SOLUTION OF DIFFERENTIAL EQUATIONS USING NONSTANDARD FINITE DIFFERENCE METHODS

STANDARD OLMAYAN SONLU FARK YÖNTEMLERİYLE DİFERANSİYEL DENKLEMLERİN ÇÖZÜMÜ

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ETHICS

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- in case of using other Works, related studies have been cited in accordance with the scientific standards
- all cited studies have been fully referenced
- I did not do any distortion in the data set
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19/06/2015

ERDİ KARA

ABSTRACT

SOLUTION OF DIFFERENTIAL EQUATIONS USING NONSTANDARD FINITE DIFFERENCE METHODS

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Differential equations are mostly used to model real life problems. However it is known that in general, these equations do not have complete solutions. For that reason, reseachers have tried to solve these problems by means of several numerical methods. Among them, standard finite difference (SFD) is a frequently used method in order to obtain numerical solutions of differential equation for a long time. However there are many mathematical problems for which the SFD models do not perform well. In recent years, nonstandard finite difference (NSFD) method which gets main motivation from SFD method has been applied to various mathematical models for the purpose of getting reliable numerical results. This topic is firstly studied in mid-1980s and nowadays is playing an important role in the construction of reliable numerical models in Science and Engineering.

In this thesis, we firstly introduced Exact Finite Difference models and NSFD models has been introduced for differential equations. Then NSFD schemes have been constructed for some models for both ordinary and partial differential equations. A NSFD models has been proposed for an automous differential equation which has three distinct fixed points. This NSFD scheme differs from the one in literature. The difference of the proposed NSFD scheme in this thesis is the discretization of the nonlinear term. For all mathematical models, numerical simulations are illustrated to see the performance of the NSFD methods. As a result, it has been seen that NSFD models give qualitatively correct behaviour for many cases.

Key words: Nonstandard Finite Difference Schemes, Exact Finite Difference Schemes,

ÖZET

STANDARD OLMAYAN SONLU FARK YÖNTEMLERİYLE DİFERANSİYEL DENKLEMLERİN ÇÖZÜMÜ

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Diferansiyel denklemler gerçek hayat problemlerinin modellenmesinde sıklıkla kullanılmaktadır. Bununla birlikte genel olarak bu denklemler tam çözüme sahip değildirler. Bu problemlerin çözümlerini elde etmek için araştırmacılar çeşitli sayısal yöntemler kullanmışlardır. Bunların içinde, standard sonlu fark yöntemi diferensiyel denklemlerin sayısal çözümlerini elde etmek için sıklıkla kullanılan bir yöntemdir. Bununla beraber bu yöntemin iyi sonuçlar vermediği birçok matematiksel model mevcuttur. Son yıllarda ise, temel motivasyonunu standard sonlu fark yöntemlerinden alan yeni bir yöntem olan standard olmayan sonlu fark yöntemi birçok matematiksel modele uygulanmış ve başarılı sayısal sonuçlar elde edilmiştir. İlk kez 1980'lerin ortalarında çalışılmaya başlanan bu method şimdilerde bilim ve mühendisliğin çeşitli dallarında güvenilir sayısal modellerin oluşturulmasında önemli bir rol oynamaktadır.

Bu tezde öncelikle diferansiyel denklemler için Tam Sonlu Fark modeli ve standard olmayan fark modeli kavramları tanıtılmışır. Hem adi hem de kısmi diferensiyel denklemler için çeşitli standard olmayan fark tasarıları oluşturulmuştur. Sonrasında üç tane sabit noktaya sahip otonom tipte bir adi diferensiyel denklem için standard olmayan bir sonlu fark modeli önerilmiştir. Bu modelde lineer olmayan terimlerin modellenmesi lokal olmayan bir ayrışımla oluşturulmuştur. Yöntemin performansını ölçmek amacı ile verilen bütün matematiksel modeller için sayısal simülasyonlar gösterildi. Sonuç olarak standard olmayan fark yönteminin incelenen modeller için niteliksel olarak doğru sonuçlar vermekte olduğu görüldü.

Anahtar Kelimeler: Standard Olmayan Fark Modelleri, Tam Sonlu Fark Modeli

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CONTENTS

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INTRODUCTION

There are many real life problems that can be modelled by means of differential equations. Especially ordinary differential equations have been used in physical sciences about 17th-18th centuries. Beginning in the middle of the 19th century, partial differential equation has also started to play an important role in modeling extremely complicated physical and biological phenomena such as population dynamics, fluid mechanics, quantum field theory etc. Therefore in order to understand these natural phenomena, it is very important to find the analytical or approximate solutions of that differential equations. Although there are some methods to find the exact solutions of them, it is a well-known fact that any differential equation -regardless of the choice of ordinary or partial- does not have general solution. For that reason approximate or numerical solutions of differential equation have been studied by researchers for a long time. Especially in the last century, it has been found some methods related to these topic. Perturbation methods, finite difference method, interpolation method, finite element methods etc. are some of these methods.

One of the main concerns about the numerical solutions of differential equation is which of the methods works more efficient for the equation. On the other hand it is a natural expectation that the method which is used to approximate solution should have the same qualitative properties with the corresponding differential equation. For instance, if the exact solution of the differential equation is bounded or oscillatory then the related method must be also bounded and oscillatory. Therefore finding and implementing the most appropriate method for any given differential equation has been a great deal of interest.

The Finite Difference Method (FDM) is one of the most used techniques to find the approximate solution of the differential equations. This method is mainly based on the replacement of the continuous variables in the differential equation by a model including discrete variables. In fact this is a procedure for constructing approximate values of the exact solution at the mesh points [1]. In this thesis, after giving a brief information about the finite difference method and its applications to differential equation, we will deal with a special kind of this method which is called non-standard finite difference (NSFD) method.

This thesis explains why NSFD method has started to play a significant role in modeling of differential equation and what sides of this method have a superior to standard finite difference (SFD) models. Now we will make a brief summarize about the historical development of this method and some considerable works on this topic. In order to make this summary, we have taken advantage of the survey article which is done by Kailash C.Patidar [2].

It will be useful to emphasize a point about the terminology. In fact, any method which does not have a standard form can be named as non-standard. However NSFD models are those which use one of the rules submitted by R.Mickens. R.Mickens mentions in details about these construction rules in his reference book [3].We will also examine these rules in this thesis.

NSFD schemes has been introduced and constructed by R.E.Mickens for the first time in mid-1980s. In 1988, R. Mickens proved an important theorem by using the group properties. That theorem states that every ordinary differential equation has an exact finite difference scheme which means that on the computational grid, the solution to the difference equation is exactly equal to the solution to the differential equation [4]. He provided some examples of exact schemes by means of this theorem in this paper.

Then in their article [5], Mickens and Smith published a paper about the influence of the denominator function in discrete modeling of derivative term for ordinary differential equation and found some denominator functions for which the numerical instabilities do not occur in the discrete models to the differential equations .

In the following years, Mickens presented some NSFD schemes for the time-dependent Schrödinger equation [6] and Fisher partial differential equation [7]. In the paper [8], Mickens and Ramadhani constructed a NSFD schemes for two-coupled ODE system with a single real fixed point and then implemented these result for constructing NSFD schemes for some differential equation like Damped Harmonic Oscillator, Van-Der Pol Oscillator and Batch Fermentation Process.

In 1994, Mickens described several ways to construct NSFD schemes for both ordinary and partial differential equation in his book [3]. He provides lots of examples to illustrate the powerful sides of NSFD schemes when compared the other methods. One can consider this book as the first and the most important reference book related to this topic.

Zhao L. [9] used a kind of 3-dimensional difference method in non-standard form in order to calculate the frequencies of resonators and also examined boundary conditions. In addition, they reduced the numerical error from %8.6 to % 3.0 by using NSFD schemes.

In 1997, Cole J. presented a new NSFD algorithm for Maxwell's equations. This algorithm has a significant superiority to standard ones both in terms of computational and stability [10]. And then he obtained a nonstandard second order finite differences for Yee algorithm with the same accuracy as the standard ones. Additionally this new algorithm requires less iteration for solving the problem [11]. He also applied NSFD method to higher dimensional problems and acquired efficient algorithms related to these type of problem [12].

Kojouharov and Chen described a non-standard model for 1-dimensinonal transient convective transport equation which includes nonlinear term for reaction [13]. Their approach has zero local truncation error and it also eliminates the numerical instabilities in the equation .

In [14] and [15], Mickens analyze the coupled non-linear reaction-diffusion PDE and an ordinary differential equation related to travelling wave solutions to the Burger's equation respectively. He constructed NSFD schemes for these equations.

The NSFD method is successfully applied to find the approximate solution of many real life problem which are modeled by both PDEs and ODEs. In [16] Chen, B.M. and Kojouharov described a nonstandard method for simulation of reactive bacterial transport in porus media. In their model numerical instabilities which comes from the incorrect modeling of some terms is eliminated and they acquired efficient numerical results.

Marcus and Mickens found nonstandard forward Euler schemes for tripled nonlinear ODEs that have some applications in photo-conductivity and by this way they eliminated some instabilities occurring in the equation [17]. In the same year Mickens published two papers about this topic. In [18], he described some NSFD schemes for some ODEs such as logistic, cubic, Monod and these proposed method gives correct numerical results for all values of the step size. He also presented NSFD schemes for a particular class of PDEs which are 1-dimensional. And he also gave a description about how to construct NSFD schemes for several types of

PDE systems [19].

Lotka-Volterra system is a coupled nonlinear ordinary differential equation which is widely used to model biological phenomena such as population dynamics. Al-Kahby constructed a NSFD method which is dynamically consistent for Lotka-Volterra system [20].

In [21] and [22] Kantartzis, N.V presented a finite difference time domain (FDTD) schemes for the first time. He efficently used both standard and nonstandard techniques. Then this method has been applied to many problems [23,24,25,26,27].

Hassan in [28] used NSFD method for time variable to construct an algorithm for the Jager and Kacur schemes and showed that their method was stable and convergent.

Jordan employed a NSFD for an initial boundary value problem and obtained a solution which has numerical stability [29].

There are lots of numerical methods used in the solutions of systems of ODEs. One of them is θ-*method* where θ is generally between 0 and 1. For some values of θ, the linear stability properties of its fixed points does not coincide with the fixed point of the differential equation. In [30] Lubuma and Roux presented a θ-*method* in nonstandard form and also they gained elementary stabilities in their method for stiff systems.

Rucker [31] proposed an exact finite difference scheme for a nonlinear advection-reaction PDE with zero diffusion by using the exact solution of the equation.

Another interesting work done by Anguleov and Lubuma in [32], they firstly described a general finite difference schemes which has the same monotonicity with the equation. They also proposed a NSFD schemes for which the nonlinear terms are modeled in an original approach.

In [33] Chen and Gumel proposed a semi-explicit and an implicit nonstandard discretizations of the generalized Nagumo reaction-diffusion equation respectively. Their methods give relatively accurate results when compared the standard methods like the Euler and Runge-Kutta. Then they applied the proposed method for a mathematical model which is about the formation of bio-barrier.

NSFD schemes have wide range applications in numerical solutions of mathematical model of biological phenomena. For instance Alexander in [34] proposed a NSFD schemes for a biological model which is related to infectious diseases. When compared the standard methods, his model has some prior properties since it preserves some qualitative behavior of the model.

Another work done by Mickens is about the Fisher PDE which can be used so as to model some phenomena in ecology. In [35] and [36] Mickens constructed some NSFD schemes for Fisher and Burger's equations and he obtained some relationship between time and space steps.

In [37] Gumel and Moghadas examined a food-chain model which is about struggling among three-species. They studied about the stability properties of their models and then employed a nonstandard method for that problem.

Gumel, A.B., Moghadas, S.M. and Mickens, R.E. investigated a sophisticated biological model in which they focused on the behavior of HIV infection in a community and they obtained some remarkable results about the threshold values for which the disease tends to decrease in that media. In [38], they used a NSFD method in order to solve the corresponding equation .Their model are also numerically stable in many aspects such as preserving the positivity of the equation.

Moghadas at al. [39] employed a NSFD schemes for a predator-prey model in Gaussian-form. They showed that the discrete model bears the same qualitative properties with the main model. They also prove that unlike the standard finite difference models, their model reflects correct qualitative behaviour with the asymptotic behavior of the predator-prey itself.

This thesis is organized as follows:

In Chapter 1, we reviewed the standard finite difference(SFD) method for differential equations. We described finite difference models for some ordinary differential equations; decay equation, logistic equation, harmonic oscillator

In Chapter 2, numerical instability notion in finite difference models has been introduced. Various discrete models have been constructed for decay equation and harmonic oscillator and then behaviour of the solutions to discrete models and the exact solution of the corresponding differential equations have been compared. In this way, we have discussed when the numerical instabilities occur in discrete modelling of differential equations. We have also introduced the linear stability analysis of a first order autonomous ODEs. This method enable us to understand the behaviour of the solutions of some types differential equation near their fixed points. By using this approach, we have discussed the numerical instability with a different point of view.

In Chapter 3, we introduced the notion of exact finite difference and nonstandard finite difference (NSFD) schemes for some differential equations. After giving some preliminary informations, we have described the construction rules of NSFD and provided some examples about NSFD schemes for differential equations. To test the performance of the method, we have compared the numerical results obtained by NSFD and SFD method. It has been seen that NSFD method could eliminate some numerical instabilities occuring in standard discretizations . In Section 3.7, we have reviewed a NSFD method proposed for second order boundary value problems. Corresponding nonstandard method has fourth order truncation error and numerical experiments show that the method gives reliable numerical results for some mathematical models. In section 3.8, we have considered a theory which proposes a criteria for finite difference models to preserve linear stability properties of fixed points and monotonicity of solutions. Although this theorem does not provide a general procedure to construct a finite difference scheme which has correct qualitative behaviour with the cooresponding differential equation, it enable us to check wheather any proposed finite difference scheme preserves the significant proporties of the original problem. By following this work, we presented a NSFD scheme for a first order autonomous ODE which has three distinct real fixed points. In Section 3.9, we have reviewed a NSFD scheme for Lotka-Volterra differential equations. While the standard discretizations show numerical instabilities in modelling of this system, NSFD scheme preserves the monotonicity of solutions and reflects correct qualitative behaviour near the fixed points.

In Chapter 4, we have first described the construction of standard finite difference models for partial differential equations and then we have reviewed a work which proposes a prodecure to construct NSFD scheme for some class of PDEs. By this way, a NSFD scheme has been described for nonlinear Huxley equation.

1. STANDARD FINITE DIFFERENCE MODELS

In this chapter, we will introduce the standard finite difference (SFD) approximation for some ordinary differential equations.We know that it can rarely be found an analytic solutions for any differential equation. For that reason lots of methods has been found in order to obtain a numerical solutions to differential equations. Throughout this thesis, our main goal will also be to introduce a particular method which is called *nonstandard finite difference method (NSFD).* First, we will review numerical solutions of ordinary differential equations by means of some standard finite difference discretization.

1.1 Standard Finite Discretizations

A finite difference method proceeds by replacing the derivatives in the differential equations by finite difference approximations. Consider the numerical solutions of the following well-posed IVP

$$\frac{dy}{dt} = f(t, y),\ a \leq t \leq b,\ y(a) = \alpha \tag{1.1.1}$$

Main concern is to be able to obtain approximate solutions to $y(t)$ at several values, called mesh points. The mesh points can be considered as equally spaced throughout the interval $[a, b]$. To do this, choose a positive integer N and select mesh points

$$t_k = t_0 + hk, \quad k = 0, 1, 2, ..N$$

The distance between two subsequent mesh points becomes

$$h = \frac{b-a}{N} = t_{k+1} - t_k$$

which is called *step size.* Suppose that $y(t)$ is the unique solution to (1.1.1) with continuous derivatives on the interval $[a, b]$. Taylor's theorem is used to derive the finite difference models. To approximate the first derivative of the function $y(t)$, there are many known discrete approximations. Some of them are the followings:

$$y' = \frac{y_{k+1} - y_k}{h} - \frac{h}{2} y''(\eta_k) \ forsome \ \eta_k \in (t_k, t_{k+1}) \tag{1.1.2}$$

$$y' = \frac{y_k - y_{k-1}}{h} + \frac{h}{2} y''(\eta_k) \ \text{ for some } \ \eta_k \in (t_{k-1}, t_k) \tag{1.1.3}$$

$$y' = \frac{y_{k+1} - y_{k-1}}{2h} - \frac{h^2}{6} y'''(\eta_k) \ \text{ for some } \ \eta_k \in (t_{k-1}, t_{k+1}) \tag{1.1.4}$$

$$y' = \frac{-y_{k+2} + 4y_{k+1} - 3y_k}{2h} - \frac{h^2}{3} y'''(\eta_k) \ \text{ for some } \ \eta_k \in (t_k, t_{k+1}) \tag{1.1.5}$$

where

$$y(t_k) = y_k$$

These representations are known forward-difference,backward-difference,central-difference and three-point difference models, respectively. The expression η_k is the truncation term of the method and its structure has a great importance for a good approximation.

For the second derivative of the function $y(t)$, discrete model

$$y'' = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} - \frac{h^2}{12} y^{(iv)}(\eta_k) \ \text{ for some } \ \eta_k \in (t_{k-1}, t_{k+1})$$

is in generally used.

It is obvious that these representations are reduced to the conventional definition of derivative of the function $y(t)$ as $h \to 0, k \to \infty$ and $t_k = t = fixed.$

At this stage we will provide some example about the construction of standard finite difference models for the differential equations. One can propose many different finite difference model for a differential equation. The main concern will be to find which of them gives qualitatively correct behaviour with the solution of the differential equations. For this, we motivated on decay equation, logistic differential eqaution and harmonic oscillator equation.

1.2 Decay Equation

Consider the decay differential equation

$$\frac{dy}{dt} = -y \tag{1.2.1}$$

Using the approximations (1.1.2-1.1.5) for the first derivative, we can consider the following finite difference models for (1.2.1)

$$\frac{y_{k+1} - y_k}{h} = -y_k$$

$$\frac{y_k - y_{k-1}}{h} = -y_k \tag{1.2.2}$$

$$\frac{y_{k+1} - y_{k-1}}{2h} = -y_k$$

These models are known as forward Euler, backward Euler and central difference models, respectively. After making some algebraic operations, these equations can be rewriten as

$$y_{k+1} - (1-h)y_k = 0 \tag{1.2.3}$$

$$y_{k+1} - (\frac{1}{1+h})y_k = 0 \tag{1.2.4}$$

$$y_{k+2} + 2hy_{k+1} - y_k = 0 \tag{1.2.5}$$

respectively. These type of equations are known as *difference equations.*

Here equations (1.2.3) and (1.2.4) are first-order linear difference equation and (1.2.5) is the second-order linear difference equation.These equations have different constant coefficents depends on the step size h. For that reason they have different solutions. We should note that the behaviour of the solutions will depend on the values of step-size h since the coefficents are also depend on it. We will argue this phenomenon in the next chapters.

1.3 Logistic Differential Equation

Logistic differential equation is a first order linear ODE of the form

$$\frac{dy}{dt} = y(1-y) \tag{1.3.1}$$

For (1.3.1) , the forward Euler, backward Euler and central difference discretization of the equation (1.3.1) are given as

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_k)$$

$$\frac{y_k - y_{k-1}}{h} = y_k(1 - y_k) \tag{1.3.2}$$

$$\frac{y_{k+1} - y_{k-1}}{2h} = y_k(1 - y_k)$$

Note that a nonlocal represantation for the backward Euler model is used. That is, the vector field in the equation (1.3.1) is evaluated at the grid point $t = t_{k+1}$ instead of the point $t = t_k$. The equations in (1.3.2) can be rewriten as

$$y_{k+1} - (1 + h)y_k - hy_k^2 = 0$$

$$hy_{k+1}^2 + (1 - h)y_{k+1} - y_k = 0 \tag{1.3.3}$$

$$y_{k+2} - 2hy_{k+1}(1 - y_{k+1}) - y_k = 0$$

respectively. These are all non-linear difference equations. As in the discrete models of decay equation, all coefficents in (1.3.3) depends on the step size h. Therefore, we can say that these three finite difference schemes have different solutions

1.4. Harmonic Oscillator

Consider the damped harmonic oscillator equation

$$\frac{d^2y}{dt^2} + 2\epsilon\frac{dy}{dt} + y = 0 \tag{1.4.1}$$

We can construct the following discrete models for the harmonic oscillator equation:

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + 2\epsilon\left(\frac{y_{k+1} - y_k}{h}\right) + y_k = 0$$

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + 2\epsilon\left(\frac{y_k - y_{k-1}}{h}\right) + y_k = 0 \tag{1.4.2}$$

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + 2\epsilon\left(\frac{y_{k+1} - y_{k-1}}{h}\right) + y_k = 0$$

The all three models are second order-linear difference equations with different coefficents which are depend on the value of h. Hence they have all different solutions.

We implemented some standard finite difference models to three differential equations and obtained different discete models. However there exists some ambiguous points in these models. First of all, standard rules yield us different models. At this stage, we have no information about which of them can be used to obtain the numerical solutions for the corresponding differential equation. In fact we can not guarantee that one of them can be used to obtain a reliable numerical solution for the related differential equation. On the other hand we must take into account the relationship between the solutions of the difference equations and the corresponding differential equations. That is, we should find a discrete model that qualitatively gives the correct behaviour with the solution of the differential equation. Furthermore when we consider a discrete model for a particular differential equation then we need to find an optimal step-size h for a good approximation .

These arguments will lead us to some important notions like stability and instability of solutions. We will deal with these subjects in the next chapters.

2. NUMERICAL INSTABILITIES

If one uses a method to obtain the numerical solutions of a particular differential equation, one of the most important questions is wheather the numerical method is able to preserve the qualitative properties of the corresponding problem, such as monotonicity, positivity, convergency etc. For instance, if the particular solution of the differential equation has monotonicity on a domain then we expect that the finite difference scheme of the differential equation is also monotonic on the same domain. It is because preserving particular properties enable us to make profound interpretion about the qualitative structure of the differential equation. Here, the stability notion is used to express this idea [41]. Hence within this chapter, we introduce the concept of stability.

When we consider the finite difference methods, a discrete model of a differential eqaution is said to have numerical stabilities if there exist solution to the finite-difference eqautions that is accordance with the qualitative properties of any possible solutions of the differential equation. Otherwise that model is said to have numerical instabilities. Of course that the reason of numerical instabilities occur in a method can arise from lots of factors like the range of step-size, boundedness, positive definite case, etc. That is, if the finite difference model is not coincide with the related differential equation according to these factors, it will appear numerical instabilites in discrete model. Throughout this thesis, our main propose will be to construct finite difference schemes that reflect the correct qualitative behaviour of the corresponding differential equation.

There are some reasons that cause numerical instabilities in finite difference models. One of the most fundemental reason is that the discrete models contain larger paramater than the corresponding differential equations. In general, the step size h can appear in the modelling as an additional paramater . Hence numerical instabilities may occur according to the step size h and its range. For that reason, we will generally focus on eliminating the numerical instabilities which stem from the step size h.

In this chapter, we consider two ordinary differential equations, decay equation and harmonic oscillator in order to illustrate the numerical instability notion. Various discrete models were constructed for these equations and compared the properties of the discrete model and the corresponding differential equations. We specify the values of the step size h for which the numerical instabilities occur in discrete models. In this way, it has been possible to eliminate the instabilities arising from the step size h.

2.1. Decay Equation

Consider the decay differential equation with initial condition

$$\begin{aligned} \frac{dy}{dt} &= -y \\ y(t_0) &= y_0 \end{aligned} \tag{2.1.1}$$

Its exact solution is given by

$$y(t) = y_0 e^{-(t-t_0)} \tag{2.1.2}$$

From the exact solution (2.1.2), it is easy to see that solutions of decay equation monotonically approach to zero as $t \to \infty$. This can be seen in the Figure 2.1

with initial conditions $y(0) = 0.5$ and $y(0) = 0.5$

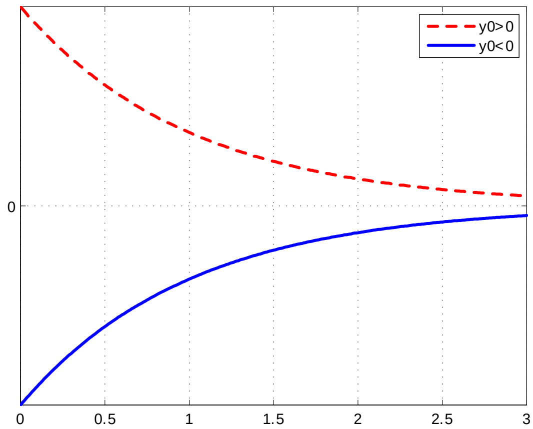

Figure 2.1: Typical trajectories for Decay equation

Now consider the forward Euler model

$$\frac{y_{k+1} - y_k}{h} = -y_k \tag{2.1.3}$$

where h is the step-size. Its reduced form, that is, the related difference equation will be

$$y_{k+1} = (1 - h)y_k$$

whose solution is

$$y_k = y_0(1 - h)^k \tag{2.1.4}$$

This solution yields the following results:

- if $0 < h < 1$ then y_k approaches monotonically to 0
- if $h = 1$ then $y_k = 0$ for $k \geq 1$
- if $1 < h < 2$ then y_k approaches to zero with an oscillating amplitude by changing sign at each step
- if $h = 2$ then y_k oscillates with a constant amplitude y_0
- if $h > 2$ then y_k oscillates with an increasing amplitude.

As a result if

$$0 < h < 1$$

then the numerical scheme (2.1.3) has same qualitative behaviour with the exact solution of the decay equation. The other values of h gives inconsistent solutions. So qualitative agreement between $y(t)$ and y_k only holds for small values of the step size h

$$0 < h << 1 \tag{2.1.5}$$

All these cases can be seen in Figure 2.2 for the initial condition $y(0) = 0.5$

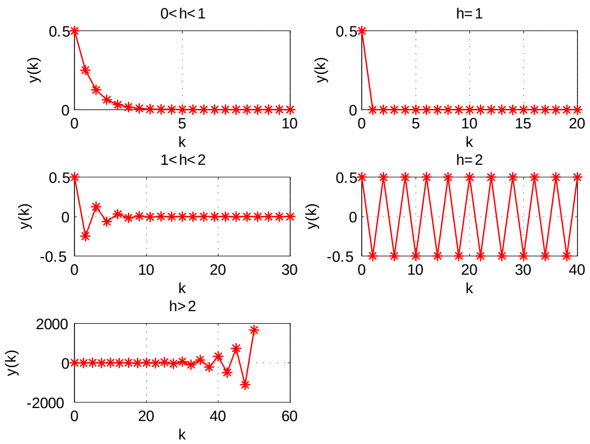

Figure 2.2 Plots of solutions for 2.1.4

We now examine the central difference form for the decay equation, that is

$$\frac{y_{k+1} - y_{k-1}}{2h} = -y_k \tag{2.1.6}$$

Some algebraic operations give the second order linear difference equation with constant coefficent.

$$y_{k+2} + 2hy_{k+1} - y_k = 0 \tag{2.1.7}$$

To solve (2.1.7), setting $y_k = r^k$ then the characteristic equation of the difference equation is given by

$$r^2 + (2h)r - 1 = 0 \tag{2.1.8}$$

By using the characteristic equation, general solution of the difference equation can be written as

$$y_k = c_1(r_1(h))^k + c_2(r_2(h))^k \tag{2.1.9}$$

where c_1, c_2 are arbitarary constants and r_1, r_2 are the solutions of the characteristic equation (2.1.8). When we observe the charcteristic equation we can write

$$r_1(h)r_2(h) = -1 \tag{2.1.10}$$

where

$$\begin{cases} r_1(h) = -h + \sqrt{1+h^2} \\ r_1(h) = -h - \sqrt{1+h^2} \end{cases} \tag{2.1.11}$$

which means that difference equation (2.1.7) yields oscillatory solutions for all values of the step size h. Therefore the central difference scheme shows numerical instabilities for all values of h. A typical plot of (2.1.7) can be seen in the Figure

2.3.

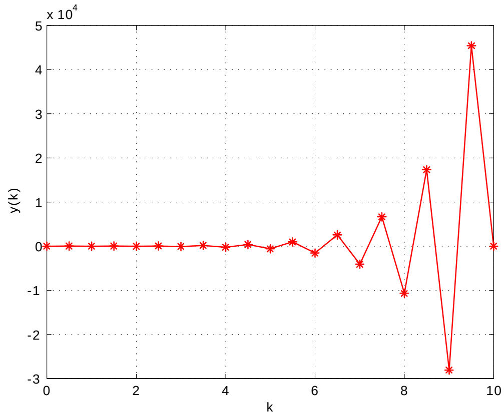

Figure 2.3 Plots of solution (2.1.7)

When we use the backward Euler scheme for the decay equation, we obtain the following discrete model

$$\frac{y_k - y_{k-1}}{h} = -y_k \tag{2.1.12}$$

which can be writen as

$$y_{k+1} = (\frac{1}{1+h})y_k$$

The solution of this difference equation can given by

$$y_k = y_0(\frac{1}{1+h})^k \tag{2.1.13}$$

It is obvious that

$$0 < \frac{1}{1+h} < 1 \qquad \text{for all } h > 0$$

Hence all solutions of (2.1.13) approach monotonically to 0 for all step-sizes. The Figure 2.4 represents a numerical solution of (2.1.12) for $h = 0.5$ and $y_0 = 0.5$.

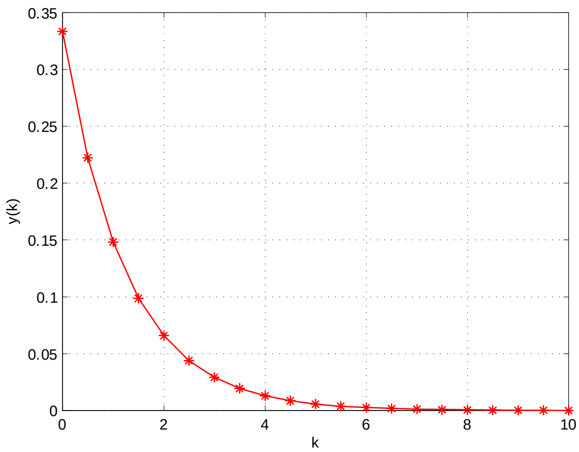

Figure 2.4 Plots of solutions (2.1.12)

As a result, three discrete models for the decay equation are implemented. For $0 < h < 1$, the forward Euler and for $h > 0$ the backward Euler gives the same qualitative behaviour with exact solution of the decay equation. However, the central difference scheme produce unexpected results for any value of h.

These results show that the central difference scheme (2.1.12) generates numerical instabilities for all step sizes h. If we put some restrictions on the value of h then the forward Euler scheme will give correct numerical results and we can use the backward Euler for any step size h.

2.2. Harmonic Oscillator

Now we will implement some finite difference methods to the Harmonic Oscillator which is a second order ordinary differential equation and will discuss for which situation of the step size h numerical instabilities occur in the discrete model for the differential equation.

The harmonic oscillator is a second order linear ordinary differential equation

$$\frac{d^2y}{dt^2}+y=0 \tag{2.2.1}$$

Its general solution is given by

$$y(t)=c_1e^{it}+c_2e^{-it} \tag{2.2.2}$$

where c_1 and c_2 are arbitrary constants.We can rewrite this expression as

$$y(t)=c_1^*\sin(t)+c_2^*\cos(t) \tag{2.2.3}$$

where c_1^* and c_2^* are arbitrary constants. From here, we conclude that solutions of the harmonic oscillator has periodic. For the harmonic oscillator we will discuss three central difference models such that one of them has local representation for the linear term y while the others has nonlocal representations for y.

The central difference scheme with local representation for the linear term is given by

$$\frac{y_{k+1}-2y_k+y_{k-1}}{h^2}+y_k=0, \tag{2.2.4}$$

which can be converted to a difference equation of the form

$$y_{k+1}-(2-h^2)y_k+y_{k-1}=0. \tag{2.2.5}$$

This is a second order linear difference equation with constant coefficent in terms of the step size h. Its characteristic equation can be written as

$$r^2-2(1-\frac{h^2}{2})r+1=0 \tag{2.2.6}$$

The roots of characteristic eqution can be easily found as

$$\begin{cases} r_1(h)=(1-\frac{h^2}{2})+(\frac{h}{2})\sqrt{(h^2-4)} \\ r_2(h)=(1-\frac{h^2}{2})-(\frac{h}{2})\sqrt{(h^2-4)}. \end{cases} \tag{2.2.7}$$

Then the general solution of the corresponding difference equation is writen as

$$y_k = C_1(r_1(h))^k + C_2(r_2(h))^k \tag{2.2.8}$$

where $C_1$and C_2 are arbitrary constants. When we observe the values of the step size h, we can make the following discussions.

Case 1 : $0 < h < 2$: In this case,
$r_1(h)$ and $r_2(h)$ will be complex valued since $h^2 - 4 < 0$. Then the roots of characteristic equation (2.2.7) becomes

$$r_1(h) = (1 - \frac{h^2}{2}) + (\frac{ih}{2})\sqrt{4 - h^2},$$

$$r_2(h) = (1 - \frac{h^2}{2}) - (\frac{ih}{2})\sqrt{4 - h^2}.$$

Modulus of these roots are equal to 1, i.e

$$|r_1(h)| = |r_2(h)| = 1$$

From here we can rewrite the roots in polar form as

$$r_1(h) = e^{i\varphi_1(h)},$$

$$r_2(h) = e^{i\varphi_2(h)},$$

where

$$\tan(\varphi_1(h)) = \frac{\frac{h}{2}\sqrt{4 - h^2}}{(1 - \frac{h^2}{2})},$$

$$\tan(\varphi_2(h)) = -\frac{\frac{h}{2}\sqrt{4 - h^2}}{(1 - \frac{h^2}{2})},$$

Using the fact that

$$\varphi_1(h) = -\varphi_2(h)$$

denote

$$\varphi(h) = \varphi_1(h) = -\varphi_2(h)$$

Then we obtain

$$r_1(h) = e^{i\varphi(h)}$$
$$r_2(h) = e^{-i\varphi(h)}$$

As a result general solution of the corresponding difference equation (2.2.5) can be writen as

$$y_k = C_1^* e^{i\varphi(h)k} + C_2^* e^{-i\varphi(h)k} \tag{2.2.9}$$

where C_1^* and C_2^* are arbitrary constants. The solution (2.2.9) is periodic for all $0 < h < 2$. Hence, this result agree with the general solution of the differential equation. Therefore the choice of the step size like

$$0 < h < 2$$

will give us consistent results for the numerical solution of the corresponding differential equation (2.2.1).

Case 2 : $h = 2$: In this case

$$r_1(h) = r_2(h) = -1$$

and since the roots are equal, the general solution becomes

$$y_k = (C_1^* + C_2^* k)(-1)^k \tag{2.2.10}$$

where C_1^* and C_2^* are arbitrary constants. The solution (2.2.10) oscillates with an increasing amplitutes by changing sign. Therefore, (2.2.10) is not consistent with the general solution of the harmonic oscillator given in (2.2.3).

Case 3 : $h > 2$: In this case, the roots of (2.2.6) will be real valued as

$$r_1(h) = (1 - \frac{h^2}{2}) + (\frac{h}{2})\sqrt{(h^2 - 4)}$$

$$r_2(h) = (1 - \frac{h^2}{2}) - (\frac{h}{2})\sqrt{(h^2 - 4)}$$

If we differentiate $r_2(h)$ with respect to h then

$$\frac{d}{dh}(r_2(h)) = -h - \frac{1}{2}\frac{h^2}{\sqrt{h^2-4}} - \frac{1}{2}\sqrt{h^2-4} < 0 \text{ for all } h > 2$$

means that $r_2(h)$ is strictly decreasing over $[2, \infty]$ and also

$$\lim_{x \to 2^+} (r_2(h)) = -1.$$

This fact leads us to the following result

$$r_2(h) < -1 \text{ for } h > 2.$$

and from the characteristic equation (2.2.6), we have

$$r_1(h) r_2(h) = 1. \tag{2.2.11}$$

which implies that

$$-1 < r_1(h) < 0 \quad \text{for } h > 2.$$

Then we can write the general solution of the corresponding difference equation for $h > 2$ in a form

$$y_k = C_1^*(r_1(h))^k + C_2^*(r_2(h))^k$$

However by using the fact that

$$r_1(h)^k = |r_1(h)|^k (-1)^k$$

$$r_2(h)^k = |r_2(h)|^k (-1)^k$$

the general solution can be described as

$$y_k = (C_1^* |r_1(h)|^k + C_2^* |r_2(h)|^k)(-1)^k \tag{2.2.12}$$

Thus for this case the solutions $\{y_k\}$ will increase exponentially with an oscillating amplitute. For example,

$$\begin{aligned} C_1^* \left| r_1(h) \right|^k &\rightarrow 0 \\ C_2^* \left| r_2(h) \right|^k &\rightarrow \infty \end{aligned}$$

as a result of (2.2.11). When we consider three possible cases, we conclude that this central difference model for harmonic oscillator gives numerically consistent result only if the value of step size is between 0 and 2, i.e, for the interval $[0, 2]$

Now we will examine two central difference schemes which has nonlocal forms for the linear term:

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + y_{k-1} = 0 \tag{2.2.13}$$

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + y_{k+1} = 0 \tag{2.2.14}$$

We can write the characteristic equation for (2.2.13) as

$$r^2 - 2r + (1 + h^2) = 0 \tag{2.2.15}$$

The solution of (2.2.15) can be written as

$$\begin{aligned} r_1(h) &= 1 + ih \\ r_2(h) &= 1 - ih \end{aligned}$$

These roots can be expressed in polar form

$$\begin{aligned} r_1(h) &= (\sqrt{1 + h^2}) e^{i\varphi_1(h)} \\ r_2(h) &= (\sqrt{1 + h^2}) e^{i\varphi_2(h)} \end{aligned}$$

where

$$\begin{aligned} \tan(\varphi_1(h) &= h \\ \tan(\varphi_2(h) &= -h \end{aligned}$$

If we denote

$$\varphi(h) = \varphi_1(h) = -\varphi_2(h)$$

we conclude that the general solution of the equation (2.2.13) becomes

$$y_k = C_1^*(r_1(h))^k + C_2^*(r_2(h))^k$$

or

$$y_k = C_1^*((\sqrt{1+h^2})e^{i\varphi(h)})^k + C_2^*((\sqrt{1+h^2})e^{-i\varphi(h)})^k \tag{2.2.16}$$

where C_1^* and C_2^* are abitrary constants.

Since the magnitude of the roots $r_1(h)$ and $r_2(h)$ are greater than 1 then we obtain oscillatory solutions with an exponentially increasing amplitute. Similarly, we can write the characteristic equation of the equation (2.2.14) as follows

$$r^2 - (\frac{2}{1+h^2})r + (\frac{1}{1+h^2}) = 0 \tag{2.2.17}$$

The solutions of (2.2.17) becomes

$$\begin{aligned} r_1(h) &= \frac{ih+1}{h^2+1} \\ r_2(h) &= \frac{ih-1}{h^2+1} \end{aligned}$$

The magnitudes of the roots are

$$|r_1(h)| = |r_2(h)| = \sqrt{\frac{1}{1+h^2}}$$

and these roots can be expressed in polar form as

$$r_1(h) = (\sqrt{\frac{1}{1+h^2}})e^{i\varphi_1(h)},$$

$$r_2(h) = (\sqrt{\frac{1}{1+h^2}})e^{i\varphi_2(h)},$$

where

$$\begin{aligned} \tan(\varphi_1(h)) &= h, \\ \tan(\varphi_2(h)) &= -h. \end{aligned}$$

From here if we denote

$$\varphi(h) = \varphi_1(h) = -\varphi_2(h)$$

then the general solution of the equation (2.2.14) is written as

$$y_k = C_1^*((\frac{1}{\sqrt{1+h^2}})e^{i\varphi(h)})^k + C_2^*((\frac{1}{\sqrt{1+h^2}})e^{-i\varphi(h)})^k \tag{2.2.18}$$

In this case, the magnitudes of the roots of the characteristic equation are less than 1. This fact leads us that the scheme (2.2.14) yields an oscillatory solution but it has an amplitude which decreases exponentially.

As a conclusion we observed that the only use of the central difference scheme with local represantation for linear term gives us a discrete model which oscillates with a constant amplitude and this model has the same qualitative behaviour with the general solution of the harmonic oscillator differential equation if we choose the step size like:

$$0 < h < 2$$

The other models also yield oscillatory solution but they have amplitudes either increasing or decreasing.

We examined two ordinary differential equation and obtain some results when the numerical instabilities occur in discrete models to the corresponding differential equation.It is obvious that the structure of the corresponding difference equation is closely related with the differential equation. When we would like to analyze the discrete models of higher order and nonlinear differential equations, we will generally obtain complicated higher-order and nonlinear discrete models. Since any general solution method for difference equations are not known as general differential equation, fully analyzing of the numerical instabilities of these types of equations are not be possible lots of time. We can get some results for only a limited range of difference equation. Hence we do not have any general method for determining when the numerical instabilities occur in the discrete models. However , for many ordinary differential equation, a linear stability analysis of the fixed points provides a determination of which values of the step size h, the numerical instabilities arise in the discrete models. For this purpose, we demostrate the linear stability analysis for the first order autonomous ordinary differential equations which have simple fixed points.

2.3. Linear Stability Analysis

Consider the following first order autonomous ordinary differential equation

$$\frac{dy}{dt} = f(y) \tag{2.3.1}$$

where $f(y)$ is a given function. Let $f(y)$ has *n-simple zeros*

$$\left\{\overline{y}^{(i)} |\ i = 1, 2, ...n\right\} \tag{2.3.2}$$

where

$$f(\overline{y}^{(i)}) = 0 \ \text{ for all } i = 1, 2..n \tag{2.3.3}$$

The points $\overline{y}^{(i)}$ are sometimes called as fixed point, equilibrium point, critical point. We will generally prefer to use fixed point in this thesis. Consider the constant functions

$$y(t) = \overline{y}^{(i)} \text{ for } i = 1, 2, ..n$$

which are also particular solutions of the equation (2.3.1). We will investigate the linear stability properties of the solutions by making small perturbations about them. For equation (2.3.1), numerical instabilities arise if the linear stability properties of the fixed-points for the discrete model are not coincide with that of differential equation.

Consider the fixed points $\overline{y}^{(i)}$ and its any sufficient smal perturbation $0 <|\epsilon(t)| << \overline{y}^{(i)}$. Our aim is to check the behaviour of the perturbed function

$$\overline{y(t)} = \overline{y}^{(i)} + \epsilon(t) \tag{2.3.4}$$

Substituting (2.3.4) into (2.3.1) yields

$$\frac{d}{dt}(\overline{y}^{(i)} + \epsilon(t)) = f(\overline{y}^{(i)} + \epsilon(t)) \Longrightarrow \frac{d\epsilon}{dt} = f(\overline{y}^{(i)} + \epsilon(t)) \tag{2.3.5}$$

If we expand f in a Taylor series about the fixed point $\overline{y}^{(i)}$, we get

$$f(\overline{y}^{(i)} + \epsilon(t)) = f(\overline{y}^{(i)}) + \left.\frac{df}{dy}\right|_{y=\overline{y}^{(i)}} \epsilon(t) + \left.\frac{d^2 f}{dy^2}\right|_{y=\eta_i} \epsilon^2(t)$$

where η_i is between $\overline{y}^{(i)}$ and $\overline{y}^{(i)} + \epsilon(t)$ then we obtain

$$f(\overline{y}^{(i)} + \epsilon(t)) = \left.\frac{df}{dy}\right|_{y=\overline{y}^{(i)}} \epsilon(t) + O(\epsilon^2)$$

since $f(\overline{y}^{(i)}) = 0$. If we ignore the non-linear terms ϵ^2 and denote $R_i = \left.\frac{df}{dy}\right|_{y=\overline{y}^{(i)}}$ then we obtain$f(\overline{y}^{(i)} + \epsilon(t)) = R_i\epsilon(t)$
By using the equation (2.3.5), we get

$$\frac{d\epsilon(t)}{dt} = R_i\epsilon(t) \ \text{ for } i = 1, 2...n \tag{2.3.6}$$

Then the general solution of (2.3.6) can be writen as

$$\epsilon(t) = \epsilon_0 e^{R_i t} \tag{2.3.7}$$

where $\epsilon(0) = \epsilon_0$. We can conclude the following results from (2.3.7).

If $R_i > 0$ then $\epsilon(t)$ is an increasing function and the perturbed function $\overline{y(t)}$ moves away from the fixed-points $\overline{y}^{(i)}$. In this case the fixed point $\overline{y}^{(i)}$ is said to be linearly unstable.
If $R_i < 0$ then $\epsilon(t)$ is a decreasing function and the perturbed function $\overline{y(t)}$ moves towards the fixed-points $\overline{y}^{(i)}$. In this case $\overline{y}^{(i)}$ is called linearly stable.

We can use the linear stability analysis of fixed points in order to identify when the numerical instabilities occur without making any deeper calculations. By using this approach, we can compare the linear stability properties of fixed points of the discrete model with the corresponding autonomous first order ODE.

Now we will discuss the numerical instabilities of the discrete models of (2.3.1) for the central difference scheme, forward Euler and backward Euler.

First construct a central difference scheme for (2.3.1) such that:

$$\frac{y_{k+1} - y_{k-1}}{2h} = f(y_k) \tag{2.3.8}$$

Small perturbation about the fixed points $y_k = \overline{y}^{(i)}$ of the equation (2.3.8), we have

$$y_k = \overline{y}^{(i)} + \epsilon_k \tag{2.3.9}$$

where $\epsilon_k > 0$.
If (2.3.9) is subsitituted into (2.3.8), we have

$$\frac{\overline{y}^{(i)} + \epsilon_{k+1} - \overline{y}^{(i)} - \epsilon_{k-1}}{2h} = f(\overline{y}^{(i)} + \epsilon_k) \Longrightarrow \frac{\epsilon_{k+1} - \epsilon_{k-1}}{2h} = R_i \epsilon_k \tag{2.3.10}$$

Then we obtain the following second order difference equation with respect to ϵ_k :

$$\epsilon_{k+1} - (2hR_i)\epsilon_k - \epsilon_{k-1} = 0 \tag{2.3.11}$$

If we write $\epsilon_k = r^k$ then the characteristic equation of the linear difference eqaution (2.3.11) becomes

$$r^2 - (2hR_i)r - 1 = 0 \tag{2.3.12}$$

If we denote the roots of (2.3.12) by r_+ and r_- and then we can write the general solution of (2.3.11) as

$$\epsilon_k = c_1(r_+)^k + c_2(r_-)^k$$

where c_1 and c_2 are arbitrary constants. It can be easily seen that

$$(r_+)(r_-) = -1 \tag{2.3.13}$$

which means that r_+ and r_- have opposite signs and one of them is larger than in magnitude. From here, we conclude that ϵ_k oscillates with an increasing amplitudes in magnitude. This means that the fixed point

$$y_k = \overline{y}^{(i)}$$

is linearly unstable. However we know that the fixed-point of the diffferential equation is stable for $R_i < 0$. Hence the using of central difference scheme for the equation (2.3.1) gives us a discrete model for which all its fixed points are linearly unstable. That is, central difference discretization (2.3.8) for the differential equation (2.3.1) yields numerical instabilities for all values of step size h.

Now we will examine the linear stability properties of the fixed points of the forward Euler scheme for the equation (2.3.1)

$$\frac{y_{k+1} - y_k}{h} = f(y_k) \tag{2.3.14}$$

In this case, substituting (2.3.4) into (2.3.14) gives

$$\frac{\epsilon_{k+1} - \epsilon_k}{h} = R_i \epsilon_k \tag{2.3.15}$$

where $R_i = \left.\frac{df}{dy}\right|_{y=\overline{y}^{(i)}}$. Equation (2.3.15) is a first order difference equation which can be written as

$$\epsilon_{k+1} - (1 + hR_i)\epsilon_k = 0$$

Its general solution is given by

$$\epsilon_{k+1} = \epsilon_0 (1 + hR_i)^k \tag{2.3.16}$$

where

$$|\epsilon_0| << |\overline{y}^{(i)}|$$

We can write the following from the equation (2.3.16).

If $Ri > 0$ then we have $1 + hR_i > 1$ so that the fixed-points $\overline{y}^{(i)}$ are linearly unstable for both (2.3.1) and the difference equation (2.3.16)

If $R_i < 0$ then we know that the fixed-point $\overline{y}^{(i)}$ is linearly stable for (2.3.1). However $\overline{y}^{(i)}$ will be linearly stable for a limited range. This is because

$$|1 + hR_i| < 1 \Longleftrightarrow -1 < 1 + hR_i < 1 \Longleftrightarrow -2 < hR_i < 0$$

From here, we conclude

$$|hR_i| < 2 \Longleftrightarrow h < \frac{2}{|R_i|}$$

If we define R^* as

$$R^* = \sup\{|R_i| : i = 1, 2, ..n\}$$

we obtain the following results:

$$0 < h < \frac{2}{R^*} \implies y_k = \overline{y}^{(i)} \text{ is linearly stable}$$

$$h \geq \frac{2}{R^*} \implies y_k = \overline{y}^{(i)} \text{ is linearly unstable}$$

As a result, the fixed-points of the forward Euler scheme (2.3.14) and the (2.3.1) have the same linear stability properties under the following restriction on the step size h.

$$0 < h < \frac{2}{|R^*|}$$

This type of instabilities are called "threeshold instability" [3].

As a last example, we consider an implicit model for (2.3.1). The backward Euler scheme for (2.3.1) is given

$$\frac{y_k - y_{k-1}}{h} = f(y_k) \tag{2.3.17}$$

where $R_i = \frac{df}{dy}\Big|_{y=\overline{y}^{(i)}}$ .Small perturbation about the fixed point $y_k = \overline{y}^{(i)}$ leads us to the following equation

$$\frac{\epsilon_k - \epsilon_{k-1}}{h} = R_i \epsilon_k$$

which can be written as

$$\epsilon_k - \epsilon_{k-1} = hR_i\epsilon_k \Longrightarrow \epsilon_k = \epsilon_{k-1}(\frac{1}{1 - hR_i})^k \tag{2.3.18}$$

whose general solution is

$$\epsilon_k = \epsilon_0(\frac{1}{1 - hR_i})^k \tag{2.3.19}$$

then we conclude the following results:
If $R_i < 0$ then

$$0 < -hR_i \Longleftrightarrow 1 < 1 - hR_i \Leftrightarrow 0 < \frac{1}{1 - hR_i} < 1$$

since $h > 0$. So that from the equation (2.3.19) $\epsilon_k \to 0$ as $k \to \infty$.Thus the fixed-points of the equation (2.3.17) are linearly stable for all values of the step size $h > 0$. Hence the linear stability properties of the equation (2.3.17) and (2.3.1) are compatiblewith each other.
If $R_i > 0$ then we know that the fixed point $\overline{y}^{(i)}$ for (2.3.1) will be linearly unstable. However since $h > 0$ then we conclude the following results:

$$hR_i > 0 \Longleftrightarrow -hR_i < 0 \Longleftrightarrow \textit{1-h}R_i < 1 \tag{2.3.20}$$

Note that $1 - hR_i$ can not be zero since

$$1 - hR_i = 0 \Longrightarrow R_i = \frac{1}{h}$$

then from the left part of the eqaution (2.3.18) we obtain

$$\epsilon_k - \epsilon_{k-1} = h\frac{1}{h}\epsilon_k$$

then $\epsilon_{k-1} = 0$. This results contradicts with the assumption which is

$$0 < |\epsilon_k| \text{ for all } k$$

That is

$$1 - hR_i \neq 0, \text{ for all } h > 0$$

If we consider the the inequality (2.3.20), we can examine it in two cases

$$1 - hR_i < -1 \text{ or } -1 < 1 - hR_i < 1$$

If

$$1 - hR_i < -1$$

then we obtain

$$1 < \frac{1}{1 - hR_i} < 0 \tag{2.3.21}$$

We conclude from the equation (2.3.19), ϵ_k approaches to zero with a decreasing amplitude. So that the fixed points $y_k = \overline{y}^{(i)}$ are linearly stable. If

$$-1 < 1 - hR_i < 1$$

then we have

$$\frac{1}{1 - hR_i} < -1 \text{ or } \frac{1}{1 - hR_i} > 1$$

We conclude from the equation (2.3.19), ϵ_k oscillates with an incresing amplitude as $k \to \infty$. Hence the fixed points $y_k = \overline{y}^{(i)}$ are linearly unstable. If we reconsider the situtaion (2.3.1) for which the fixed points $y_k = \overline{y}^{(i)}$ becomes linearly stable, we can write

$$1 - hR_i < -1 \Leftrightarrow hR_i > 2$$

If we define $\overline{R}$ as

$$\overline{R} = \inf\{|R_i| : \ i = 1, 2, ...n\}$$

Then for

$$h > \frac{2}{\overline{R}}$$

all the fixed points of the implicit scheme (2.3.17) are linearly stable. This is an interesting result. The corresponding ordinary differential equation has unstable behaviour for some ranges of the step size h but the related discrete models for it has stable behaviour for some values of h in the same range. This phenomena is known as super-stability [42]. There are lots of papers about the structure of super-stability and its applications to real life problem [42].

In this section, we obtained some results about the linear stability properties of the fixed points for a certain type of ODEs. In the next chapters, we will use these results in order to construct discrete models which gives correct qualitative behaviour with the corresponding differential equation.

3. NONSTANDARD FINITE DIFFERENCE MODELS

In the previous chapter we have presented some standard difference methods for some ordinary differential equations. Even if using SFD models in differential equation give some consistent result for some values of the step size h, numerical instabilities occur in modelings for a broad range of h. When we use SFD method for numerical solutions of differential equation, numerical instabilities generally stem from the modelling of terms including derivatives.

There are some ways to prevent the instabilities in computations. One of them is to use a smaller step size h. Nonetheless lots of time this reducing increases the computational requirements. For example computational load is proportional to $\dfrac{1}{h^{d+1}}$ for standard discretization of $d-$dimensional wave equation. This means that halving the step size h for $3-$dimensional wave equation requires 8 times as much computer memory and 16 times as many computations [43]. Hence using smaller step size for this equation causes compuational costs. This result is observed in lots of finite difference models.Another way that can be used for decreasing the errors in methods is to use higher order methods for derivative terms. For instance we generally use the following form for the modelling of second order derivative term of a function $y = y(t)$:

$$y''(t_k) = \frac{y(t_{k+1}) - 2y(t_k) + y(t_{k-1})}{h^2} - \frac{h^2}{12}y''''(\eta_k)$$

where $\eta_k \in [t_{k-1}, t_{k+1}]$. This model has a second order accuracy. By using Taylor series representation we can also obtain the fourth order approximation

$$y''(t_k) = \frac{-y(t_{k+2}) + 16y(t_{k+1}) - 30y(t_k) + 16y(t_{k-1}) - y(t_{k-2})}{12h^2} - \frac{1}{90}h^4 y^{(6)}(t_k)$$

where $\eta_k \in [t_{k-2}, t_{k+2}]$. Higher order models generally give more accurate results. However they require much more operations in computations. Here our aim is to present nonstandard finite difference(NSFD) models which is firstly constructed by Mickens R.E. [3]. This method enables us to obtain reliable numerical results by means of low order finite differences without changing the step size or using the higher order methods for derivative terms. In this chapter we will provide some information about the general structure of NSFD models and introduce some rules given by Mickens in [3]

In the first chapter we have given a brief information about the SFD models for differential equations. The structure of denominator function and the modeling of nonlinear terms are two main differences between the SFD and NSFD. Nonstandard models generally include more complicated denominator functions. In this chapter, we will construct NSFD schemes for some particular ODEs. The rules of NSFD schemes are consequence of exact finite difference schemes of ODEs.

3.1 Exact Difference Schemes and Nonstandard Finite Difference Schemes

In this section, we will construct exact finite difference scheme for some ordinary differential equations. They are obtained from the exact solution of differential equation. Exact finite difference model has a great importance for numerical computations since they do not have any numerical instablities.

Consider the first order ODE

$$\frac{dy}{dt} = f(t, y) \tag{3.1.1}$$

where $f(t, y)$ is a given function for which the equation (3.1.1) has a unique solution. One of the standard modelings of this general equation can be

$$\frac{y_{k+1} - y_k}{h} = f(t_k, y_k)$$

where the denominator function $\varphi(h) = h$ is in the classical form. However we can make the following observations.

Assume that φ_1 and φ_2 be two function satisfying the conditions

$$\begin{cases} \varphi_1(h) = h + O(h^2) \\ \varphi_2(h) = h + O(h^2) \end{cases} \tag{3.1.2}$$

As $h \to 0$, we can retain the terms $O(h^2)$ and then we can write:

$$\lim_{h \to 0} \frac{y(t+\varphi_1(h)) - y(t)}{\varphi_2(h)} = \lim_{h \to 0} \frac{y(t+h+O(h^2)) - y(t)}{h+O(h^2)} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} = \frac{dy}{dt} \tag{3.1.3}$$

as definition of derivative. This observation is valid for $h \to 0$. However behaviour of the left hand side of the eqaution (3.1.3) will be different from the classical definiton of derivative for the finite values of the step size h.With an inspration from here, we will generally consider denominator functions such as $\varphi_1(h)$ and $\varphi_2(h)$ instead of classical forms and then we will focus on discrete forms:

$$\frac{dy}{dt} = \frac{y_{k+1} - y_k}{\varphi(h)} \tag{3.1.4}$$

where $\varphi(h)$ is a function satisfying the condition $\varphi(h) = h + O(h^2)$. We will firstly deal with first order ODEs in the form (3.1.1) whose exact solution can be written as

$$y(t) = \varphi(y_0, t_0, t) \tag{3.1.5}$$

with the initial condition

$$y_0 = \varphi(y_0, t_0, t_0)$$

If we consider a discrete scheme

$$y_{k+1} = u(h, y_k, t_k) \qquad t_k = hk \tag{3.1.6}$$

for (3.1.1), the solution to this explicit model can be expressed in a form

$$y_k = \phi(h, y_0, t_0, t_k) \tag{3.1.7}$$

We will say that equation (3.1.1) and (3.1.6) have the same general solution if and only if

$$y(t_k) = y_k \tag{3.1.8}$$

for all values of the step size h.

Definition 1. [3] If the solution of a discrete model has the same general solution with the corresponding differential equation then the difference scheme is called an exact finite difference scheme for corresponding differential equation

Theorem [4] The equation (3.1.1) has an exact finite-difference scheme given by

$$y_{k+1} = \phi(h, y_k, t_k, t_{k+1})$$

where the function ϕ is defined in (3.1.7) [3]. It would be useful to emphasize that this theorem is an existance theorem. In other word, it does not provide any information about how to construct the exact difference schemes for the related differential equations. Actually, there is no general procedure in order to construct an exact schemes. Now we will provide some examples for NSFD schemes discussed in [3,5,32]. When the exact scheme is obtained, by using some algebraic manipulation, one can reformulate the exact scheme to obtain a NSFD scheme. Using the exact finite difference scheme, Mickens [3] list some rules to construct a NSFD scheme for some certain ODEs and PDEs.

3.2. Logistic Differential Equation

Consider the logistic differential equation with two parameters λ_1 and λ_2 such that

$$\frac{dy}{dt} = \lambda_1 y - \lambda_2 y^2 \tag{3.2.1}$$

with the initial condition

$$y(t_0) = y_0 \tag{3.2.2}$$

Some algebric manipulations yield that

$$\frac{1}{\lambda_1 y - \lambda_2 y^2} = \frac{1/\lambda_1}{y} + \frac{\lambda_2/\lambda_1}{\lambda_1 - \lambda_2 y}$$

Then we can write

$$\frac{dy}{\lambda_1 y - \lambda_2 y^2} = dt \Rightarrow \int \frac{dy}{\lambda_1 y - \lambda_2 y^2} = \int dt \Rightarrow \int (\frac{1/\lambda_1}{y} + \frac{\lambda_2/\lambda_1}{\lambda_1 - \lambda_2 y})dy = \int dt$$

from here

$$\frac{1}{\lambda_1}\ln(y) - \frac{\lambda_2}{\lambda_1}(\frac{-1}{\lambda_2})\ln(\lambda_1 - \lambda_2 y) = t + c \Rightarrow \ln(y^{1/\lambda_1}(\lambda_1 - \lambda_2 y)^{-1/\lambda_1}) = t + C$$

$$\frac{1}{\lambda_1}\ln(\frac{y}{\lambda_1 - \lambda_2 y}) = t + C$$

Some additional operation gives us the following expression

$$\frac{ye^{-\lambda_1 t}}{\lambda_1 - \lambda_2 y} = e^{\lambda_1 C}$$

where c is a real constant. If we impose the initial condition (3.2.2), then we obtain

$$e^{\lambda_1 C} = \frac{y_0 e^{-\lambda_1 t_0}}{\lambda_1 - \lambda_2 y_0}$$

Consequently we will obtain an expression for a particular solution of (3.2.1-3.2.2) such that

$$\frac{ye^{-\lambda_1 t}}{\lambda_1 - \lambda_2 y} = \frac{y_0 e^{-\lambda_1 t_0}}{\lambda_1 - \lambda_2 y_0} \Rightarrow \frac{y}{\lambda_1 - \lambda_2 y} = \frac{y_0 e^{-\lambda_1 (t-t_0)}}{\lambda_1 - \lambda_2 y_0}$$

Then the general solution for the initial value problem (3.2.1-3.2.2) can be written as

$$y(t) = \frac{\lambda_1 y_0}{(\lambda_1 - \lambda_2 y_0)e^{-\lambda_1 (t-t_0)} + \lambda_2 y_0} \tag{3.2.3}$$

At this step we make the following subsitutions in order to obtain a difference scheme for (3.2.1) [3].

$$\begin{cases} y(t) \to y_{k+1} \\ y_0 \to y_k \\ t_0 \to t_k \\ t \to t_{k+1} \end{cases} \tag{3.2.4}$$

These subtitutions comes from the group properties of the solutions of the ordinary differential equations [48]. Then by making these subsitutions into (3.2.3) we obtain the following discete model

$$y_{k+1} = \frac{\lambda_1 y_k}{(\lambda_1 - \lambda_2 y_k)e^{-\lambda_1 h} + \lambda_2 y_k} \tag{3.2.5}$$

since $t_{k+1} - t_k = h$. Now by using this expression, we make some algebraic operations like

$$\lambda_1 y_{k+1} - \lambda_2 y_{k+1} y_k = \lambda_1 y_k e^{\lambda_1 h} - \lambda_2 y_{k+1} y_k e^{\lambda_1 h}$$

or equivalently we have the following equations:

$$\lambda_1 y_{k+1} - \lambda_2 y_{k+1} y_k + \lambda_2 y_{k+1} y_k e^{\lambda_1 h} - \lambda_1 y_k e^{\lambda_1 h} = 0 \tag{3.2.6}$$

Adding and substracting the term $\lambda_1 y_k$ to (3.2.6), we obtain

$$(y_{k+1} - y_k) = (\lambda_1 y_k - \lambda_2 y_{k+1} y_k)(\frac{1 - e^{\lambda_1 h}}{\lambda_1})$$

As a result we obtain the following NSFD scheme for the initial value problem (3.2.1-3.2.2)

$$\frac{y_{k+1} - y_k}{\frac{1-e^{\lambda_1 h}}{\lambda_1}} = \lambda_1 y_k - \lambda_2 y_{k+1} y_k \tag{3.2.7}$$

It is obviously seen that this nonstandard scheme is different from the standard ones for (3.2.1)-(3.2.2)

$$\begin{aligned} \frac{y_{k+1} - y_k}{h} &= \lambda_1 y_k - \lambda_2 y_k^2 \\ \frac{y_k - y_{k-1}}{h} &= \lambda_1 y_k - \lambda_2 y_k^2 \end{aligned} \tag{3.2.8}$$

The discrete model is exact finite difference scheme for the equation (3.2.5), i.e,

$$y_{k+1} = y(t_{k+1}), \quad k = 0, 1, 2... \text{ for all } h \tag{3.2.9}$$

To show this, we use mathematical induction for k. If $k = 0$, then from (3.2.7)

$$y_1 = \frac{\lambda_1 y_0}{(\lambda_1 - \lambda_2 y_0)e^{-\lambda_1 h} + \lambda_2 y_0} = \frac{\lambda_1 y_0}{(\lambda_1 - \lambda_2 y_0)e^{-\lambda_1 (t_1 - t_0)} + \lambda_2 y_0} = y(t_1)$$

So that

$$y_1 = y(t_1)$$

Suppose that the assumption (3.2.9) is true for all $k < k+1$, i.e,

$$y_k = y(t_k),\ \ k = 1, 2...k+1 \ \text{ for all } h \tag{3.2.10}$$

Then

$$y_{k+1} = \frac{\lambda_1 y_k}{(\lambda_1 - \lambda_2 y_k)e^{-\lambda_1 h} + \lambda_2 y_k} = \frac{\lambda_1 y(t_k)}{(\lambda_1 - \lambda_2 y(t_k))e^{-\lambda_1 h} + \lambda_2 y(t_k)}$$

by the assumption (3.2.10), we write

$$y_{k+1} = \frac{\lambda_1 \frac{\lambda_1 y_0}{(\lambda_1 - \lambda_2 y_0)e^{-\lambda_1 (t_k - t_0)} + \lambda_2 y_0}}{(\lambda_1 - \lambda_2 \frac{\lambda_1 y_0}{(\lambda_1 - \lambda_2 y_0)e^{-\lambda_1 (t_k - t_0)} + \lambda_2 y_0})e^{-\lambda_1 h} + \lambda_2 \frac{\lambda_1 y_0}{(\lambda_1 - \lambda_2 y_0)e^{-\lambda_1 (t_k - t_0)} + \lambda_2 y_0}}$$

$$y_{k+1} = \frac{\lambda_1 y_0}{((\lambda_1 - \lambda_2 y_0)e^{-\lambda_1 (t_k - t_0)} + \underbrace{\lambda_2 y_0 - \lambda_2 y_0})e^{-\lambda_1 h} + \lambda_2 y_0}$$

since then we obtain

$$y_{k+1} = \frac{\lambda_1 y_0}{(\lambda_1 - \lambda_2 y_0)e^{-\lambda_1 (t_k - t_0 + h)} + \lambda_2 y_0}$$

since $t_{k+1} = t_k + h$ then we write

$$y_{k+1} = \frac{\lambda_1 y_0}{((\lambda_1 - \lambda_2 y_0)e^{-\lambda_1 (t_{k+1} - t_0)} + \lambda_2 y_0} = y(t_{k+1})$$

implies that the claim is true, i.e

$$y_{k+1} = y(t_{k+1}),\ \ k = 0, 1, 2... \text{ for all } h$$

This shows that the NSFD scheme (3.2.6) or its equivalent statement (3.2.5) is an exact finite difference scheme for the initial value problem (3.2.1-3.2.2). It should be noted that the nonlinear term y^2 in (3.2.1) is modelled nonlocally by

$$y^2 \rightarrow y_{k+1}y_k$$

Indeed these type of representations are one of the most important properties of the NSFD schemes that differ from the standard forms. However we can not guarrente that nonlocal modelling of nonlinear terms for any differential equation yields consistent discrete models. There are some numerically stable models that use both local and nonlocal modelling for some nonlinear terms [40]. Main reason for this point is that there are lots of nonlocal representations of a nonlinear term. For instance

$$\begin{aligned} y^2 &\rightarrow y_{k+1}y_k \ , \ y_k(\frac{y_{k+1}+y_k}{2}), .. \\ y^3 &\rightarrow y_{k+1}y_k^2 \ , \ y_{k+1}y_k(\frac{y_{k+1}+y_k}{2}), .. \end{aligned}$$

Hence in addition to identifying a suitable denominator function, finding an appropriate model for nonlinear terms in a differential equation is very important to obtain consistent results. However we do not have a general method to overcome this point.

The following figure presents numerical solutions obtained from the equation (3.2.7). We use $y(0) = y_0 = 0.5$ as initial condition and three step sizes $h =$

$0.01, 1.5, 2.5$. We choose $\lambda_1 = \lambda_2 = 1$ for simplicity.

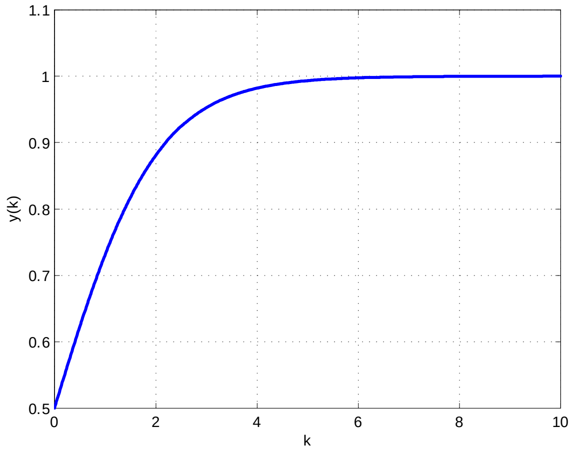

Figure 3.1. NSFD scheme (3.2.7) for $h = 0.01$.

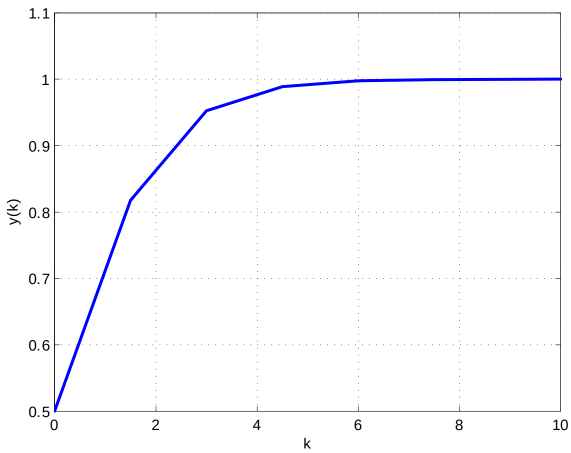

Figure 3.2. NSFD scheme (3.2.7) for $h = 1.5$

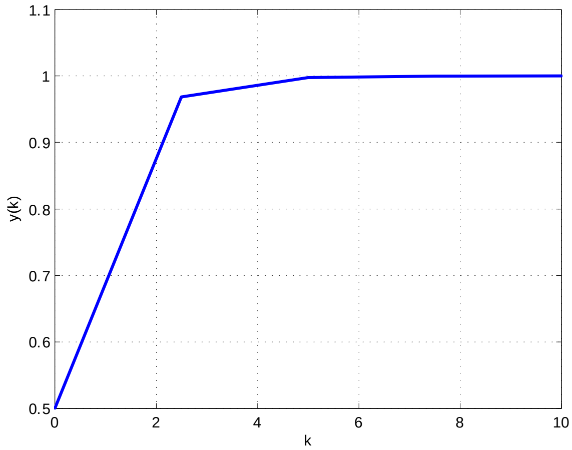

Figure 3.3. NSFD scheme (3.2.7) for $h = 2.5$

The following graphs shows some SFD schemes for the equation (3.2.1). We use $y(0) = y_0 = 0.5$ as initial condition and same step sizes $h = 0.01, 1.5, 2.5$

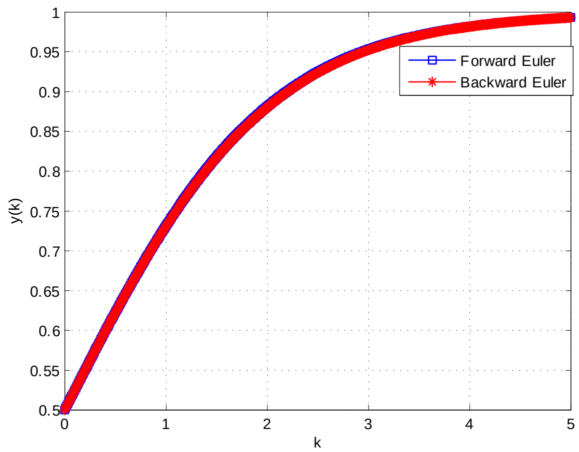

Figure 3.4. SFD schemes (3.2.8) for $h = 0.01$

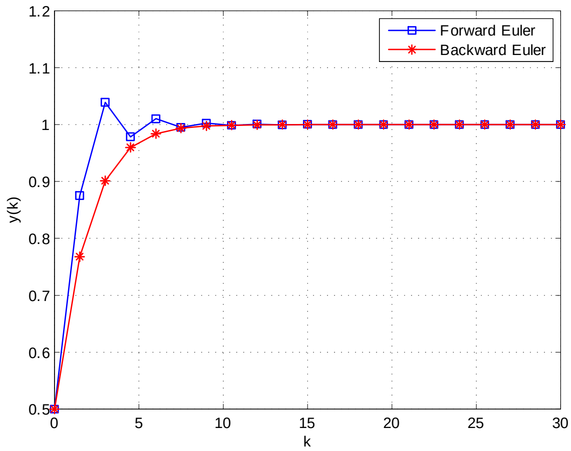

Figure 3.5. SFD schemes (3.2.8) for $h = 1.5$

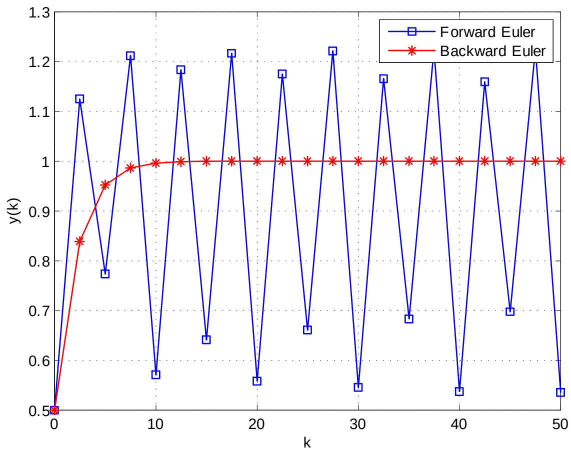

Figure 3.6. SFD schemes (3.2.8) for $h = 2.5$

Figure 3.7 represent the exact solution (3.2.3) of the differential equation (3.2.1) with the initial condition $y_0 = 0.5$. From the figure we see that the trajectory of the solution approaches to the fixed point $y = 1$. This behaviour is seen in NSFD scheme (3.2.7) for all step sizes $h = 0.01, 1.5, 2.5$. However, forward Euler scheme in (3.2.8) fails to approach the fixed point $y = 1$ for large step sizes, while backward Euler scheme perform well for all step sizes (See Figure (3.4-3.6))

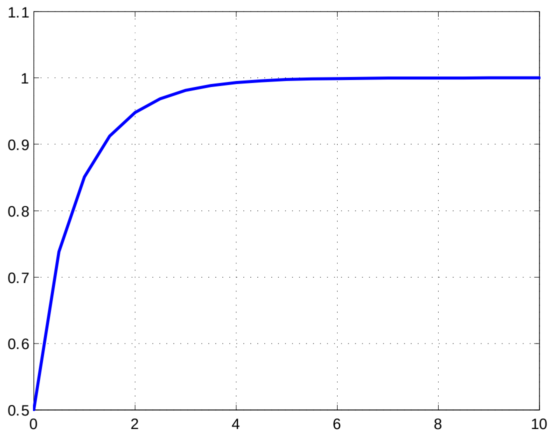

Figure 3.7. Exact solution of (3.2.1)

3.3. A First Order Nonlinear ODE

Consider the following ordinary differential eqaution

$$2\frac{dy}{dt} + y = \frac{1}{y} \tag{3.3.1}$$

$$y(t_0) = y_0 \tag{3.3.2}$$

Note that as $t \to \infty$, the solution approaches to the fixed point $y = \pm 1$. The solution of (3.3.1)-(3.3.2) is given by

$$y^2(t) = 1 - ce^{-t}, \ c \in \mathbb{R} \tag{3.3.3}$$

If we make the subsitutions in (3.2.4) then we obtain the following discrete model

$$y_{k+1}^2 = 1 - (1 - y_k^2)e^{-h} \tag{3.3.4}$$

It can be seen that this model is an exact finite difference scheme for the IVP (3.3.1)-(3.3.2). Adding and substracting y_k^2 to (3.3.4) and making some simple algebraic manipulation we get

$$\frac{(y_{k+1} - y_k)(y_{k+1} + y_k)}{1 - e^{-h}} + y_k^2 = 1$$

Multiplying both sides by $\dfrac{1}{(\frac{y_{k+1}+y_k}{2})}$ we obtain the following NSFD schemes for the equation (3.3.1)

$$2(\frac{y_{k+1} - y_k}{1 - e^{-h}}) + \frac{y_k^2}{\frac{y_{k+1} + y_k}{2}} = \frac{1}{\frac{y_{k+1} + y_k}{2}} \tag{3.3.5}$$

For (3.3.1), the scheme

$$\begin{aligned} 2(\frac{y_{k+1} - y_k}{h}) + y_k &= \frac{1}{y_k} \\ 2(\frac{y_k - y_{k-1}}{h}) + y_k &= \frac{1}{y_k} \end{aligned} \tag{3.3.6}$$

can be proposed as a SFD scheme. Note that the equation (3.3.5) has denominator function $\varphi(h) = 1 - e^{-h}$ and different nonlocal representations for the nonlinear terms. Figure (3.8) represents the solution of (3.3.1) with the initial condition $y(0) = 0.5$. Note that solution approaches to the fixed point $y = 1$.This behaviour can be seen for NSFD scheme (3.3.5) in the Figure (3.9-3.11) for the step sizes $h = 0.01, 1.5$ and 2.5 respectively. However, forward Euler scheme (3.3.6) represents the correct qualitative behaviour only for small step size $h = 0.01$.(See Figure 3.12). Figure (3.13) and (3.14) represent the failure of forward Euler scheme in (3.3.6) for large step sizes $h = 1.5$ and $h = 2.5$. However we observe that backward Euler scheme in (3.3.6) gives realible numerical results for all step sizes. This fact can be seen in Figures (3.12-3.14).

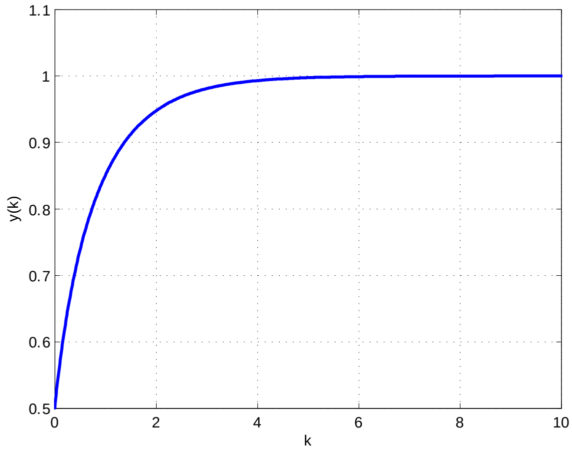

Figure 3.8. Exact solution of (3.3.1) with $y_0 = 0.5$

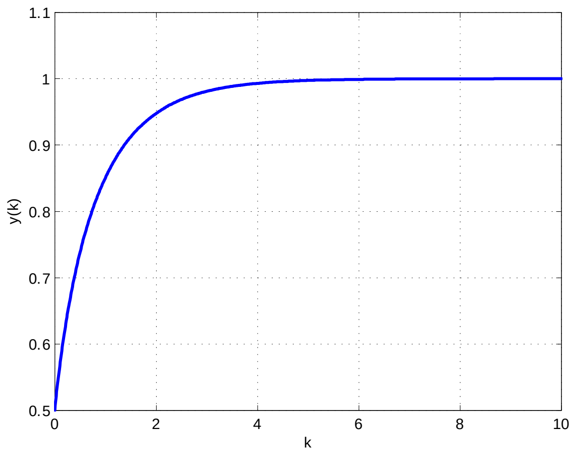

Figure 3.9. Plots of (3.3.5) for $h = 0.01$

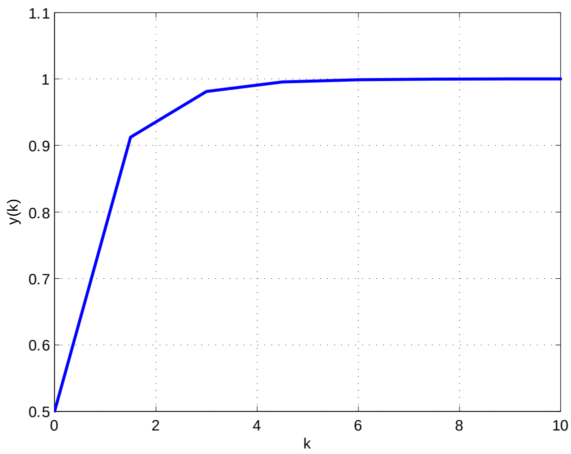

Figure 3.10. Plots of (3.3.5) for $h = 1.5$

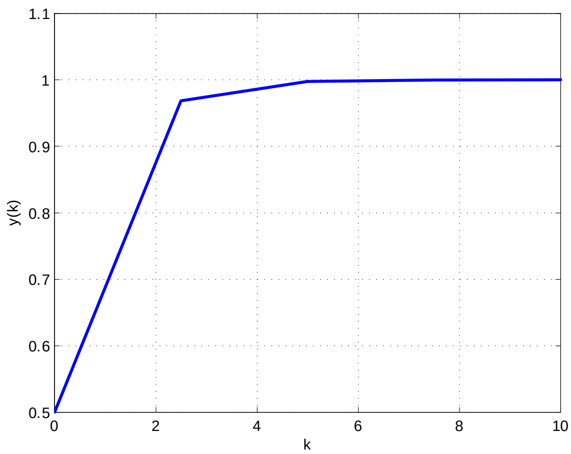

Figure 3.11. Plots of (3.3.5) for $h = 2.5$

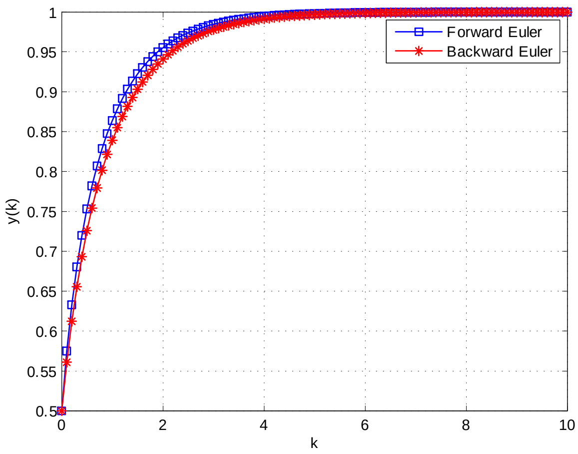

Figure 3.12. Plots of (3.3.6) for h=0.1

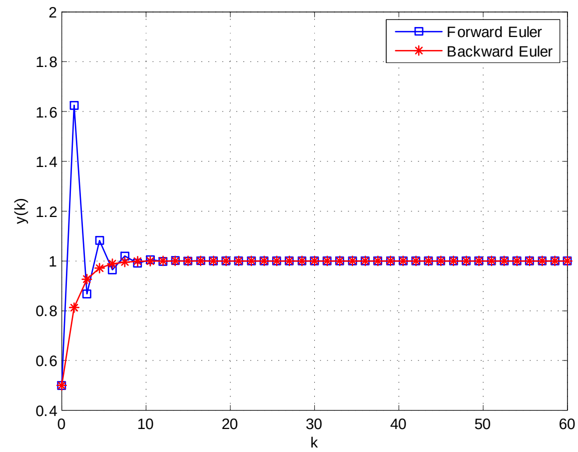

Figure 3.13. Plots of (3.3.6) for h=1.5

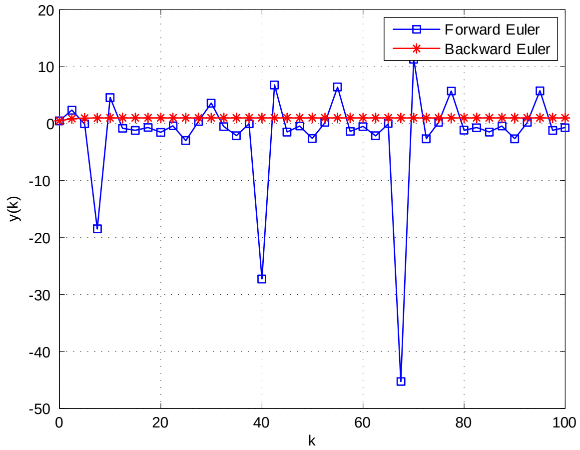

Figure 3.14. Plots of (3.3.6) for h=2.5

It is obvious that it can be diffucult to find a NSFD schemes for a differential equation only by making such algebraic operations.To overcome this difficulty, some new approches are developed in [3,5,44,45].

3.4. A New Finite Difference Scheme

In the first chapter we have investigated the linear stability properties of fixed points of differential equations and related difference equations.We have discussed that the elementary numerical instabilities occur in discrete modelling if the linear stability properties of fixed points of differential and difference equations are not same. Therefore if this point could ovecome then it would be possible to construct a consistent discrete models. In this section we will find a finite difference scheme which has correct linear stability properties for all step size h. Consider the IVP

$$\frac{dy}{dt} = f(y), \; y(t_0) = y_0 \tag{3.4.1}$$

Assume (3.4.1) has a uniqe solution. Let us denote the set of fixed point of (3.4.1) as

$$\{\overline{y}^{(i)}|\ f(\overline{y}^{(i)}) = 0,\ i = 1, 2..n\}$$

where we assume that all $\overline{y}^{(i)}$'s are real and distinct. Define

$$R_i = \left.\frac{df}{dy}\right|_{y=\overline{y}^{(i)}} \tag{3.4.2}$$

and

$$R^* = \sup\{|R_i|\ |\ i = 1, 2..n\}$$

also we have obtained the following results from the section (2.4) by means of linear stability analysis of fixed points mentioned in the Section 2.4:
If $R_i < 0$ then the fixed-point $y(t) = \overline{y}^{(i)}$ is linearly stable
If $R_i > 0$ then the fixed-point $y(t) = \overline{y}^{(i)}$ is linearly unstable

Now pick an arbitrary function $\varphi(s)$ satisfying the conditions:

$$\varphi(s) = s + O(s^2) \quad \text{as } s \to 0 \tag{3.4.3}$$

$$0 < \varphi(s) < 1 \quad \text{for } s > 0 \tag{3.4.4}$$

Theorem: [**3**] The fixed point of the finite discrete model

$$\frac{y_{k+1} - y_k}{\frac{\varphi(hR^*)}{R^*}} = f(y_k) \tag{3.4.5}$$

has the same linear stability properties with the differential equation (3.4.1).
Proof: Making a small perturbation about the fixed point $\overline{y}^{(i)}$ such that

$$y_k = \overline{y}^{(i)} + \varepsilon_k \tag{3.4.6}$$

where $\varepsilon_k > 0$. Then substituting (3.4.6) into (3.4.5) and using the Taylor's series expansion of $f(\overline{y}^{(i)} + \varepsilon_k)$ around $f(\overline{y}^{(i)})$, we get

$$\frac{\varepsilon_{k+1} - \varepsilon_k}{\frac{\varphi(hR^*)}{R^*}} = R_i\varepsilon_k \tag{3.4.7}$$

Equation (3.4.7) can be rewriten as

$$\varepsilon_{k+1} = (1 + \frac{R_i}{R^*}\varphi(hR^*))^k \varepsilon_0 \tag{3.4.8}$$

If $R_i > 0$ we know that the fixed points $\overline{y}^{(i)}$ of (3.4.1) are linearly unstable. Since

$$1 + \frac{R_i}{R^*}\varphi(hR^*) > 0$$

From (3.4.8) we get $\varepsilon_{k+1} \to \infty$ as $k \to \infty$. Therefore the fixed points $\overline{y}^{(i)}$ of (3.4.5) are also linearly unstable.
If $R_i < 0$ then the fixed points $\overline{y}^{(i)}$ of (3.4.1) are linearly stable. Using the fact that $0 < |R_i| \leq R^*$ we can write $-R^* \leq -|R_i| < 0$ hence $-R^* \leq R_i < 0$ (since $R_i < 0$). Then we have $-1 \leq \frac{R_i}{R^*} < 0$. On the other hand since $hR^* > 0$ then by using (3.4.4) we get $0 < \varphi(hR^*) < 1$ which implies that

$$-1 < \frac{R_i}{R^*}\varphi(hR^*) < 0 \Rightarrow 0 < 1 + \frac{R_i}{R^*}\varphi(hR^*) < 1$$

Then from (3.4.7), we get $\varepsilon_{k+1} \to 0$ as $k \to \infty$. This means that the fixed-point $\overline{y}^{(i)}$ of (3.4.5) are also linearly stable for all h which completes the proof.

This theorem enables us to construct discrete models in which the elemantary numerical instabilities do not appear.

Now we will apply this theorem to some differential equations.

3.5. Decay Equation

Consider the decay differantial equation

$$\frac{dy}{dt} = -\lambda y, \; y(t_0) = y_0 \tag{3.5.1}$$

where $\lambda > 0$ is a parameter. Here $f(y) = -\lambda y$ and $\overline{y}^{(1)} = 0$ is the only fixed point. Then from (3.4.2)
$R_1 = \left.\frac{df}{dy}\right|_{y=0} = -\lambda$ and so $R^* = \lambda$. The function

$$\varphi(h) = 1 - e^{-h} \tag{3.5.2}$$

satisfies the conditions (3.4.3)- (3.4.4). Then the denominator function in (3.4.5) becomes

$$\frac{\varphi(hR^*)}{R^*} = \frac{1 - e^{-\lambda h}}{\lambda}$$

We can construct the following NSFD scheme for (3.5.1)

$$\frac{y_{k+1} - y_k}{\frac{1-e^{-\lambda h}}{\lambda}} = -\lambda y_k \tag{3.5.3}$$

It is an exact finite difference for the equation (3.5.1). In fact (3.5.3) is reduced to the following difference equation

$$y_{k+1} = (e^{-\lambda h})y_k$$

whose general solution can be writen as

$$y_k = (e^{-\lambda h})^k y_0$$

The solution of (3.5.1) can be expressed as

$$y(t) = y_0 e^{-\lambda(t-t_0)} \tag{3.5.4}$$

Then

$$y(t_k) = y_0 e^{-\lambda(t_k-t_0)} = y_0 e^{-\lambda(hk)} = y_0(e^{-\lambda h})^k = y_k \quad \text{for all } h$$

means that the discrete scheme (3.5.3) is an exact finite difference scheme for the equation (3.5.1). One of the SFD schemes for the equation (3.5.1) can be written as

$$\frac{y_{k+1} - y_k}{h} = -\lambda y_k \tag{3.5.5}$$

The following numerical computations enable us to see the comparison of two methods (3.5.3) and (3.5.5) for various step sizes with the exact solution. We impose the initial condition $y(0) = y_0 = 0.5, \lambda = 2$, and $h = 0.1, 0.75, 1.2$ in each computation.

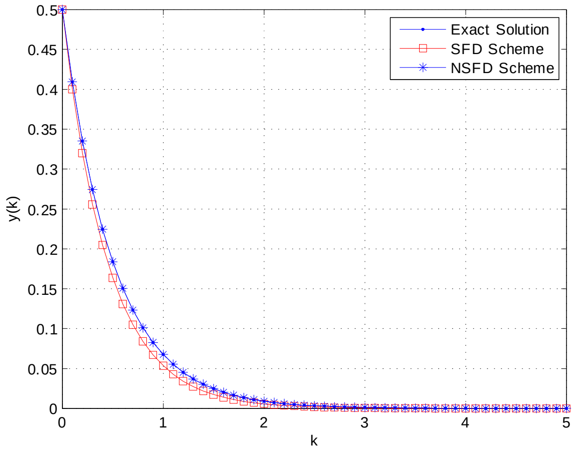

Figure 3.15. Comparison of methods for (3.5.1)
h=0.1

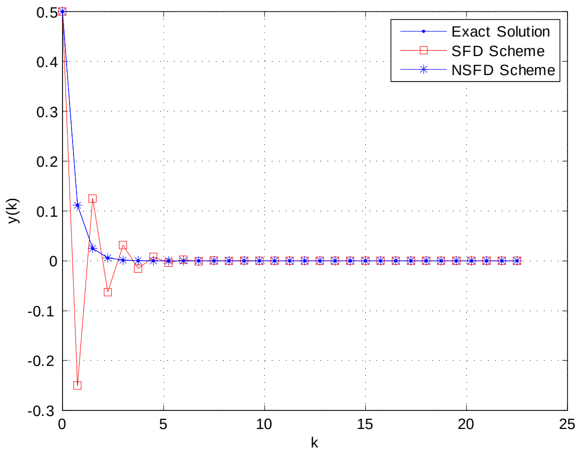

Figure 3.16. Comparison of methods for (3.5.1)
h=0.75

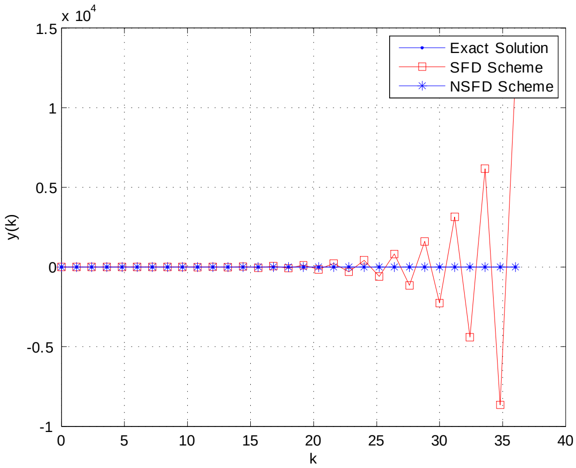

Figure 3.17. Comparison of methods for (3.5.1)
h=1.2

As a result, we see that the nonstandard scheme (3.5.2) has correct qualitative behaviours with the differential equation (3.5.1) for all step sizes. However standard scheme (3.5.4) fails for large step sizes, $h = 0.75, 1.2$

3.6 ODE with three fixed-point

Consider the following ODE

$$\frac{dy}{dt} = y(a - y^2),\ a > 0 \tag{3.6.1}$$

which has three fixed points. The equation (30) has three fixed points which are

$$\overline{y}^{(1)} = 0,\ \overline{y}^{(2)} = \sqrt{a},\ \overline{y}^{(3)} = -\sqrt{a}$$

and then we have

$$\begin{aligned} R_1 &= \left.\frac{df}{dy}\right|_{y=\overline{y}^{(1)}=0} = a \\ R_2 &= \left.\frac{df}{dy}\right|_{y=\overline{y}^{(2)}=\sqrt{a}} = R_3 = \left.\frac{df}{dy}\right|_{y=\overline{y}^{(2)}=-\sqrt{a}} = -2a \end{aligned}$$

So that

$$R^* = 2a$$

Using $\varphi(h)$ from (3.5.2) and equation (3.4.5) we obatin the NSFD scheme

$$\frac{y_{k+1} - y_k}{\frac{1-e^{-2ha}}{2a}} = y_k(a - y_k^2) \tag{3.6.2}$$

for the equation (3.6.1). This is an exact finite difference for the equation (3.6.1). Note that the NSFD scheme has the denominator function $\varphi(h) = \frac{1-e^{-2ha}}{2a}$ and its nonlinear term is modelled locally. For the purpose of comparison, consider a SFD scheme

$$\frac{y_{k+1} - y_k}{h} = y_k(a - y_k^2) \tag{3.6.3}$$

for the equation (3.6.1). When we compare these two schemes for various step sizes, we see that NSFD scheme has exactly same qualitative behaviour as the solution of (3.6.1). Numerical solutions of (3.6.2), (3.6.3) and the exact solution of (3.6.1) are displayed in Figures (3.18-3.20) for $y_0 = 0.5$, $a = 2$ and various step-sizes $h = 0.1, 0.75$ and $h = 1$. We see that the NSFD scheme (3.6.2) approaches monotonically to the fixed point $y = \sqrt{2}$. This is exactly the same qualitative behaviour with the exact solution of (3.6.1). Note that the SFD scheme (3.6.3) fails to preserve monotonicity of the solution for all selected step sizes.

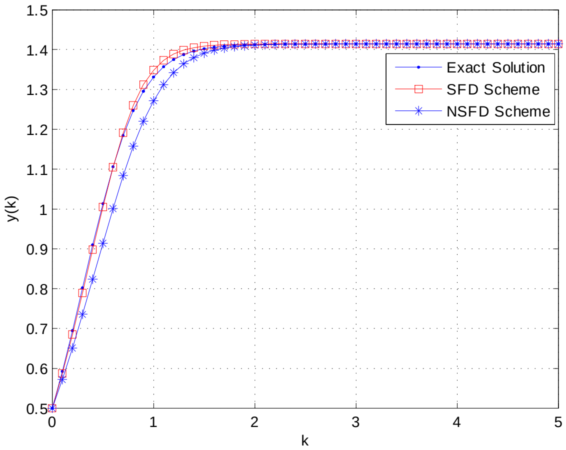

Figure 3.18. Comparison of methods (3.6.2) and (3.6.3) for (3.6.1) with h=0.1

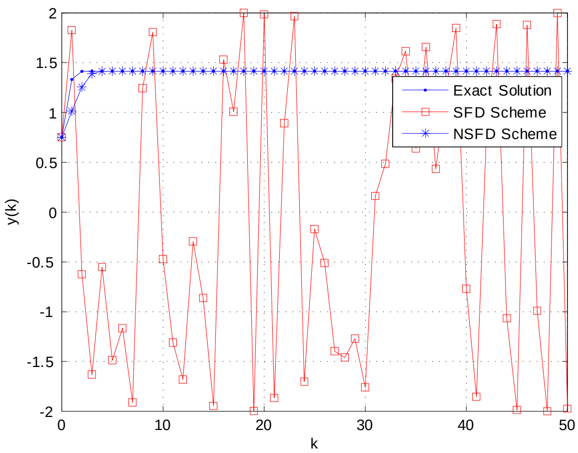

Figure 3.19. Comparison of methods (3.6.2) and (3.6.3) for (3.6.1) with h=0.75

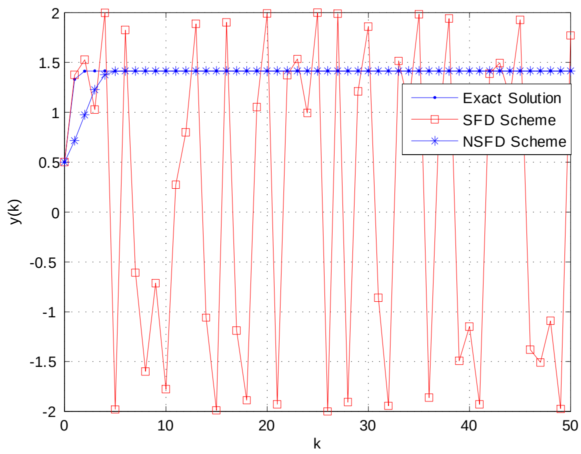

Figure 3.20. Comparison of methods (3.6.2) and (3.6.3) for (3.6.1) with h=1

We would like to emphasize that this theorem proposes the local representation for both linear and nonlinear terms in the right hand side of the equation (3.4.1). In Section 3.8, we will present a work which focuses on the nonlocal representation of the nonderivative terms in the equation (3.4.1).

We discussed four ordinary differantial equations whose general form is

$$\frac{dy}{dt} = f(y)$$

and obtained NSFD models these four differential equations. Numerical calculations have shown that even if the SFD schemes work well for small sizes-as expected, it does not reflect the correct qualitative behaviour of the exact solution of the corresponding differential equations for large step sizes. Here, we have seen that NSFD schemes resolve this point. On the other hand, we have seen that the use of nonstandard denominator function (it is sometimes called renormalized function) has an important influence on the behaviour of the discrete model.

As said before, there are no general procedure for finding a NSFD models of a differential equation. As a result of those analytical and numerical studies, the following rules are generally used in order to construct a consistent NSFD scheme [2,3]

Rule 1. The order of the discrete derivatives must be the same with the corresponding differential equation.

Rule 2. Instead of classical used, sophisticated denominator functions must be choosen for discrete models.

Rule 3. Nonlinear term must be modelled nonlocally.

Rule 4. Special conditions that hold for the solution of the differential equation should also hold for the solutions of the finite difference scheme.

Rule 5. The scheme should not introduce irrevelant solutions.

In the following sections, we will review two recent papers about NSFD modelling of differential equations . One of them is about a boundary value problem and the other one is about a general discrization of ODEs with three fixed points [44,45].

3.7. A NSFD Scheme for Second Order BVP

In this section, we will review the work [44] about nonstandard finite discretization of any second order BVPs. Consider the following BVP

$$y^{''} = f(x, y), \quad a < x < b \tag{3.7.1}$$

with boundary conditions are

$$y(a) = \alpha,\ y(b) = \beta \tag{3.7.2}$$

Here $f(x, y)$ is a given function for which the BVP (3.7.1) - (3.7.2) has a unique solution in the interval (a, b). We consider the equally spaced points $\{x_0, x_{1,}...x_n\}$ with step size $h = \frac{b-a}{n}$ for the interval $[a, b]$ such that

$$a = x_0 < x_1.. < x_{n-1} < x_n = b.$$

The main motivation of this work is based on an approximation in which the frozen coefficients are used. First of all, first order ODE

$$y' = f(x, y) \tag{3.7.3}$$

will be exaimed. Let $\widetilde{y} = \widetilde{y}(x)$ be an approximating function to $y = y(x)$ in the subinterval (x_{i-1}, x_{i+1}) such that

$$\frac{d\widetilde{y}}{dx} = f_i \tag{3.7.4}$$

where f_i is a constant number which is called frozen coefficent at $x = x_i$ and

$$f_i = f(x_i, \widetilde{y}_i), \qquad \widetilde{y}_i = \widetilde{y}(x_i). \tag{3.7.5}$$

The general solution of (3.7.4) is

$$\widetilde{y}(x) = f_i x + c \tag{3.7.6}$$

Imposing the continuity of the function $\widetilde{y} = \widetilde{y}(x)$ at the mesh points $x = x_{i-1}$ and $x = x_{i+1}$, set

$$\widetilde{y}_{i-1} = \widetilde{y}(x_{i-1}) \ , \ \widetilde{y}_{i+1} = \widetilde{y}(x_{i+1})$$

Then by using (3.7.6), it is obtained

$$\begin{aligned} \widetilde{y}_{i+1} &= f_i x_{i+1} + c \\ \widetilde{y}_{i-1} &= f_i x_{i-1} + c \end{aligned}$$

from here, we write

$$f_i = \frac{\widetilde{y}_{i+1} - \widetilde{y}_{i-1}}{x_{i+1} - x_{i-1}} = \frac{\widetilde{y}_{i+1} - \widetilde{y}_{i-1}}{2h}$$

This characterize an approximation such that

$$\frac{d\widetilde{y}}{dx} = \frac{\widetilde{y}_{i+1} - \widetilde{y}_{i-1}}{2h} \tag{3.7.7}$$

which is the central difference formula for $\dfrac{d\widetilde{y}}{dx}$ in the subinterval $[x_{i-1}, x_{i+1}]$. Forward and backward Euler formulas can be obtained by using the subintervals $[x_i, x_{i+1}]$ and $[x_{i-1}, x_i]$ respectively. Now freezing f at the mesh point $x = x_i$ in the interval (x_i, x_{i+1}) and adding a parameter $w \in (x_i, x_{i+1})$ it can be obtained

$$\widetilde{y}' - w\widetilde{y} = f_i - w\widetilde{y}_i \tag{3.7.8}$$

which is a linear first order ODE. Its general solution can be written as

$$\widetilde{y}(x) = \frac{-1}{w}(f_i - w\widetilde{y}_i) + ce^{wx} \tag{3.7.9}$$

By using the conditions

$$\widetilde{y}_i = \widetilde{y}(x_i) \ , \ \widetilde{y}_{i+1} = \widetilde{y}(x_{i+1})$$

f_i can be obtained as

$$f_i = \frac{\widetilde{y}_{i+1} - \widetilde{y}_{i-1}}{\frac{e^{wh}-1}{2}} = \frac{d\widetilde{y}}{dx} \tag{3.7.10}$$

Note that (3.7.10) has a denominator function $\varphi(h) = \frac{e^{wh}-1}{2}$ which is the same with function described by Mickens [3]. Now let $\widetilde{y} = \widetilde{y}(x)$ be an approximation to $y = y(x)$ in the interval (x_{i-1}, x_{i+1}) and

$$\frac{d^2\widetilde{y}}{dx^2} = f_i \tag{3.7.11}$$

where f_i is the frozen coefficient at $x = x_i$. With an inspiration from (3.7.8), it is possible to construct a NSFD scheme for the BVP (3.7.1-3.7.2). Construct the following differential equation

$$\widetilde{y}'' - w^2\widetilde{y} = f_i - w^2\widetilde{y}_i \tag{3.7.12}$$

Equation (3.7.12) can be rewritten as

$$\widetilde{y}'' - w^2(\widetilde{y} + \frac{f_i}{w^2} - \widetilde{y}_i) = 0 \tag{3.7.13}$$

If we denote $u = \widetilde{y} + \frac{f_i}{w^2} - \widetilde{y}_i$ then the equation (3.7.13) becomes

$$u'' - w^2 u = 0$$

whose general solution is

$$u(x) = c_1 e^{wx} + c_2 e^{-wx}$$

Then the general solution of the equation (3.7.13) is

$$\widetilde{y}(x) = c_1 e^{wx} + c_2 e^{-wx} - \frac{f_i}{w^2} + \widetilde{y}_i \tag{3.7.14}$$

If we use the conditions

$$\widetilde{y}_{i-1} = \widetilde{y}(x_{i-1}),\ \widetilde{y}_i = \widetilde{y}(x_i)\ ,\ \widetilde{y}_{i+1} = \widetilde{y}(x_{i+1})$$

f_i can be found as

$$f_i = \frac{\widetilde{y}_{i+1} - 2\widetilde{y}_i + \widetilde{y}_{i-1}}{\varphi(w,h)} \tag{3.7.15}$$

where

$$\varphi(w,h) = 2(\frac{\cosh(wh) - 1}{w^2}) \tag{3.7.16}$$

As a result the following NSFD scheme is presented for the BVP (3.7.1-3.7.2)

$$\begin{aligned} \frac{y_{i+1} - 2y_i + y_{i-1}}{\varphi(w,h)} - f(x_i, y_i) &= 0,\ i = 0, 1, ..n-1 \\ y_0 &= \alpha,\ y_n = \beta \end{aligned} \tag{3.7.17}$$

It should be noted that w can be in pure imaginary form, i.e,

$$w \to iw \text{ or } w \to -iw$$

However $\varphi(w,h)$ will be a real number again. Since $\cos(wh) = \frac{e^{wh} - e^{-wh}}{2}$ then

$$\cosh((iw)h) = \frac{e^{iwh} - e^{-iwh}}{2} = \frac{\cos(wh) + i\sin(wh) + \cos(wh) - i\sin(wh)}{2} = \cos(wh)$$

where $i^2 = -1$.
Now if we use Taylor series expansion for the hyperbolic function $\cosh(wh)$

$$\cosh(wh) = 1 + \frac{1}{2}w^2h^2 + \frac{1}{24}w^4h^4 + \frac{1}{720}w^6h^6 + O(w^8h^8) \tag{3.7.18}$$

Using (3.7.18), Taylor series for (3.7.16) will be

$$\varphi(w,h) = h^2 + \frac{h^4w^2}{12} + \frac{h^6w^4}{360} + \frac{h^8w^6}{20160} + ... \tag{3.7.19}$$

for small h and w. It can be observed that

$$\varphi(w,h) \to h^2 \text{ as } w \to 0.$$

This situation is the conventional finite difference form of the problem (3.7.1-3.7.2). On the other hand it can be find an expression for the local truncation error of the equation (3.7.17). If we use the Taylor series for the function $y = y(x)$ in the subinterval (x_{i-1}, x_{i+1}), it can be found that

$$y(x_{i+1}) - 2y(x_i) + y(x_{i-1}) = h^2y''(x_i) + \frac{h^4}{12}y^{(iv)}(x_i) + \frac{h^6}{360}y^{(6)}(\eta_i) \tag{3.7.20}$$

where $\eta_i \in (x_{i-1}, x_{i+1})$. If we divide both sides by $\varphi(w,h)$ then substract $f(x_i, y_i)$ from both sides

$$\frac{y(x_{i+1})-2y(x_i)+y(x_{i-1})}{\varphi(w,h)} - f(x_i,y_i) = \frac{h^2y''(x_i)}{\varphi(w,h)} + \frac{h^4}{12}\frac{y^{(iv)}(x_i)}{\varphi(w,h)} + \frac{h^6y^{(vi)}(\eta_i)}{360\varphi(w,h)} - f(x_i,y_i)$$

since $y''(x_i) = f(x_i, y_i)$, we obtain

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{\varphi(w,h)} - f(x_i,y_i) = f(x_i,y_i)(\frac{h^2}{\varphi(w,h)} - 1) + \frac{h^4}{12}\frac{y^{(iv)}(x_i)}{\varphi(w,h)} + \frac{h^6}{360}\frac{y^{(vi)}(\eta_i)}{\varphi(w,h)} \tag{3.7.21}$$

As a result, local truncation error of the scheme (3.7.17) can be written as

$$\tau_i = f(x_i, y_i)(\frac{h^2}{\varphi(w,h)} - 1) + \frac{h^4}{12}\frac{y^{(iv)}(x_i)}{\varphi(w,h)} + \frac{h^6}{360}\frac{y^{(vi)}(\eta_i)}{\varphi(w,h)} \tag{3.7.22}$$

Now for small w , $\varphi(w,h) \approx h^2$ and then $\tau_i \to 0$ as $h \to 0$. This shows that the method (3.7.17) is second order. However it is possible to obtain an optimal w which depends on the values of mesh points. Hence it can be possible to find a local truncation error which behaves like fourth order.

If the local truncation error is chopped in the fourth order then the local truncation error reduced to

$$\tau_i = f(x_i, y_i)(\frac{h^2}{\varphi(w,h)} - 1) + \frac{h^4}{12}\frac{y^{(iv)}(\eta_i)}{\varphi(w,h)} \tag{3.7.23}$$

If we impose

$$\tau_i = 0$$

then

$$f(x_i, y_i)(\frac{h^2}{\varphi(w,h)} - 1) + \frac{h^4}{12}\frac{y^{(iv)}(\eta_i)}{\varphi(w,h)} = 0 \Rightarrow f(x_i, y_i)(\varphi(w,h) - h^2) = \frac{h^4}{12} y^{(iv)}(\eta_i)$$

implies that

$$\varphi(w,h) - h^2 = \frac{h^4}{12}\frac{y^{(iv)}(\eta_i)}{f(x_i, y_i)} \Rightarrow 2(\frac{\cosh(wh) - 1}{w^2}) - h^2 = \frac{h^4}{12}\frac{y^{(iv)}(\eta_i)}{f(x_i, y_i)}$$

If we use the expansion (3.7.18) up to the fourth order then it is obtained

$$2(\frac{1 + \frac{1}{2}w^2h^2 + \frac{1}{24}w^4h^4 - 1}{w^2}) - h^2 \approx \frac{h^4}{12}\frac{y^{(iv)}(\eta_i)}{f(x_i, y_i)} \Rightarrow h^2 + \frac{1}{12}w^2h^2 - h^2 \approx \frac{h^4}{12}\frac{y^{(iv)}(\eta_i)}{f(x_i, y_i)}$$

then

$$\frac{1}{12}w^2h^4 \approx \frac{h^4}{12}\frac{y^{(iv)}(\eta_i)}{f(x_i, y_i)} \tag{3.7.24}$$

Since w is depends on the values of the mesh points in the interval (x_{i-1}, x_{i+1}), we denote it by w_i and then get

$$w_i^2 \approx \frac{y^{(iv)}(x_i)}{f(x_i, y_i)} \tag{3.7.25}$$

where we imposed the condition $\eta_i \approx x_i$. As a result, the NSFD scheme (3.7.17) can be updated as

$$\begin{aligned} \frac{y_{i+1} - 2y_i + y_{i-1}}{\varphi(w_i, h)} - f(x_i, y_i) &= 0, \; i = 0, 1, ..n-1 \\ y_0 &= \alpha, \; y_n = \beta \end{aligned} \tag{3.7.26}$$

where w_i is of the form (3.7.25). In this way, (3.7.26) behaves like fourth order.

3.8. A NSFD Scheme Preserving the Linear Stability Properties of Fixed Points and Monotonicity of Solutions

One of the most important reason for using NSFD method is to be able construct discrete models which have corrrect qualitative behaviour with the corresponding differential equation. Although there is no general prodecure to achieve this point, there has been some powerful results for some types of differential equation. For instance when considered first order autonomous differential equation, numerical instabilities occur in discrete modelling if the linear stability properties of any fixed-points of difference equation is not concordance with those differential equation. This means that if it can be found any method which enable to achieve this point, it would be possible to eliminate numerical instabilities which stem from this fact. On the other hand it is a natural expectation that any discrete scheme should preserve the qualitative properties of differential equation such as positivity, monotonicity, asymptotic behaviour of the solution. In this section, we will cover some material in [32] which proposes criteria for finite difference models to preserve linear stability properties of fixed points and monotonicity of solutions.

Consider the following IVP

$$\begin{aligned} \frac{dy}{dt} &= f(y) \\ y(t_0) &= y_0 \end{aligned} \tag{3.8.1}$$

where $t \in [t_0, T]$ (T is allowed to be ∞) and also assume that (3.8.1) has a unique solution. At the mesh points $t_k := t_0 + hk$ ($h > 0$), we will consider the following explicit model for (3.8.1)

$$y_{k+1} = F(h, y_k) \tag{3.8.2}$$

where $y_k \approx y(t_k)$. Now supppose that the solution of (3.8.1) $y(t)$ satisfies any property $\wp$(monotonicity, positivity etc). If the discerete scheme (3.8.2) also satisfies $\wp$ for all step sizes $h > 0$ then we will say that the solutions $\{y_k\}$ of (3.8.2) is stable with respect to $\wp$. Assume that the function $F(h, y)$ has continuous first partial derivatives w.r.t h and y. Also assume $F(h, y)$ satisfy

$$\frac{\partial F}{\partial h}(0; y) = f(y) \text{ and } F(0; y) = y \tag{3.8.3}$$

Definition 2. Let $\mathcal{F}(U)$ be a set of functions such that

$$\mathcal{F}(U) := \{f \mid f : U \subseteq \mathbb{R} \to \mathbb{R}\}$$

and let $f, g \in \mathcal{F}(U)$. We say that the set $\mathcal{F}(U)$ is monotonically depend on the initial value at $t_0 \in U$, if

$$f(t_0) \leq g(t_0) \text{ implies } f(t) \leq g(t),\ \forall t \in U$$

If we consider the solution set of the equation (3.8.1), then we know that it is monotonically depend on the initial value at t_0, because of the assumption of uniqueness. Now set $U := \{t_k : k = 0, 1, ..\}$ i.e. $\mathcal{F}(U) := \{y_k \mid y_k : U \subseteq \mathbb{R} \to \mathbb{R}\}$.

Theorem 1. [32] The scheme (3.8.2) is stable w.r.t monotone dependence on initial value if

$$\frac{\partial F}{\partial y}(h; y) \geq 0, y \in \mathbb{R}, h > 0 \tag{3.8.4}$$

Proof . Let $y_k,\ z_k \in \mathcal{F}(\mho)$. $\frac{\partial F}{\partial y}(h; y) \geq 0$ means that the function $F(h, y)$ is increasing in y. If $y_0 \leq z_0$ then $F(h; y_0) \leq F(h; z_0)$ this gives $y_1 \leq z_1$ implies that $F(h; y_1) \leq F(h; z_1)$ then $y_2 \leq z_2$. If we continue in this manner we obtain $y_k \leq z_k$. This completes the proof.

Definition 3. [32] The scheme (3.8.2) is stable w.r.t monotonicity of solutions if the solution (y_k) for all $y_0 \in \mathbb{R}$ is increasing or decreasing whereever the solution $y(t)$ of (3.8.1) is incresing or decreasing.

Theorem 2. [32] Suppose that the scheme (3.8.2) is stable w.r.t monotone dependence on initial value and suppose that the following equations

$$y = F(h; y) \text{ and } f(y) = 0 \tag{3.8.6}$$

have the same roots in y according to their multiplicity.Then the scheme (3.8.2) is stable w.r.t monotonicity of solutions.

Proof . Let $y_0 \in \mathbb{R}$ be an initial condition for (3.8.1) .If y_0 is a fixed point then the unique solution of (3.8.1) becomes the constant function $y(t) = y_0$. By the assumtion of the theorem, y_0 is also a fixed point of the scheme (3.8.2), that is, y_0 satisfies

$$F(h, y_0) = y_0$$

This yields that $y_1 = F(h, y_0) = y_0$. If we continue in this way, we obtain

$$y_k = y_0, \ k = 1.2...$$

This shows that the monotonicity is preserved in discrete scheme (3.8.2). For that reason, assume that $f(y_0) > 0$. Now let $\widetilde{y}$ be the smallest fixed point greater than y_0.Then we know that the solution $y(t)$ of (3.8.1) for the initial value $y(t_0) = y_0$ is increasing on $[t_0, \infty]$ such that $y(t) \in [y_0, \widetilde{y}]$.(If there is no fixed point greater than y_0, we can set $\widetilde{y} = \infty$ then $f(y) > 0$ means that the solution $y(t)$ is increasing over the interval $[t_0, \infty]$ and $y(t) \in [y_0, \infty]$). Our aim is to prove the solution (y_k) of (3.8.2) is increasing sequence. First of all we prove the following claim

$$F(h, y) > y \text{ for } h > 0, \ y \in [y_0, \widetilde{y}] \tag{3.8.7}$$

Assume the contrary, i.e, $\exists \overline{h} > 0$ and $\exists \overline{y} \in [y_0, \widetilde{y})$ such that

$$F(\overline{h}, \overline{y}) < \overline{y} \tag{3.8.8}$$

From (3.8.3) , $\dfrac{\partial F}{\partial h}(0;\overline{y}) = f(\overline{y}) > 0$ and $F(0;\overline{y}) = \overline{y}$. Since $F(0;y)$ is increasing in $y \in [y_0, \overline{y})$ then this means that for very small h

$$F(h, \overline{y}) > \overline{y} \tag{3.8.9}$$

(3.8.8) and (3.8.9) implies that $\exists \hat{h} \in (0, \overline{h})$ such that

$$F(\hat{h}, \overline{y}) = \overline{y}$$

means that $\overline{y} \in [y_0, \widetilde{y})$ is a fixed point for the scheme (3.8.2). This contradicts our assumption since the scheme (3.8.2) has no fixed point on $[y_0, \widetilde{y})$.That is, the claim (3.8.7) is true. Now consider the solution (y_k) of the scheme (3.8.2). We know that $y_k = \widetilde{y}$ is also a solution for the scheme (3.8.2) such that $y_0 < \widetilde{y}$ since we accept that (y_k) is stable w.r.t monotone dependence on initial values then $y_k \leq \widetilde{y}, k = 1, 2...$
(3.8.2-3.8.7) implies that

$$\widetilde{y} > y_k = F(h, y_{k-1}) > y_{k-1}$$

From here it can be shown by means of induction that

$$y_0 < y_1 < ... < y_{k-1} < y_k < ... < \widetilde{y}$$

means that (y_k) is an increasing sequence. The other parts of the proof can be made with a similar way. That is, (y_k) will be a decreasing sequence if it is initiated at y_0 as to be $f(y_0) < 0$. This completes the proof.

In Chapter 2, we have examined the behaviour of the solutions of (3.8.1) about the single-real distinct fixed points $\widetilde{y}$, i.e,

$$f(\widetilde{y}) = 0 \text{ and } f'(\widetilde{y}) \neq 0 \tag{3.8.10}$$

These points are sometimes called hyperbolic fixed points. We will also use this notion for this rewiew. We obtained that behaviour of the solutions of (3.8.1) with initial value near the fixed point $\widetilde{y}$ can be characterized by the behaviour of the solution of the following equation

$$\frac{d\epsilon}{dt} = R\epsilon \tag{3.8.11}$$

where

$$R = \left.\frac{df}{dy}\right|_{y=\widetilde{y}} \quad \text{and } |\epsilon(t)| << |\widetilde{y}| \tag{3.8.12}$$

for the hyperbolic fixed point $\widetilde{y}$. We called $\widetilde{y}$ as linearly stable fixed point if $\epsilon(t) \to 0$ as $t \to \infty$. In fact this condition is satisfied if

$$R = \left.\frac{df}{dy}\right|_{y=\widetilde{y}} < 0 \tag{3.8.13}$$

since the general solution of (3.8.11) can be written as

$$\epsilon(t) = \epsilon_0 e^{Rt} \tag{3.8.14}$$

Otherwise we called the fixed point $\widetilde{y}$ as linearly unstable. Now we can interpret the equation (3.8.11) in discrete form. When we consider the first order difference equation (3.8.2), its fixed point will be numbers $\widetilde{y}$ satisfying the following equation

$$\widetilde{y} = F(h; \widetilde{y})$$

Let us consider the perturbed solution

$$y_k = \widetilde{y} + \epsilon_k \tag{3.8.15}$$

where $|\epsilon_k| << |\widetilde{y}|$. Direct subsitution of (3.8.15) into (3.8.2) yields the result

$$\widetilde{y} + \epsilon_{k+1} = F(h; \widetilde{y} + \epsilon_k) \tag{3.8.16}$$

Using the linerization of the right hand side of (3.8.16) about the fixed point $\widetilde{y}$, we can obtain

$$F(h; \widetilde{y} + \epsilon_k) \approx F(h; \widetilde{y}) + \epsilon_k \frac{\partial F}{\partial y}(h; \widetilde{y}) = \widetilde{y} + \epsilon_k \frac{\partial F}{\partial y}(h; \widetilde{y}) \tag{3.8.17}$$

Equation (3.8.16) and (3.8.17) imply that

$$\epsilon_{k+1} = \epsilon_k \frac{\partial F}{\partial y}(h; \widetilde{y}) \tag{3.8.18}$$

which is a first order difference equation whose general solution can be written as

$$\epsilon_k = \epsilon_0 (R_h)^k \tag{3.8.19}$$

where

$$R_h = \frac{\partial F}{\partial y}(h; \widetilde{y}) \tag{3.8.20}$$

Thus, we will call $\widetilde{y}$ as a linearly stable fixed point of the discrete scheme (3.8.2) if $\epsilon_k \to 0$ as $k \to \infty$ which is satisfied if

$$|R_h| = |\frac{\partial F}{\partial y}(h; \widetilde{y})| < 1 \tag{3.8.21}$$

Otherwise $\widetilde{y}$ is called a linearly unstable fixed point.

Definiton 4. [32] Let $\widetilde{y}$ be an hyperbolic fixed point of (3.8.1) and also be a solution of the difference equation (3.8.2). If linear stability properties of all $\widetilde{y}$'s are same for both (3.8.1) and the scheme (3.8.2) for all step size $h > 0$, then the scheme (3.8.2) is called elementary stable.

Theorem 3. [32] The scheme (3.8.2) is elemenary stable under the assumptions of the Theorem 2.

Proof . First we can say that condition (3.8.6) implies that the equation (3.8.1) and the scheme (3.8.2) has the same fixed points for all $h > 0$. Let $\widetilde{y}$ be a linearly stable fixed point of (3.8.1). From (3.8.13), $\frac{df}{dy}\Big|_{y=\widetilde{y}} = f_y(\widetilde{y}) < 0$. Also $F(h; \widetilde{y}) = \widetilde{y}$ since $\widetilde{y}$ is a fixed point of (3.8.2). Our aim is to show

$$0 \leq \frac{\partial F}{\partial y}(h; \widetilde{y}) < 1, \text{ for all } h > 0$$

which will mean that $\widetilde{y}$ is a linearly stable fixed point of (3.8.2). Since $f_y(\widetilde{y}) < 0$

then for small $\Delta y > 0$

$$\begin{cases} f_y(\widetilde{y}) = \lim_{\Delta y \to 0} \frac{f(\widetilde{y})-f(\widetilde{y}-\Delta y)}{\Delta y} < 0 \text{ implies } f(\widetilde{y}-\Delta y) > f(\widetilde{y}) = 0 \\ f_y(\widetilde{y}) = \lim_{\Delta y \to 0}(\frac{f(\widetilde{y}+\Delta y)-f(\widetilde{y})}{\Delta y}) < 0 \text{ implies } f(\widetilde{y}+\Delta y) < f(\widetilde{y}) = 0 \end{cases} \tag{3.8.23}$$

(3.8.23) means that neither $\widetilde{y} - \Delta y$ nor $\widetilde{y} + \Delta y$ are fixed points for (3.8.1). For that reason they are also not fixed points for the scheme (3.8.2). From (3.8.7) we can write

$$F(h; \widetilde{y} - \Delta y) > \widetilde{y} - \Delta y \tag{3.8.24}$$

In addition, it can be shown that

$$F(h; \widetilde{y} + \Delta y) < \widetilde{y} + \Delta y \tag{3.8.25}$$

(3.8.24) and (3.8.25) implies that

$$\frac{F(h; \widetilde{y} + \Delta y) - F(h; \widetilde{y} - \Delta y)}{2\Delta y} < 1 \tag{3.8.26}$$

then as $\Delta y \to 0$, we can obtain

$$\frac{\partial F}{\partial y}(h; \widetilde{y}) < 1 \tag{3.8.27}$$

which means that $\widetilde{y}$ is a linearly stable fixed point for the scheme (3.8.2). It can be shown that if $\widetilde{y}$ is a linearly unstable fixed point for (3.8.1) then $\widetilde{y}$ is also a linearly unstable fixed point for the scheme (3.8.2). In that case we use $\dfrac{\partial F}{\partial y}(h; \widetilde{y}) > 1$. This completes the proof .

Theorem 3 enable us to make a direct investigation under which conditions a corresponding discrete scheme is elementary stable for (3.8.1). However there exist some difficulties in application of this theorem. For instance, any discrete scheme must enable us to express the term y_{k+1} in terms of y_k. In other words, representation of nonlinear terms in the equation (3.8.1) must be done in such a way that y_{k+1} can be written explicitly in terms of y_k. It is not easy to obtain such a representation for higher order nonlinear term since there can be found lots of ways to model nonlinear terms. However by using this theorem we can choose an appropriate form among the candidates for discrete modelling of (3.8.1). To do this, it is useful to use additional parameters in modelling of nonlinear terms. After finding a discrete scheme for (3.8.1) in a form

$$y_{k+1} = F(h; y_k)$$

we will try to put some restrictions on free parameters so that we can obtain an elementary stable discrete scheme. We provide an example from to clarify this point [45].

Consider the following first-order differential equation which has three fixed-points

$$\frac{dy}{dt} = f(y) = y(y-a)(1-y) = y^3 - (1+a)y^2 - ay, \ 0 \le a \le 1, \ y \ge 0 \quad (3.8.28)$$

The following discrete scheme is proposed in [45] for (3.8.28)

$$\tfrac{y_{k+1}-y_k}{\varphi(h)} = y_k^2[(\theta_2+1)y_k - \theta_2 y_{k+1}] - (1+a)y_k[(\theta_1+1)y_{k+1} - \theta_1 y_k] + a[(\theta_0+1)y_k - \theta_0 y_{k+1}] \quad (3.8.29)$$

where $\theta_0, \theta_1, \theta_2$ are positive parameters and $\varphi(h) = h + O(h^2)$. The equation (3.8.29) can be expressed in a form like (3.8.2):

$$y_{k+1} = F(h; y_k) = y_k + \frac{\varphi(h) y_k (y_k - a)(y_k - 1)}{1 + \varphi(h)(\theta_2 y_k^2 + (1+\theta_1)(1+a)y_k + a\theta_0)} \quad (3.8.30)$$

Then

$$F(h; y) = y + \frac{\varphi(h) f(y)}{1 + \varphi(h) g(y)} \quad (3.8.31)$$

where $f(y) = y(y-a)(y-1)$ and $g(y) = \theta_2 y^2 + (1+\theta_1)(1+a)y + a\theta_0$. Then

$$\frac{\partial F}{\partial y} = \frac{(1+\varphi g)^2 + \varphi f' + \varphi^2(f'g - fg')}{(1+\varphi g)^2} = \frac{b_4 y^4 + b_3 y^3 + b_2 y^2 + b_1 y + b_0}{(1+\varphi g)^2} \quad (3.8.32)$$

where b_i's are constant numbers such that

$$\begin{cases} b_0 = 1 + a\varphi + a\varphi\theta_0 \\ b_1 = 2\varphi\theta_1(1+a)(1+a\varphi\theta_0) \\ b_2 = [a(3\theta_0 + 2\theta_0\theta_2 + 2\theta_1 + 2\theta_1^2 - \theta_2) + \theta_1(1+a^2)(1+\theta_1)]\varphi^2 + (\theta_2+2)\varphi \\ b_3 = 2(1+\theta_2)(1+\theta_1)(1+a)\varphi^2 \\ b_4 = \theta_2(1+\theta_2)\varphi^2 \end{cases} \tag{3.8.33}$$

We will try to justify the conditon (3.8.4) and (3.8.6). Note that, the denominator of (3.8.32) is positive. Numerator of (3.8.32) will be positive if

$$b_2 \geq 0$$

since other coefficents are positive. If $b_2 \geq 0$ is positive then

$$[a(3\theta_0 + 2\theta_0\theta_2 + 2\theta_1 + 2\theta_1^2 - \theta_2) + \theta_1(1+a^2)(1+\theta_1)]\varphi^2 + (\theta_2+2)\varphi \geq 0$$

which is satisfied when the first term

$$3\theta_0 + 2\theta_0\theta_2 + 2\theta_1 + 2\theta_1^2 - \theta_2 \geq 0 \tag{3.8.34}$$

is positive. Under the condition that (3.8.34) is satisfied, the scheme (3.8.29) will be monotonically depend on initial value. On the other hand, for the scheme (3.8.29)

$$F(h;y) = y \Leftrightarrow f(y) = 0 \tag{3.8.35}$$

since

$$y = F(h;y) = y + \frac{\varphi(h)f(y)}{1+\varphi(h)g(y)} \text{ which gives}$$

$$\frac{\varphi(h)f(y)}{1+\varphi(h)g(y)} = 0 \Leftrightarrow f(y) = 0, \text{ for all } h > 0$$

this means that the equation (3.8.28) and the scheme (3.8.29) have the same fixed points. Therefore the scheme satisfies the conditios of Theorem 3. Hence if we choose appropriate parameters satisfying (3.8.34), the scheme (3.8.29) will be elementary stable and stable w.r.t monotonicity of solutions according to Theorem 2 and 3. For instance if $\theta_0 = \theta_1 = \theta_2 = 0$ then the NSFD scheme (3.8.29) will be in a form

$$\frac{y_{k+1} - y_k}{\varphi(h)} = y_k^3 - (1+a)y_k y_{k+1} + a y_k \tag{3.8.36}$$

where $\varphi(h) = h + O(h^2)$. Some other applications of this theorem can be found in [45,46,47].

We see that theorem 3 enables us to check whether the proposed scheme preserves some properties (such as monotonicity, stability of fixed points) of the corresponding autonomous differential equation in the form (3.8.1).

Another discrete scheme can be proposed for the autonomous differential equation which has three distinct fixed points. Consider the following differential equation

$$\frac{dy}{dt} = f(y) = y(1-y)(y-p) = -y^3 + y^2(1+p) - yp \tag{3.8.37}$$

where p is a paramater such that $p > 1$. Equation (3.8.37) has three distinct fixed points whose linear stability properties are given as follows:

$$\begin{aligned} y &= 0 \text{ and } y = p \text{ are linearly stable fixed points,} \\ y &= 1 \text{ is linearly unstable fixed point.} \end{aligned}$$

These properties can also be seen in the Figure 3.21.

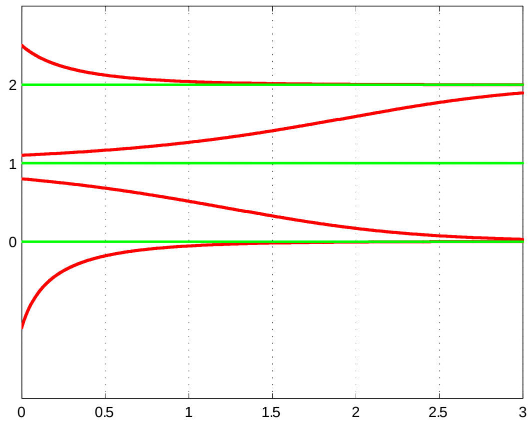

Figure 3.21 Typical trajectories for (3.8.37)

We will propose a nonstandard scheme for the equation (3.8.37). We use the following nonlocal representations for non-derivative terms in (3.8.37) [32].

$$\begin{aligned} y^3 &\rightarrow ay_k^3 + (1-a)y_k^2 y_{k+1} \\ y^2 &\rightarrow by_k^2 + (1-b)y_k y_{k+1} \end{aligned} \tag{3.8.38}$$

where a and b are arbitrary parameters that we specify their values according to the condition (3.8.4). By using these representations the following discrete scheme is proposed for (3.8.37)

$$\frac{y_{k+1} - y_k}{\varphi(h)} = -ay_k^3 - (1-a)y_k^2 y_{k+1} + (by_k^2 + (1-b)y_k y_{k+1})(1+p) - py_k \tag{3.8.39}$$

where $\varphi(h) = h + O(h^2)$. We can explicitly express (3.8.39) as follows

$$y_{k+1} = F(h, y_k) = \frac{y_k(a\varphi y_k^2 - bp\varphi y_k - b\varphi y_k + p\varphi - 1)}{a\varphi y_k^2 - bp\varphi y_k - b\varphi y_k + p\varphi y_k - \varphi y_k^2 + \varphi y_k - 1} \tag{3.8.40}$$

If we consider the right hand side of the equation as a function of y, then (3.8.40) can be written as

$$F(h,y) = \frac{y(a\varphi y^2 - bp\varphi y - b\varphi y + p\varphi - 1)}{a\varphi y^2 - bp\varphi y - b\varphi y + p\varphi y - \varphi y^2 + \varphi y - 1} \tag{3.8.41}$$

We can make the following observation about this equation

$$\begin{aligned} F(h,y) - y &= \frac{\varphi y(1-y)(y-p)}{y^2\varphi - y\varphi + by\varphi - py\varphi - ay^2\varphi + bpy\varphi + 1} \\ &= \frac{\varphi f(y)}{y^2\varphi - y\varphi + by\varphi - py\varphi - ay^2\varphi + bpy\varphi + 1} \end{aligned}$$

implies that

$$F(h,y) - y = 0 \Leftrightarrow f(y) = 0$$

This means that the scheme (3.8.40) and the equation (3.8.37) share the same fixed points. We will find appropriate values for the parameters a and b such that

$$\frac{\partial F}{\partial y}(h,y) \geq 0 \tag{3.8.43}$$

for all $y \in \mathbb{R}$ and $h > 0$. When we consider the denominator portion of (3.8.43), it will be positive. Then for positivity of the numerator of (3.8.43), the following condition should be satisfied

$$(a(a-1)y^2 + (-2abp + 2ap + 2a)y + b^2p^2 + 2b^2p - bp^2 - ap + b^2 - 2bp - b + p)\varphi^2y^2 + ((-2a-1)y^2 + 2b(p+1))y - p)\varphi \geq 0$$

If we impose the positive solutions of the equation (3.8.37), we obtain the following restrictions for the parameters a and b which enable us to justify the condition (3.8.43)

$$\begin{cases} -2 \leq a \leq -0.5 \\ \\ b \geq \dfrac{a+p}{p+1} \end{cases} \tag{3.8.44}$$

If the parameters a and b are chosen in this way then the scheme (3.8.39) reflects correct qualitative behaviour with the differential equation of the form (3.8.37). We can make some comparison between the standard and nonstandard finite difference schemes of the equation (3.8.37). Let us impose the following conditions

$$\begin{aligned} y(0) &= 1.1 \\ p &= 2 \\ a &= -0.5, -1.5 \\ b &= 0.6,\ 3 \\ \varphi(h) &= 1 - e^{-h} \end{aligned} \tag{3.8.45}$$

The following SFD scheme can be proposed for (3.8.37)

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_k)(y_k - p) \tag{3.8.46}$$

Now let's compare the two NSFD schemes obtained by using the specified values in (3.8.45) and the SFD scheme (3.8.46) for the equation (3.8.37) using the initial condition $y(0) = 1.1$ with two different step sizes $h = 0.5$, 1.5. From the pictures, we see that NSFD schemes gives qualitatively correct behaviour for all step sizes while SFD schemes work only for small step size $h = 0.5$. The same results are obtained for other values of p and initial conditions.

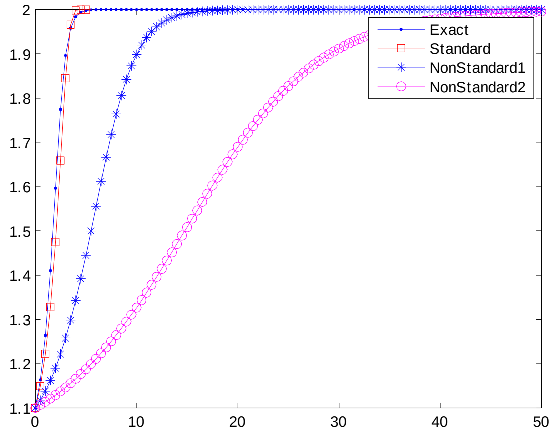

Figure 3.22 Comparison of the schemes (3.8.39) and (3.8.45) for $h = 0.5$

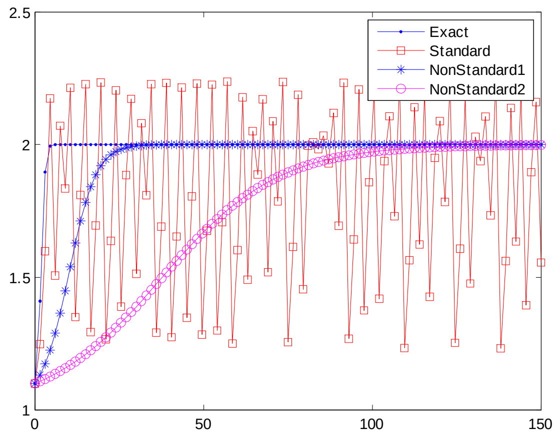

Figure 3.23 Comparison of the schemes (3.8.39) and (3.8.45) for $h = 1.5$

3.9 A NSFD Scheme for Lotka-Volterra System

In this section we will consider a nonstandard discretization of Lotka-Volterra system, proposed by Mickens in [50]. Lotka-Volterra system is the following non-linear systems of differential equation

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned} \tag{3.9.1}$$

where (a, b, c, d) are positive parameters. This system is mostly used in biological phenomena especially for modelling the interactions between two species living in same habitat. An exact solution of Lotka-Volterra system is given in [52] in quadrature form. That form can be used to identify the qualitative properties of the system. When considered the positive initial conditons, all the solutions of the system (3.9.1) are periodic except the fixed point solution $(x^*, y^*) = (c/d, a/b)$ [51]. It would be useful to note that the standard finite difference modellings of (3.9.1) produce numerical solution that does not reflect the same qualitative behaviours with the system (3.9.1). Hence nonstandard modelling of (3.9.1) has a remarkable importance since it preserves the periodicity of solutions. Now without lose the generality, the following normalized form can be used

$$\begin{aligned} \frac{dx}{dt} &= x - xy \\ \frac{dy}{dt} &= -y + xy \end{aligned} \tag{3.9.2}$$

The following nonstandard scheme is proposed for (3.9.1)

$$\frac{x_{k+1} - x_k}{\phi} = 2x_k - x_{k+1} - x_{k+1}y_k \tag{3.9.3}$$

$$\frac{y_{k+1} - y_k}{\phi} = -y_{k+1} + 2x_{k+1}y_k - x_{k+1}y_{k+1} \tag{3.9.4}$$

where $\phi = h + O(h^2)$. If positive initial conditions are imposed for (3.9.2) such as

$$\begin{aligned} x(0) &= x_0 > 0 \\ y(0) &= y_0 > 0 \end{aligned} \tag{3.9.5}$$

then the scheme (3.9.3-3.9.4) will guarentee the positivity property of the Lotka-Volterra system. We can explicitly express the corresponding nonstandard scheme as follows

$$x_{k+1} = [\frac{1+2\phi}{1+\phi+\phi y_k}]x_k \tag{3.9.6}$$

$$y_{k+1} = [\frac{1+2\phi x_{k+1}}{1+\phi+\phi x_{k+1}}]y_k \tag{3.9.7}$$

Numerical procedure can be proceeded first by selecting the initial values (x_0, y_0). Then we can compute x_1 and lastly use x_1, y_0 in order to compute y_1.Typical solutions of the scheme (3.9.6) and (3.9.7) are periodic and also $x_k y_k$ phase space curves are closed surrounding about the fixed point $(1, 1)$.These results are fullly suited with the behaviour of the solutions of the Lotka-Volterra system. However, when we consider a standard discretization of the system (3.9.2) as

$$\begin{aligned} \frac{x_{k+1} - x_k}{h} &= x_k - x_k y_k \\ \frac{y_{k+1} - y_k}{h} &= -y_k + x_k y_k \end{aligned} \tag{3.9.8}$$

we observe that standard schemes produce numerical results that spiral into or away from the fixed point $(1, 1)$. These cases can be seen in the following figures. It would be useful to emhasize that although we use a small step size, SFD scheme

(3.9.8) shows numerical instabilities in modelling of Lotka-Volterra equation.

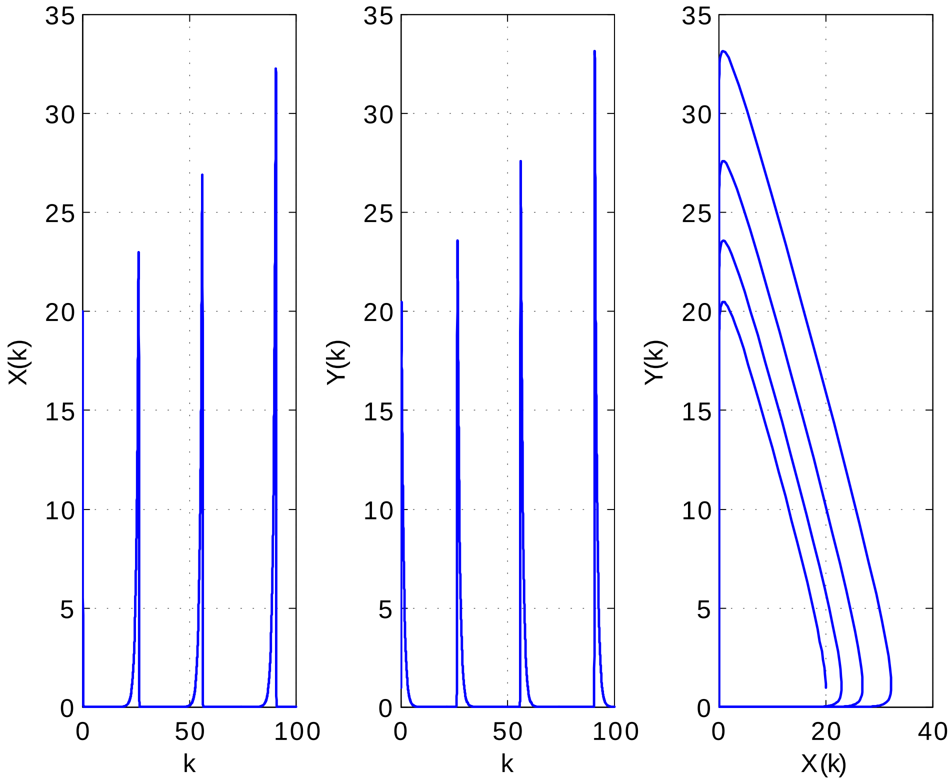

Figure 3.24 Numerical solution of (3.9.8) with $h = 0.01$ $x_0 = 20$ $y_0 = 1$

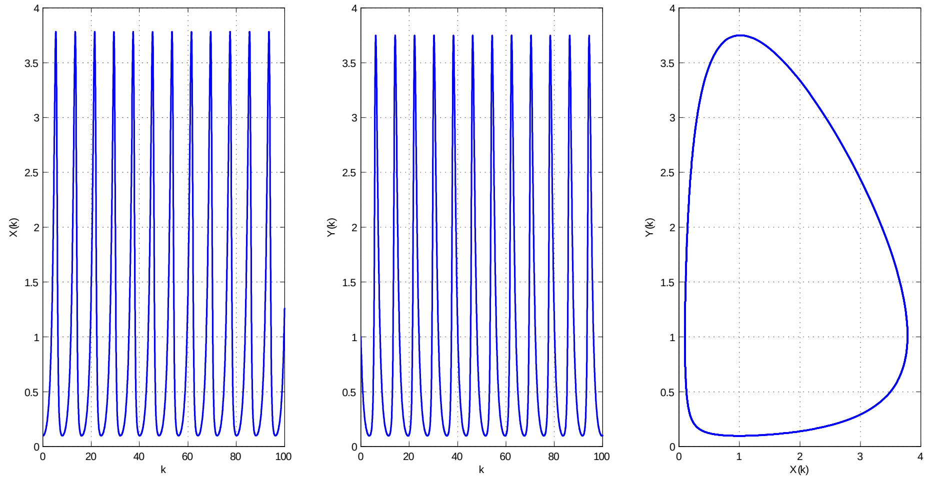

Figure 3.25 Numerical solution of (3.9.6) and (3.9.7) with $h = 0.01$ $x_0 = 20$ $y_0 = 1$

4. PARTIAL DIFFERENTIAL EQUATIONS

Partial Differential Equations (PDE) have broad usage in modelling of biological and psyhical phenomena depends on time and space variables. These equations have some special solutions such as solutions under some conditions. When the exact solution is not avaliable, numerical solutions of PDE is necessary to understand the behaviour of its solutions such as evaluation of solitons. Finite difference method (FDM), finite element method (FEM), discontinous galarkin method (DGM) are some of the known methods for numerical solutions of PDEs. It is significantly difficult to find exact schemes for PDEs, when compared the ordinary differential equation. Unlike the ODEs, we have no theorem to guarantee the existence of exact schemes for partial differental equations. However, there are many studies about exact schemes for PDEs, such as advection-reaction equation in [31], Burgers and Burgers-Fisher equations in [49].

4.1 Standard Difference Models for Partial Differential Equations

In this chapter, we study the construction of NSFD schemes for some type of partial differential equations. We start with the standard finite disceretization of a PDE. The application of Finite-Difference method in two dimension is very similar to one dimensional case. Consider the function

$$u = (x, t)$$

which depends on two independent variables x and t and assume that $u(x, t)$ is defined on a rectangle

$$R = \{(x, t)|a < x < b,\ c < t < d\}$$

with continuous partial derivatives up to an appropriate order. We define two positive integers n and m. Then divide the interval $[a, b]$ into n equal partitions whose width $h = \frac{b-a}{n}$ and the interval $[c, d]$ into m equal partitions whose width

$\Delta t = \frac{c-d}{m}$. Hence we can define the following points on the coordinate axes

$$\begin{aligned} x_i &= a + ih, \; i = 0, 1, ...n-1 \\ t_j &= b + j\Delta t, \; j = 0, 1, ...m-1 \end{aligned} \tag{4.1.1}$$

Here $x = x_i$, $t = t_j$ are called grid lines and their intersection points (x_i, t_j) are called mesh points. Consider the mesh point

$$(x_i, t_j), \; i = 0, 1, ..n-1, \; j = 0, 1, ..m-1 \tag{4.1.2}$$

If we use the Taylor series in x about the point $x = x_i$ and in t about the point $t = t_j$, we can obtain the following finite difference formulas for the first and second partial derivatives of the function $u = u(x, t)$.

$$\frac{\partial u}{\partial x}(x_i, t_j) = \frac{u(x_{i+1}, t_j) - u(x_i, t_j)}{h} - \frac{h}{2}\frac{\partial^2 u}{\partial x^2}(\eta_i, t_j) \tag{4.1.3}$$

for some $\eta_i \in (x_i, x_{i+1})$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\eta_i, t_j) \tag{4.1.4}$$

for some $\eta_i \in (x_i, x_{i+1})$

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\Delta t} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j) \tag{4.1.5}$$

for some $\mu_j \in (t_j, t_{j+1})$

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) = \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{\Delta t^2} - \frac{\Delta t^2}{12}\frac{\partial^4 u}{\partial t^4}(x_i, \mu_j) \tag{4.1.6}$$

for some $\mu_j \in (t_j, t_{j+1})$. If we retain the local trucation errors in expressions above then we can obtain the following abbreviated forms for partial derivatives:

$$\frac{\partial u}{\partial x}(x_i, t_j) \to \frac{u_{i+1}^j - u_i^j}{h} \tag{4.1.7}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \to \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} \tag{4.1.8}$$

$$\frac{\partial u}{\partial t}(x_i, t_j) \to \frac{u_i^{j+1} - u_i^j}{\Delta t} \tag{4.1.9}$$

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) \to \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\Delta t^2} \tag{4.1.10}$$

where u_i^j is an approximation to $u(x_i, t_j)$. Using these formulas, we firstly replace the terms in the given partial differential equation by the discrete terms described in (4.1.7 - 4.1.10). We will obtain a partial difference equation after these operations. Similar to the ordinary differential equation, we have more than one finite-difference representation for any partial differential equation. Here, main concern is to be able to construct the most suitable finite-difference form which enable us to obtain consistent numerical results with the corresponding partial differential equation. However, this process is very sophisticated when compared ordinary differential equations. In this section, we will focus on the nonstandard discretization of Burgers and Burgers-Fisher equations [49].

Consider the Burgers partial differential equation

$$u_t = u_{xx} - u u_x \tag{4.1.11}$$

The solitary wave solution of (4.1.11) is given

$$u(x,t) = \frac{1}{1 + e^{(1/2)(x - t/2)}} \tag{4.1.12}$$

Note that this solution satisfies the condition $0 \leq u(x,t) \leq 1$. A nonstandard finite difference scheme can be constructed based on the exact solitary wave solution (4.2.12). The following discrete derivatives in nonstandard form can be used for this scheme.

$$\begin{aligned} u_t &\to \frac{u_j^{n+1} - u_j^n}{\phi(\Delta t, \lambda)} \\ u_x &\to \frac{u_{j+1}^n - u_j^n}{\psi(h, \chi)} \end{aligned}$$

where $\phi(\Delta t, \lambda) = h + O(h^2)$, $\psi(h, \chi) = h + O(h^2)$ and λ,χ are various parameters than can be appeared in the equation. Use the approximations

$$u_j^n \approx u(x_j, t_n)$$

where $x_j = jh$, $t_n = n\Delta t$. A standard finite diffference scheme for (4.1.11) can be written as follows

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} - u_j^{n+1}\frac{u_j^n - u_{j-1}^n}{h} \tag{4.1.13}$$

By using this discretization, the NSFD scheme

$$\frac{u_j^{n+1} - u_j^n}{\phi} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Psi} - u_j^{n+1}\frac{u_j^n - u_{j-1}^n}{\Gamma} \tag{4.1.14}$$

is proposed in [49] where ϕ, Γ and Ψ are denominator functions (step functions) such that $\Psi = \Gamma^2$. An appropriate form for this step function ϕ can be found based on the selection of Γ. We can explicitly express ϕ as follows

$$\phi = \frac{(u_j^{n+1} - u_j^n)\Psi\Gamma}{(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\Gamma - u_j^{n+1}(u_j^n - u_{j-1}^n)\Psi}$$

Denote $s \to s_j^n = e^{(1/2)(x_j - t_n/2)}$ for simplicity and choose $\Gamma = 2(e^{h/2} - 1)$ so $\Psi = 4(e^{h/2} - 1)^2$ then the following expression is obtained by means of the exact solution (4.1.12).

$$\phi = \frac{4(1 - e^{-\Delta t/4})(1 + e^{h/2}s)(e^{h/2} + s)}{(1 + e^{-\Delta t/4})(s - 1) + 2(1 + e^{h/2}s)}$$

We can observe that $s \to 1$ as $\Delta t \to 0, h \to 0$ implies that $\phi \to 4(1 - e^{-\Delta t/4})$. In this case, $\phi = \Delta t + O(\Delta t^2)$. As a result, the following nonstandard scheme can be obtained

$$\frac{u_j^{n+1} - u_j^n}{\phi} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Psi} - u_j^{n+1}\frac{u_j^n - u_{j-1}^n}{\Gamma} \tag{4.1.15}$$

where

$$
\begin{aligned}
\phi &= 4(1-e^{-\Delta t/4}) \\
\Psi &= 4(e^{h/2}-1)^2 \\
\Gamma &= 2(e^{h/2}-1)
\end{aligned}
$$

Denote $R = \frac{\phi}{\Psi}$ and $r = \frac{\phi}{\Gamma}$ then the nonstandard scheme (4.1.15) can be explicitly explained as

$$u_j^{n+1} = \frac{R(u_{j+1}^n + u_{j-1}^n) + (1-2R)u_i^n}{1 + r(u_j^n - u_{j-1}^n)} \tag{4.1.16}$$

Note that $1 - 2R - r \geq 0$ then the positivity and boundedness properties of the exact solution (4.2.12) are preserved in numerical modelling, i.e, the numerical solutions u_j^n satisfies

$$0 \leq u_j^n \leq 1 \Rightarrow 0 \leq u_j^{n+1} \leq 1 \tag{4.1.17}$$

for all possible n and j. This approach enable us to construct a nonstandard schemes for some partial differential equations.

4.2 NSFD Schemes for Huxley Equation

In this section we will describe a nonstandard scheme for Huxley equation [57]

$$u_t - u_{xx} = \beta u(1-u)(u-\gamma) \tag{4.2.1}$$

where β and γ are constants, $\beta \geq 0$. It is used to understand the how action potential in neurons are initiated and propagated [55]. Solutary wave solution of (4.2.1) is given [56]

$$u(x,t) = \frac{\gamma}{2} + \frac{\gamma}{2}\tanh[\frac{\beta^{\frac{1}{2}}}{8}(x + \frac{(2-\gamma)\sqrt{2}}{2}\beta^{\frac{1}{2}}t)] = \frac{\gamma}{1 + e^{-\sqrt{\frac{\beta}{2}}x - (\frac{2-\gamma}{2})\beta t}} \tag{4.2.2}$$

Typical standard finite difference model for (4.2.1) can be given as

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \beta u_i^j(1-u_i^j)(u_i^j - \gamma) \tag{4.2.3}$$

If we consider the nonstandard discrization rules proposed in [3], we can use the following denominator functions instead of conventional ones.

$$u_t \to \frac{u_i^{j+1} - u_i^j}{\phi(\Delta t, \lambda)}, \ \phi(\Delta t, \lambda) = \Delta t + O(\Delta t^2) \tag{4.2.4}$$

$$u_{xx} \to \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\psi(h, \chi)}, \ \psi(h, \chi) = h^2 + O(h^4) \tag{4.2.5}$$

$$u_x \to \frac{u_{i+1}^j - u_i^j}{\psi(h, \chi)} \text{ or } u_x \to \frac{u_{i+1}^j - u_{i-1}^j}{2\psi(h, \chi)}, \ \psi(h, \chi) = h + O(h^2) \tag{4.2.6}$$

where λ and χ are several parameters appearing in equation. Now retaining the nonlinear term in the equation (4.2.1), we consider the general NSFD scheme for (4.2.1)

$$\frac{u_i^{j+1} - u_i^j}{\phi} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\psi} \tag{4.2.7}$$

where ϕ and ψ are denominator functions. If we solve (4.2.7) for ϕ then we will obtain

$$\phi = \frac{(u_i^{j+1} - u_i^j)\psi}{u_{i+1}^j - 2u_i^j + u_{i-1}^j} \tag{4.2.8}$$

Based on the solutary wave solution (4.2.1) and the approximation $u_i^j \approx u(x_i, t_j)$, we can compute a value for ϕ. For simplicity we define

$$s \to s_i^j = e^{-\sqrt{\frac{\beta}{2}}x_i - (\frac{2-\gamma}{2})\beta t_j} \tag{4.2.9}$$

then by using (4.2.1), we have

$$\phi = \frac{(1 - e^{-(\frac{2-\gamma}{2})\beta\Delta t})\psi(1 + se^{-\sqrt{\frac{\beta}{2}}h})(1 - se^{\sqrt{\frac{\beta}{2}}h})}{(1 - se^{-(\frac{2-\gamma}{2})\beta\Delta t})s(s-1)(1 - e^{-\sqrt{\frac{\beta}{2}}h})^2 e^{\sqrt{\frac{\beta}{2}}h}} \tag{4.2.10}$$

If we choose

$$\psi = \frac{2}{\beta}(e^{\sqrt{\frac{\beta}{2}}h} - 1)^2 \tag{4.2.11}$$

then by using a similar approach with Burgers equation mentioned above, we can obtain the following form for ϕ.

$$\phi = \frac{2}{(2-\gamma)\beta}(1 - e^{-(\frac{2-\gamma}{2})\beta\Delta t}) \tag{4.2.12}$$

Space independent form of the Huxley equation is

$$u_t = \beta u(1-u)(u-\gamma) \tag{4.2.13}$$

Note that it is an ODE having three fixed points. In the previous chapters, we have shown that NSFD schemes give qualitatively correct results comparing with SFD schemes. Here we propose the NSFD

$$\frac{u_i^{j+1} - u_i^j}{\phi} = \beta u_i^j(1-u_i^j)(u_i^j - \gamma) \tag{4.2.15}$$

for (4.2.12) where ϕ is of the form (4.2.10). Combining (4.2.7) and (4.2.15) we propose the following NSFD schemes for (4.3.1)

$$\begin{cases} \frac{u_i^{j+1}-u_i^j}{\phi} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\psi} + \beta u_i^j(1-u_i^j)(u_i^j-\gamma) \\ \phi = \frac{2}{(2-\gamma)\beta}(1 - e^{-(\frac{2-\gamma}{2})\beta\Delta t}) \\ \psi = \frac{2}{\beta}(e^{\sqrt{\frac{\beta}{2}}h} - 1)^2 \end{cases} \tag{4.2.16}$$

To test the performance of the scheme, we use the following initial conditions and set of parameters. Numerical experiments also show that the scheme gives accurate results for the corresponding equation (4.3.16). Initial conditions are the followings

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh(\sigma\gamma x)\ \beta = 1,\ \sigma = \frac{\sqrt{8\beta}}{8},\ \gamma = 0.001$$

where

$$0 \le x \le 1,\ \Delta x = 0.1,\ 0 \le t \le 5,\ \Delta t = 0.005$$

We used the L_∞ error accuracy

$$|Err|_\infty = \max_{0 \le j \le M}\{|u(x_j, t_n) - u_j^n|\}$$

and the absolute error at each mesh point

$$AbsErr = |u(x_j, t_n) - u_j^n| \; j = 1, 2..M, \;\; n = 0, 1, ..N$$

Numerical results is obtained as follows.

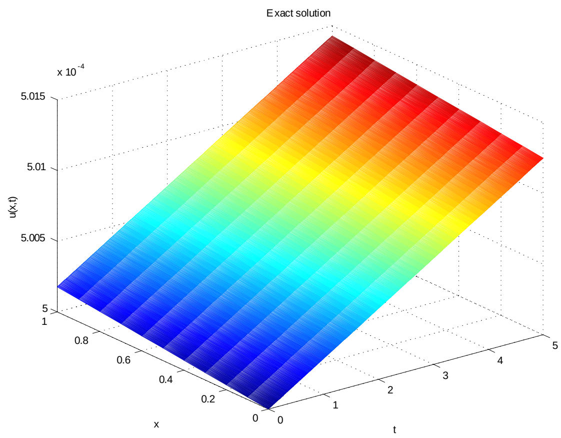

Figure 4.1 Plot of exact solution (4.2.2)

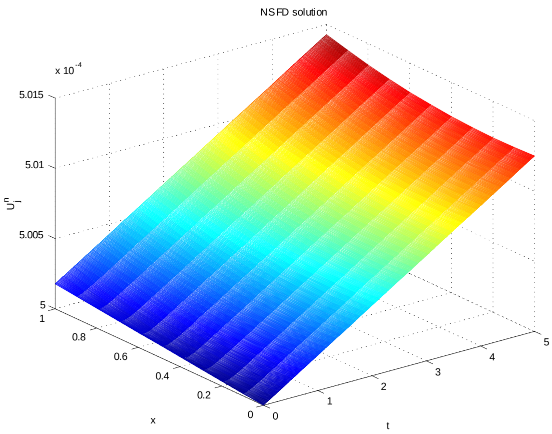

Figure 4.2 Plots of NSFD scheme (4.2.16)

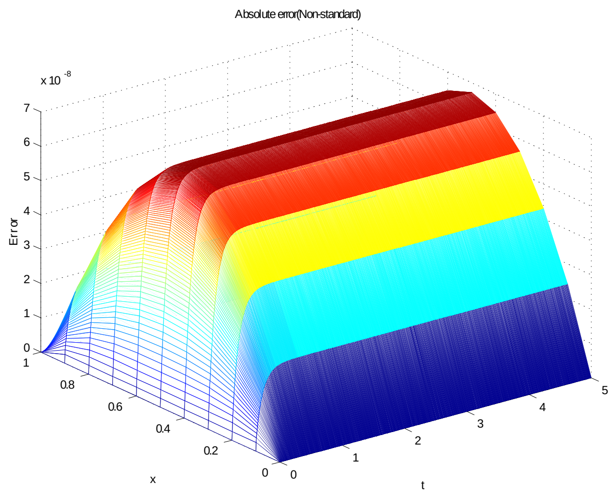

Figure 4.3 Absolute error for NSFD scheme (4.2.16)

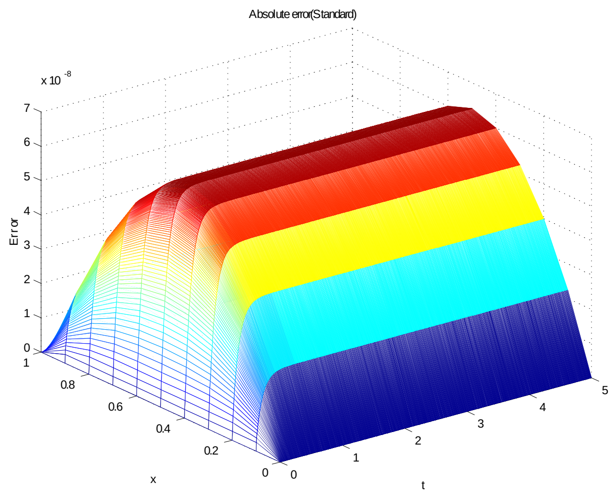

Figure 4.4 Absolute error for SFD scheme (4.2.3)

Analytical and numerical solution of Huxley Equation (4.2.1) with standard scheme (4.2.3) and NSFD scheme (4.2.16) are seen in the Figure 4.1-4.4

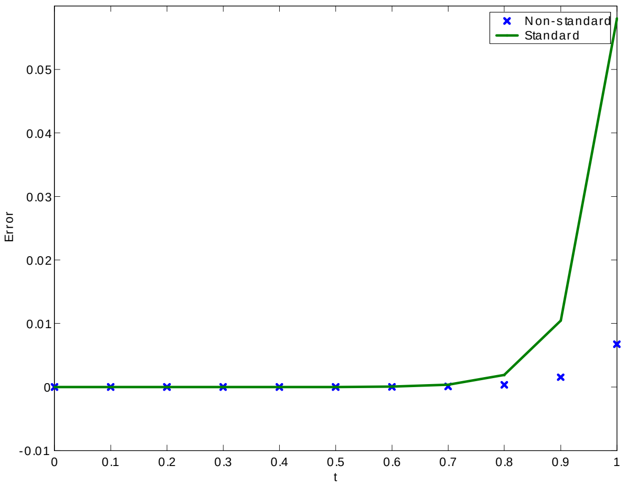

Figure 4.5 L_∞ error for numerical solutions (4.2.3) and (4.2.16)

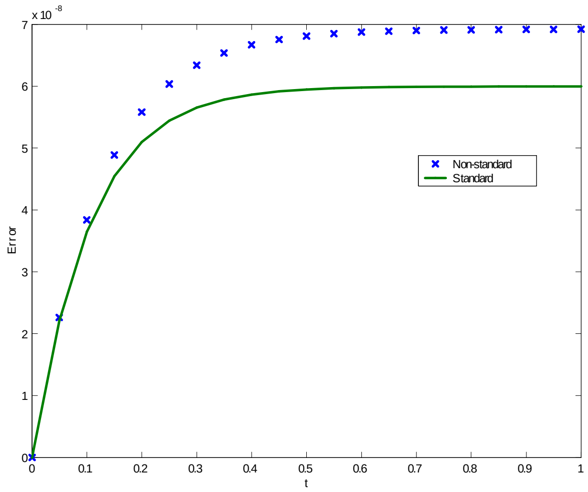

Figure 4.6 L_∞ error for numerical solutions (4.2.3) and (4.2.16)

L_∞ error for numerical solutions of the equation (4.2.1) using standard and NSFD schemes are seen in the Figures 4.5 and 4.6. Numerical experiments also show that the scheme gives accurate results for the NSFD scheme (4.2.16).

CONCLUSION

Throughout this thesis, we considered the numerical solutions of some differential equations by using finite difference methods. First, we observed that if one uses the standard discretizations to obtain numerical solution of any differential equation, more than one model (or difference equation) emerge for the corresponding differential equation. Second observation is that standard finite difference schemes contain a larger parameter set than the corresponding differential equation. For instance, step size h appears as an additional parameter in all SFD schemes. Because of two situations, we encounter some difficulties to construct discrete schemes that reflect correct qualitative behaviour with the original problem. For the discretization of the decay equation, we have seen in the section 2.1 that backward Euler scheme gives reliable numerical for all step sizes while the forward Euler scheme perform well under some restrictions on step size h and central difference scheme shows numerical instabilities for all possible step sizes.

Hence after choosing an appropriate finite difference representation for a differential equation, we have mainly focused on the refinement of step size h by means of various ways. Indeed, it is possible to construct SFD schemes which do not arise numerical instabilities for some differential equation by putting restrictions on the step size h. It is known that the standard finite difference models generally perform well for smaller step size. However lots of time this reducing increases the computational requirements. On the other hand, some mathematical models requires long time computation. These kind of arguments lead us to seek discrete models which gives correct qualitative behaviour with the original problem for all step sizes. For this purpose, the notion of exact finite difference scheme is introduced in the third chapter. Local truncation error of these schemes are zero [sec 3.1] and they give qualitatively correct numerical results regardless of the choice of the step size h. It has been proven that if a first order differential equation has a unique solution, then it has an exact finite difference scheme. The corresponding theorem does not provide any information how to find exact scheme. Hence there exist some difficulties here. Firstly, exact solution of the differential equation should be known to construct an exact finite difference scheme. On the other hand, even if we have the analytical solutions of the corrseponding differential equation, it can be very diffucult to find an appropriate finite difference representation. Nevertheless there are many exact finite difference schemes for both ordinary and partial differential equations in literature. We reviewed some of them in Section 3.1 and 4.2 . Some exact finite difference scheme has been described in the third chapter. For example, logistic differential equation

$$\frac{dy}{dt} = \lambda_1 y - \lambda_2 y^2$$

has the exact finite difference scheme

$$\frac{y_{k+1} - y_k}{\frac{1-e^{\lambda_1 h}}{\lambda_1}} = \lambda_1 y_k - \lambda_2 y_{k+1} y_k$$

One of the standard discretization of logistic differential equation can be written as

$$\frac{y_{k+1} - y_k}{h} = \lambda_1 y_k - \lambda_2 y_k^2$$

Note that the exact scheme has denominator function $\varphi(h) = \dfrac{1 - e^{\lambda_1 h}}{\lambda_1}$ which is remarkably different from the conventional usage $\varphi(h) = h$. On the other hand, nonlinear term y^2 in the equation is modeller nonlocally as to be $y^2 \rightarrow y_{k+1} y_k$ instead of $y^2 \rightarrow y_k^2$ in exact scheme. While the NSFD scheme performs well for all step sizes, SFD gives correct numerical results if the step size h is chosen as to be $0 < h < 1$.

When we examine the general structure of the exact finite difference scheme, we observe that they generally differs from the SFD schemes at two points: first is the denominator function and second is the representation of nonlinear term. Many experiments show that using nonlocal representation and different denominator functions enable to construct discrete schemes which do not show numerical instabilities. Here the main topic of this thesis NSFD schemes draw its inspiration from the exact finite difference scheme. Nonstandard discretization is mainly based on using unconventional denominator function and nonlocal modelling of nonlinear terms. Throughout this thesis, we have provided many examples for which the nonstandard discretization eliminate the numerical instabilities that occur in standard modelling. Even if there is no general prodecure to find a NSFD scheme that gives correct qualitative behaviour with the original problem, there are some powerful tools which enable use to find such schemes. Section 3.4, 3.7, 3.8, 4.3 all provide important strategies to construct NSFD schemes.

In the last chapter, we briefly discussed some applications of NSFD method for partial differential equations. It is obvious that numerical solutions of PDEs by finite difference method require much more effort than the ODEs. Moreover unlike the first order ODEs, we can not guarantee that any PDE has an exact difference scheme. In this thesis, we reviewed NSFD scheme for Burger's PDE and then following the similar way, a NSFD scheme is described for Huxley equation.

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