of M has a minimal member.*

Corollary 2.6.4. *[2, Corollary 10.11] Let M be a right R-module. If M is Artinian, then M contains a simple module. Moreover,* $\operatorname{Soc}(M) \leq_e M$.

Corollary 2.6.5. *[2, Corollary 10.16] For a semisimple right R-module S the following statements are equivalent:*

(i) *S is Artinian;*

(ii) *S is Noetherian;*

(iii) *S is finitely generated.*

Proposition 2.6.6. *[2, Proposition 10.15] For a right R-module S the following assertions are equivalent:*

(i) $\operatorname{rad} S = 0$ *and S is Artinian;*

(ii) *S is the direct sum of a finite set of simple submodules;*

(iii) *S is semisimple and finitely generated;*

(iv) *S is semisimple and Noetherian.*

Lemma 2.6.7. *Every nonzero right ideal is essential in a right Noetherian domain.*

Proof. Let I be a non zero right ideal in R. Assume on the contrary that $I \cap aR = 0$ for $0 \neq a \in R$. Since $I \neq 0$, there exists $0 \neq b \in I$ such that $bR \cap aR = 0$. We shall prove $\sum_{n=1}^{\infty} a^n bR$ is a direct sum. Let

$$a^k br_0 + a^{k+1} br_1 + \cdots + a^{k+t} br_t = 0 \text{ with } a^k \neq 0 \text{ and } r_i \in R \text{ for } i = 0, \ldots, t.$$

Then we can write $br_0 + abr_1 + \cdots + a^t br_t = 0$ and see that

$$br_0 = -a(br_1 + \cdots + a^{t-1} = 0) \in aR \cap bR.$$

Consequently, $br_0 = 0$. An inductive argument shows that $br_1, \ldots, br_t$ are all zero. Hence, $\sum_{n=1}^{\infty} a^n bR$ is a direct sum. Then we have

$$abR \subset abR \oplus a^2bR \subset abR \oplus a^2bR \oplus a^3bR \subset \cdots .$$

But this is a contradiction since R is right Noetherian. Hence, I must be essential in R_R. □

Lemma 2.6.8. *A commutative Artinian integral domain is a field.*

Proof. Let u be a nonzero element in R. It is enough to show that the inverse of u exists in R. We can write a descending chain of ideals of R

$$(u) \supseteq (u^2) \supseteq \cdots \supseteq (u^i) \supseteq (u^{i+1}) \supseteq \cdots .$$

Then this chain must terminate by assumption. Namely, there is an integer n such that $(u^n) = (u^{n+1}) = \cdots$. Then there is $b \in R$ such that $u^n = u^{n+1}b$. This shows that $u^n(1 - ub) = 0$. Since R is an integral domain and u is nonzero, we have $1 - ub = 0$. This shows that b is an inverse of u, which completes the proof. □

2.7 The Singular Submodule

Definition 2.7.1. Let M be a right R-module, the singular submodule of M consists of elements whose annihilators are essential right ideals of R and we denote it by $\mathrm{Z}(M)$, this means that $\mathrm{Z}(M) = \{m \in M \mid \mathrm{ann}_R(m) \leq_e R\}$. A right R-module M is called singular (respectively, nonsingular) if $\mathrm{Z}(M) = M$ (respectively, $\mathrm{Z}(M) = 0$).

We would like to note that nonsingular $\neq$ not singular. If a right R-module M is both singular and nonsingular, then it must be zero.

Example 2.7.2. *[4, Example 7.6] The following are examples of singular and nonsingular modules:*

(i) Let R be a simple ring, then it is nonsingular.

(ii) *For a commutative domain R, all nonzero ideals of R are essential. Therefore, for every R-module A,*

$$\mathrm{Z}(A) = \{a \in A : \mathrm{ann}(a) \neq 0\}$$

is just the torsion submodule of A. Moreover, A is singular if and only if A is torsion, and A is nonsingular if and only if A is torsion-free.

Proposition 2.7.3. *[8, Proposition 1.5] For right R- modules A, B and C the following statements satisfy:*

(a) *A module C is singular if and only if there is a right R-module B and an essential submodule A of B such that $C \cong B/A$.*

(b) *If $A \leq B$ and B is nonsingular, then B/A is singular if and only if $A \leq_e B$.*

Proposition 2.7.4. *[8, Proposition 1.6] If R is a right nonsingular ring, then for any right R-module M, we have $\mathrm{Z}(M/\mathrm{Z}(M)) = 0$.*

Proposition 2.7.5. *[8, Proposition 1.7] Let R be a right nonsingular ring. For right R-modules M and N, the following statements satisfy:*

(i) *A module M is singular if and only if $\mathrm{Hom}_R(M, N) = 0$ for all nonsingular modules N.*

(ii) *A module N is nonsingular if and only if $\mathrm{Hom}_R(M, N) = 0$ for all singular modules M.*

(iii) *The class of singular right R-modules is closed under taking submodules, factors, direct sums, and extensions.*

(iv) *The class of nonsingular right R-modules is closed under taking submodules, direct products, and extensions.*

2.8 Semi-artinian Rings and Modules

Definition 2.8.1. An R-module M is said to be semi-artinian if every nonzero homomorphic image of M has an essential socle. We call the ring R right (resp., left) semi-artinian in case R_R (resp., ${}_RR$) is semi-artinian.

The following result is well-known.

Proposition 2.8.2. *The following conditions are equivalent for a ring R.*

(i) R *is a right semi-artinian ring.*

(ii) Any right R-module is semi-artinian.

(iii) Any cyclic right R-module is semi-artinian.

(iv) Every cyclic right R-module contains a simple right R-module.

Proposition 2.8.3. *[9, Proposition 5.2] Let R be a ring. If R is right Noetherian and right or left semi-artinian, then it is right Artinian.*

The property of being semi-artinian for a module is closely related to the concept of the socle series (see [9]). The terms of this series associated with ordinals are constructed iteratively. Now the socle series (or Loewy series) of M is the chain of submodules

$$\mathrm{Soc}_0(M) \leq \mathrm{Soc}_1(M) \leq \cdots \leq \mathrm{Soc}_\alpha(M) \leq \mathrm{Soc}_{\alpha+1}(M) \leq \cdots,$$

where $\mathrm{Soc}_{\alpha+1}(M)/\mathrm{Soc}_\alpha(M) = \mathrm{Soc}(M/\mathrm{Soc}_\alpha(M))$ for every ordinal α, and if α is a limit ordinal, then

$$\mathrm{Soc}_\alpha(M) = \bigcup_{\beta<\alpha} \mathrm{Soc}_\beta(M).$$

By contsruction, there must exist a smallest ordinal $\lambda \leq |2^M|$ such that $\mathrm{Soc}_\lambda(M) = \mathrm{Soc}_{\lambda+1}(M)$. The following well-known result shows how we can use the socle series of a module to determine if it is semi-artinian.

Proposition 2.8.4. *[10, Proposition 1] Let R be any ring. Then the following statements are equivalent for any right R-module M:*

(i) M is a semi-artinian right R-module.

(ii) Every nonzero homomorphic image of M has nonzero socle.

(iii) $\mathrm{Soc}_\lambda(M) = M$ *for some ordinal λ.*

It is clear that any right Artinian ring is right semi-artinian; but not vice versa (for interesting nontrivial examples, we refer the reader to §4 of [11]). More generally, every Artinian right R-module is semi-artinian. Indeed, if M is Artinian, then every factor of M is also

Artinian. Hence, every nonzero submodule of nonzero factor of M is also Artinian, and so contains a simple submodule. However, the class of semi-artinian modules is broader since it contains arbitrary direct sums of Artinian modules since the class of semi-artinian module is closed under taking arbitrary direct sums. Note also that the class of semi-artinian modules is closed under taking submodules and factor modules, a fact which can be easily deduced from the definition.

2.9 Semiprime Rings

Before giving the definition of a semiprime ring it is useful to define semiprime ideals.

Definition 2.9.1. A proper ideal I of R is called semiprime if, for an ideal J of R and some positive integer k, $J^k \subseteq I$ implies that $J \subseteq I$.

Note that semiprime ideals are precisely those ideals which are intersections of prime ideals. In particular, every prime ideal is semiprime.

Definition 2.9.2. A ring R is called a semiprime ring provided (0) is a semiprime ideal. Equivalently, if R has no nonzero nilpotent ideals then R is called semiprime.

Theorem 2.9.3. *[12, Theorem 10.24] The following conditions are equivalent for any ring R.*

(i) R is semiprime and right Artinian.

(ii) R is semisimple.

(iii) R is semiprime and satisfies the descending chain condition on principal right ideals.

2.10 Local and Semilocal Rings

Definition 2.10.1. A ring R is called local if R has a unique maximal right ideal.

Note that if R is a local ring then R has no nontrivial idempotents.

Definition 2.10.2. An idempotent element e of the ring R is called local if eRe is a local ring.

Proposition 2.10.3. *[12, Proposition 2.18] Let e be an idempotent in R and let $J = \operatorname{rad} R$. Then the following conditions are equivalent:*

(i) *e is a local idempotent in R.*

(ii) *eR/eJ is a simple right R-module.*

(iii) *eJ is the unique maximal submodule of eR.*

Proposition 2.10.4. *[12, Corollary 23.12] Let R be a commutative ring. Then the following conditions are equivalent;*

(i) *R is an Artinian ring;*

(ii) *R is a finite direct product of Artinian local rings;*

(iii) *R is Noetherian, with Krull dimension zero (i.e., all prime ideals of R are maximal ideals).*

Definition 2.10.5. A ring R is said to be semilocal if $R/\operatorname{rad} R$ is a right Artinian ring, or, equivalently, if $R/\operatorname{rad} R$ is a semisimple ring.

Proposition 2.10.6. *[12, Proposition 20.2] Consider the following two conditions for a ring R:*

(i) *R is semilocal.*

(ii) *R has finitely many maximal right ideals.*

We have, in general $(ii) \Rightarrow (i)$. The converse holds when $R/\operatorname{rad} R$ is commutative.

Notice that, any local or right Artinian ring is semilocal.

2.11 Categories and Functors

Definition 2.11.1. A category $\mathscr{C}$ is given by;

1. a class of objects, $Ob(\mathscr{C})$,
2. a set of morphisms $\operatorname{Mor}(M, N)$ for every ordered pair (M, N) of objects,

3. a composition of morphisms, $\mathrm{Mor}(M,N) \times \mathrm{Mor}(N,K) \longrightarrow \mathrm{Mor}(M,K)$, denoted $(\alpha,\beta) \longrightarrow \beta\alpha$ where $\alpha \in \mathrm{Mor}(M,N)$, $\beta \in \mathrm{Mor}(N,K)$ such that

   (i) every $\alpha \in \mathrm{Mor}(M,N)$ has a unique domain M and target N,

   (ii) composition is associative, that is, given morphisms $\alpha \in \mathrm{Mor}(M,N)$, $\beta \in \mathrm{Mor}(N,K)$, $\gamma \in \mathrm{Mor}(K,L)$, then $\gamma(\beta\alpha) = (\gamma\beta)\alpha$,

   (iii) for any object $M \in \mathscr{C}$, there exists an identity morphism $1_M \in \mathrm{Mor}(M,M)$ such that $\alpha 1_M = \alpha$ and $1_N\alpha = \alpha$ for all $\alpha : M \longrightarrow N$.

Example 2.11.2. *The category of right R-modules, denoted* Mod-R*, is the category whose objects are all right R-modules and whose morphisms are all right R-module homomorphisms. The composition of the morphisms is the usual composition.*

Example 2.11.3. *The category* Ab *contains the abelian groups as objects and group homomorphisms as morphisms. The composition of the morphisms is the usual composition.*

Definition 2.11.4. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a function such that

(i) if $M \in Ob(\mathscr{C})$, then $F(M) \in Ob(\mathscr{D})$;

(ii) if $\alpha : M \longrightarrow N$ in $\mathscr{C}$, then $F(\alpha) : F(M) \longrightarrow F(N)$ in $\mathscr{D}$;

(iii) if $M \xrightarrow{\alpha} N \xrightarrow{\beta} K$ in $\mathscr{C}$, then $F(M) \xrightarrow{F(\alpha)} F(N) \xrightarrow{F(\beta)} F(K)$ in $\mathscr{D}$ and

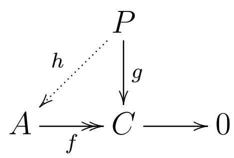
$$F(\beta\alpha) = F(\beta)F(\alpha);$$

(iv) $F(1_M) = 1_{F(M)}$ for every $M \in Ob(\mathscr{C})$.

Definition 2.11.5. A functor $F : \mathrm{Mod\text{-}R} \longrightarrow \mathrm{Ab}$ is called an additive functor if $F(\alpha+\beta) = F(\alpha) + F(\beta)$ for every pair of R-morphisms $\alpha, \beta : A \longrightarrow B$.

2.12 Projective Modules

Definition 2.12.1. A right R-module P is called projective if for every module epimorphism $f : A \longrightarrow C$ and module homomorphism $g : P \longrightarrow C$, there exists a homomorphism $h : P \longrightarrow A$ such that the below diagram commute, that is, $f \circ h = g$.



Proposition 2.12.2. *[13, Proposition 3.3] A right R-module P is projective if and only if every short exact sequence* $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{g} P \longrightarrow 0$ *splits.*

Definition 2.12.3. A ring R is called hereditary if every submodule of projective R-module is also projective.

Lemma 2.12.4. *[2, Corollary 16.11] Let* $\{M_\alpha\}_{\alpha \in A}$ *be an indexed set of right R-modules. Then* $\bigoplus_A M_\alpha$ *is projective module if and only if each* M_α *is projective.*

Proposition 2.12.5. *[3, Proposition 1.24] If S is a simple right R-module, then S is either singular or projective, but not both.*

Proof. Let S be a simple right R-module. For some maximal right ideal M of R, we have $S \cong R/M$. We know that S is singular if and only if M is an essential submodule in R. If S is not singular, we must have $N \cap M = 0$ for some nonzero right ideal N of R. Since M is a maximal right ideal, we have $N \oplus M = R$ and hence S is projective. $\square$

Corollary 2.12.6. *[3, Corollary 1.25] Every nonsingular semisimple right R-module is projective.*

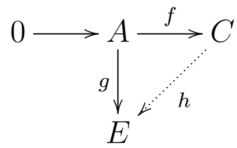
Proof. Let $S = \bigoplus S_i$ be a semisimple right R-module such that S_i is simple module. If $\mathrm{Z}(S) = 0$, then each S_i is nonsingular and hence projective by Proposition 2.12.5. Then by Lemma 2.12.4, S is projective. $\square$

Lemma 2.12.7. *[12, Lemma 19.27] Let R be any ring and* $\overline{R} = R/J$, *where J is an ideal of R contained in* $\operatorname{rad} R$. *Let A, B be finitely generated projective right R-modules. Then* $A \cong B$ *as R-modules if and only if* $A/AJ \cong B/BJ$ *as* $\overline{R}$*-modules.*

2.13 Injective Modules and Related Concepts

In this chapter, we will focus on injective modules that are of particular importance for this thesis and give some auxiliary notions that will be needed in the next chapter.

Definition 2.13.1. A right R-module E is said to be injective if for every module monomorphism $f : A \longrightarrow C$ and module homomorphism $g : A \longrightarrow E$ there is a homomorphism $h : C \longrightarrow E$ such that the following diagram commute, that is, $h \circ f = g$.



Definition 2.13.2. A ring R is said to be self-injective provided R_R is an injective R-module.

Lemma 2.13.3. *Every self-injective integral domain is a field.*

Proposition 2.13.4. *[13, Proposition 3.25] A right R-module E is injective if and only if $\mathrm{Hom}_R(-, E)$ is an exact functor.*

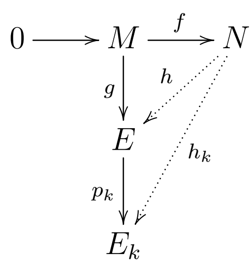
Proposition 2.13.5. *[13, Proposition 3.40] A right R-module E is injective if and only if every short exact sequence $0 \longrightarrow E \longrightarrow M \longrightarrow N \longrightarrow 0$ splits.*

Proposition 2.13.6. *[13, Proposition 3.28] The following statements satisfy for any ring R:*

(i) If $(E_k)_{k \in K}$ is a family of injective right R-modules, then $\prod_{k \in K} E_k$ is also an injective right R-module.

(ii) Any direct summand of an injective right R-module E is injective.

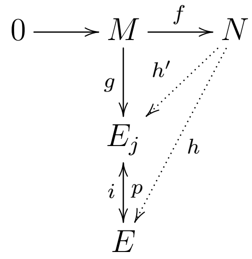
Proof. (i) Let E denote the product $\prod E_k$ and let $f : M \longrightarrow N$ be a monomorphism and $g : M \longrightarrow E$ be a homomorphism. Since E_k is an injective module, then for the composition homomorphism $p_k \circ g : M \longrightarrow E_k$ where p_k denotes the usual projection of E onto E_k, we can find $h_k : N \longrightarrow E_k$ such that $p_k \circ g = h_k \circ f$.



We can define a map $h : N \longrightarrow E$ with $h : n \longrightarrow (h_k(n))$.

The map h does extend g, for if $n = f(m)$ with $m \in M$ and $n \in N$, then $h(fm) = (h_k(fm)) = (p_k gm) = gm$, since $x = p_k(x)$ for every $x \in E$.

(ii) Let E_j be a direct summand of an injective R-module E and $f : M \longrightarrow N$ be a monomorphism and $g : M \longrightarrow E_j$ be a homomorphism. Since E is an injective module, then for the composition homomorphism $i \circ g : M \longrightarrow E$ there is $h : N \longrightarrow E$ such that $i \circ g = h \circ f$.



Consider the homomorphism $h' = p \circ h : N \longrightarrow E_j$, then we obtain

$$h' \circ f = p \circ h \circ f = p \circ i \circ g = 1_{E_j} \circ g = g.$$

□

Corollary 2.13.7. *The direct sum of finitely many injective right R-modules is injective.*

Theorem 2.13.8. *[2, Theorem 25.1] The following statements are equivalent for an injective right R-module E:*

(i) $E^{(U)}$ *is injective for all sets U.*

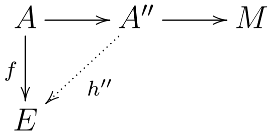
(ii) $E^{(\mathbb{N})}$ *is injective.*

Theorem 2.13.9. *[13, Theorem 3.30](Baer Criterion) A right R-module E is injective if and only if every R-homomorphism $f : I \longrightarrow E$, where I is an ideal in R, can be extended to a homomorphism $R \longrightarrow E$.*

Proof. Let A be a submodule of an R-module M and $f : A \longrightarrow E$. Let

$$\psi = \{(A', h') : A \subseteq A' \subseteq M, h' \in \mathrm{Hom}_R(A', E) \text{ such that } h'_{|A} = f\}$$

be a partially ordered set ordered by $(A', h') \leq (A'', h'')$ if and only if $A' \subseteq A''$ and $h''_{|A'} = h'$. We know that ψ has a maximal element by Zorn's Lemma, say (A'', h''). Consider the following diagram:



It is enough to show that $A'' = M$. Assume $A'' \neq M$. Then there exist $x \in M \setminus A''$. Let $I = \{r \in R : xr \in A''\}$ be a right ideal. We define $\lambda : I \longrightarrow E$ with $\lambda(r) = h''(xr)$. By assumption there exists $\bar{\lambda} : R \longrightarrow E$ which extends λ. Now define $\bar{h} : A'' + xR \longrightarrow E$ with

$$\bar{h}(y + xr) = h''(y) + \bar{\lambda}(r)$$

where $y \in A''$ and $r \in R$. It is not difficult to see that $\bar{h}$ is well defined because $y + xr = 0$ implies that $xr \in A''$ and $\bar{h}_{|A''} = h''$. Therefore, $(A'' + xR, \bar{h}) \in \psi$. However this is a contradiction. In this case, $A'' = M$, and so E is injective. The converse follows from the definition. □

Theorem 2.13.10. *[12, Theorem 2.9] The following statements on a ring R are equivalent:*

(i) R is a semisimple ring.

(ii) All finitely generated right R-modules are injective.

(iii) All right R-modules are injective.

(iv) All cyclic right R-modules are injective.

Proposition 2.13.11. *[13, Proposition 3.43] A right R-module E is injective iff E has no proper essential extensions.*

Proposition 2.13.12. *[13, Lemma 3.44] Let M be a right R-module. Then the following statements are equivalent for a module E containing M.*

(i) E is a maximal essential extension of M; i.e, no proper extension of E is an essential extension of M.

(ii) E is an injective module and M is essential in E.

(iii) E is an injective module and there is no proper injective submodule E'; that is, there is no injective E' with $M \subseteq E' \subsetneq E$.

Definition 2.13.13. A right R-module E containing a right R-module M is said to be an injective envelope (hull) of M, denoted $\mathrm{E}(M)$, if one of the equivalent conditions in Proposition 2.13.12 satisfies.

Theorem 2.13.14. *[2, Theorem 18.10] Every module has an injective envelope. It is unique up to isomorphism.*

Proposition 2.13.15. *[2, Proposition 18.12] In the category of right R-modules for a ring R, the following conditions hold:*

(i) I *is injective if and only if* $I = \mathrm{E}(I)$.

(ii) If $I \leq_e N$, *then* $\mathrm{E}(I) = \mathrm{E}(N)$.

(iii) If $I \leq K$, *with* K *injective, then* $K = \mathrm{E}(I) \oplus E'$ *for some* $E' \leq K$.

(iv) If $\bigoplus_A \mathrm{E}(M_\alpha)$ *is injective (for instance, if* A *is finite), then*

$$\bigoplus_A \mathrm{E}(M_\alpha) = \mathrm{E}(\bigoplus_A M_\alpha).$$

Proposition 2.13.16. *[2, Proposition 18.13] For any ring R, the following statements are equivalent:*

(i) Every direct sum of injective right R-modules is injective;

(ii) If $(M_\alpha)_{\alpha \in A}$ *is an indexed set of right R-modules, then*

$$\bigoplus_A \mathrm{E}(M_\alpha) = \mathrm{E}(\bigoplus_A M_\alpha);$$

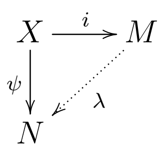
(iii) R *is a right Noetherian ring.*

Lemma 2.13.17. *[3, Exercise 17, Ch.1, Sec. B] Let R be a commutative Noetherian ring, and let P, Q be prime ideals of R. Then, $P \subseteq Q$ iff* $\mathrm{Hom}_R(\mathrm{E}(R/P), \mathrm{E}(R/Q)) \neq 0$.

Proposition 2.13.18. *[5, Proposition 3.16] Let M be a right R-module. Then* $\mathrm{Soc}(M) = \mathrm{Soc}(\mathrm{E}(M))$.

2.13.1 Relative Injectivity

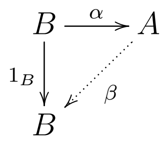
Definition 2.13.19. Let M be a right R-module. A module N is called M-injective if for any $X \leq M$, any homomorphism $\psi : X \longrightarrow N$ can be extended to a homomorphism $\lambda : M \longrightarrow N$.



Notice that E is an injective module if and only if it is injective relative to every R-module.

Lemma 2.13.20. *[14, Lemma 1.2] If B is A-injective, then any monomorphism $\alpha : B \longrightarrow A$ splits. If, also, A is indecomposable, then α is an isomorphism.*

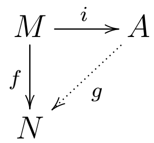
Proof. There is an R-homomorphism $\beta : A \longrightarrow B$ such that the following diagram commutes.



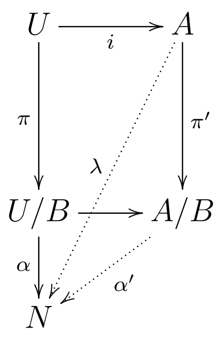
Now $\beta \circ \alpha = 1_B$, and so $A = \alpha(B) \oplus \ker \beta$. The last statement is clear. □

Proposition 2.13.21. *[14, Proposition 1.3] Let N be an A-injective module. If $B \leq A$, then N is both B-injective and A/B-injective.*

Proof. For a right R-module M, let $f : M \longrightarrow N$ and let M be a submodule of B. Since N is A-injective, then there is $g : A \longrightarrow N$ such that $g \circ i = f$.



The restriction $g|_B : B \longrightarrow N$ extends f. This shows that N is B-injective. To prove N is A/B-injective, let $B \subseteq U \subseteq A$ and let $\alpha : U/B \longrightarrow N$ be a homomorphism. Now consider the projection maps $\pi : U \longrightarrow U/B$ and $\pi' : A \longrightarrow A/B$. We can build the following diagram:



It means that there exists $\lambda : A \longrightarrow N$ which extends $\alpha \circ \pi$. In this case $\lambda(B) = (\alpha \circ \pi)(B) = \alpha(0) = 0$. This shows that $\ker \pi' \leq \ker \lambda$, and so there is $\alpha' : A/B \longrightarrow N$ such that $\alpha' \circ \pi' = \lambda$. Therefore for all $u \in U$, we have

$$\alpha'(u + B) = (\alpha' \circ \pi')(u) = \lambda(u) = (\alpha \circ \pi)(u) = \alpha(u + B).$$

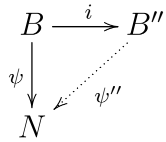
Hence, α' extends α, and this shows that N is A/B-injective. $\square$

Lemma 2.13.22. *[15, Lemma 1.11] Let $A = \prod_{i\in I} A_i$ and B be right R-modules. Then A is B-injective if and only if A_i is B-injective for each $i \in I$.*

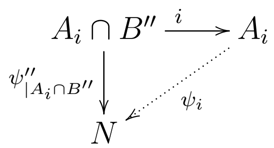
Proof. Follows by using similar arguments given in the proof of the Proposition 2.13.6. $\square$

Proposition 2.13.23. *[14, Proposition 1.5] A module N is $(\bigoplus_{i\in I} A_i)$-injective if and only if N is A_i-injective for every $i \in I$.*

Proof. Assume that N is A_i-injective for each $i \in I$. We shall denote $A = \bigoplus_{i\in I} A_i$. Let $B \subseteq A$ and let $\psi : B \longrightarrow N$ be a homomorphism. Let $\Gamma = \{(B', \psi') | B \subseteq B' \subseteq A$, $\psi' \in \mathrm{Hom}_R(B', N)$ such that $\psi'_{|B} = \psi\}$ ordered by $(B', \psi') \leq (B'', \psi'')$ if and only if $B' \subseteq B''$ and $\psi''_{|B'} = \psi'$. Let (B'', ψ'') be a maximal member of Γ guaranteed by Zorn's Lemma.



It is enough to show that $B'' = A$. Since N is A_i-injective, then there is $\psi_i : A_i \longrightarrow N$ such that $\psi_{i|A_i\cap B''} = \psi''_{|A_i\cap B''}$.



We define $\psi'_i : A_i + B'' \longrightarrow N$ by $\psi'_i(a_i + b) = \psi_i(a_i) + \psi''_{|A_i\cap B''}(b)$, where $a_i \in A_i$ and $b \in B''$.

$$a_i + b = a'_i + b' \Rightarrow$$
$$a_i - a'_i = b' - b \in A_i \cap B'' \Rightarrow$$
$$\psi_i(a_i - a'_i) = \psi_i(b' - b) = \psi''_{|A_i\cap B''}(b' - b) \Rightarrow$$
$$\psi_i(a_i) + \psi''_{|A_i\cap B''}(b) = \psi_i(a'_i) + \psi''_{|A_i\cap B''}(b') \Rightarrow$$
$$\psi'_i(a_i + b) = \psi'_i(a'_i + b').$$

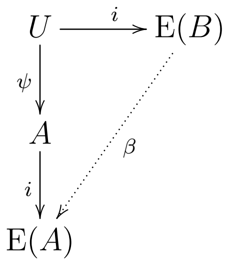
This shows that ψ'_i is well-defined. We now check that ψ'_i is an extension of ψ, as follows:

$$\psi'_i(b) = \psi'_i(0 + b) = \psi_i(0) + \psi''_{|A_i\cap B''}(b) = 0 + \psi(b) = \psi(b)$$

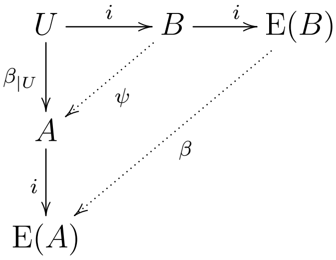
Since (B'', ψ'') is maximal, we obtain $A_i + B'' = B''$, and so $A_i \subseteq B''$ for all $i \in I$. Hence, $A = B''$. To prove the converse, Proposition 2.13.21 together with Lemma 2.13.22 is sufficient. □

Proposition 2.13.24. *[15, Lemma 1.14] Let A and B be right R-modules. Then A is B-injective if and only if $\beta(B) \subseteq A$ for every R-homomorphism $\beta : \mathrm{E}(B) \longrightarrow \mathrm{E}(A)$.*

Proof. Suppose that $\beta(B) \subseteq A$ for every R-homomorphism $\beta : \mathrm{E}(B) \longrightarrow \mathrm{E}(A)$. Let $U \subseteq B$ and $\psi : U \longrightarrow A$ be an R-homomorphism.



Since $\mathrm{E}(A)$ is an injective module, then there exists $\beta : \mathrm{E}(B) \longrightarrow \mathrm{E}(A)$ such that $\beta_{|U} = \psi$. It follows that, by hypothesis, $\beta_{|B} : B \longrightarrow A$ extends ψ, which implies that A is B-injective. Conversely, assume that A is B-injective and let $\beta : \mathrm{E}(B) \longrightarrow \mathrm{E}(A)$ be a nonzero R-homomorphism. Now consider the set $U = \{b \in B | \beta(b) \in A\}$. Since $A \leq_e \mathrm{E}(A)$, $\operatorname{Im}\beta \cap A \neq 0$; hence $U \neq 0$ as $B \leq_e \mathrm{E}(B)$ and $U = B \cap \beta^{-1}(A)$.



Since A is B-injective, there is $\psi : B \longrightarrow A$ such that $\psi_{|U} = \beta_{|U}$. It is enough to show that $(\beta - \psi)(B) = 0$. Let us show that $M \cap ((\beta - \psi)(B)) = 0$. If $a = (\beta - \psi)(b)$, where $a \in A$ and $b \in B$, then $\beta(b) = \psi(b) + a \in A$. Hence, $b \in U$. Consequently, $a = (\beta - \psi)(b) = 0$ for all $a \in A$ and $b \in B$. In this case, $A \cap (\beta - \psi(B)) = 0$. But since A is essential in $\mathrm{E}(A)$, we have $(\beta - \psi)(B) = 0$. This shows $\beta(B) \subseteq A$.

□

Definition 2.13.25. Let A and B be right R-modules. We define the trace of B in A by $\mathrm{Tr}(B, A) := \sum\{f(B) : f \in \mathrm{Hom}_R(B, A)\}$. If $\mathscr{U}$ is a class of right R-modules, then we define the trace of $\mathscr{U}$ in A as $\mathrm{Tr}(\mathscr{U}, A) := \sum_{B \in \mathscr{U}} \mathrm{Tr}(B, A)$.

Lemma 2.13.26. *Let A and B be right R-modules. A module A is B-injective if and only if* $\mathrm{Tr}(B, \mathrm{E}(A)) \subseteq A$.

Proof. Easily follows from Proposition 2.13.24. □

Lemma 2.13.27. *Let M be a right R-module and let E be an injective module. Then* $\mathrm{Tr}(M, E)$ *is M-injective.*

Proof. Let $T = \mathrm{Tr}(M, E)$. It is enough to show that $\mathrm{Tr}(M, \mathrm{E}(T)) \subseteq T$. Let $f : M \to \mathrm{E}(T)$ be a homomorphism. We already know that $\mathrm{E}(T)$ is the minimal injective module containing T by Proposition 2.13.12, and so $\mathrm{E}(T)$ is embedded in the injective module E. We can regard E as containing $\mathrm{E}(T)$. Thus, we have $f : M \longrightarrow E$. This gives that $\mathrm{Im}\, f \subseteq Tr(M, E) = T$, and so $Tr(M, E)$ is M-injective. □

2.13.2 Quasi-Injective Modules

Definition 2.13.28. A right R-module M is said to be quasi-injective if it is injective relative to itself.

Lemma 2.13.29. *[2, Corollary 1.14] A right R-module M is quasi-injective if and only if M is a fully invariant submodule of* $\mathrm{E}(M)$.

There are natural example of quasi-injective modules, namely injective modules. Now we give some other examples.

Example 2.13.30. *Any simple right R-module is quasi-injective. However, it need not always be injective. Moreover, any semisimple module is quasi-injective.*

Example 2.13.31. *[4, Example 6.72] Let R be a commutative PID. Then any proper cyclic module is quasi injective.*

Lemma 2.13.32. *[15, Lemma 1.17] If M is quasi-injective, then every direct summand N of M is also quasi-injective.*

Proof. Suppose that $M = N \oplus S$ is quasi- injective. Then by Lemma 2.13.22, N is M-injective. By Proposition 2.13.21, N is quasi-injective. □

Note that a direct sum of two quasi-injective modules need not always be quasi-injective in general. Now we give the following counterexample for this situation, which is given in [4].

Example 2.13.33. *Let $R = \mathbb{Z}$, $A = \mathbb{Q}$ and $A' = \mathbb{Z}_n$ for any natural number n. Note that A and A' are quasi-injective $\mathbb{Z}$-modules. However, $B = A \oplus A'$ is not quasi-injective. Indeed, let $C = \mathbb{Z} \oplus (0) \subseteq B$, and take $f \in \mathrm{Hom}_{\mathbb{Z}}(C, B)$ such that f takes $\mathbb{Z}$ to $\mathbb{Z}_n$ by the natural projection map. f cannot be extended to an endomorphism of B, since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_n) = 0$. Therefore, B cannot be quasi-injective.*

2.14 Some Rings Characterized by Homological Properties

Now, we will give some important classes of rings that are determined by the homological properties of their modules.

Definition 2.14.1. A right nonsingular ring R is called right SI provided every singular right R-module is injective.

Proposition 2.14.2. *[8, Proposition 3.1] Let R be a ring. Then the following properties are equivalent:*

(i) R is a right SI-ring.

(ii) All singular right R-modules are semisimple.

(iii) R/I is semisimple for all essential right ideals I of R.

Proof. $(i) \Rightarrow (ii)$: Let M be a singular right R-module. Then by assumption, all submodules of M are injective, and hence are direct summands of M. It follows that M is semisimple by Proposition 2.3.3.

$(ii) \Rightarrow (iii)$: If I is an essential right ideal of R, then R/I is singular by Proposition 2.7.3. Hence, it is semisimple.

$(iii) \Rightarrow (i)$: Suppose that M is a singular right R-module and that I is a right ideal of R. Let $f : I \to M$ be R-homomorphism. Note that, $I/\ker f$ is singular. Then $\ker f$ is essential by Proposition 2.7.3, and so $R/\ker f$ is semisimple. Observe that $I/\ker f$ is a direct summand of $R/\ker f$, it follows f extends to a homomorphism $g : R \to M$. This implies that M is injective by Theorem 2.13.9. Therefore R is right SI-ring. □

Proposition 2.14.3. *[8, Proposition 3.3] The following statements satisfy for a right SI-ring R.*

(i) $\operatorname{rad}(R) \subseteq \operatorname{Soc}(R_R)$.

(ii) $[\operatorname{rad}(R)]^2 = 0$.

(iii) $I^2 = I$ *for every essential right ideal* I *of* R.

(iv) R *is right hereditary.*

Proposition 2.14.4. *[8, Proposition 3.6] Let* R *be a right* SI*-ring, then* $R/\operatorname{Soc}(R_R)$ *is right Noetherian.*

Theorem 2.14.5. *[8, Theorem 3.11] A ring* R *is right* SI *if and only if there exists a ring decomposition* $R = U \times R_2 \times \cdots \times R_n$ *such that* $U/\operatorname{Soc}(U_U)$ *is a semisimple ring and each* R_i *is Morita equivalent to a right* SI*-domain.*

Definition 2.14.6. A ring R is called a right PCI-ring if every proper cyclic right R-module is injective.

Theorem 2.14.7. *[16, Theorem 5.2] A right* PCI*-ring* R *is either semisimple Artinian or a simple right semi-hereditary right Ore domain.*

Proposition 2.14.8. *[17] Let* R *be a right* PCI- *domain, then* R *is right Noetherian.*

Proposition 2.14.9. *For a domain, the* SI- *and* PCI- *conditions are equivalent.*

Proof. Let R be a right SI-domain and let R/I be a proper cyclic ideal of R for some right ideal I of R. If we show that R/I is singular, then we are done. If R is semisimple Artinian, then there is nothing to prove. Thus we suppose that R is not semisimple Artinian. First, let us prove that R is a Noetherian domain. Since R is a domain by Theorem 2.14.7, we have $\operatorname{Soc}(R) = 0$. Then, by Proposition 2.14.4, $R/\operatorname{Soc}(R_R)$ is Noetherian, implying that R is Noetherian. It follows that, by Lemma 2.6.7, I must be essential in R_R, and so R/I is singular. Since R is a right SI-ring, R/I is injective. Therefore, R is a right PCI-domain. Conversely, let R be a right PCI-domain. It is enough to show that every singular cyclic module is injective. Let S be a singular cyclic module. Then S must be proper because R_R cannot be singular. Hence, S is injective. □

Lemma 2.14.10. *Every injective cyclic right* R*-module is semisimple over right* PCI*-domains.*

Proof. Let R be a right PCI-domain and let U be an injective cyclic right R-module which is not isomorphic to R. Then we can write $U \cong R/I$ for some nonzero right ideal I of R. Note that I is essential in R by Lemma 2.6.7, and so U must be singular by Proposition 2.7.3. So every nonzero submodule of U is singular. Since the SI- and PCI- conditions are equivalent for a domain by Proposition 2.14.9, every submodule of U is injective, and thus a direct summand. □

Definition 2.14.11. A ring R is said to be right QI provided every quasi-injective right R-module is injective.

Definition 2.14.12. A ring R is called a right V-ring if every simple right R- module is injective.

Example 2.14.13. *By definition,* QI*-rings and right* PCI *domains arise as natural examples of right* V*-rings.*

Indeed, we have the following implications (see [18] and [19, Theorem 7])

$$Right\,\mathrm{PCI} \Longrightarrow Right\,\mathrm{QI} \Longrightarrow right\,\mathrm{V}\,.$$

Moreover, Boyle proved in [19, Theorem 5] that if R is a hereditary Noetherian ring, then R is a right QI-ring if and only if it is a right V-ring.

Lemma 2.14.14. *[15, Lemma 8.12] Let R be a ring. Then the following statements are equivalent:*

(i) R *is a right* V*- ring.*

(ii) $\mathrm{rad}(M_R) = 0$ *for any nonzero right R-module M.*

(iii) Any proper right ideal T of R is an intersection of maximal right ideals.

Corollary 2.14.15. *[15, Corollary 8.13] Let R be a right* V*-ring. Then the following statements satisfy:*

(i) $\mathrm{J}(R) = 0$.

(ii) If I is any ideal of R, then R/I is also a right V*- ring.*

Corollary 2.14.16. *Right or left* V*-rings are semiprime.*

Lemma 2.14.17. *Right Artinian right* V*-rings are semisimple.*

Proof. Let R be a right Artinian right V-ring. Then $\mathrm{Soc}(R_R)$ is essential in R_R, and it is injective. Therefore, $\mathrm{Soc}(R_R) = R_R$, completing the proof. $\square$

Definition 2.14.18. A ring R is called semiperfect if R is semilocal and idempotents of $R/radR$ can be lifted to R in the sense that given an idempotent η of $R/\operatorname{rad} R$, there is an idempotent $e \in R$ such that $\eta = e + \operatorname{rad} R$.

Example 2.14.19. *Local rings and one-sided Artinian rings can be given as examples of semiperfect rings.*

The following result shows how semiperfect rings are determined homologically.

Proposition 2.14.20. *[12, Theorem 24.16] Let R be any ring. R is a semiperfect ring if and only if every finitely generated right R-module M has a projective cover.*

Theorem 2.14.21. *[12, Theorem 23.6] A ring R is semiperfect if and only if the identity element 1 can be decomposed as $1 = e_1 + \cdots + e_n$, where the e_i's are mutually orthogonal local idempotents.*

2.15 Some Torsion Theory

Definition 2.15.1. Let $\mathscr{C}$ and $\mathscr{D}$ be module categories and let $\sigma : \mathscr{C} \longrightarrow \mathscr{D}$ and $\tau : \mathscr{C} \longrightarrow \mathscr{D}$ be functors. A functor τ is called a subfunctor of σ if it satisfies the following properties:

(i) $\tau(C) \leq \sigma(C)$ for all $C \in \mathscr{C}$,

(ii) $\tau(f) = \sigma(f)_{|\tau(C)}$ for all $f \in \mathrm{Hom}_{\mathscr{C}(C,D)}$.

By above definition, a functor $\tau : \mathscr{C} \longrightarrow \mathscr{C}$ will be a subfunctor of the identity functor on $\mathscr{C}$ provided the following conditions hold:

(i) $\tau(C) \leq C$ for all $C \in \mathscr{C}$, and

(ii) $\tau(f) = f_{|\tau(C)}$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(C, D)$.

Definition 2.15.2. If an additive functor τ : $\mathrm{Mod\text{-}R} \longrightarrow \mathrm{Mod\text{-}R}$ is a subfunctor of the identity functor, then we say that τ is a preradical on $\mathrm{Mod\text{-}R}$.

Definition 2.15.3. Let M be a right R-module, then a preradical τ is called a left exact preradical if and only if $\tau(N) = \tau(M) \cap N$ for any submodule N of M.

Definition 2.15.4. A preradical τ is called idempotent provided $\tau^2 = \tau$. Also, τ is said to be a radical if $\tau(M/\tau(M)) = 0$ for every right R-module M.

Definition 2.15.5. For a ring R, a subclass $\mathscr{C}$ of Mod-R is called a hereditary pretorsion class if it is closed under taking submodules, factor modules, and arbitrary direct sums. If, in addition, $\mathscr{C}$ is closed under taking extensions, then we call that $\mathscr{C}$ is a hereditary torsion class.

Remark 2.15.6. One important and useful property of hereditary pretorsion classes is that they are determined by the cyclic modules in them.

We remark that, to a left exact preradical τ one can associate the following two classes of right R-modules, namely,

$$\mathscr{T}_\tau = \{M | \tau(M) = M\}$$
$$\mathscr{F}_\tau = \{M | \tau(M) = 0\}.$$

Note that $\mathscr{T}_\tau$ is a hereditary pretorsion class. Conversely let $\mathscr{C}$ be a hereditary pretorsion class of right R-modules. If M is an any right R-module, and $\tau(M)$ denotes the sum of all submodules of M belonging to $\mathscr{C}$, then $\tau(M) \in \mathscr{C}$. Therefore any module M contains a largest submodules $\tau(M)$ belonging to $\mathscr{C}$. In this way $\mathscr{C}$ gives rise to a preradical τ on Mod-R. Connecting this procedure with the former assignment $\tau \longrightarrow \mathscr{T}_\tau$we obtain the following

Proposition 2.15.7. *[9, Proposition 3.1] There is a one-to-one correspondence between hereditary pretorsion (respectively, hereditary torsion) classes of right R-modules and left exact preradicals (respectively, left exact radicals) on* Mod-R.

Note that, if $\tau(M) = M$ (respectively, $\tau(M) = 0$) for some right R-module M, where τ is a left exact preradical on Mod-R, then we say that M is τ-torsion (respectively, τ-torsion-free). Also note that, from now on, to avoid constant need for reffering to the correspondence given in Proposition 2.15.7, we will use the same notation for both a hereditary pretorsion class and its associated left exact preradical.

Remark 2.15.8. There exist a partial ordering $\leq$ on the class of left exact preradicals (equivalently, hereditary pretorsion classes) where $\mathscr{T}_1 \leq \mathscr{T}_2$ means $\mathscr{T}_1(M) \subseteq \mathscr{T}_2(M)$ for all right R-modules M.

Proposition 2.15.9. *[9, Ch. vi Proposition 1.5] For every preradical $\mathscr{T}$, there exists a largest idempotent preradical $\tilde{\mathscr{T}}$ smaller than $\mathscr{T}$, and there is a smallest radical $\overline{\mathscr{T}}$ larger than $\mathscr{T}$.*

We can construct the smallest radical $\overline{\mathscr{T}}$ larger than preradical $\mathscr{T}$ by transfinite induction as mentioned in [9] as follows. We have an increasing sequence of preradicals $\mathscr{T}_\alpha$ for an ordinal α such that given any R-module M there exists an ascending chain

$$\mathscr{T}_0(M) \leq \mathscr{T}_1(M) \leq \ldots \leq \mathscr{T}_\alpha(M) \leq \mathscr{T}_{\alpha+1}(M) \leq \ldots,$$

of submodules of M where $\mathscr{T}_{\alpha+1}(M)/\mathscr{T}_\alpha(M) = \mathscr{T}(M/\mathscr{T}_\alpha(M))$, for every ordinal α, and if α is a limit ordinal, then

$$\mathscr{T}_\alpha(M) = \bigcup_{\beta<\alpha} \mathscr{T}_\beta(M).$$

Now we define $\overline{\mathscr{T}}$ by $\overline{\mathscr{T}}(M) = \bigcup_\alpha \mathscr{T}_\alpha(M)$ for every R-module M.

Remark 2.15.10. Combining Propositions 2.15.7 and 2.15.9, one can conclude that given a hereditary pretorsion class $\mathscr{T}$, there exists a smallest hereditary torsion class $\overline{\mathscr{T}}$ containing $\mathscr{T}$.

Corollary 2.15.11. *[9, Ch. vi. Corollary 3.5] Let M be a right R-module. If $\mathscr{T}$ is a left exact preradical , then we have $\mathscr{T}(M) \leq_e \overline{\mathscr{T}}(M)$.*

Example 2.15.12. Soc *is a left exact preradical on* Mod-R. *It turns out that any right R-module M is* Soc*-torsion modules are precisely semisimple modules. Note that the class of all semisimple right R-modules, denoted* SSMod-R*, is a hereditary pretorsion class, which corresponds to the left exact preradical* Soc *under the correspondence given in Proposition 2.15.7.*

Example 2.15.13. *The class of semi-artinian right R-modules is a hereditary torsion class. If we consider the preradical* Soc*, then* $\overline{\mathrm{Soc}}$ *is the smallest radical larger than* Soc*, and so any* $\overline{\mathrm{Soc}}$*-torsion module is nothing but a semi-artinian module. Therefore,* $\overline{\mathrm{Soc}}$ *is a left exact radical corresponding to the hereditary torsion class of semi-artinian modules. It turns out that the class of semi-artinian modules is the smallest hereditary torsion class containing* SSMod-R.

Example 2.15.14. *Let M be a right R-module and let $\mathrm{Z}(M)$ denote the singular submodule of M, then Z is a left exact preradical. Observe that a module M is Z-torsion (respectively, Z-torsion-free) if and only if M is singular (respectively, nonsingular). Also note that the class of all singular right R-modules, denoted* $\mathrm{Sing\text{-}R}$, *is a hereditary pretorsion class, corresponding to the left exact preradical Z under the correspondence given in Proposition 2.15.7.*

Proposition 2.15.15. *[20, Proposition 1.11] If $\mathscr{T}$ is a torsion class on* $\mathrm{Mod\text{-}R}$*, then:*

(i) a right R-module M is $\mathscr{T}$-torsion if and only if $\mathrm{Hom}_R(M, \mathrm{E}(N)) = (0)$ *for every $\mathscr{T}$-torsion-free right R-module N;*

(ii) a right R-module N is $\mathscr{T}$-torsion-free if and only if $\mathrm{Hom}_R(M, \mathrm{E}(N)) = (0)$ *for every $\mathscr{T}$-torsion right R-module M.*

Proposition 2.15.16. *[20, Proposition 1.12] If $\mathscr{T}$ is a torsion class on* $\mathrm{Mod\text{-}R}$*, then the class of all $\mathscr{T}$-torsion-free right R-modules is closed under taking extensions.*

Definition 2.15.17. A hereditary torsion class $\mathscr{T}$ is said to be stable if $\mathscr{T}$ is closed under taking injective hulls.

Example 2.15.18. *Let M be a right R-module. Then* $\mathrm{Z}_2 : \mathrm{Mod\text{-}R} \longrightarrow \mathrm{Mod\text{-}R}$ *is a left exact preradical defined by*

$$\mathrm{Z}_2(M)/\,\mathrm{Z}(M) = \mathrm{Z}(M/\,\mathrm{Z}(M))$$

for all $M \in \mathrm{Mod\text{-}R}$. *In particular,* Z_2 *is the smallest left exact radical larger than* Z.

Proposition 2.15.19. *[9, Proposition 6.2] Let M be a right R-module. Then the class of* Z_2*-tosion modules* $\mathscr{G} = \{M \in \mathrm{Mod\text{-}R} \mid \mathrm{Z}_2(M) = M\}$ *holds the following conditions:*

(i) $\mathscr{G}$ *is the smallest hereditary torsion class containing* $\mathrm{Sing\text{-}R}$.

(ii) $\mathrm{Z}(M) \leq_e \mathrm{Z}_2(M)$

(iii) $\mathrm{Z}(M/\,\mathrm{Z}_2(M)) = 0$

(iv) $\mathscr{G}$ *is a stable torsion class.*

Lemma 2.15.20. *The following statements are equivalent for a hereditary pretorsion class $\mathscr{T}$:*

(i) *$\mathscr{T}$ is a torsion class.*

(ii) *$M/\mathscr{T}(M)$ is $\mathscr{T}$-torsion-free for every right R-module M.*

(iii) *If M is $\mathscr{T}$-torsion-free, then so is $\mathrm{E}(M)$.*

Now we define a particular class of modules that is of importance in our thesis introduced by Robert Wisbauer.

Definition 2.15.21. Let $\mathscr{C}$ be a nonempty subclass of Mod-R. Then $\sigma[\mathscr{C}]$ consists of submodules of factors of direct sums of a right R-modules in $\mathscr{C}$. If $\mathscr{C}$ consists of only a right R-module M, then one can write $\sigma[M]$ instead of $\sigma[\mathscr{C}]$.

Example 2.15.22. *Let M be a right R-module. Then $\mathrm{Tr}(\sigma[M], \_) : \mathrm{Mod}\text{-}\mathrm{R} \longrightarrow \mathrm{Mod}\text{-}\mathrm{R}$ is a left exact preradical, and so this preradical corresponds to the hereditary pretorsion class $\sigma[M]$ under the correspondance given in Proposition 2.15.7.*

Proposition 2.15.23. *[21, Proposition 15.2] Let A and B be right R-modules. Then the following statements are equivalent:*

(i) $\sigma[A] = \sigma[B]$*;*

(ii) $B \in \sigma[A]$ *and* $A \in \sigma[B]$.

Proposition 2.15.24. *[21, Proposition 15.3] For any right R-module M, the following statements are equivalent:*

(i) *R is subgenerated by M (i.e, $R \in \sigma[M]$);*

(ii) $\sigma[M] = \mathrm{Mod}\text{-}\mathrm{R}$*;*

(iii) *R can be embedded in M^n for some $n \in \mathbb{N}$.*

Lemma 2.15.25. *Let A and B be right R-modules. If $\mathrm{Tr}(\sigma[A], B) = 0$, then B is A-injective.*

Proof. Suppose that $\mathrm{Tr}(\sigma[A], B) = 0$. It is enough to show that $\mathrm{Tr}(A, \mathrm{E}(B)) \subseteq B$. Suppose that $f : A \longrightarrow \mathrm{E}(B)$ is a nonzero R-homomorphism. Since $B \leq_e \mathrm{E}(B)$, we obtain $0 \neq f(A) \cap B \subseteq B$. Now let $0 \neq g : C \longrightarrow B$ be the restriction of f to the nonzero submodule C of A, where $C = f^{-1}(f(A) \cap B)$. Since C is in $\sigma[A]$, g is a nonzero map in $\mathrm{Tr}(\sigma[A], B)$, a contradiction. In this case, $\mathrm{Tr}(A, \mathrm{E}(B)) = 0 \subseteq B$. Hence B is A-injective. □

Lemma 2.15.26. *Let M be a right R-module and let E be an injective right R-module. Then,* $\mathrm{Tr}(M, E) = \mathrm{Tr}(\sigma[M], E)$.

Proof. Clearly, $\mathrm{Tr}(M, E) \subseteq \mathrm{Tr}(\sigma[M], E)$. For the reverse inclusion, let $f : A \longrightarrow E$ be an R-homomorphism, where $A \in \sigma[M]$. Note that A is a homomorphic image of a submodule B of a direct sum $\bigoplus_{\lambda\in\lambda} M_\lambda$, where $M_\lambda \cong M$ for each $\lambda \in \Lambda$. Let $g : B \longrightarrow A$ be an epimorphism. Since E is injective, fg extends to a homomorphism $h : \bigoplus_{\lambda\in\Lambda} M_\lambda \longrightarrow E$. Note that $\mathrm{Im}\, h = \sum_{\lambda\in\Lambda} \mathrm{Im}(hi_\lambda) \subseteq \mathrm{Tr}(M, E)$, where $i_\lambda : M_\lambda \longrightarrow \bigoplus_{\lambda\in\Lambda} M_\lambda$ is the natural injection. Also, note that $\mathrm{Im}(fg) \subseteq \mathrm{Im}\, h$. Since g is onto, we have $\mathrm{Im}\, f = \mathrm{Im}(fg)$. This gives that

$$\mathrm{Im}\, f = \mathrm{Im}(fg) \subseteq \mathrm{Im}\, h \subseteq \mathrm{Tr}(M, E),$$

and so $\mathrm{Tr}(\sigma[M], E) \subseteq \mathrm{Tr}(M, E)$. □

Before introducing the lattice structure of hereditary pretorsion classes, let us briefly define the notion of a lattice.

Definition 2.15.27. A partially ordered set L is said to be a lattice provided that any two elements $a, b \in L$ have greatest lower bound $a \wedge b$ (which is also called a meet of a and b), and least upper bound $a \vee b$ (which is also called a join of a and b). If every subset of a lattice L has least upper bound and greatest lower bound, then L is called a complete lattice. Let m be an element in a lattice L. If $x \leq m$ for all $x \in L$ then m is called a greatest element in L. If there exist a greatest element, then it is unique and we denote it by 1. Let L be a lattice with greatest element. If given any $m \in L$ with $m \neq 1$, there exists a coatom x such that $m \leq x$, then L is called coatomic. We say that L is a modular lattice if $(c \wedge b) \vee a = (c \vee a) \wedge b$ for all $a, b, c \in L$ with $a \leq b$.

Now, we define meet and join in the partially ordered set of hereditary pretorsion classes. If $\{\tau_i : i \in I\} \subseteq \text{Mod-R}$ is a set of hereditary pretorsion classes, then a right R-module A is $(\bigwedge_{i\in I} \tau_i)$-torsion if and only if A is τ_i-torsion for each $i \in I$, and A is $(\bigvee_{i\in I} \tau_i)$-torsion if and only if A is $\sigma[\mathscr{C}]$-torsion, where

$$\mathscr{C} = \{A \in \text{Mod-R} : \tau_i(A) = A \text{ for some } i \in I\}.$$

Then we can say that hereditary pretorsion classes form a lattice which is denoted $\mathrm{hptors\text{-}R}$, and hence left exact preradicals form a lattice, too, denoted $\mathrm{lep\text{-}R}$, which is equivalent to $\mathrm{hptors\text{-}R}$.

Now we introduce some notions that are going to be needed later.

Definition 2.15.28. Let $\mathscr{F}$ be a set of right ideals of R. If $\mathscr{F}$ holds the following conditions, then it is called a linear filter of right ideals.

(i) $R \in \mathscr{F}$.

(ii) $I, J \in \mathscr{F}$ implies that $I \cap J \in \mathscr{F}$.

(iii) If $I \in \mathscr{F}$ and $I \leq J$, then $J \in \mathscr{F}$.

(iv) $r^{-1}I = \{x \in R : rx \in I\} \in \mathscr{F}$ for all $I \in \mathscr{F}$ and $r \in R$.

Proposition 2.15.29. *[9, Proposition 4.2] There is a one-to-one correspondance between a pair of the following classes:*

(i) Linear filters of right ideals of R.

(ii) Hereditary pretorsion classes of right R-modules.

(iii) Left exact preradicals on Mod-R.

Therefore, we can put a lattice structure on linear filters of right ideals of R, denoted fil-R, using the correspondence of the above proposition.

Proposition 2.15.30. *[22, Proposition 1.1] The following lattices are equivalent for all rings R.*

(i) The lattice hptors-R *of hereditary pretorsion classes in* Mod-R.

(ii) The lattice fil-R *of linear filters of right ideals of R.*

(iii) The lattice lep-R *of left exact preradicals in* Mod-R.

Chapter 3

Injectivity Domains and Extent of Injectivity

3.1 Opposite of Injectivity: Poor Modules

Definition 3.1.1. For a right R-module M, the class $\mathfrak{In}^{-1}(M) = \{N \in \text{Mod-R} : M \text{ is } N\text{-injective}\}$ is said to be the injectivity domain of M.

Note that M is injective module if and only if it has the largest injectivity domain, that is, $\mathfrak{In}^{-1}(M) = \text{Mod-R}$. Now we consider the opposite case, that is, the case when an injectivity domain is as small as possible.

Definition 3.1.2. [1] A right R-module M is said to be a poor module provided that injectivity domain of M consists of only semisimple modules, that is, $\mathfrak{In}^{-1}(M) = \text{SSMod-R}$.

The notion of a poor module has been introduced in [1]. It is natural to ask whether every ring has a poor module . Before answering this question affirmatively, we give the following proposition.

Proposition 3.1.3. *[1, Proposition 3.1]*

$$\bigcap_{M \in \text{Mod-R}} \mathfrak{In}^{-1}(M) = \text{SSMod-R}$$

Proof. Let $A \in \bigcap_{M \in \text{Mod-R}} \mathfrak{In}^{-1}(M)$ and let $B \leq A$. It is sufficient to see that B is a direct summand of A. Since A is in the injectivity domain of any module, then B is A-injective. In this case, B is a direct summand of A by Lemma 2.13.20, and so $A \in \text{SSMod-R}$. The converse is obvious. □

Existence of poor modules states even more; at least one injectivity domain in this intersection is directly equal to SSMod-R.

Proposition 3.1.4. *[23, Proposition 1] Every ring possesses a poor module.*

Proof. Let R be a ring and let $\{M_\beta | \beta \in B\}$ denote the complete set of representatives of isomorphism classes of non-semisimple cyclic right R-modules. It means that we can choose a proper essential submodule N_β of M_β for each $\beta \in B$. Now set $K = \oplus_{\beta \in B} N_\beta$. Let A be a cyclic module which is not semisimple, and suppose that K is A-injective. Then there exist a $\beta \in B$ such that $A \cong M_\beta$. Hence A contains a proper essential submodule U, which isomorphic to N_β. Since K is A-injective, any direct summand of K is also A-injective. This gives that U is A-injective, which is a contradiction. Therefore, K is a poor module. $\square$

Now we consider the probability that all right R-modules are the poor case.

Lemma 3.1.5. *If every right R-module is poor then any right R-module is injective; and so R is a semisimple Artinian ring.*

Proof. Assume that every right R-module is poor and consider the injective envelope of R_R as a poor module. It is easy to see that

$$\text{Mod-R} = \mathfrak{In}^{-1}(\text{E}(R_R)) = \text{SSMod-R}$$

This gives that R is semisimple. By Theorem 2.13.10, any R-module is injective. $\square$

Corollary 3.1.6. *[23, Corollary 1] The following statements are equivalent for a ring R:*

(i) R is a semisimple Artinian ring.

(ii) Every nonzero factor of poor right R-module is poor.

(iii) Every poor right R-module is semisimple.

(iv) Every nonzero direct summand of poor right R-module is poor.

Proof. If R is semisimple Artinian, statements (ii), (iii), and (iv) follow easily. Now suppose that any of the statements (ii), (iii) or (iv) holds. We remark that the direct sum of any module with a poor module is also poor. We already know the existence of poor modules by Proposition 3.1.4. Then (ii) or (iii) implies that every module is poor. In particular every injective module is poor. This shows that R is a semisimple Artinian ring. On the other hand, (iv) implies that every module is semisimple; hence again R is semisimple Artinian. $\square$

The following proposition provides us with an alternative construction of a poor module. Note that its proof is based on a result given in [24] and [25].

Proposition 3.1.7. *[23, Proposition 2] Let R be any ring and let $A = \bigoplus_{B\in\Gamma} B$, where Γ is any complete set of representatives of cyclic right R-modules. Then A is poor.*

Proof. Let A and N be right R-modules. If A is N-injective, then all cyclic submodules of all factors of N must be N-injective. It follows that N is semisimple (see [24] and [25]). $\square$

Now we give a useful fact about poor modules which is given in [1].

Lemma 3.1.8. *A right R-module M is poor if and only if all cyclic right R-modules contained in $\mathfrak{In}^{-1}(M)$ is semisimple.*

Proof. Let $A \in \mathfrak{In}^{-1}(M)$ and assume that aR is semisimple for every $a \in A$. Since $A = \sum_{a\in A} aR$ and the sum of semisimple modules is semisimple, A is semisimple; thus M is poor. The converse is clear. $\square$

Theorem 3.1.9. *[1, Theorem 3.3] Let R be a right Artinian ring, then the cyclic right R-module $M = R/J$ is poor, where J is Jacobson radical of R.*

Proof. Let $M = R/J$ and let $N = aR$ be a nonzero cyclic right R-module in the injectivity domain of M. It is enough to show that aR is semisimple. Let S_1 be a simple submodule of aR. Then S_1 is isomorphic to a direct summand of M, and so S_1 is aR-injective. Then we may write $aR = S_1 \oplus K_1$ for some submodule K_1 of aR. If $K_1 = 0$, then aR must be semisimple. Otherwise, let S_2 be a simple submodule of K_1. Then, similarly, $K_1 = S_2 \oplus K_2$ and so $aR = S_1 \oplus S_2 \oplus K_2$, where $S_1 \oplus S_2$ is semisimple and $K_1 \supseteq K_2$. This process must terminate after a finite step. Thus, aR is semisimple. Hence, R/J is poor. $\square$

Before we give the following lemma, we shall define the notion of uniserial and serial modules. Let M be a right R-module, then it is called a uniserial module if its submodules are linearly ordered by inclusion. Also, we say that an R- module M is serial if it is a direct sum of uniserial submodules. A ring R which satisfies the minimum condition on both sides is called a generalized uniserial ring if for any primitive idempotent e of R the right (left) ideal eR (Re) has unique composition series (see [18]).

Lemma 3.1.10. *[1, Lemma 2.1]*

(i) *[18, Lemma 1] Any finitely generated torsion module over a hereditary Noetherian prime ring R is a direct sum of finitely many uniserial modules.*

(ii) *[18, Lemma 2] If $x \in M$ is a torsion element, then xR is a torsion submodule over a hereditary Noetherian prime ring R with nonzero annihilator.*

(iii) *[18, Theorem 1] Every R-module is a direct sum of uniserial modules, where R is a generalized uniserial ring.*

(iv) *[26, Theorem 1] Every proper factor ring of a hereditary Noetherian prime ring is generalized uniserial.*

In view of Theorem 3.1.9, as we consider further examples of poor modules, it is reasonable to focus on semisimple modules. In this regard we give the following

Proposition 3.1.11. *[1, Proposition 3.4] Let R be a hereditary Noetherian domain and let M be a semisimple module that contains a copy of each simple R-module. Then M is either poor or injective. In particular, if R has only one simple module (up to isomorphism), then that module M is either injective or poor. Moreover, for a ring R and a module M over R satisfying the above hypotheses, M is poor unless R is a* V*-ring.*

Proof. Let R be a hereditary Noetherian domain and let M be as described in the statement of the proposition. Assume that M is not injective and let $xR \in \mathfrak{In}^{-1}(M)$. Since M is not injective, $\operatorname{ann}_R(x)$ is nonzero. Then xR is serial by (i) and (ii) of Lemma 3.1.10. It follows that $xR = U_1 \oplus \cdots \oplus U_n$, where each U_i is uniserial. In this case, M is U_i-injective for each i. It is enough to show that each U_i is simple. If U_i is not simple, then it must contain a simple submodule, say S. Then the inclusion map from S to M can be extended to a monomorphism from U_i to M. However, this is a contradiction. Therefore, U_i must be simple. This gives that xR is semisimple. □

In this example below, we will see that both possibilities in Proposition 3.1.11 are possible.

Example 3.1.12. *Let $R = \mathbb{Z}$. Then $U = \bigoplus_p(\mathbb{Z}/p\mathbb{Z})$, where the sum runs through prime numbers p, is a poor $\mathbb{Z}$- module, however, there exists no proper poor summand of U (see Theorem 3.1 of [27]).*

3.2 The Injective Profile of a Ring

Definition 3.2.1. Let R be any ring. The class of right R-modules $\mathscr{A}$ is said to be an i-portfolio if $\mathscr{A} = \mathfrak{In}^{-1}(M)$ for some $M \in \text{Mod-R}$. The class $\{\mathscr{A} \subseteq \text{Mod-R} : \mathscr{A}$ is an i-portfolio$\}$ is said to be the right injective profile of R (or right i-profile) and we denote it by $i\mathcal{P}_r(R)$. The left i-profile, denoted $i\mathcal{P}_l(R)$, is defined similarly.

Remark that any i-portfolio is a hereditary pretorsion class of right R-modules containing SSMod-R, as it is closed under taking submodules, arbitrary direct sums and factor modules (see Propositions 2.13.21 and 2.13.23).

Lemma 3.2.2. *[22, Lemma 2.2] Let R be any ring. Let $\mathbb{X} \subseteq i\mathcal{P}(R)$. Then, $\bigcap \mathbb{X}$ is an i-portfolio.*

Proof. We shall think of $\mathbb{X}$ as a set. Let $M_{\mathscr{A}}$ be a module such that $\mathscr{A} = \mathfrak{In}^{-1}(M_{\mathscr{A}})$ for each $\mathscr{A} \in \mathbb{X}$. By Lemma 2.13.22 we obtain

$$\bigcap \mathbb{X} = \mathfrak{In}^{-1}(\textstyle\prod_{\mathscr{A} \in \mathbb{X}} M_\alpha).$$

This completes the proof. $\square$

In view of the above lemma, one can see that $i\mathcal{P}(R)$ is a complete lattice and is, indeed, a sublattice of hptors-R. Furthermore, since any right R-module is injective relative to any semisimple module, $i\mathcal{P}(R)$ is a sublattice of the interval $[\text{SSMod-R}, \text{Mod-R}] \subseteq \text{hptors-R}$.

Proposition 3.2.3. *[22, Proposition 2.3] Let R and S be rings. Then there exists a lattice isomorphism $i\mathcal{P}(R \times S) \cong i\mathcal{P}(R) \times i\mathcal{P}(S)$.*

Now our aim is to show that any hereditary pretorsion class including SSMod-R is necessarily a portfolio. To prove this, we need to define the following notion.

Definition 3.2.4. Let A and B be right R-modules. If any A-injective module is B-injective, we say that A rises to B and in this situation we write $A \uparrow B$.

Remark 3.2.5. If $B \in \sigma[A]$, then $A \uparrow B$, indeed, if any module is A-injective, then it is also injective relative to its direct sums, factors, and submodules. Hence it is injective relative to B. Also note that if $A \uparrow B$ and $B \uparrow U$, then we have $A \uparrow U$. Moreover, if $A \uparrow B$ and $U \leq B$, then $A \uparrow U$.

We show that, under some conditions, the situation $A \uparrow B$ is indeed equivalent to $B \in \sigma[A]$. Note that, since any right R-module is injective relative to any semisimple module, if B is a semisimple module, then any module A rises to B.

Lemma 3.2.6. *[22, Lemma 2.7] Let A and B be right R-modules, and suppose that $A \uparrow B$. Then, either B is semisimple or* $\mathrm{Tr}(\sigma[A], B) \neq 0$.

Proof. Suppose that B is not semisimple, and that $\mathrm{Tr}(\sigma[A], B) = 0$. Since B is not semisimple, we can find a submodule of B which is not a direct summand of B, say K. This shows that K is not B-injective. Since $\mathrm{Tr}(\sigma[A], B) = 0$, then K is A-injective by Lemma 2.15.25. But this is a contradiction. Hence, we obtain $\mathrm{Tr}(\sigma[A], B) \neq 0$. □

Theorem 3.2.7. *[22, Theorem 2.8] Let R be any ring and let A be a module which subgenerates every semisimple module. Then, for any module B, $A \uparrow B$ if and only if $B \in \sigma[A]$.*

Proof. Suppose that A subgenerates every semisimple module and that $A \uparrow B$. If B is semisimple, then we are done. Hence, we assume that B is not semisimple. We show first that $\mathrm{Tr}(\sigma[A], B)$ is essential in B. Let $T \leq B$ be a nonzero submodule. Since $\mathrm{Tr}(\sigma[A], -)$ is a left exact preradical corresponding to the hereditary pretorsion class $\sigma[A]$, then $T \cap \mathrm{Tr}(\sigma[A], B) = \mathrm{Tr}(\sigma[A], T)$. If T is semisimple, then $T \in \sigma[A]$ and this implies that $\mathrm{Tr}(\sigma[A], T) = T$. If T is not semisimple, then by Lemma 3.2.6 and considering that $A \uparrow T$, we have $\mathrm{Tr}(\sigma[A], T) \neq 0$. Since $\mathrm{Tr}(\sigma[A], B) \leq_e B \leq_e \mathrm{E}(B)$, we obtain $\mathrm{Tr}(\sigma[A], B) \leq_e \mathrm{E}(B)$ and this means that $\mathrm{E}(\mathrm{Tr}(\sigma[A], \mathrm{E}(B))) = \mathrm{E}(B)$ by Proposition 2.13.15. Also note that $\mathrm{Tr}(\sigma[A], \mathrm{E}(B))$ is A-injective (see Lemmas 2.13.27 and 2.15.26). Hence, it is B-injective by our assumption. Therefore, every morphism $B \longrightarrow \mathrm{E}(\mathrm{Tr}(\sigma[A], \mathrm{E}(B))) = \mathrm{E}(B)$ has its image in $\mathrm{Tr}(\sigma[A], \mathrm{E}(B))$. Now considering the inclusion map we get that $B \leq \mathrm{Tr}(\sigma[A], \mathrm{E}(B))$. Therefore $B \in \sigma[A]$. □

Theorem 3.2.8. *[22, Theorem 2.9] Let R be any ring, and let $\mathscr{C}$ be a hereditary pretorsion class in* Mod-R, *containing* SSMod-R. *Then, $\mathscr{C}$ is an i-portfolio, that is, $\mathfrak{In}^{-1}(M) = \mathscr{C}$ for some $M \in$* Mod-R. *In other words, the following lattices are the same:*

1. *$i\mathcal{P}(R)$.*

2. *The interval* $[\mathrm{SSMod\text{-}R}, \mathrm{Mod\text{-}R}] \subseteq \mathrm{hptors\text{-}R}$.

Moreover, the following three lattices are isomorphic:

(i) $i\mathcal{P}(R)$*.*

(ii) The lattice of linear filters of right ideals $\mathscr{F}$ *with* $I \in \mathscr{F}$ *for any maximal right ideal* I*.*

(iii) The lattice of left exact preradicals τ *with* $Soc \leq \tau$*.*

Proof. Let N be a right R-module such that $\sigma[N] = \mathscr{C}$ and let $\mathscr{U} = \{\mathscr{A} : N \in \mathscr{A}$ where $\mathscr{A}$ is a portfolio$\}$. It is enough to show that $\sigma[N] = \bigcap \mathscr{U}$. Since the intersection of injectivity domains is again injectivity domain by Lemma 3.2.2, it is obvious that $\sigma[N] \subseteq \mathscr{U}$. Now let K be a module such that $K \notin \sigma[N]$. Since N subgenerates every semisimple module, Theorem 3.2.7 implies that there is a portfolio $\mathscr{B}$ such that $N \in \mathscr{B}$ and $K \notin \mathscr{B}$. Since $K \notin \sigma[N]$, then $K \notin \bigcap \mathscr{U}$ and we obtain $\sigma[N] = \bigcap \mathscr{U}$. Hence, by Lemma 3.2.2 $\sigma[N]$ is a portfolio. It follows that the lattices $i\mathcal{P}(R)$ and $[\text{SSMod-R}, \text{Mod-R}] \subseteq \mathscr{C}$ are isomorphic. The last part follows from Proposition 2.15.30. □

Corollary 3.2.9. *[22, Corollary 2.10] For a ring* R*, the lattice* $i\mathcal{P}(R)$ *is a modular and coatomic lattice.*

Proof. Follows from Theorem 3.2.8 and [28, Theorem 2]. □

Proposition 3.2.10. *[22, Proposition 2.13] Let* R *be a right Artinian ring. Then* $i\mathcal{P}(R)$ *is anti-isomorphic to the lattice of ideals contained in* $\mathrm{J}(R)$*. Also,* $i\mathcal{P}(R)$ *is an Artinian and Noetherian lattice. Hence,* $i\mathcal{P}(R)$ *is also atomic.*

Proof. Since R is right Artinian, then any linear filter of right ideals $\mathscr{F}$ of R is closed under arbitrary intersections. Then, for every linear filter $\mathscr{F}$ there is a two-sided ideal I of R such that $\mathscr{F} = \{J \leq R_R : I \subseteq J\}$ (see [29]). Since $i\mathcal{P}(R)$ and the lattice of linear filters of right ideals which contain every maximal right ideal are isomorphic, $i\mathcal{P}(R)$ is anti-isomorphic to the lattice of ideals contained in $\mathrm{J}(R)$. This completes the proof. □

Chapter 4

Rings without a Middle Class

The main focus of the present chapter is on the rings without a right middle class. We will give some new properties of this ring and characterize them with respect to hereditary pretorsion classes.

4.1 Definition and some examples

Definition 4.1.1. We say that a ring R has no right middle class (or alternatively R is a right NMC-ring) provided every element in Mod-R is either poor or injective. Equivalently, R is a right NMC-ring if the sublattice $[\mathrm{SSMod}\text{-}\mathrm{R}, \mathrm{Mod}\text{-}\mathrm{R}]$ of hptors-R consists only of SSMod-R and Mod-R. Moreover, if R is both right and left NMC, then we call that R is an NMC-ring.

Example 4.1.2. *The first trivial example of rings without a right middle class is semisimple Artinian rings because every module over this ring is injective by Theorem 2.13.10.*

Just as with the case that every module is injective if every right R-module is poor, then R is semisimple Artinian. On the other hand, there exist plenty of rings with no middle class which are not semisimple Artinian, for example, any right PCI-domain as the following proposition shows. Also, the proposition below provides us with the first non-trivial example of rings with no middle class.

Proposition 4.1.3. *[1, Proposition 3.2] If R is a right* PCI*-domain, then it is a right* NMC*-ring and R_R is a poor module.*

Proof. If R is a division ring, then there is nothing to prove. Suppose that R is not a divison ring. Then every cyclic right R-module is injective except those which are isomorphic to R.

We already know that the injective cyclics are all semisimple by Lemma 2.14.10. Therefore, if M is a right R-module which is not injective, then it is a poor module. $\square$

Theorem 4.1.4. *[22, Corollary 2.14] Let R be a right Artinian ring. Then no nontrivial ideals of R are contained in $\mathrm{J}(R)$ properly if and only if R is a right NMC-ring.*

Proof. This follows easily from Proposition 3.2.10. $\square$

Example 4.1.5. *[30, Example 2.9] Let $M = \mathbb{Z}_3$, and let $\alpha = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Note that $M' = \{0, 1, \alpha, \alpha^2, \dots, \alpha^7\}$ is a field. It follows that the ring (M, M^2, M') is right NMC.*

Example 4.1.6. *[30, Example 2.10] Let $F \leq F_1$ be an extension of fields, and let K be a division subring of $\mathbb{M}_p(F)$. Suppose that K properly contains the field of scalar matrices in $\mathbb{M}_p(F)$, where p is a prime number. Then the ring*

$$R = \begin{pmatrix} F_1 & 0 \\ F_1^p & K \end{pmatrix}$$

is right NMC. In particular, if $F = \mathbb{Q}$, then all scalar matrices are contained in any division subring of $\mathbb{M}_p(\mathbb{Q})$. In this case, R is a right NMC-ring if K is any division subring of $\mathbb{M}_p(\mathbb{Q})$ which is not the field of scalar matrices.

Example 4.1.7. *[30, Example 2.12] Given a field E and an irreducible polynomial $q(x)$ over E of prime degree, the ring*

$$\begin{pmatrix} E & 0 \\ E[x]/(q(x)) & E[x]/(q(x)) \end{pmatrix}$$

is right NMC.

Example 4.1.8. *[30, Example 2.13] The ring*

$$R = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \\ \mathbb{C} & M \end{pmatrix}, \text{ where } M = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} \cong \mathbb{C},$$

is right NMC.

Example 4.1.9. *[30, Example 2.20] Let $E = \mathbb{Q}(\sqrt{2})$ be the field and let $\alpha : E \longrightarrow \mathbb{Q}$ be the mapping defined by $\alpha(a + b\sqrt{2}) = a$. Regarding $R = E \times E$ as additive abelian group we define a multiplication on R by*

$$(x, y)(t, z) = (xt, xz + y\alpha(t))$$

and turn R into a ring. Observe that the composition length of $_RR$ is 2. Thus $\mathrm{J}(R)$ does not contain nonzero ideal of R properly, and so R is a right NMC*-ring by Theorem 4.1.4.*

The property of having no middle class for a ring is closely related to its quasi-injective modules as the following proposition shows.

Proposition 4.1.10. *[22, Proposition 3.1] If R is a right* NMC*-ring, then every non-semisimple quasi-injective right R-module is injective.*

Proof. Assume that A is quasi-injective module that is not semisimple. It follows that SSMod-R $\subsetneq \mathfrak{In}^{-1}(A)$.Then we have $\mathfrak{In}^{-1}(A) = \text{Mod-R}$ because R is a right NMC-ring. Hence A is an injective module. □

The converse of the above proposition was proved to be true when R is semi-artinian in [22, Proposition 3.1].

Lemma 4.1.11. *[23, Lemma 1] Having no middle class property is closed under the formation of factor rings.*

Proof. Let R be any right NMC-ring and let A be an ideal of R. Take a right $\frac{R}{A}$-module $M_{\frac{R}{A}}$, which is not poor. It means that we can choose a non-semisimple $N_{\frac{R}{A}}$ such that $M_{\frac{R}{A}}$ is $N_{\frac{R}{A}}$-injective. This gives that M_R is N_R-injective. Then M_R must be injective because $N_R \notin$ SSMod-R and R is a right NMC-ring. Thus, $M_{\frac{R}{A}}$ is also injective, which completes the proof. □

4.2 An approach to rings without a middle class from a torsion theoretic view point

We will see in the sequel that when we study rings without a right middle class it is very useful to consider some specific hereditary pretorsion classes that are not necessarily portfolios. But, the following lemma shows that these hereditary pretorsion classes can be extended to portfolios, which are large enough.

Lemma 4.2.1. *[31, Lemma 2.1] Suppose that R is a ring and that $\mathscr{T}$ is a hereditary pretorsion class of right R-modules. Define the class*

$$\overline{\mathscr{T}} = \{A \in \text{Mod-R} : A/\mathscr{T}(A) \textit{ is } \mathscr{T}\textit{-torsion free and semisimple }\}.$$

Then $\overline{\mathscr{T}}$ is a portfolio such that $\mathscr{T} \subseteq \overline{\mathscr{T}}$.

Proof. We first show that $\overline{\mathscr{T}}$ contains both $\mathscr{T}$ and SSMod-R. Let $A \in \mathscr{T}$, then we obtain $\mathscr{T}(A) = A$. In this case, $(0) = A/\mathscr{T}(A)$ is semisimple and $\mathscr{T}$-torsion free. This implies that $\mathscr{T} \subseteq \overline{\mathscr{T}}$. Now let $A \in$ SSMod-R. This gives that $A = \mathscr{T}(A) \oplus A'$ for some submodule A' of A. Then we obtain $\mathscr{T}(A) = \mathscr{T}(\mathscr{T}(A)) \oplus \mathscr{T}(A')$. Since any left exact preradical is idempotent, we have $\mathscr{T}(\mathscr{T}(A)) = \mathscr{T}(A)$, and so $\mathscr{T}(A') = 0$. It follows that $A/\mathscr{T}(A) \cong A'$ is $\mathscr{T}$-torsion free and semisimple. Hence SSMod-R $\subseteq \overline{\mathscr{T}}$. If we see that $\overline{\mathscr{T}}$ is a hereditary pretorsion class, then we are done. Now we show that $\overline{\mathscr{T}}$ is closed under submodules, factor modules and direct sums. Let $A \in \overline{\mathscr{T}}$ and let B be a submodule of A. Then we obtain

$$\frac{B}{\mathscr{T}(B)} = \frac{B}{B \cap \mathscr{T}(A)} \cong \frac{B + \mathscr{T}(A)}{\mathscr{T}(A)} \leq \frac{A}{\mathscr{T}(A)}.$$

Since $A \in \overline{\mathscr{T}}$, $A/\mathscr{T}(A)$ is semisimple $\mathscr{T}$-torsion free. The property of being semisimple and $\mathscr{T}$-torsion free is inherited by submodules. Then $B/\mathscr{T}(B)$ is semisimple $\mathscr{T}$-torsion free and so $B \in \overline{\mathscr{T}}$. Let $f : A \to A'$ be an epimorphism. In this case, there exists an epimorphism $\bar{f} : A/\mathscr{T}(A) \to A'/\mathscr{T}(A')$ induced by f. Since an epimorphic image of semisimple modules is also semisimple, $A'/\mathscr{T}(A')$ is semisimple, and hence isomorphic to a submodule of $A/\mathscr{T}(A)$. Then $A'/\mathscr{T}(A')$ is $\mathscr{T}$-torsion free. Lastly, let $\{A_i\}_{i\in I}$ be an arbitrary family of right R-modules contained in $\overline{\mathscr{T}}$. Note that $\mathscr{T}(\bigoplus_{i\in I} A_i) = \bigoplus_{i\in I} \mathscr{T}(A_i)$. Then we have

$$\frac{\bigoplus A_i}{\mathscr{T}(\bigoplus A_i)} = \frac{\bigoplus A_i}{\bigoplus \mathscr{T}(A_i)} \cong \bigoplus(\frac{A_i}{\mathscr{T}(A_i)}).$$

Therefore, $\bigoplus_{i\in I} A_i \in \overline{\mathscr{T}}$. This completes the proof. □

As we mentioned before, semisimple Artinian rings are trivial examples of rings without a right middle class. Therefore, we will focus on rings that are not semisimple Artinian in our study of rings which have no middle class. The proposition below is crucial in this direction.

Proposition 4.2.2. *[31, Proposition 2.2] Suppose that R is a right* NMC*-ring. Then any singular right R-module is semisimple. If, in addition, R is right nonsingular, then R is a right* SI*-ring.*

Proof. In case R is semisimple Artinian, there is nothing to prove. Thus we may suppose that R is not semisimple Artinian. Let $\mathscr{A}$ be the class $\overline{\text{Sing-R}}$, as constructed in Lemma 4.2.1. It follows that $\mathscr{A}$ is a portfolio. Then we obtain either $\mathscr{A} = \text{SSMod-R}$ or $\mathscr{A} = \text{Mod-R}$ because R is a right NMC-ring.

First we assume that $\mathrm{Z}(R_R) \neq 0$, and that $\mathscr{A}$ is not equal to $\mathrm{SSMod\text{-}R}$. Then we have $\mathscr{A} = \mathrm{Mod\text{-}R}$. Note that $\mathscr{A}$ is defined as follows:

$$\mathscr{A} = \{M \in \mathrm{Mod\text{-}R} : M/\,\mathrm{Z}(M) \text{ is semisimple and nonsingular}\}$$

In this case $R/\mathrm{Z}(R_R)$ must be semisimple nonsingular right R-module. This means that it is a projective right R-module by Corollary 2.12.6. Thus $\mathrm{Z}(R_R)$ is a direct summand of R, a contradiction. Hence $\mathscr{A} = \mathrm{SSMod\text{-}R}$, implying that any singular right R-module is semisimple.

Now suppose that $\mathrm{Z}(R_R) = 0$. If $R \in \mathscr{A}$, then $R/\mathrm{Z}(R_R)$ must be nonsingular semisimple. But this contradicts our assumption that R is not semisimple Artinian. It follows that we have $R \notin \mathscr{A}$, and so $\mathscr{A} = \mathrm{SSMod\text{-}R}$. Note that, in case $\mathrm{Z}(R_R) = 0$, the property of being singular is closed under injective hulls. Suppose that M is a singular right R-module, in which case $\mathrm{E}(M)$ is also singular. This shows that $\mathrm{E}(M)$ is semisimple. Since M is essential in $\mathrm{E}(M)$, we obtain $M = \mathrm{E}(M)$ by Corollary 2.5.11. Hence, every singular module is injective, and so R is right SI-ring. In particular, singular right R-modules are semisimple. □

Suppose that R is a ring and that $\mathscr{T}$ is a torsion class of right R-modules. We say that $\mathscr{T}$ is a splitting torsion class if $\mathscr{T}(M)$ is a direct summand of each right R-module M.

Theorem 4.2.3. *[31, Theorem 2.3] A ring R is right* NMC *if and only if given any hereditary pretorsion class $\mathscr{T}$ of right R-modules with* $\mathrm{Sing\text{-}R} \subseteq \mathscr{T}$*, $\mathscr{T}$ satisfies either one of the following statements:*

(i) Every $\mathscr{T}$-torsion module is semisimple (equivalently, $\mathscr{T} \subseteq \mathrm{SSMod\text{-}R}$), or

(ii) Every $\mathscr{T}$-torsion-free module is semisimple and injective and $\mathscr{T}$ is a (splitting) torsion class.

Proof. Suppose that R is a right NMC-ring. It follows from Proposition 4.2.2 that we have $\mathrm{Sing\text{-}R}$ is contained in $\mathrm{SSMod\text{-}R}$. Let $\mathscr{T}$ be a hereditary pretorsion class of right R-modules with $\mathrm{Sing\text{-}R} \subseteq \mathscr{T}$. In this case, we have either $\overline{\mathscr{T}} = \mathrm{Mod\text{-}R}$ or $\overline{\mathscr{T}} = \mathrm{SSMod\text{-}R}$, where $\overline{\mathscr{T}}$ is defined as in Lemma 4.2.1. In the latter case, we obtain $\mathscr{T} \subseteq \mathrm{SSMod\text{-}R}$. Hence, let $\overline{\mathscr{T}} = \mathrm{Mod\text{-}R}$. It follows that $M/\mathscr{T}(M)$ is $\mathscr{T}$-torsion-free semisimple for any right R-module M. By Lemma 2.15.20, this means that $\mathscr{T}$ is a torsion class. On the other hand $\mathrm{Z}(M) \subseteq$

$\mathscr{T}(M)$ for all $M \in$ Mod-R because Sing-R $\subseteq \mathscr{T}$. It follows that $\mathrm{Z}(M/\mathscr{T}(M)) = 0$ since $\mathscr{T}(M/\mathscr{T}(M)) = 0$. Thus, $M/\mathscr{T}(M)$ is semisimple nonsingular (and hence projective) for any $M \in$ Mod-R. In this case we see that $\mathscr{T}(M)$ is a direct summand of M; hence $\mathscr{T}$ is splitting. In particular, we obtain that $R/\mathscr{T}(R)$ is a semisimple Artinian ring. Now we will show that every $\mathscr{T}$-torsion free module is semisimple injective. Assume that M is a $\mathscr{T}$-torsion-free module. Note that if $\mathscr{T}$ is regarded as a preradical, then $\mathscr{T}(M)$ is a two-sided ideal of R. Now consider the homomorphism $f_m : R \to M$ with $f_m(r) = mr$ for all $m \in M$. As $f_m(\mathscr{T}(R)) \subseteq \mathscr{T}(M)$ for all $m \in M$, we have $M\mathscr{T}(R) \subseteq \mathscr{T}(M)$. It follows that $M\mathscr{T}(R) = 0$, implying M is a semisimple module since $R/\mathscr{T}(R)$ is a semisimple Artinian ring. This applies to $\mathrm{E}(M)$, i.e, $E(M)$ is semisimple and $\mathscr{T}$-torsion-free by Lemma 2.15.20 because $\mathscr{T}$ is a hereditary torsion class. This gives that $M = \mathrm{E}(M)$, implying M is injective.

Conversely, assume that hereditary pretorsion class $\mathscr{T}$ with Sing-R $\subseteq \mathscr{T}$ satisfies either (i) or (ii). We may suppose that R is not semisimple Artinian . First we shall show that Sing-R is contained in SSMod-R. Assume the contrary that Sing-R $\nsubseteq$ SSMod-R. Then (ii) holds for $\mathscr{T} =$ Sing-R. In this case, every nonsingular module is semisimple and Sing-R is a splitting torsion class. Therefore, $\mathrm{Z}(R_R)$ is a direct summand of R_R. This shows that R is a right nonsingular ring, and hence semisimple Artinian. However, this is a contradiction. Hence, every singular right R-module is semisimple. Now take a hereditary pretorsion class $\mathscr{T}$ such that SSMod-R $\subsetneq \mathscr{T}$. Since $\mathscr{T}$ contains Sing-R and $\mathscr{T} \nsubseteq$ SSMod-R, by assumption, $\mathscr{T}$ is a torsion class and every $\mathscr{T}$-torsion free module is semisimple. Therefore, $\mathscr{T}$-torsion-free modules are also $\mathscr{T}$-torsion. Let $M \in$ Mod-R, since $\mathscr{T}$ is a torsion class, $M/\mathscr{T}(M)$ is $\mathscr{T}$-torsion-free, and hence $\mathscr{T}$-torsion. Then M is also $\mathscr{T}$-torsion by Proposition 2.15.16 since $\mathscr{T}$ is a hereditary torsion class. It follows that $\mathscr{T} =$ Mod-R. Thus R is a right NMC-ring. □

As we shall see later, every ring without a right middle class can be decomposed into an indecomposable ring without a right middle class and a semisimple Artinian ring. The theorem below provides a characterization for indecomposable rings to have no middle class by means of their lattices of hereditary pretorsion classes. Once we comprehend the structure of these rings, we can reduce the study of rings without a right middle class to the case of indecomposable rings.

Theorem 4.2.4. *[31, Theorem 2.4] The following statements are equivalent for a ring R that* *is not semisimple Artinian:*

(i) R *is an indecomposable right* NMC*-ring.*

(ii) The lattice hptors-R *contains* SSMod-R *as a unique coatom.*

(iii) We have $\sigma[C] = \text{Mod-R}$*, for every cyclic non-semisimple right* R*-module* C*.*

(iv) If C *is a cyclic right* R*-module which is not semisimple, then* R_R *is isomorphic to a submodule in a finite direct sum of copies of* C*.*

Proof.

$(i) \Rightarrow (ii)$ Suppose that R is an indecomposable right NMC-ring. By definition hptors-R contains SSMod-R as a coatom. Now we shall show that it is a unique coatom. Let $\mathscr{T}$ be a hereditary pretorsion class of right R-modules such that $\mathscr{T} \nsubseteq \text{SSMod-R}$. Now assume that $\text{Sing-R} \subseteq \mathscr{T}$. Then by Theorem 4.2.3, every $\mathscr{T}$-torsion-free module is semisimple and $\mathscr{T}$ is a splitting torsion class. This gives that $R = \mathscr{T}(R) \oplus B$ for some right ideal B of R. Consider the homomorphism $f : \mathscr{T}(R) \to B$. Since $\mathscr{T}$ is a torsion class, $R/\mathscr{T}(R) \cong B$ is $\mathscr{T}$-torsion-free. Then $f(\mathscr{T}(\mathscr{T}(R))) \subseteq \mathscr{T}(B) = 0$. So we have $f = 0$ and $\text{Hom}_R(\mathscr{T}(R), B) = 0$. Now let $g : B \to \mathscr{T}(R)$ be a homomorphism. Since B is semisimple, $\ker g$ must be a direct summand of B. Then we obtain $B = \ker g \oplus C$. It follows that $C \cong B/\ker g \hookrightarrow \mathscr{T}(R)$. Since B is $\mathscr{T}$-torsion-free, C is also $\mathscr{T}$-torsion-free. In this case, C is both $\mathscr{T}$-torsion-free and $\mathscr{T}$-torsion, and so it must be equal to zero. Therefore, $B = \ker g$ and $g = 0$. It follows $\text{Hom}_R(B, \mathscr{T}(R)) = 0$. By Lemma 2.3.2, B is an ideal of R. By the assumption that R is an indecomposable ring, it must be equal to either B or $\mathscr{T}(R)$. As R is not a semisimple Artinian ring, we must have $R = \mathscr{T}(R)$. Hence, $\mathscr{T} = \text{Mod-R}$. Now we suppose that $\mathscr{T}$ is arbitrary (not necessarily containing Sing-R). Set $\mathscr{C} = \mathscr{T} \bigvee \text{Sing-R} \subseteq \text{hptors-R}$. Then by above arguments we obtain that $\mathscr{C} = \text{Mod-R}$. It follows that there is a monomorphism $f : R \to S \oplus T$ of right R-modules where $S \in \text{Sing-R}$, $T \in \mathscr{T}$. Let $f(1) = (s, t) \in S \oplus T$ and $f(r) = f(1)r = (sr, tr)$, where $s \in S$, $r \in R$ and $t \in T$. Since f is monic, we have $0 = \ker f = \text{ann}(s) \cap \text{ann}(t)$. If $s \neq 0$, then $\text{ann}(s) \leq_e R_R$ by definition of singular element; hence $\text{ann}(t) = 0$. In this case, we can define a monomorphism $R \to T$ with $1 \longmapsto t$. On the other hand, if $s = 0$, then f is a monomorphism $R \to T$. Therefore, $R \in \mathscr{T}$, implying that $\mathscr{T} = \text{Mod-R}$. This shows that any hereditary pretorsion class $\mathscr{T} \neq \text{Mod-R}$ must be contained in SSMod-R, completing the proof.

$(ii) \Rightarrow (i)$ Assume that hptors-R contains SSMod-R as a unique coatom. This shows that R is a right NMC-ring. Let $0 \neq I$ be a proper ideal of R and let $\mathscr{T}_I = \{N \in \text{Mod-R} : NI = 0\}$. Then $\mathscr{T}_I$ is a hereditary pretorsion class. Since $RI = I$ is nonzero, we obtain that $R \notin \mathscr{T}_I$. By assumption, $\mathscr{T}_I$ is contained in SSMod-R , then $R/I \in \text{SSMod-R}$. Assume that R can be decomposed as $R = I \oplus J$, where $I \neq 0$ and $J \neq 0$. Then $I \cong R/J$ and $J \cong R/I$, where both of them are semisimple by above arguments, a contradiction because R is not semisimple Artinian.

$(ii) \Rightarrow (iii)$ Let C be a non-semisimple cyclic right R-module. In this case, $\sigma[C] \nsubseteq \text{SSMod-R}$. Since SSMod-R is the unique coatom in hptors-R and hptors-R is a coatomic lattice, we obtain $\sigma[C] = \text{Mod-R}$.

$(iii) \Rightarrow (ii)$ Suppose that $\mathscr{T}$ is a hereditary pretorsion class of right R-modules. Let $\mathscr{T} \nsubseteq \text{SSMod-R}$. Then there is a right R-module C such that $C \in \mathscr{T} \setminus \text{SSMod-R}$, and so by (iii) we obtain $\mathscr{T} \supseteq \sigma[C] = \text{Mod-R}$. This implies that $\mathscr{T} = \text{Mod-R}$.

$(iii) \Leftrightarrow (iv)$ This easily follows from Proposition 2.15.24. □

Proposition 4.2.5. *[31, Proposition 2.5] Suppose that R is an indecomposable right* NMC*-ring. Then every hereditary pretorsion class of right R-modules (except possibly* SSMod-R*) is a torsion class. Moreover, either one of the following statements holds.*

(i) $\text{Z}(M)$ *is an injective direct summand of M and* $\text{Z}(M) \subseteq \text{Soc}(M)$ *for every right R-module M, or*

(ii) $\text{Soc}(M) = \text{Z}(M) \leq_e M$ *for every right R-module M.*

Also, the statement (i) is satisfied if $\text{Z}(R_R) = 0$ *while the statement (ii) is satisfied otherwise.*

Proof. Suppose that $\mathscr{T}$ is a hereditary pretorsion class in Mod-R. If $\mathscr{T} = \text{Mod-R}$, then there is nothing to prove. So, assume that $\mathscr{T} \neq \text{Mod-R}$. Then by Theorem 4.2.4, $\mathscr{T} \subseteq \text{SSMod-R}$. Now suppose that $\mathscr{T} \neq \text{SSMod-R}$. In this case, we can find the smallest hereditary torsion class containing $\mathscr{T}$ by Proposition 2.15.9, say $\mathscr{T}'$. If $\mathscr{T}' \nsubseteq \text{SSMod-R}$, then we have $\mathscr{T}' = \text{Mod-R}$ by Theorem 4.2.4. This implies that, $\mathscr{T}'(M) = M$, and so $\mathscr{T}(M) \leq_e M$ for every right R-module M by Corollary 2.15.11. Now let S be a semisimple right R-module, then $\mathscr{T}(S)$ is both a direct summand and an essential submodule of S, so we have $\mathscr{T}(S) = S$ by Corollary 2.5.11. But this is a contradiction since S is arbitrary and $\mathscr{T} \neq \text{SSMod-R}$. By Theorem 4.2.4 again, $\mathscr{T}' \subseteq \text{SSMod-R}$. It follows that, we have $\mathscr{T} \subseteq \mathscr{T}' \subseteq \text{SSMod-R}$, and so $\mathscr{T}(M) \leq_e \mathscr{T}'(M) \leq \mathrm{Soc}$. Since $\mathscr{T}'$ is semisimple, $\mathscr{T}(M) = \mathscr{T}'(M)$ for any right R-module M again by Corollary 2.5.11 and we obtain $\mathscr{T} = \mathscr{T}'$. Hence $\mathscr{T}$ is a torsion class.

If $\mathrm{Z}(R_R) = 0$, then R is a right SI-ring and $\mathrm{Z}(M) \subseteq \mathrm{Soc}(M)$ by Proposition 4.2.2. Hence (i) holds. By Theorem 4.2.4, there are two cases: $\mathrm{Sing\text{-}R} \subsetneq \mathrm{SSMod\text{-}R}$ and $\mathrm{Sing\text{-}R} = \mathrm{SSMod\text{-}R}$. In the former case, $\mathrm{Sing\text{-}R}$ is a hereditary torsion class by the first part of the proposition. If $\mathrm{Z}(R_R) = 0$, then we are done. Assume the contrary. Since $\mathrm{Sing\text{-}R}$ is a hereditary torsion class, $R/\mathrm{Z}(R_R)$ is nonsingular as a right R-module. Then it cannot be semisimple because $\mathrm{Z}(R_R)$ is not a direct summand of R_R. In this case, by Theorem 4.2.4 (iv), R_R can be embedded in a finite direct sum of copies of $R/\mathrm{Z}(R_R)$. But this is impossible because R_R cannot be annihilated by $\mathrm{Z}(R_R)$.

Now consider the case $\mathrm{Sing\text{-}R} = \mathrm{SSMod\text{-}R}$. If $\mathrm{Z}(R_R) = 0$, then we are done. Suppose that $\mathrm{Z}(R_R) \neq 0$. In this case, there exists no nonzero nonsingular right R-modules. If there were a nonzero nonsingular right R-module C, then we would have $\sigma[C] = \mathrm{Mod\text{-}R}$. But this is a contradiction since $\mathrm{Z}(R_R) \neq 0$. Thus $\mathrm{Sing\text{-}R}$ cannot be a hereditary torsion class. It follows that $\mathrm{Mod\text{-}R}$ is the smallest hereditary torsion class containing $\mathrm{Sing\text{-}R}$. Then $\mathrm{Z}(M) \leq_e M$ for every nonzero right R-module M by Proposition 2.15.19. Thus (ii) holds. □

4.3 A decomposition theorem

In this section, we focus on decomposition of rings without a right middle class. We also give several important properties of such rings.

Proposition 4.3.1. *[31, Proposition 3.1] Suppose that R is any right* NMC*-ring. Then R is either a right semi-artinian ring or a right Noetherian, right* V*-ring.*

Proof. Let $\mathscr{U} = \{M \in \mathrm{Mod\text{-}R} : M_R \text{ is semi-artinian}\}$. Note that $\mathscr{U}$ is a portfolio by Example 2.15.13 and Theorem 3.2.8. Suppose that R is not right semi-artinian. In this case, $\mathscr{U} \neq \mathrm{Mod\text{-}R}$. Hence $\mathscr{U} = \mathrm{SSMod\text{-}R}$ by Theorem 4.2.4. We shall show that all semisimple right R-modules are injective. Assume the contrary. Let U be a semisimple right R-module which is not injective. Since U is not injective, $\mathrm{E}(U)/U$ must be nonzero. Note that $\mathrm{E}(U)/U$ is singular by Proposition 2.7.3, and so semisimple by Proposition 4.2.2. It follows that $\mathrm{E}(U) \in \mathscr{U} \setminus \mathrm{SSMod\text{-}R}$, which is a contradiction. Thus we obtain that every semisimple right R-module is injective. Therefore R is a right V-ring and also right Noetherian (see [32]). □

Proposition 4.3.2. *[31, Proposition 3.2] If R is a right* NMC*-ring, then R is either a right* V*-ring or a right Artinian ring. If R is a right* NMC*-ring such that* $\mathrm{Z}(R_R) \neq 0$*, then R is right Artinian.*

Proof. Let $\mathscr{U} = \{M \in \text{Mod-R} : M$ is a semi-artinian module$\}$ and $\mathscr{N} = \{M \in \text{Mod-R} :$ finitely generated submodules of M are Noetherian$\}$. Notice that both $\mathscr{N}$ and $\mathscr{U}$ are portfolios. Now, assume that R is not a right V-ring. Then we can find a simple right R-module V which is not injective. As in the proof of Proposition 4.3.1, $\mathrm{E}(V)/V$ is semisimple, and hence it contains a simple submodule, say U/V. It follows that we obtain $0 \subset V \subset U$, where V is the unique simple in $\mathrm{E}(V)$. Then $V = \mathrm{Soc}(U)$, so we have $0 \subset V \subset U$ as a socle series of U. Since this socle series terminates, U is semi-artinian and also Noetherian. Note that $\mathscr{U} \nsubseteq \text{SSMod-R}$ because $U \in \mathscr{U} \setminus \text{SSMod-R}$. Now we obtain that $\mathscr{N} = \mathscr{U} = \text{Mod-R}$ since R is a right NMC-ring. This gives that R is right Artinian by Proposition 2.8.3.

For the last statement, suppose that R is a right NMC-ring with $\mathrm{Z}(R_R) \neq 0$ and that R is not right semi-artinian. Then by Proposition 4.3.1, R is a right Noetherian, right V-ring. This means that $\mathrm{Soc}(R_R)$ is injective, and so it is a direct summand of R_R. But since $\mathrm{Z}(R_R) \subseteq \mathrm{Soc}(R_R)$ by Proposition 4.2.5, $\mathrm{Z}(R_R)$ is a nonzero direct summand of R_R, a contradiction. Hence, R is right semi-artinian. Moreover, if R is a right V-ring, then it is a regular ring (see [11]). But this is a contradiction because in the regular ring R, $\mathrm{Z}(R_R)$ is zero. Therefore, R is right Artinian. □

The following proposition can be regarded as a consequence of Proposition 3.2.3.

Proposition 4.3.3. *[31, Proposition 3.3] Let R be a ring and $R = A \times B$ be a ring decomposition. If R has a linearly ordered profile, then either A or B is semisimple Artinian.*

Proof. We already know that $i\mathcal{P}(A \times B) \cong i\mathcal{P}(A) \times i\mathcal{P}(B)$ by Proposition 3.2.3. If both $i\mathcal{P}(A)$ and $i\mathcal{P}(B)$ have at least two elements, then $i\mathcal{P}(A \times B)$ cannot be linearly ordered. Hence, one of these portfolios must have single element. Assume that $i\mathcal{P}(A)$ is a singleton. Then $\text{SSMod-A} = \text{Mod-A}$, and so A is semisimple Artinian. □

Lemma 4.3.4. *[30, Lemma 2.4] Let $R = A \times B$ be a ring decomposition, where A is semisimple. Then R is a right* NMC*-ring if and only if B is a right* NMC*-ring.*

Proof. First suppose that B is a right NMC-ring. Let N be a non-semisimple and cyclic right R-module. Assume that M is N-injective. Note that any right R-module can be written as $X \oplus Y$, where X is an A-module and Y is a B-module. Then, we have $N = N_1 \oplus N_2$ such that N_1 is an A-module and N_2 is a B-module. Since N_1 and N_2 are also cyclic, we obtain that $N \cong A/I \oplus B/J$, where $I \subseteq A$ and $J \subseteq B$ for some right ideals I and J. Also note that B/J is not semisimple (as both R- and B-modules). Since $1 = a+b$ for some $a \in A, b \in B$, we have $m = ma + mb$, which implies that $M = MA + MB$. Now let $x \in MA \cap MB$. It follows that $x = ma = m'b$ for some $m, m' \in M$, so we obtain that $x = 0$. This gives that $M = MA \oplus MB$. Since M is N-injective, then it is also B/J-injective by Proposition 2.13.23; hence MB is B/J-injective as both R- and B-modules. MB is also injective as B-module by assumption. It is not difficult to see that it is also injective as an R-module. On the other hand, one can easily see that MA is an injective right R-module. In this case, $\mathfrak{In}^{-1}(M) = \text{Mod-R}$, and so R is a right NMC-ring. Now the converse follows from Lemma 4.1.11. □

Lemma 4.3.5. *[31, Lemma 3.4] Let R be a right* NMC*-ring and I be a nonzero ideal of R. Then either R/I or $R/\text{ann}_l I$ is a semisimple Artinian ring.*

Proof. It is not difficult to see that the functor r defined by $r(M) = \text{ann}_M(I)$ is a left exact preradical on Mod-R for any right R-module M. Then, the class $\mathscr{C} = \{M \in \text{Mod-R} : M/r(M)$ is r-torsion-free and semisimple$\}$, containing Mod-(R/I), is a portfolio by Lemma 3.1.10. It follows that we have either $\mathscr{C} = \text{Mod-R}$ or $\mathscr{C} = \text{SSMod-R}$. In the latter case, we obtain that R/I is semisimple. If $\mathscr{C} = \text{Mod-R}$, then $R/r(R)$ is semisimple. Therefore $R/\text{ann}_l(I)$ is semisimple because $\text{ann}_l(I) = \text{ann}_{R_R}(I) = r(R_R)$. □

Proposition 4.3.6. *[31, Proposition 3.5] Let R be a right* NMC*-ring that is not semisimple Artinian. Let A_1 and A_2 be nonzero ideals of R such that $A_1 \cap A_2 = 0$. Then $R = A_i \oplus \text{ann}_l(A_i)$ for at least one $i = 1, 2$. In particular, either A_1 or A_2 is semisimple.*

Proof. Let $f : R \longrightarrow (R/A_1) \oplus (R/A_2)$ be a homomorphism, where A_1 and A_2 are nonzero two-sided ideals. Since $\ker f = A_1 \cap A_2 = 0$, R can be embedded in $(R/A_1) \oplus (R/A_2)$. In this case, R/A_i is not semisimple for at least one of $i = 1, 2$. Otherwise we have a contradiction because R is not semisimple Artinian. Say R/A_1 is not semisimple. It follows that $R/\text{ann}_l(A_1)$ is semisimple by Lemma 4.3.5.

We first assume that R is not right Artinian. By Proposition 4.3.2 R is a right V-ring, and so it is semiprime by Corollary 2.14.16. Since

$$(A_i \cap (\operatorname{ann}_l(A_i))) \cdot (A_i \cap (\operatorname{ann}_l(A_i))) \subseteq \operatorname{ann}_l(A_i) \cdot A_i = 0,$$

and R is semiprime, we obtain that $A_i \cap \operatorname{ann}_l(A_i) = 0$ for each $i = 1, 2$.

Now suppose that R is a right Artinian ring. Since $(A_i \cap (\operatorname{ann}_l(A_i))^2 = 0$, we see that $D := A_1 \cap \operatorname{ann}_l(A_1) \subseteq \mathrm{J}(R)$. It follows that $\mathrm{J}(R) \nsubseteq A_1$ because R/A_1 is not semisimple. Therefore we obtain that $D \subsetneq J(R)$. In this case $R/\mathrm{J}(R)$ is a factor of R/D, and so it is in $\sigma[R/D]$. Thus we have $\mathrm{SSMod\text{-}R} \subseteq \sigma[R/D]$ because all simples have an isomorphic copy contained in $R/\mathrm{J}(R)$. It follows that $\sigma[R/D]$ is a portfolio. Then $\sigma[R/D]$ must be equal to $\mathrm{Mod\text{-}R}$ since R/D is not semisimple. Indeed, otherwise we would have $\mathrm{J}(R) \subseteq D$, a contradiction. Then R is subgenerated by R/D. Hence R must be annihilated by D, i.e, $RD = D = 0$.

Thus, in any case, $R/\operatorname{ann}_l(A_1)$ is semisimple as a right R-module and $A_1 \cap \operatorname{ann}_l(A_1) = 0$. Set $X = \operatorname{ann}_l(A_1)$. As $0 \neq \frac{A_1 \oplus X}{X} \leq \frac{R}{X}$, where R/X is a semisimple right R-module, we can write $\frac{A_1 \oplus X}{X} \oplus \frac{Y}{X} = \frac{R}{X}$ such that $(A_1 \oplus X) \cap Y = X$ and $(A_1 \oplus X) + Y = R$, where $Y \leq R_R$ containing X. Then we have $(A_1 \cap Y) + X = X$. It follows that $A_1 \cap Y \subseteq X \cap A_1 = 0$, and so $A_1 \cap Y = 0$. Since A_1 is a two-sided ideal of R, we have $YA_1 \subseteq Y \cap A_1 = 0$ and $Y \subseteq X$. This gives that $R = A_1 \oplus \operatorname{ann}_l(A_1)$. Therefore, either A_1 or $\operatorname{ann}_l(A_1)$ is semisimple by Proposition 4.3.3. Since $A_2A_1 \subseteq A_1 \cap A_2 = 0$, we obtain that $A_2 \subseteq \operatorname{ann}_l(A_1) = X$. So, if $\operatorname{ann}_l(A_1)$ is semisimple, then A_2 is semisimple. □

Lemma 4.3.7. *[31, Lemma 3.6] Let R be a right* NMC*-ring. Then there exists only a finitely many isomorphism classes of simple right ideals of R.*

Proof. Suppose, without loss of generality, that R is not right Noetherian. It follows that R must be a right nonsingular ring by Proposition 4.3.2. Assume contrarily that R contains infinite number of nonisomorphic simple right ideals. In this case, one can find two semisimple right ideals S_1 and S_2 having infinite length such that $S_1 \cap S_2 = 0$, and $\operatorname{Hom}_R(S_i, S_j) = 0$ (for $1 \leq i \neq j \leq 2$). Now suppose that S_i is injective for at least one of $i = 1, 2$. Say S_1 is injective. This means that S_1 is a direct summand of R, a contradiction. Hence, neither S_1 nor S_2 is injective. Let $\mathscr{T}$ be the hereditary torsion class $\{M \in \mathrm{Mod\text{-}R} : \operatorname{Hom}_R(M, \mathrm{E}(S_2)) = 0\}$. Note that $\mathrm{Sing\text{-}R} \subseteq \mathscr{T}$ by Proposition 2.7.5 (i) because $\mathrm{E}(S_2)$ is nonsingular. Since $\mathrm{E}(S_2)$ is $\mathscr{T}$-torsion-free and not semisimple, by Theorem 4.2.3, every $\mathscr{T}$-torsion module must be semisimple. But this is a contradiction because $\mathrm{E}(S_1)$ is a $\mathscr{T}$-torsion module that is not semisimple. This completes the proof. □

Lemma 4.3.8. *[31, Lemma 3.7] Any ring without a right middle class satisfies the ascending chain condition on its ideals.*

Proof. Suppose that R is a right NMC-ring and that $M_1 \subsetneq M_2 \subsetneq \cdots$ is a strictly ascending chain of ideals of R. Then R/M_i is not semisimple Artinian for every $i = 1, 2, \ldots$. Without loss of generality, we suppose that R is not right Artinian. In view of the proof of Proposition 4.3.6, it can be seen that $R/\operatorname{ann}_l(M_i)$ is semisimple Artinian and $R = M_i \oplus \operatorname{ann}_l(M_i)$ for every $i = 1, 2, \ldots$. Then we have $M_{i+1} = M_i \oplus (M_{i+1} \cap \operatorname{ann}_l(M_i))$ for every $i = 1, 2, \ldots$. Observe that $N_i := M_{i+1} \cap \operatorname{ann}_l(M_i)$ is a nonzero ideal of R for every $i = 1, 2, \ldots$. It follows that $R = M_1 \oplus N_1 \oplus \cdots \oplus N_i \oplus \operatorname{ann}_l(M_{i+1})$ for every $i = 1, 2, \ldots$. Hence $M_1 \oplus N_1 \oplus \cdots \oplus N_i$ is semisimple as a right R-module because $R/\operatorname{ann}_l(M_{i+1})$ is semisimple. But in this case, there are infinitely many nonisomorphic simple right ideals of R. By Lemma 4.3.7, we have a contradiction. This completes the proof. □

Theorem 4.3.9. *[31, Theorem 3.8] Let R be any ring, then R is right* NMC *if and only if R can be decomposed as a ring as $R = A \times B$, where A is semisimple Artinian, and $B = 0$ or B is an indecomposable right* NMC*-ring.*

Proof. The "if" part follows from Lemma 4.3.4. To prove the converse, suppose that R is a right NMC-ring. If R is semisimple Artinian or indecomposable, then there is nothing to prove. Hence assume that R is not semisimple Artinian and that there is a ring decomposition $R = A_1 \times B_1$, where A_1 and B_1 are nonzero. Then either A_1 or B_1 is semisimple by Proposition 4.3.3. Say A_1 is semisimple. By Lemma 4.3.4, B_1 is a right NMC-ring. Then B_1 must be either indecomposable or has a decomposition $B_1 = A_2 \oplus B_2$ such that A_2 and B_2 are nonzero ideals of R and one of them is semisimple. Say A_2 is semisimple. Continuing this process, we get a strictly ascending chain $A_1 \subset A_1 \oplus A_2 \subset \cdots$ of ideals of R. By Lemma 4.3.8, this chain must stop after finite steps, say after n steps. Hence we can find a semisimple ideal $A_1 \oplus \cdots \oplus A_n$ and an indecomposable ideal B_n such that $R = A_1 \oplus \cdots \oplus A_n \oplus B_n$, completing the proof. □

4.4 Noetherian rings with no middle class

The main focus of the present section is on the right Noetherian rings without a right middle class. We will give some important information about these rings, focusing on indecomposable rings.

Proposition 4.4.1. *[31, Proposition 4.1] Let R be an indecomposable ring. If R is a right* NMC*-ring, then R does not contain a pair of nonzero two-sided ideals which meet at zero. Moreover, either* $\mathrm{Soc}(R_R) = 0$ *or* $\mathrm{Soc}(R_R)$ *is homogeneous and* $\mathrm{Soc}(R_R) \leq_e R_R$*.*

Proof. We may suppose that R is not a semisimple Artinian ring. If R has independent two nonzero ideals A_1 and A_2 (i.e, $A_1 \cap A_2 = 0$), then we can write $R = A_i \oplus \mathrm{ann}_l(A_i)$ for at least one $i = 1, 2$ by Proposition 4.3.6. But this is a contradiction because R is indecomposable.

Assume that $\mathrm{Soc}(R_R) \neq 0$. Then it must be homogeneous since R does not contain independent nonzero two-sided ideals. If R is right Artinian, then $\mathrm{Soc}(R_R) \leq_e R_R$ by Corollary 2.6.4. Therefore we assume that R is not right Artinian. In this case, R must be a right V-ring by Proposition 4.3.2. Let R be right Noetherian. Then every simple R-module is injective, and arbitrary direct sum of simple R-modules is also injective. It means that $\mathrm{Soc}(R_R)$ is a direct summand of R_R. So we may write $R = U \oplus \mathrm{Soc}(R_R)$, where U is a right ideal of R. In this case, U is nonzero because R is not a semisimple Artinian ring. Since $U \cdot \mathrm{Soc}(R_R) \subseteq U \cap \mathrm{Soc}(R_R) = 0$, we have $U \cdot \mathrm{Soc}(R_R) = 0$. It follows that $U \subseteq \mathrm{ann}_l(\mathrm{Soc}(R_R))$, and so $\mathrm{ann}_l(\mathrm{Soc}(R_R))$ is a nonzero ideal of R. Since R is semiprime by Corollary 2.14.16, we obtain that $\mathrm{Soc}(R_R) \cap \mathrm{ann}_l(\mathrm{Soc}(R_R)) = 0$ in view of the the proof of Proposition 4.3.6, a contradiction. Thus R is not right Noetherian, and so R must be right semi-artinian by Proposition 4.3.1. This means that $\mathrm{Soc}(R_R) \leq_e R_R$, and we are done. □

We can easily deduce from the proof of Proposition 4.4.1 that , if a ring R without a right middle class with $\mathrm{Soc}(R_R) \neq 0$ is right Noetherian, then it is right Artinian. Furthermore, examination of indecomposable rings R without a right middle class splits into two cases with respect to whether $\mathrm{Soc}(R_R) = 0$ or not. We begin with considering the case when the right socle of the ring is zero. We should be noted that any ring without a right middle class with zero right socle must be a right Noetherian, right V-ring by Proposition 4.3.1.

Theorem 4.4.2. *[31, Theorem 4.2] Let R be an indecomposable ring with* $\mathrm{Soc}(R_R) = 0$*. Then R is a right* NMC*-ring if and only if R is Morita equivalent to a right* SI*-domain.*

Furthermore, if R is an indecomposable ring without a right middle class that is not simple Artinian, we obtain $\mathrm{Soc}(R_R) = 0$ *if and only if every hereditary pretorsion class of right R-modules is a torsion class.*

Proof. First assume that R is a right NMC-ring. Then by Proposition 4.3.2, R must be right nonsingular, and so by Proposition 4.4.2 R is a right SI-ring. By Theorem 2.14.5, R has a ring decomposition $R = U \times R_2 \times \ldots \times R_n$ such that $U/\mathrm{Soc}(U_U)$ is a semisimple ring and each R_i is Morita equivalent to a right SI-domain. Since R is an indecomposable ring, it must be equal to one of the summands. If $R = U$, we obtain a contradiction. Therefore, R must be equal to one of the $R_2, \ldots, R_n$. This means that R is a Morita equivalent to a right SI-domain.

Conversely, we can suppose, without loss of generality, that R is a right SI-domain because having no right middle class property is Morita invariant property. Hence R is right Noetherian by Proposition 2.14.4. In this case, every nonzero right ideal of R is essential in R_R by Lemma 2.6.7. Then R/I is semisimple for any nonzero right ideal I of R by Proposition 2.14.2. Let A be a non-semisimple right R-module. Since $\mathrm{Z}(A)$ is injective, it must be a direct summand of A. Then we can write $A = \mathrm{Z}(A) \oplus B$ for some submodule B of A. By Proposition 2.14.2, $\mathrm{Z}(A)$ is semisimple. Thus we may find a nonzero element $b \in B$ such that bR is not semisimple. It follows that B is nonsingular. In domains, nonsingular elements are precisely torsion-free elements. This means that b cannot be annihilated by any element of R, and so $bR \cong R$. Then R can be embedded in A and we may write $\sigma[A] = \mathrm{Mod\text{-}R}$ by Proposition 2.15.24. Therefore R is a right NMC-ring by Theorem 4.2.4.

To prove the last statement, suppose that $\mathrm{Soc}(R_R) = 0$. It means that R cannot be a right semi-artinian ring. We already know that the class of semi-artinian modules is the smallest torsion class that contains $\mathrm{SSMod\text{-}R}$. In this case, the smallest torsion class containing $\mathrm{SSMod\text{-}R}$ is $\mathrm{SSMod\text{-}R}$ itself since it is a coatom in $\mathrm{hptors\text{-}R}$. Then by Proposition 4.2.5, we obtain that every hereditary pretorsion class is a torsion class. Conversely, suppose that every hereditary pretorsion class is a torsion class. In this case, $\mathrm{SSMod\text{-}R}$ is a torsion class. If $\mathrm{Soc}(R_R) \neq 0$, we have $\mathrm{Soc}(R_R) \leq_e R_R$ by Proposition 4.4.1. Hence $R/\mathrm{Soc}(R_R)$ is semisimple. But this is a contradiction by Lemma 2.15.20. Therefore $\mathrm{Soc}(R_R) = 0$. $\square$

Remark 4.4.3. We already know that every hereditary pretorsion class of right R-modules (except possibly $\mathrm{SSMod\text{-}R}$) is a torsion class by Proposition 4.2.5. Now it is worth emphasising here that $\mathrm{SSMod\text{-}R}$ is also a hereditary torsion class under the conditions of the above theorem.

Now that Theorem 4.4.2 has shed light on the case when the indecomposable right NMC-ring has zero right socle, we can turn our attention to the case when the socle is nonzero. Towards this end we begin with the following theorem.

Theorem 4.4.4. *[31, Theorem 4.3] Let R be an indecomposable right NMC-ring that is not semisimple Artinian. If $\mathrm{Z}(R_R) = 0$ and $\mathrm{Soc}(R_R)$ is nonzero, then there exists exactly one isomorphism class of singular simple right R-modules. In particular, the isomorphism classes of singular simple right R-modules are precisely those of simple right ideals of the ring $R/\mathrm{Soc}(R_R)$ as R-modules and $R/\mathrm{Soc}(R_R)$ is a simple Artinian ring.*

Proof. Note that $\mathrm{Soc}(R_R) \leq_e R_R$ by Proposition 4.4.1. Since $R/\mathrm{Soc}(R_R)$ is singular, it is also a semisimple right R-module by Proposition 4.2.2. Assume that U is a singular simple right R-module. In this case, there is some maximal ideal $\mathfrak{M}$ of R such that $U \cong R/\mathfrak{M}$. Then by Proposition 2.7.3, $\mathfrak{M}$ is essential in R_R and we have $\mathrm{Soc}(R_R) \subseteq \mathfrak{M}$. This shows that $\frac{R}{\mathfrak{M}} \cong \frac{R/\mathrm{Soc}(R_R)}{\mathfrak{M}/\mathrm{Soc}(R_R)}$. Therefore U is isomorphic to a simple right R-submodule of $R/\mathrm{Soc}(R_R)$. This gives, in particular, that isomorphic copies of every singular simple right R-module is contained in $R/\mathrm{Soc}(R_R)$. Thus $R/\mathrm{Soc}(R_R)$ contains only a finitely many singular simple right R-modules up to isomorphism. Let $\{U_1 = U, \ldots, U_n\}$ be a complete set of representatives of isomorphism classes of singular simple right R-modules. Now suppose that $n > 1$. Let B be the proper ideal of R such that $B/\mathrm{Soc}(R_R)$ is the sum of all simple right ideals of the semisimple Artinian ring $R/\mathrm{Soc}(R_R)$ isomorphic to U. Then we have $B \neq \mathrm{Soc}(R_R)$. Notice that every non-singular simple right R-module must be projective by Proposition 2.12.5. This gives that it can be embedded in R, and so isomorphic to a simple right ideal of R (necessarily contained in B). In this case, the right R-module $M = B \oplus U_2 \oplus \cdots \oplus U_n$ contains isomorphic copies of every simple right R-module. Since $\frac{M}{\mathrm{Soc}(R_R) \oplus U_2 \oplus \cdots \oplus U_n} \cong \frac{B}{\mathrm{Soc}(R_R)}$, we have $U \subseteq B/\mathrm{Soc}(R_R) \subseteq \sigma[M]$. Also, since B_R is not semisimple, SSMod-R is strictly contained in $\sigma[M]$. Then we obtain that $\sigma[M] = \mathrm{Mod\text{-}R}$. Thus R can be embedded in a finite direct sum of copies of M by Proposition 2.15.24. Since R is right nonsingular, $R/\mathrm{Soc}(R_R)$ can be embedded in a finite direct sum of copies of $B/\mathrm{Soc}(R_R)$. But this is a contradiction because the simple modules $U_2, \ldots, U_n$ are isomorphic to a submodule of $R/\mathrm{Soc}(R_R)$ and they cannot be embedded in a direct sum of copies of $B/\mathrm{Soc}(R_R)$. Therefore, we have $n = 1$. The last statement is obvious. $\square$

Proposition 4.4.5. *[31, Proposition 4.4] The following statements are equivalent for a ring R without a right middle class that is not semisimple Artinian.*

(i) There exists an indecomposable right R-module of composition length two.

(ii) R is right Artinian.

(iii) R is not a right V*-ring.*

Proof.

$(i) \Rightarrow (ii)$ Let T be an indecomposable right R-module of composition length two such that $0 \subset U \subset T$, where U is the simple submodule of T. Assume that R is not right Artinian, then it is a right V-ring by Proposition 4.3.2. It follows that U must be an injective module, and so it is a direct summand of T. Since U is also an essential submodule of T, it must be equal to T. But this is a contradiction.

$(ii) \Rightarrow (iii)$ Let R be a right Artinian ring. Assume that R is a right V-ring. Then by Lemma 2.14.17, we have a contradiction. Thus R is not a right V-ring.

$(iii) \Rightarrow (i)$ Suppose that R is not a right V-ring. Then there is a simple right R-module W, that is not injective. Since $W \leq_e \mathrm{E}(W)$, $\frac{\mathrm{E}(W)}{W}$ is a nonzero singular module by Proposition 2.7.3. Then it is also semisimple by Proposition 4.2.2. It follows that there exist a simple right R-module U/W in $\mathrm{E}(W)/W$, where $W \leq U$. Hence we have an indecomposable right R-module $0 \subset W \subset U$ of composition length two. □

We now study right Noetherian rings without a right middle class with nonzero right socle. It is worth mentioning that an indecomposable right Noetherian ring having no right middle class with nonzero right socle must be right Artinian. The proposition below reveals that a majority of significant properties of right Artinian rings with no right middle class, stated in [23], are shared by rings subgenerated by an indecomposable module of composition length two.

Proposition 4.4.6. *[31, Proposition 4.5] Let R be a ring. Assume that there exists an indecomposable right R-module U of composition length two and S be its simple submodule. Assume also that $\sigma[U] = \mathrm{Mod\text{-}R}$. Then the following conditions are satisfied:*

(i) R is a right Artinian ring.

(ii) $\mathrm{Soc}(R_R)$ is homogeneous.

(iii) *Singular right R modules are semisimple. In particular, as a right R-module $R/\operatorname{Soc}(R_R)$ is semisimple, that is,* $\operatorname{J}(R) \subseteq \operatorname{Soc}(R_R)$.

(iv) *S and U/S are the only simple right R-modules up to isomorphism.*

(v) *If a right R-module M contains no copy of S, then M is injective.*

(vi) *If S is a nonsingular module, then*

(a) *U/S is an injective module, and*

(b) *R is a right* SI*-ring.*

(vii) *If S is a singular module, then we have* $\operatorname{J}(R) = \operatorname{Soc}(R_R) = \operatorname{Z}(R_R)$.

Proof. Let U be an indecomposable right R-module of composition length two such that $\sigma[U] = \text{Mod-R}$. In this case, R can be embedded in a direct sum $W = U_1 \oplus \cdots \oplus U_n$ by Proposition 2.15.24, where $U_i \cong U$ for every $i = 1, \ldots, n$. Since length of U_i's is 2, length of W must be $2n$. It follows that W is a right Artinian module, that implies that R is also right Artinian. Since $\operatorname{Soc}(R_R) = \operatorname{Soc}(W) \cap R$, we obtain that $\operatorname{Soc}(R_R)$ is homogeneous. Hence (i) and (ii) hold.

Now let S_i denote the unique simple submodule of U_i for each $i = 1, \ldots, n$. There is an epimorphism $\sigma : W \longrightarrow W/\operatorname{Soc}(W)$ with $\ker \sigma = \operatorname{Soc}(W)$. In this case, σ can be restricted to R, and we have a homomorphism $\sigma' = R \longrightarrow W/\operatorname{Soc}(W)$ such that $\ker \sigma' = \operatorname{Soc}(W) \cap R = \operatorname{Soc}(R_R)$. This implies that $R/\operatorname{Soc}(R_R)$ can be embedded in $W/\operatorname{Soc}(W) \cong (U_1/S_1) \oplus \cdots \oplus (U_n/S_n)$. Since $W/\operatorname{Soc}(W)$ is semisimple, $R/\operatorname{Soc}(R_R)$ is also semisimple. Therefore $\operatorname{J}(R/\operatorname{Soc}(R_R))$ must be equal to zero, which implies that $\operatorname{J}(R) \subseteq \operatorname{Soc}(R_R)$. Now let A be a right ideal of R such that R/A is a singular right R-module. In this case, A must be essential in R_R. This gives that $A \supseteq \operatorname{Soc}(R_R)$. Since $R/A \cong \frac{R/\operatorname{Soc}(R_R)}{A/\operatorname{Soc}(R_R)}$, R/A is semisimple. Hence every nonzero singular right R-module is semisimple.

Let T be a simple right R-module such that $T \cong R/\mathfrak{M}$, where $\mathfrak{M}$ is a maximal right ideal of R. We suppose that T is singular. Then $\mathfrak{M}$ must be essential in R_R by Proposition 2.7.3. This means that $\operatorname{Soc}(R_R) \subseteq \mathfrak{M}$. As $R/\operatorname{Soc}(R_R)$ is semisimple such that its simple submodules are isomorphic to U/S, we have $T \cong R/\mathfrak{M} \cong U/S$. If T is a nonsingular simple right R-module, then it must be projective by Corollary 2.12.6. Thus $\mathfrak{M}$ is a direct summand of R_R and T is isomorphic to a simple submodule of R_R which is necessarily isomorphic to S. In this case, R has only S and U/S as a simple right R-modules, up to isomorphism.

To prove (v), let M be a right R-module that contains no copy of S. Let $0 \neq f : U \longrightarrow \mathrm{E}(M)$ be an R-homomorphism. Since $M \leq_e \mathrm{E}(M)$, we have $f(U) \cap M \neq 0$, and so $f^{-1}(M) \neq 0$. In this case, we obtain that either $f^{-1}(M) = U$ or $f^{-1}(M) = S$. In the latter case, f maps S isomorphically onto a simple submodule of M, a contradiction. Then we must get that $f^{-1}(M) = U$; hence $\mathrm{Im}\, f \subseteq M$. This shows that M is U-injective by Proposition 2.13.24. It follows that $\mathrm{Mod\text{-}R} = \sigma[U] \subseteq \mathfrak{In}^{-1}(M)$, and so we have $\mathfrak{In}^{-1}(M) = \mathrm{Mod\text{-}R}$. Therefore M is an injective module.

Let M be a nonzero singular right R-module. Suppose that S is nonsingular. Then a copy of S is not contained in M. This means that M is an injective module by (v), and so U/S is injective. Moreover, R is a right SI-ring. This proves (vi).

To prove (vii), let S be a singular module. In this case, any simple right R-module is singular by (iv). It follows that $\mathfrak{M} \leq_e R_R$, where $\mathfrak{M}$ is any maximal ideal of R. Thus we have $\mathrm{Soc}(R_R) \subseteq \mathfrak{M}$. This gives that $\mathrm{J}(R) = \mathrm{Soc}(R_R)$. Then we obtain $\mathrm{Soc}(R_R) \subseteq \mathrm{Z}(R_R)$ because every simple right R-module is singular. By (iii), $\mathrm{Z}(R_R) \subseteq \mathrm{Soc}(R_R)$, completing the proof. □

Lemma 4.4.7. *[31, Lemma 4.6] Let R be a semilocal ring and let* $\mathrm{J}(R) \subseteq \mathrm{Soc}(R_R)$. *Assume that R has an indecomposable right module U of composition length two. In this case, the simple submodule of U can be embedded in R_R.*

Proof. Let $0 \subset S \subset U$, where S is a simple submodule of U. Now take $0 \neq u \in U \setminus S$. Then uR must be equal either S or U. Since $u \notin S$, we have $uR = U$. Hence U is cyclic. It follows that there is an R-homomorphism $\psi : R \longrightarrow U$. Assume that $\mathrm{Soc}(R_R) \subseteq \ker \psi$. Then we obtain that $U \cong \frac{R}{\ker \psi} \cong \frac{R/\mathrm{Soc}(R_R)}{\ker \psi / \mathrm{Soc}(R_R)}$. In this case, $R/\mathrm{Soc}(R_R)$ is semisimple because $R/\mathrm{J}(R)$ is semisimple. Thus U must be semisimple. But this is impossible because S is not a direct summand of U. Therefore $\mathrm{Soc}(R_R) \nsubseteq \ker \psi$, and so there is a simple right ideal T of R such that $\psi(T) \neq 0$. It follows that $\psi(T)$ must be isomorphic to S. This completes the proof. □

Theorem 4.4.8. *[31, Theorem 4.7] The following statements are equivalent for a ring R that is not semisimple Artinian.*

(i) R is an indecomposable right Artinian ring without a right middle class.

(ii) R has an indecomposable right R-module of composition length two and we obtain $\sigma[U] = \text{Mod-R}$ for any such module U.

Proof. We suppose that R is an indecomposable, right Artinian ring and has no right middle class. In this case, we can find an indecomposable right R-module of composition length two by Proposition 4.4.5, say U. Let S be the simple submodule of U. Since $\operatorname{Soc}(R_R) \leq_e R_R$, $R/\operatorname{Soc}(R_R)$ is singular as a right R-module by Proposition 2.7.3. Hence it is a semisimple right R-module by Proposition 4.2.2, implying $\mathrm{J}(R) \subseteq \operatorname{Soc}(R_R)$. By Proposition 4.4.1 $\operatorname{Soc}(R_R)$ is homogeneous. It follows that any simple right ideal of R is isomorphic to S by Lemma 4.4.7. Now let $\mathscr{C} = \sigma[U \oplus T]$, where T is the direct sum of a complete set of isomorphism classes of simple right R-modules except for that of S (note that if there exist no such simple right R-modules we take $T = 0$). Since U contains S and T contains simple modules that are not isomorphic to S, $\text{SSMod-R} \subseteq \mathscr{C}$. Then $\mathscr{C}$ is a portfolio by Theorem 3.2.8. Thus we obtain that $\mathscr{C} = \text{Mod-R}$ because U is not semisimple. Hence R_R is a submodule of the direct sum $(U_1 \oplus \cdots \oplus U_n) \oplus (T_1 \oplus \cdots \oplus T_m)$ by Proposition 2.15.24, where $U_i \cong U$ and $T_j \cong T$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Let $\pi : (U_1 \oplus \cdots \oplus U_n) \oplus (T_1 \oplus \cdots \oplus T_m) \longrightarrow (U_1 \oplus \cdots \oplus U_n)$ be the natural projection with $\ker \pi = (T_1 \oplus \cdots \oplus T_m)$. As $R \cap \ker \pi$ is zero, R can be embedded in $(U_1 \oplus \cdots \oplus U_n)$. Therefore $\sigma[U] = \text{Mod-R}$ by Proposition 2.15.24.

Assume that there is an indecomposable right R-module of composition length two. In case U is such a module, we may write $\sigma[U] = \text{Mod-R}$. It follows, by Proposition 4.4.6, that R is right Artinian and $\operatorname{Soc}(R_R)$ is homogeneous; so R is an indecomposable ring. Suppose that M is a right R-module that is not semisimple. Let $f : R \longrightarrow M$ be an R-homomorphism with $f(r) = mr$ for $m \in M$. Then we have $MJ \subseteq \operatorname{rad} M$ by Proposition 2.4.3. Since R is semilocal, R/J is semisimple, and so M/MJ is an R/J-module. Therefore, M/MJ is a semisimple R-module and MJ can be written as an intersection of maximal submodules. It means that $\operatorname{rad} M \subseteq MJ$. If $MJ = 0$, then M must be semisimple, a contradiction. Thus, we obtain that $\operatorname{rad} M = MJ \neq 0$. In this case, there is $0 \neq m \in M$ such that mR is a small submodule of M by Corollary 2.5.10. Then we can choose a submodule L of M by Zorn's lemma that is maximal amongst every submodule of M such that $m \notin L$. Then $(L + mR)/L$ is the simple submodule of M/L. Note that if any nonzero submodule of M/L contains $L/(L + mR)$, then $L/(L + mR)$ is the unique simple submodule. Now let $L \subsetneq A$, where A is a right R-module. If $m \notin A$, then we have a contradiction with the maximality of L. Hence $m \in A$ and $(L + mR) \subseteq A$. This gives that $\frac{L+mR}{L} \subseteq \frac{A}{L} \subseteq \frac{M}{L}$, and so $L + mR/L$ is the unique (essential) simple submodule of M/L. Then $L + mR \neq M$ because $mR \ll M$. Since M is semi-artinian, every nonzero factor of M contains a simple submodule. In this case, $\frac{M}{L+mR}$ has a simple submodule, say $K/(L + mR)$. Then we have $0 \subset \frac{L+mR}{L} \subset \frac{K}{L}$, and so K/L is an indecomposable right R-module of composition length two. In case a right R-module A is M-injective, then it must be U-injective. It follows that $\sigma[U] = \mathfrak{In}^{-1}(A) = \text{Mod-R}$. Hence R is a right NMC-ring. This completes the proof. □

Corollary 4.4.9. *[31, Corollary 4.8] Let R be a right Artinian ring. Then R is a right* NMC-*ring if only if R can be decomposed as a ring as $R = A \times B$, where A is semisimple Artinian, and $B = 0$ or B is an indecomposable ring that holds the following statements:*

(i) $\text{Soc}(B_B)$ *is homogeneous.*

(ii) $\text{J}(B) \subseteq \text{Soc}(B_B)$.

(iii) $\text{Tr}_B(U, \text{E}_B(U)) = \text{E}_B(U)$ *for any indecomposable right B-module of composition length two.*

Proof. Assume that R is a right Artinian ring without a right middle class. By Theorem 4.3.9, R can be decomposed as a ring as $R = A \times B$, where A is semisimple Artinian, and $B = 0$ or B is an indecomposable ring without a right middle class. Suppose that B is nonzero. It follows by Proposition 4.4.6 and Theorem 4.4.8, that $\text{Soc}(B_B)$ is homogeneous and $\text{J}(B) \subseteq \text{Soc}(B_B)$. It is well-known that the injective hull of a uniform module is indecomposable. If U is an indecomposable right B-module of composition length two, then $\text{E}_B(U)$, an injective hull of the right B-module U, is indecomposable. Since U is essential in $\text{E}_B(U)$, U is also essential in $\text{Tr}_B(U, \text{E}_B(U))$. Hence we have $\text{E}_B(U) = \text{E}_B(\text{Tr}_B(U, \text{E}_B(U)))$ by Proposition 2.13.15. Consider the inclusion $U \subseteq \text{Tr}_B(U, \text{E}_B(U)) \subseteq \text{E}_B(U)$. It follows that $\text{Tr}_B(U, \text{E}_B(U)) \subseteq \text{E}(\text{Tr}_B(U, \text{E}_B(U))) \subseteq \text{E}_B(U)$. Therefore $\text{Tr}_B(U, \text{E}_B(U))$ is quasi-injective by Lemma 2.13.29. Since $\text{Tr}_B(U, \text{E}_B(U))$ contains U, it is not semisimple. Also since B is a right NMC-ring, $\text{Tr}_B(U, \text{E}_B(U))$ must be injective B-module. It follows that $\text{Tr}_B(U, \text{E}_B(U)) = \text{E}_B(U)$.

Now suppose that R can be decomposed as a ring as $R = A \times B$, where A is semisimple Artinian, and $B = 0$ or B is an indecomposable ring which satisfies the conditions $(i) - (iii)$. If $B = 0$, there is nothing to prove. Hence suppose that B is nonzero. By Theorem 4.3.9, we need to show that B is a right NMC-ring. Then we may suppose that $R = B$ by Theorem 4.3.9. Let U be an indecomposable right R-module of composition length two with its simple submodule S. As $\mathrm{Soc}(R_R)$ is homogeneous, by Lemma 4.4.7, any simple right ideal of R is isomorphic to S. This means that there exist simple submodules $S_1, \ldots, S_n$ such that $S_1 \oplus \cdots \oplus S_n = \mathrm{Soc}(R_R) \leq_e R_R$. Then we have $\mathrm{E}(S_1) \oplus \cdots \oplus \mathrm{E}(S_n) = \mathrm{E}(R_R)$ by Proposition 2.13.15. Therefore, R_R can be embedded in a finite direct sum of copies of $\mathrm{E}(S) = \mathrm{E}(U)$. It follows, by (iii), that U generates $\mathrm{E}(S)$. Then U generates R_R. This implies that $\sigma[U] = \mathrm{Mod\text{-}R}$ by Proposition 2.15.24. Hence R is a right NMC-ring by Theorem 4.4.8. □

Corollary 4.4.10. *[31, Corollary 4.9] Let R be a ring. Then R is a right Noetherian ring without a right middle class if and only if R can be decomposed as a ring as $R = A \times B$, where A is semisimple Artinian, and $B = 0$ or B is an indecomposable ring which satisfies one of the following statements:*

(i) B is Morita equivalent to a right SI*-domain, or*

(ii) B is a right Artinian ring such that its Jacobson radical does not properly contain a nonzero ideal.

Proof. Let R be a right Noetherian ring with no right middle class. Then R can be decomposed as a ring as $R = A \times B$, where A is semisimple Artinian, and $B = 0$ or B is an indecomposable ring having no right middle class by Theorem 4.3.9. If $B = 0$, then we are done. We may assume that $R = B$ by Theorem 4.3.9. If $\mathrm{Soc}(B_B) = 0$, then the statement (i) holds by Theorem 4.4.2. Otherwise, $\mathrm{Soc}(B_B)$ is essential in B_B by Proposition 4.4.1. Assume that B is a right V-ring. Then $\mathrm{Soc}(B_B)$ is an injective module and it is a direct summand of B_B. But this is a contradiction. Therefore, B is a right Artinian ring and no nontrivial ideal of B is contained in its Jacobson radical by Theorem 4.1.4. The converse easily follows from Theorem 4.1.4, Theorem 4.3.9 and 4.4.2, Corollary 4.4.9. □

Corollary 4.4.11. *[31, Corollary 4.10] Assume that R is an indecomposable ring. If R is a right Artinian ring without a right middle class, then R satisfies the following statements:*

(i) $\mathrm{Soc}(R_R)$ *is homogeneous.*

(ii) If $\mathrm{Z}(R_R) = 0$*, R has a unique simple singular right R-module up to isomorphism.*

(iii) *There exist at most two simple right R-modules up to isomorphism.*

(iv) *Any singular right R-module is semisimple. Also,* $\mathrm{J}(R) \subseteq \mathrm{Soc}(R_R)$.

(v) *If* $\mathrm{Z}(R_R)$ *is nonzero, then* $\mathrm{Soc}(R_R) = \mathrm{J}(R) = \mathrm{Z}(R_R)$.

Proof. Follows easily from Propositions 4.2.2, 4.4.1, 4.4.4, 4.4.5 and 4.4.6 and Theorem 4.4.8. □

Now we give a decomposition theorem of rings without a right middle class, which can be regarded as a summary of this section.

Theorem 4.4.12. *[23, Theorem 2] Let R be a right* NMC*-ring. Then R can be decomposed as a ring as $R = A \times B$, where A is a semisimple Artinian ring, and B is zero or it belongs to one of the following classes:*

(i) *B is Morita equivalent to a right* SI*-domain (equivalently right* PCI*-domain), or*

(ii) *B is an indecomposable right* SI*-ring such that:*

(a) *B is either a right* V*-ring or a right Artinian ring,*

(b) $\mathrm{Soc}(B_B)$ *is homogeneous and* $\mathrm{Soc}(B_B) \leq_e B_B$*,*

(c) *B has a unique singular simple right module up to isomorphism, or*

(iii) *B is a right Artinian indecomposable ring such that:*

(a) $\mathrm{Soc}(B_B) = J(B) = \mathrm{Z}(B_B)$*,*

(b) $\mathrm{Soc}(B_B)$ *is homogeneous.*

Proof. Let R be a right NMC-ring. Then by Theorem 4.3.9, R can be decomposed as a ring as $R = A \times B$, where A is a semisimple Artinian ring, and B is zero or B is an indecomposable right NMC-ring. If $B = 0$, then we are done. Thus suppose that B is nonzero. By Theorem 4.3.9, we can suppose that $R = B$. Let $\mathrm{Soc}(B_B) = 0$. Then by Theorem 4.4.2, B must be Morita equivalent to a right SI-domain. Since SI- and PCI- conditions are equivalent by Proposition 2.14.9, B is Morita equivalent to a right PCI-domain. Therefore (i) holds.

Assume that $\mathrm{Soc}(B_B)$ is nonzero. We split our proof into two cases: $Z(B_B) = 0$ or $\mathrm{Z}(B_B) \neq 0$. In the former case, B is a right SI-ring by Proposition 4.2.2. Then by Theorem 4.4.4, B has a unique simple singular right B-module up to isomorphism. By Proposition

4.4.1, B has a homogeneous right socle. Also by Proposition 4.3.2, B must be either a right V-ring or right Artinian. Thus (ii) holds. Now suppose that $\mathrm{Z}(B_B) \neq 0$. It follows that B is right Artinian by Proposition 4.3.2. Then by Proposition 4.4.1, $\mathrm{Soc}(B_B)$ is homogeneous. Also, we can say that $\mathrm{Soc}(B_B) = \mathrm{J}(B) = \mathrm{Z}(B_B)$ by Proposition 4.4.6 and Theorem 4.4.8. Hence (iii) holds. This completes the proof. □

So far in this section, we have investigated the structure of right Noetherian rings with no right middle class. However, we still do not know if there exists a right non-Noetherian ring with no right middle class. Now we give a survey of some interesting properties of such rings in the following proposition considering that they exist. Before giving the proposition, we remark that when studying a non-semisimple ring R with no right middle class, it is sufficient to suppose that R is an indecomposable ring in light of Theorem 4.3.9.

Note that a right R-module M is called faithful if its annihilator is equal to zero. A ring R is called right primitive if it has a faithful simple right R-module.

Proposition 4.4.13. *[31, Proposition 5.1] The following statements are satisfied for any indecomposable ring R without a right middle class that is not right Noetherian.*

(i) *R is a right* SI*-ring.*

(ii) *$R/\operatorname{Soc}(R_R)$ is a simple Artinian ring.*

(iii) *R is a right semi-artinian, right* V*-ring, and so a von Neumann regular ring (see [11]).*

(iv) *R has homogeneous right socle that has countably infinite length and $\operatorname{Soc}(R_R) \leq_e R_R$.*

(v) *R is a right primitive ring.*

(vi) *There is exactly one non-trivial ideal of R, namely $\operatorname{Soc}(R_R)$.*

Proof. Note that since the right Artinian ring is right Noetherian, R is right nonsingular by Proposition 4.3.2. It follows that R is a right SI-ring by Proposition 4.2.2. By Proposition 4.3.1 and 4.3.2 it is a right semi-artinian, right V-ring. Then by Theorem 4.4.4, $R/\operatorname{Soc}(R_R)$ is a simple Artinian ring. Therefore (i), (ii), and (iii) hold. By statement (iii), $\operatorname{Soc}(R_R)$ is nonzero. Then $\operatorname{Soc}(R_R) \leq_e R_R$ and $\operatorname{Soc}(R_R)$ is homogeneous by Theorem 4.4.1. Hence to prove (iv), it is sufficient to see that $\operatorname{Soc}(R_R)$ is isomorphic to a direct sum of countably many simple modules. Let T be a simple right ideal of R. Since R is not semisimple Artinian, $\operatorname{Soc}(R_R)$ is not injective, and so $U = T^{(\mathbb{N})}$ is not injective by Theorem 2.13.8. It follows that

$\mathrm{E}(U)$ is a non-semisimple right R-module; hence $\mathrm{E}(U)/U$ is (singular) nonzero semisimple. Then we have $\sigma[\mathrm{E}(U)] \supsetneq$ SSMod-R by Theorem 4.4.4. Thus, by assumption, $\sigma[\mathrm{E}(U)] = \text{Mod-R}$. In this case, R can be embedded in a finite direct sum $W := \mathrm{E}(U) \oplus \cdots \oplus \mathrm{E}(U)$ by Proposition 2.15.24. As $\mathrm{Soc}(W)$ is isomorphic to a direct sum of countably infinite simple modules, so is $\mathrm{Soc}(R_R)$. This proves (iv).

To prove (v), it is enough to show that R has a faithful simple right ideal. Let S be a simple right ideal of R. Assume that $Sa = 0$ for some nonzero $a \in R$. As $\mathrm{Soc}(R_R) \leq_e R_R$, aR must contain a simple right ideal of R, say S_1. Then we obtain that $S_1^2 \neq 0$ because R is regular. It follows that there exists $x \in S_1$ such that $S_1 x \neq 0$. Since $Sa = 0$ and $x \in aR$, we get $Sx = 0$. However, we have a contradiction because $S \cong S_1$. Hence R is a right primitive ring.

Lastly, to prove (vi), let A be a nonzero ideal of R. Assume that A is not an essential right ideal of R. Then $A \cap B = 0$ for some right ideal B. In this case, there exists a simple right ideal S such that $SA \subseteq A \cap B = 0$. But this contradicts the proof of (v). Hence we have $\mathrm{Soc}(R_R) \subseteq A$. This implies that $R/A \subseteq R/\mathrm{Soc}(R_R)$. Since $R/\mathrm{Soc}(R_R)$ is simple ring by (ii), A must be equal $\mathrm{Soc}(R_R)$. This completes the proof of (vi). □

4.5 In search of a converse

In this section, we concern ourselves with conditions under which R is a right NMC-ring.

The following proposition deals with the case when the ring is Artinian as in Theorem 4.4.12 (ii) and shows that the converse is satisfied under a uniqueness condition that is stronger than that of Theorem 4.4.12 $(ii)(c)$. Also remark that if R is a right SI-ring then any non-simple local right R-module, is nonsingular.

Proposition 4.5.1. *[23, Proposition 6] Let R be a right Artinian right* SI*-ring. Suppose that there exists a unique indecomposable module of composition length two up to isomorphism. Also suppose that R has homogeneous socle. Then R is a right* NMC*-ring.*

Proof. We already know that one-sided Artinian rings are semiperfect. Then by Theorem 2.14.21, we have a ring decomposition $R_R = e_1R \oplus \cdots \oplus e_kR \oplus f_1R \oplus \cdots \oplus f_nR$, where e_iR are simple right ideals and f_jR are local modules of length ≥ 2. Notice that since R is right Artinian right SI, $\mathrm{Soc}(f_tR) \leq_e f_tR$ and $\mathrm{Soc}(f_tR) \subseteq f_t\,\mathrm{J}$ for each t. This gives that $\mathrm{Soc}(f_tR) \leq_e f_t\,\mathrm{J}$. It follows that $\frac{f_t\,\mathrm{J}}{\mathrm{Soc}(f_tR)}$ is singular by Proposition 2.7.3. Hence it is injective and splits in $\frac{f_tR}{\mathrm{Soc}(f_tR)}$. This means that $\frac{f_tR}{\mathrm{Soc}(f_tR)} = \frac{f_tJ}{\mathrm{Soc}(f_tR)} \bigoplus \frac{C'}{\mathrm{Soc}(f_tR)}$ for some $C' \subseteq f_tR$. Assume that $\frac{C'}{\mathrm{Soc}(f_tR)} \neq \frac{f_tR}{\mathrm{Soc}(f_tR)}$. Since ftR is local for each t, we have $C' \subseteq ft\,\mathrm{J}$. However, this is a contradiction. Thus $\frac{C'}{\mathrm{Soc}(f_tR)}$ must be equal to $\frac{f_tR}{\mathrm{Soc}(f_tR)}$. In this case, we obtain that $f_tJ = \mathrm{Soc}(f_tR)$. For any t, t', we can choose two right ideals $A_t \subseteq f_tR$ and $A'_t \subseteq f_{t'}R$ such that $\mathrm{cl}(\frac{f_tR}{A_t}) = \mathrm{cl}(\frac{f_{t'}R}{A'_t}) = 2$, where cl denotes the composition length. Then by assumption, we have an isomorphism $\frac{f_tR}{A_t} \cong \frac{f_{t'}R}{A'_t}$. This means that $\frac{f_tR}{f_t\,\mathrm{J}} \cong \frac{f_{t'}R}{f'_t\,\mathrm{J}}$ by Lemma 2.4.5. Therefore $f_tR \cong f_{t'}R$ by Lemma 2.12.7.

Let M and A be right R-modules such that A is non-semisimple cyclic and M is A-injective. If we see that M is injective, then we are done. Let $\lambda : R \longrightarrow A$ be an epimorphism. If $\lambda(f_iR)$ is semisimple for each i, then $\lambda(R)$ must be semisimple, a contradiction. In this case, there exists a non-semisimple local submodule in A, which is an image of some f_iR, say A'. Then we can choose some local factor B of A' with composition length two. It follows that M is B-injective. Fix any f_j and suppose that $\mathrm{Soc}(f_jR) = S_1 \oplus \cdots \oplus S_l$ for some simple right ideals S_i. For any i, set $V_i = \bigoplus_{t\neq i} S_t$ (if $l = 1$ we take $V_i = 0$). It follows that $\bigcap_{i=1}^{l} V_i = 0$ and $\mathrm{cl}(\frac{f_jR}{V_i}) = 2$ for each i. Since $\mathrm{Soc}(f_tR)$ is maximal, we have $V_i = \bigoplus_{t\neq i} S_t \subset \mathrm{Soc}(f_jR) \subset f_jR$. Note that $\mathrm{Soc}(f_jR)/V_i \cong S_i$ is simple. Let $\alpha : f_jR \longrightarrow \oplus_{i=1}^{l} \frac{f_jR}{V_i} \cong B^l$ be an R-homomorphism. Then f_jR can be embedded in B^l. Since M is B^l-injective, it is also injective relative to f_jR. Then M is injective because $\mathrm{Soc}(R_R)$ is homogeneous and all f_jR are isomorphic. Hence R is a right NMC-ring. □

Proposition 4.5.2. *[23, Proposition 7] Let R be a right Artinian ring with homogeneous* $\mathrm{Soc}(R_R) = \mathrm{J}(R)$ *and unique indecomposable module of composition length two up to isomorphism. Then R is a right* NMC*-ring. Moreover, R is a ring of Theorem 4.4.12* (iii).

Proof. Since $\mathrm{Soc}(R_R) = \mathrm{J}(R)$, in view of the proof of Proposition 4.5.1 it is enough to prove the last statement. Note that R can be decomposed as a ring as $R = A \times B$, where A and B are as described in Theorem 4.4.12. If both A and B are nonzero, R must contain two simple right ideals (one in A and one in B) with distinct annihilators. However, this contradicts the fact that R has a homogeneous socle. Also, since R has an indecomposable module of composition length two, R cannot be equal to A. Otherwise, we would have a contradiction by our assumption that an indecomposable module of composition length two exists. Hence R must be equal to B, where B is not semisimple Artinian. Then B cannot be Morita equivalent to a domain because B is right Artinian but not semisimple. Assume that S is a simple right ideal of R which can split in R. Then we have $R = S \oplus X$ for some right ideal X of R. In this case, S must be generated by an idempotent. Since $\mathrm{Soc}(R_R) = \mathrm{J}(R)$, we obtain that $S \subseteq \mathrm{J}(R)$. But this is a contradiction because $\mathrm{J}(R)$ does not contain any nonzero idempotent. Thus no simple right ideal in R can split. Similarly, also no maximal ideal in R can split. Let $V = vR$ be any simple right ideal of R for $v \in V$. So we have $V = R/\operatorname{ann}(v)$ by Lemma 2.2.5. Hence $\operatorname{ann}(v)$ must be an essential right ideal of R. In this case, $R/\operatorname{ann}(v)$ is singular by Proposition 2.7.3. This implies that the simple right ideals of R must be singular submodules of R_R. It follows that $\mathrm{Z}(R_R) \neq 0$; hence R cannot be a right SI ring. Thus R must be as in Theorem 4.4.12. □

The following example exemplifies the case in Proposition. 4.5.2.

Example 4.5.3. *[23, Example 6] Let $R = \frac{\mathbb{Z}}{p^2\mathbb{Z}}$, where p is a prime number. Then R is a right* NMC*-ring.*

Example 4.5.4. *[23, Example 7] Let $R = \frac{\mathbb{Z}}{p^3\mathbb{Z}}$, where p is a prime number. Then R does not have no right middle class.*

The example 4.5.4 indicates that the condition $\mathrm{Soc}(R_R) = \mathrm{J}(R)$ is necessary in Proposition 4.5.2. Moreover, if R is right SI, the condition $\mathrm{Soc}(R_R) = \mathrm{J}(R)$ is not superfluous.

Lemma 4.5.5. *[23, Lemma 11] Let R be a (non-semisimple) right* SI*-ring with homogeneous right socle and let $\mathrm{Soc}(R_R) \leq_e R_R$. Suppose that every proper essential submodule of $\mathrm{E}(R_R)$ is poor. Then R is a right* NMC*-ring.*

Proof. Assume that M is injective relative to A such that A is cyclic but not semisimple. Our aim is to show that M is injective. Since R is a right SI-ring, in view of the Proposition 2.14.2, we may assume that A is nonsingular. It follows that $A \cong R/I$ for some essentially closed right ideal I of R. Otherwise, if there is a right ideal I' of R such that $I \leq_e I' \leq R$, I'/I must be singular. In this case, $\mathrm{Z}(R/I) = 0$ because $R/I \cong A$. This implies that I'/I is contained in a nonsingular module R/I, as a singular module, a contradiction. Then by Proposition 2.5.6, there exists a right ideal B of R such that I is complement to B. By Proposition 2.5.5, we obtain that $I \oplus B \leq_e R$. In this case, we have $B \cong \frac{I \oplus B}{I} \leq_e \frac{R}{I}$ by Proposition 2.5.4. Hence, there exists an essential submodule of A which is isomorphic to a right ideal of R. It follows that $\mathrm{Soc}(A)$ can be embedded in $\mathrm{Soc}(R_R)$. Since R has an essential right socle, every nonzero cyclic submodule of A contains some simple right R-module. This gives that $\mathrm{Soc}(A) \leq_e A$.

Let S be a simple submodule of $\mathrm{E}(R_R)$ and let S' be a (nonsingular) simple submodule in A. Then S' can be embedded in R by Corollary 2.12.6. Since $\mathrm{Soc}(R_R)$ is homogeneous, we obtain that $S' \cong S$. Also since $\mathrm{E}(R_R)$ is injective we have an R-homomorphism $f : A \longrightarrow \mathrm{E}(R_R)$. It follows that $S \subseteq \mathrm{Im}\, f \subseteq \mathrm{Tr}(A, \mathrm{E}(R_R))$. Since S is arbitrary, we get $\mathrm{Soc}(R_R) \subseteq \mathrm{Tr}(A, \mathrm{E}(R_R))$. It is not difficult to see that $\mathrm{Tr}(A, \mathrm{E}(R_R)) \leq_e \mathrm{E}(R_R)$. Since $\mathrm{Tr}(A, \mathrm{E}(R_R))$ is a fully invariant submodule of $\mathrm{E}(R_R)$, it is also quasi-injective by Lemma 2.13.29.

Now let $f : \mathrm{Soc}(A) \longrightarrow R_R$ be any monomorphism, which is mentioned above. A monomorphism f can be extended to some homomorphism $g : A \longrightarrow \mathrm{E}(R_R)$. Then we have $0 = \ker f = \mathrm{Soc}(A) \cap \ker g$. Since $\mathrm{Soc}(A)$ is essential in A, g is a monomorphism. In this case, A can be embedded in $\mathrm{Tr}(A, \mathrm{E}(R_R))$. Since A is non-semisimple, $\mathrm{Tr}(A, \mathrm{E}(R_R))$ cannot be semisimple. Suppose now that $\mathrm{Tr}(A, \mathrm{E}(R_R))$ is a proper submodule of $\mathrm{E}(R_R)$. Then by assumption, it is a poor module and also quasi-injective. This means that $\mathrm{Tr}(A, \mathrm{E}(R_R))$ is semisimple. But this is a contradiction. Hence $\mathrm{Tr}(A, \mathrm{E}(R_R))$ must be equal $\mathrm{E}(R_R)$. Since M is A-injective, it is also injective relative to $\mathrm{Tr}(A, \mathrm{E}(R_R))$; hence $\mathrm{E}(R_R)$-injective. This implies that M is R_R-injective. Thus R is a right NMC-ring. □

Lastly, we give an example to show that converse of Theorem 4.4.12 (iii), in general, is not true.

Example 4.5.6. *[30, Example 2.23] Let* $R = \begin{pmatrix} \mathbb{Z}/4\mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$. *Then*

(i) $\mathrm{Soc}(R_R) \leq_e R_R$ *and* $\mathrm{Soc}(R_R)$ *is homogeneous.*

(ii) $\mathrm{Soc}(R_R) = \mathrm{J}(R) = \mathrm{Z}(R_R)$*,*

(iii) R_R *is a poor module, and*

(iv) R *has right middle class.*

4.6 Commutative rings

In this section, we turn our attention to commutative rings. Notice that a commutative ring R is right NMC if and only if it is left NMC. We complete our thesis by giving a full characterization of commutative NMC-rings.

Proposition 4.6.1. *[30, Proposition 4.2] If R is a commutative Noetherian* NMC*-ring, then R is a Artinian ring.*

Proof. By Proposition 2.10.4, it is enough to show that any prime ideal of R is maximal. Since this is the case for any commutative V-ring we may assume that R is not a V-ring. In this case, we can choose a maximal ideal $\mathfrak{M}$ such that $R/\mathfrak{M}$ is not injective. Then $\mathrm{E}(R/\mathfrak{M})$ cannot be semisimple. Let P be any prime ideal such that $P \neq \mathfrak{M}$. Our aim is to show that P is a maximal ideal of R. Now $\mathrm{Hom}_R(\mathrm{E}(R/\mathfrak{M}), \mathrm{E}(R/P)) = 0$ by Lemma 2.13.17 since $\mathfrak{M}$ is a maximal ideal and $\mathfrak{M}$ is not contained in P. Therefore R/P is $\mathrm{E}(R/\mathfrak{M})$-injective by Proposition 2.13.24. Since $\mathrm{E}(R/\mathfrak{M})$ is not semisimple and R is an NMC-ring, R/P is injective. Therefore, R/P is a self-injective domain; hence a field by Lemma 2.13.3. This completes the proof. □

Theorem 4.6.2. *[30, Theorem 4.3] A commutative ring R is* NMC *if and only if R can be decomposed as a ring as $R = A \oplus B$, where A is semisimple Artinian, and $B = 0$ or B is a local ring with exactly one nonzero proper ideal.*

Proof. Assume first that R is NMC-ring. By Theorem 4.4.12, R can be decomposed as a ring as $R = A \oplus B$, where A is semisimple Artinian, and $B = 0$ or it belongs to one of the following cases:

Case I: B must be Morita equivalent to a right PCI-domain B'. It follows that B' is right Noetherian by Proposition 2.14.8, then so is B. In this case, B' is an Artinian domain by Proposition 4.6.1, and so it is a simple ring. This implies that B is also simple; hence it is a field.

Case II: B is an indecomposable SI-ring. Then it must be either a V-ring or Artinian by Proposition 4.3.2. Suppose first that B is Artinian. It follows that B is a finite product of local rings by Proposition 2.10.4. Since B is an indecomposable ring, it must be a commutative local Artinian ring. Also since B is an SI-ring, it is nonsingular; hence semiprime. It is well-known that Artinian semiprime rings are semisimple. It follows that B is a field. Now assume that B is a V-ring. If B is Noetherian then it should be Artinian and by above arguments B is a field. So we may suppose that B is not Noetherian. Then B must be a semi-artinian by Proposition 4.3.1. This implies that $\mathrm{Soc}(B) \neq 0$. Let S be a nonzero minimal ideal of B. As $\mathrm{J}(B) = 0$ by Lemma 2.14.14, there is a maximal ideal $\mathfrak{M}$, which does not contain S. Hence $S \oplus \mathfrak{M} = B$. Since B is indecomposable, we obtain that $S = B$. Therefore, B is a field.

Case III: B is an indecomposable Artinian ring such that $\mathrm{Soc}(B) = \mathrm{J}(B)$. Notice that as mentioned in Case II, B is a local ring. In this case, by Theorem 4.1.4, B is a ring whose maximal ideal $\mathrm{J}(B)$ is minimal.

Conversely, suppose that B is a commutative local ring with exactly one nonzero proper ideal. This implies that B has a unique indecomposable module of composition length two up to isomorphism, that is B itself. Then it has homogeneous $\mathrm{Soc}(B) = \mathrm{J}(B)$. Therefore, B is an NMC-ring by Proposition 4.5.2, completing the proof. □

Corollary 4.6.3. *Let R be a commutative* NMC*-ring. Then it is Artinian.*

Corollary 4.6.4. *Let R be a commutative ring. Then R is a local ring with exactly one nonzero proper ideal if and only if R is an indecomposable Artinian* NMC*-ring with* $\mathrm{Soc}(R) = \mathrm{J}(R)$.

Chapter 5

CONCLUSION

In this thesis, we mainly studied rings with no right middle class. The following theorem is the most comprehensive conclusion we have reached.

Theorem 5.0.1. *[23, Theorem 2] Let R be a right* NMC*-ring. Then R can be decomposed as a ring as $R = A \times B$, where A is a semisimple Artinian ring, and B is zero or it belongs to one of the following classes:*

(i) B is Morita equivalent to a right SI*-domain (equivalently right* PCI*-domain), or*

(ii) B is an indecomposable right SI*-ring such that:*

(a) B is either a right V*-ring or a right Artinian ring,*

(b) $\mathrm{Soc}(B_B)$ *is homogeneous and* $\mathrm{Soc}(B_B) \leq_e B_B$*,*

(c) B has a unique simple singular right B-module up to isomorphism, or

(iii) B is a right Artinian indecomposable ring such that:

(a) $\mathrm{Soc}(B_B) = J(B) = \mathrm{Z}(B_B)$*,*

(b) $\mathrm{Soc}(B_B)$ *is homogeneous.*

Also, we studied the structure of the right Noetherian rings without a right middle class. However, we need to investigate the open problem whether there is not right Noetherian ring without a right middle class. The following result gives some interesting properties of such rings considering that they exist.

Proposition 5.0.2. *[31, Proposition 5.1] The following statements are satisfied for any indecomposable ring R without a right middle class that is not right Noetherian:*

(i) R *is a right semi-artinian, right* V*-ring, and so a von Neumann regular ring (see [11]).*

(ii) R *is a right* SI*-ring.*

(iii) $R/\operatorname{Soc}(R_R)$ *is a simple Artinian ring.*

(iv) R *has homogeneous essential right socle which has countably infinite length.*

(v) R *is a right primitive ring.*

(vi) There is exactly one non-trivial ideal of R*, namely* $\operatorname{Soc}(R_R)$.

Bibliography

[1] A. N. Alahmadi, M. Alkan, and S.R. Lòpez-Permouth, Poor modules: the opposite of injectivity, Glasg. Math. J. 52, no.A, 7-17, (**2010**).

[2] F.W. Anderson and K.R. Fuller, Rings and categories of modules, Springer-Verlag, New-York, **1992**.

[3] K.R. Goodearl, Ring theory, Marcel Dekker, Inc., New York-Basel,**1976**.

[4] T.Y. Lam, Lectures On Modules and Rings, Graduate Texts in Mathematics, Vol. 189, Springer-Verlag, Berlin-Heidelberg-New York, **1999**.

[5] D.W. Sharpe, and P. Vamos, Injective Modules, Cambridge University. Press, **1972**.

[6] C. Faith, Lectures on Injective Modules and Quotient Rings, Springer, **1967**.

[7] J. A. Beachy, Introductory Lectures on Rings and Modules, Cambridge University Press, **1999**.

[8] K.R. Goodearl, Singular torsion and the splitting properties, Memoirs of the American Mathematical Society, No. 124, American Mathematical Society, Providence, R.I., **1972**.

[9] B. Stenström, Rings of Quotients, Springer-Verlag, Berlin - Heidelberg, **1975**.

[10] N. V. Dung and P. F. Smith, On semi-artinian V-modules, J. Pure and Appl. Alg., **82**, 27-37, (**1992**).

[11] G. Baccella, Semiartinian V-rings and Semiartinian Von Neumann Regular Rings, J. Algebra, **173**, 587-612, (**1995**).

[12] T.Y. Lam, A first course in noncommutative rings, Springer-Verlag, New York,**1991**.

[13] J. Rotman, An introduction to homological algebra, Academic Press, New York,**2008**.

[14] S. Mohamed and B. Müller, Continuous and discrete modules, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge vol. 147, **1990**.

[15] W. K. Nicholson and M. F. Yousif, Quasi-Frobenius rings, Cambridge University Press, Cambridge vol.158, **2003**.

[16] S. K. Jain, A. K. Srivastava and A. A. Tuganbaev, Cyclic Modules and the Structure of Rings, Oxford University Press, **2012**.

[17] R.F. Damiano, A right PCI rings is right Noetherian, Proc. Amer. Math. Soc., Vol 77, No. 1, 11-14, **(1997)**.

[18] S. Singh, Quasi-injective and quasi-projective modules over hereditary Noetherian prime rings, Can. J. Math. 26, 1173–1185, (**1974**).

[19] A. K. Boyle, Hereditary QI-rings, Transactions of the American Mathematical Society, Vol. 192, **(1974)**.

[20] J.S. Golan, Torsion Theories, Pitman Monographs and Surveys in Pure and Applied Mathematics, 29. Longman Scientific and Technical, Harlow, Inc., New York, **1986**.

[21] R. Wisbauer, Foundations of Module and Ring Theory Gordon and Breach Science Publishers, Philadelphia, PA.,**1991**.

[22] S.R. Lopez-Permouth and J. E. Simental, Characterizing rings in terms of the extent of the injectivity and projectivity of their modules, J. Alg.,56-69, 362 (**2012**).

[23] N. Er, S.R. Lòpez-Permouth and N. Sökmez, Rings whose modules have maximal or minimal injectivity domains, J. Algebra 330, 404-417,**(2011)**.

[24] B. Osofsky, P.F. Smith, Cyclic modules whose quotients have all complement submodules direct summands, J. Algebra 139, 342–354, **(1991)**.

[25] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, Extending Modules, Pitman Res. Notes in Math. Ser., vol. 313, **1994**.

[26] S. Singh, Modules over hereditary Noetherian prime rings, Can. J. Math., 867–883, (**1975**).

[27] R. Alizade, E. Büyükaşık, Poor and pi-poor Abelian groups, Communications in Algebra, 45:1, 420-427, **(2017)**.

[28] J. E. van den Berg, Primeness described in the language of torsion preradicals, Semigroup Forum 64, no. 3, 425-442 **(2002)**.

[29] J. Golan, Linear Topologies on a Ring: An Overview, Pitman Res. Notes Math. Ser., vol. 159, **1987**.

[30] P. Aydoğdu and B. Saraç , On Artinian rings with restricted class of injectivity domains, J. Algebra 377, 49-6, **(2013)**.

[31] B. Saraç, S. R. Lopez- Permouth and S. Zamora- Erazo, Rings without a middle class from a lattice theoretic perspective, Mediterranean Journal of Mathematics, 17, 55, **(2020)**.

[32] R. P. Kurshan, Rings whose cyclic modules have finitely generated socles, J. Algebra 15, 376-386, **(1970)**.