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Research Article

Analytical Solution for Free Vibration Analysis of Beam on Elastic Foundation with Different Support Conditions

Baki Ozturk[1] and Safa Bozkurt Coskun[2]

[1] *Department of Civil Engineering, Faculty of Engineering, Hacettepe University, 06800 Ankara, Turkey*
[2] *Department of Civil Engineering, Faculty of Engineering, Kocaeli University, 41380 Kocaeli, Turkey*

Correspondence should be addressed to Baki Ozturk; bakiozturk1@yahoo.com

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Analytical solutions for free vibration analyses of a beam on elastic foundation are obtained for different support conditions. The analytical solutions are applied on three different axially loaded cases, namely; (1) one end clamped, the other end simply supported; (2) both ends clamped, and (3) both ends simply supported cases. Analytical solutions and frequency factors are evaluated for different ratios of axial load, N, acting on the beam to Euler buckling load, N_r. The analytical solutions give results which are in excellent agreement with the variational iteration method (VIM) and homotopy perturbation method (HPM) results available in the literature for the particular problem considering all the cases provided in this study and the differential transform method (DTM) results available in the literature for the clamped-pinned case.

1. Introduction

The free vibration equation of an axially loaded beam on elastic foundation is a fourth-order partial differential equation. For this particular engineering problem, the analytical solutions will be implemented in this study. Although the governing equation seems to be a linear one, finding the eigenvalues for the free vibration analysis is still challenging. One may not simply obtain the eigenvalues sequentially and their corresponding eigen vectors even with a software.

Free vibration equation of the beam on partially elastic foundation including only bending moment effect was analytically solved [1] while the eigenvalues for free vibration of beam-column systems on elastic foundation were obtained using a numerical approach [2]. The separation of variables technique was used to obtain the free vibration circular frequencies of piles partially embedded in soils [3]. In addition, differential transform method (DTM) has been proposed to solve eigenvalue problems for free and transverse vibration problems of a rotating twisted Timoshenko beam under axial loading [4, 5]. Furthermore, the DTM was also used to find the nondimensional natural frequencies of tapered cantilever Bernoulli-Euler beam [6]. Free vibration equations for one end clamped and other end simply supported beam on elastic foundation were solved by using the DTM for various axial loads acting on the beam [7].

Meanwhile, both the variational iteration method (VIM) and homotopy perturbation method (HPM) were used to solve the free vibration equations of beam on elastic foundation for support conditions of one end clamped, and other end simply supported, both ends clamped and both ends simply supported considering various case studies [8–11]. The beam on elastic foundation was investigated for these three different support conditions considering various N_r values.

Recently, there have been also other studies which are helpful to better understand dynamic behavior of both infinite beams resting on elastic foundation [12] and tapered column with pinned ends embedded in Winkler-Pasternak elastic foundation [13]. In this study, the analytical solutions and analytical results for free vibration analysis of beam on elastic foundation are provided. The analytical results are in excellent agreement with the results of the particular problem solved using the VIM method and the HPM method available in the literature [8–10].

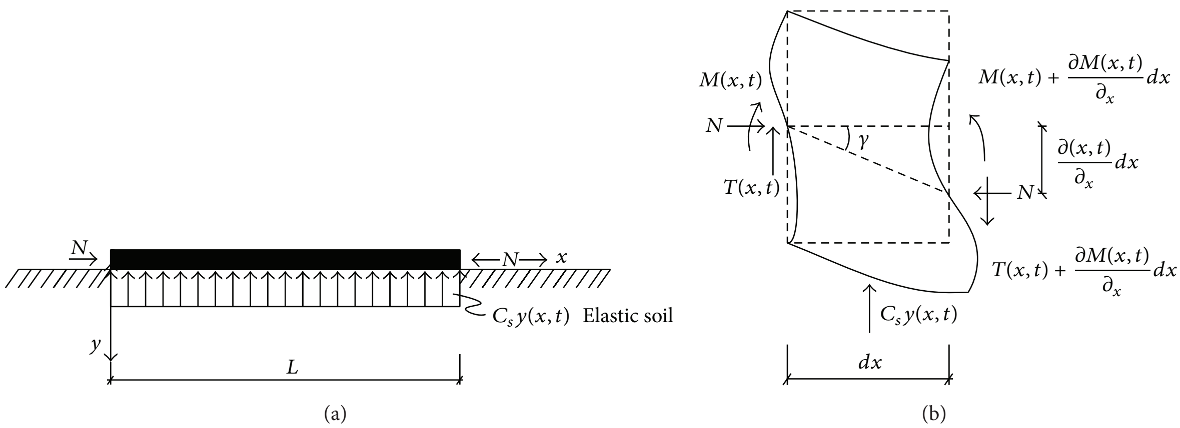

Figure 1: (a) beam on elastic foundation and (b) internal forces and deformations of the beam on elastic foundation.

2. Problem Formulation

Beam on elastic foundation and internal forces and deformations of differential segment of the beam having a length of dx are depicted in Figures 1(a) and 1(b), respectively. The elastic foundation is idealized by Winkler model. Hence, the relationship between displacement function $y(x,t)$ of the beam on elastic foundation and the distributed force $q(x,t)$ applied on the elastic foundation beneath the beam can be written by $q(x,t) = C_s y(x,t)$. In this equation $C_s = C_o b$, where C_o is the modulus of subgrade reaction while b is the beam width.

The equilibrium equations of the internal forces acting on differential beam segment, dimensionless parameter z instead of position variable x with $0 \leq z \leq x/L$ and neglecting the second-order terms are used in order to define the motion equation of the beam on elastic foundation [3]:

$$\left\{\phi_1^{iv}(z) + \left[\pi^2 N_r + \frac{\left(\overline{m}\omega^2 - C_S\right)\overline{k}L^2}{AG}\right]\phi_1^{ii}(z) + \frac{\left(C_S - \overline{m}\omega^2\right)L^4}{EI}\phi_1(z)\right\}\sin(\omega t + \theta) = 0. \tag{1}$$

In (1), $\phi_1(z)$ is dimensionless displacement function of the beam considering axial and shear forces; t is the time variable; θ is the phase angle; $N_r = NL^2/(\pi^2 EI)$ is the ratio of axial load N acting on the beam to Euler buckling load; $\overline{m}$ is the distributed mass of the beam; ω is the beam circular frequency; $\overline{k}$ is the shape factor for the shape of the beam section considered; L is the beam length; A, G, E, and I are cross-section area, shear modulus, elastic modulus, and moment of inertia of the beam; and $\phi_1^{ii}(z) = d^2\phi_1(z)/dt^2$, $\phi_1^{iv}(z) = d^4\phi_1(z)/dt^4$, respectively. If the axial and shear force effects are neglected, the dimensionless equation of motion for the beam on elastic foundation becomes as follows [14]:

$$\left\{\phi_2^{iv}(z) + \frac{\left(C_S - \overline{m}\omega^2\right)L^4}{EI}\phi_2(z)\right\}\sin(\omega t + \theta) = 0. \tag{2}$$

In (2), $\phi_2(z)$ is the dimensionless displacement function of the beam neglecting axial and shear forces. Division of both sides of (1) and (2) by $\sin(\omega t+\theta)$ gives the following equations:

$$\left\{\phi_1^{iv}(z) + \left[\pi^2 N_r + \frac{\left(\overline{m}\omega^2 - C_S\right)\overline{k}L^2}{AG}\right]\phi_1^{ii}(z) + \frac{\left(C_S - \overline{m}\omega^2\right)L^4}{EI}\phi_1(z)\right\} = 0, \tag{3}$$

$$\left\{\phi_2^{iv}(z) + \frac{\left(C_S - \overline{m}\omega^2\right)L^4}{EI}\phi_2(z)\right\} = 0. \tag{4}$$

3. Analytical Solutions of the Problem

The equation of motion in (3) can be rearranged as follows:

$$\phi_1^{iv}(z) + \left[\pi^2 N_r - \frac{EI\overline{k}}{AGL^2}\left(\lambda - \gamma^4\right)\right]\phi_1''(z) + \left(\lambda - \gamma^4\right)\phi_1(z) = 0, \tag{5}$$

where λ is relative stiffness, $\lambda = C_S L^4/EI$, and γ is frequency factor, $\gamma = \sqrt[4]{\overline{m}\omega^2 L^4/EI}$.

In order to simplify the symbolic representation of solution process, (5) is rewritten as

$$\phi_1^{iv}(z) + \xi_1\phi_1''(z) + \xi_2\phi_1(z) = 0, \tag{6}$$

where

$$\xi_1 = \pi^2 N_r - \frac{EI\overline{k}}{AGL^2}\left(\lambda - \gamma^4\right), \quad \xi_2 = \lambda - \gamma^4. \tag{7}$$

Assuming a solution of the form of e^{rz}, characteristic equation for (6) becomes

$$r^4 + \xi_1 r^2 + \xi_2 = 0. \tag{8}$$

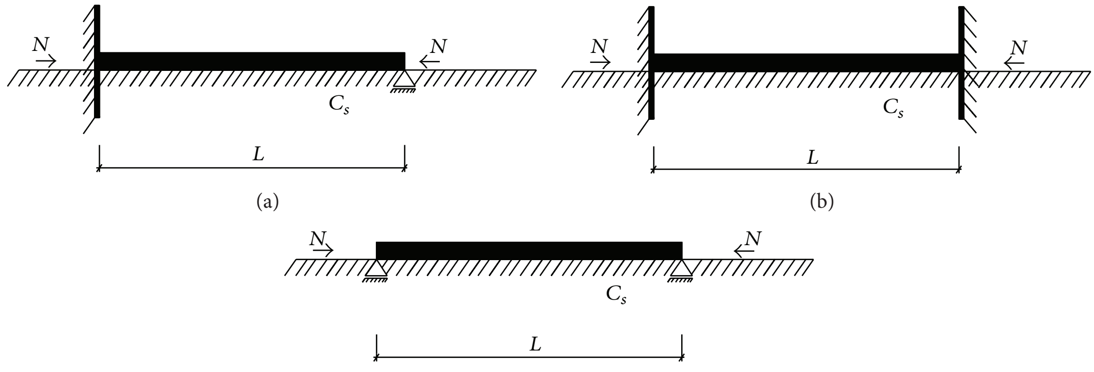

FIGURE 2: (a) CS beam, (b) CC beam, and (c) SS beam.

Equation (8) is a quadratic equation in terms of r^2 and its solution is

$$\left(r^2\right)_{1,2} = -\frac{\xi_1}{2} \pm \sqrt{\left(\frac{\xi_1}{2}\right)^2 - \xi_2}. \tag{9}$$

By taking the square root of (9), roots of (8) would be obtained as follows:

$$\begin{aligned} s_1 &= \sqrt{(r^2)_1} \\ s_2 &= -\sqrt{(r^2)_1} \\ s_3 &= \sqrt{(r^2)_2} \\ s_4 &= -\sqrt{(r^2)_2}. \end{aligned} \tag{10}$$

Hence the solution finally takes the following form:

$$\phi_1(z) = C_1 e^{s_1 z} + C_2 e^{s_2 z} + C_3 e^{s_3 z} + C_4 e^{s_4 z}. \tag{11}$$

Coefficients C_1, C_2, C_3, and C_4 would be obtained by inserting the boundary conditions for the beam. In this study three different support configurations for the beam will be considered as shown in Figure 2. These are SS for *Simply Supported*, CC for *Clamped-Clamped,* and CS for *Clamped-Simply Supported* beams. The boundary conditions related to these supports are given below.

For a simple support:

$$\phi_1\big|_z = 0, \qquad \phi_1''\big|_z = 0. \tag{12}$$

For a clamped support:

$$\phi_1\big|_z = 0, \qquad \phi_1'\big|_z = 0. \tag{13}$$

Inserting the boundary conditions given in (12)-(13) into (11) for SS, CC, and CS beams, a nonlinear system of equations in the following form would be produced:

$$[K(\gamma)]\{C\} = \{0\}, \tag{14}$$

where $\{C\}^T = \{C_1 \;\; C_2 \;\; C_3 \;\; C_4\}$ and includes the coefficients of (11). Coefficient matrix $[K(\gamma)]$ is dependent on frequency factor γ and equating the determinant to zero gives the frequency factors for vibration modes of the corresponding beam:

$$\det[K(\gamma)] = 0. \tag{15}$$

Positive real roots of this equation are free vibration frequencies of the beams on elastic foundations shown in Figure 2.

4. Numerical Study

Analytical solution for free vibration of a one end clamped, and one end pinned beam on elastic foundation was previously given by Çatal [7] with additional analyses by DTM. However, Çatal [7] gives analytical solutions for only CS beam, and in this study it is observed that these results fail to coincide with the exact results.

Previously, both the variational iteration method (VIM) and homotopy perturbation method (HPM) were used to solve the free vibration equations of beam on elastic foundation for support conditions of one end clamped, and other end simply supported, both ends clamped and both ends simply supported considering various case studies [8–10]. In this study, analytical solutions for three cases are provided which are shown in Figure 2. Each case was previously analyzed by the use of both VIM and HPM with the previously explained procedure [8–10]. Numerical values in this study are chosen as the same used in Çatal [7]. Hence, an IPB 500 steel profile resting on a Winkler foundation having a modulus of subgrade reaction of 50,000 kN/m^2 is considered. Other numerical values are as follows:

$I = 107.2 * 10^{-5}$ m^4; $A = 2.39 * 10^{-2}$ m^2; $\overline{m} = 0.19$ kN s^2/m; $\overline{k} = 3.705$;

$E = 2.1 * 10^8$ kN/m^2; $G = 8.1 * 10^7$ kN/m^2.

Frequency factors $\gamma = \sqrt[4]{\overline{m}\omega^2 L^4/EI}$ are calculated taking bending moment, shear, and axial effects into consideration for $N_r = 0.25$, $N_r = 0.5$, and $N_r = 0.75$ due to circular frequencies of the beam.

Table 1: Variation of frequency factor, γ, with relative stiffness, λ, for a CS beam on elastic foundation. (The analytical results of Çatal [7] are provided below in bold characters.)

	γ_1	γ_2	γ_3
N_r = 0.25			
λ = 1	2.81620011	3.97739569	4.82021485
	2.81574011	**3.97658348**	**4.81918573**
λ = 10	3.37458660	5.04748828	6.25843195
	3.37451696	**5.04723072**	**6.25809765**
λ = 100	4.05117292	6.09258233	7.83201825
	4.05107170	**6.09248874**	**7.83184147**
λ = 1000	5.87924217	7.32194493	9.34675005
	5.87935543	**7.32217598**	**9.34584181**
N_r = 0.5			
λ = 1	2.71457148	3.93377748	4.79443677
	2.71413994	**3.93298292**	**4.79341412**
λ = 10	3.26137662	4.99222355	6.22484874
	3.26133704	**4.99199057**	**6.22451830**
λ = 100	3.96066525	6.02920759	7.79040674
	3.96064973	**6.02918911**	**7.79025650**
λ = 1000	5.84614526	7.26848984	9.30219043
	5.84624672	**7.26853895**	**9.30226231**
N_r = 0.75			
λ = 1	2.59996785	3.88867219	4.76823873
	2.59956455	**3.88787082**	**4.76721819**
λ = 10	3.13469732	4.93509845	6.19072130
	3.13456678	**4.93488312**	**6.19038296**
λ = 100	3.86312198	5.96380562	7.74812917
	3.86301875	**5.96338463**	**7.74796867**
λ = 1000	5.81232280	7.21383882	9.25698856
	5.81241941	**7.21407604**	**9.25701332**

Table 2: Variation of frequency factor, γ, with relative stiffness, λ, for a CC beam on elastic foundation.

	γ_1	γ_2	γ_3
N_r = 0.25			
λ = 1	3.39742898	4.32815705	5.16088024
λ = 10	4.05721417	5.52373153	6.68428571
λ = 100	4.66987707	6.68497843	8.36140368
λ = 1000	6.15541347	7.87417668	9.94633264
N_r = 0.5			
λ = 1	3.33994125	4.29375653	5.14012431
λ = 10	3.99112373	5.48024811	6.65690121
λ = 100	4.60834095	6.63472888	8.32680496
λ = 1000	6.12479804	7.82851855	9.90820057
N_r = 0.75			
λ = 1	3.27929840	4.25850809	5.11911832
λ = 10	3.92148631	5.43569919	6.62918805
λ = 100	4.54408497	6.58329012	8.29178517
λ = 1000	6.09362560	7.78201148	9.86962997

Variation of frequency factor, γ is tabulated in Tables 1–3. Table 1 includes comparison of analytical results of this study for the CS beam with analytical results obtained by Çatal [7]. Tables 2 and 3 show the analytical results for both CC and SS beams which are not solved by Çatal [7].

The graphical presentations of analytical solutions are provided in Figures 3, 4, and 5.

Table 3: Variation of frequency factor, γ, with relative stiffness, λ, for an SS beam on elastic foundation.

	γ_1	γ_2	γ_1
N_r = 0.25			
λ = 1	2.24318643	3.57344623	4.48319716
λ = 10	2.71977348	4.53601512	5.82884027
λ = 100	3.56133061	5.49391862	7.29761546
λ = 1000	5.71760639	6.82177513	8.75859570
N_r = 0.5			
λ = 1	2.03688119	3.51272374	4.45090640
λ = 10	2.51190763	4.46032723	5.78712097
λ = 100	3.44330860	5.41052448	7.24679924
λ = 1000	5.68672681	6.75973740	8.70619682
N_r = 0.75			
λ = 1	1.73715809	3.44867815	4.41789714
λ = 10	2.23396379	4.38057743	5.74447900
λ = 100	3.31172675	5.32308548	7.19489096
λ = 1000	5.65533583	6.69594299	8.65283438

5. Conclusion

In this study, the analytical results for free vibration analysis of beam on elastic foundation are provided for three different axially loaded cases which are namely one end clamped, the other end simply supported (CS beam); both ends clamped (CC beam), and both ends simply supported (SS beam) cases. In addition, the analytical solution and the frequency factors evaluated for different ratios of axial load N acting on the beam to Euler buckling load, N_r are obtained. In Figures 3, 4, and 5, it is observed that for the same values of λ, the largest values of γ are obtained for both ends clamped (CC) beam while the lowest values of γ are obtained for both ends pinned (SS) beam.

For the CS Beam, it is observed that analytical solutions given by Çatal [7] fail to coincide with the exact results provided in this study. Previous studies [8–10] employing both the variational iteration method (VIM) and homotopy perturbation method (HPM) to solve the free vibration equations of beam on elastic foundation in this study converge to the same results obtained in this study. The reason for that is the governing equation is a linear equation in nature; and as expected analytical approximate methods converge to the exact solution. In the study by Çatal [7], the DTM converges the analytical results provided in the same study. Presumably, the difference is due to the fact that Çatal [7] used the length of the beam, L, in three-digit accuracy while in this study the exact values of length, L, are used in all calculations conducted.

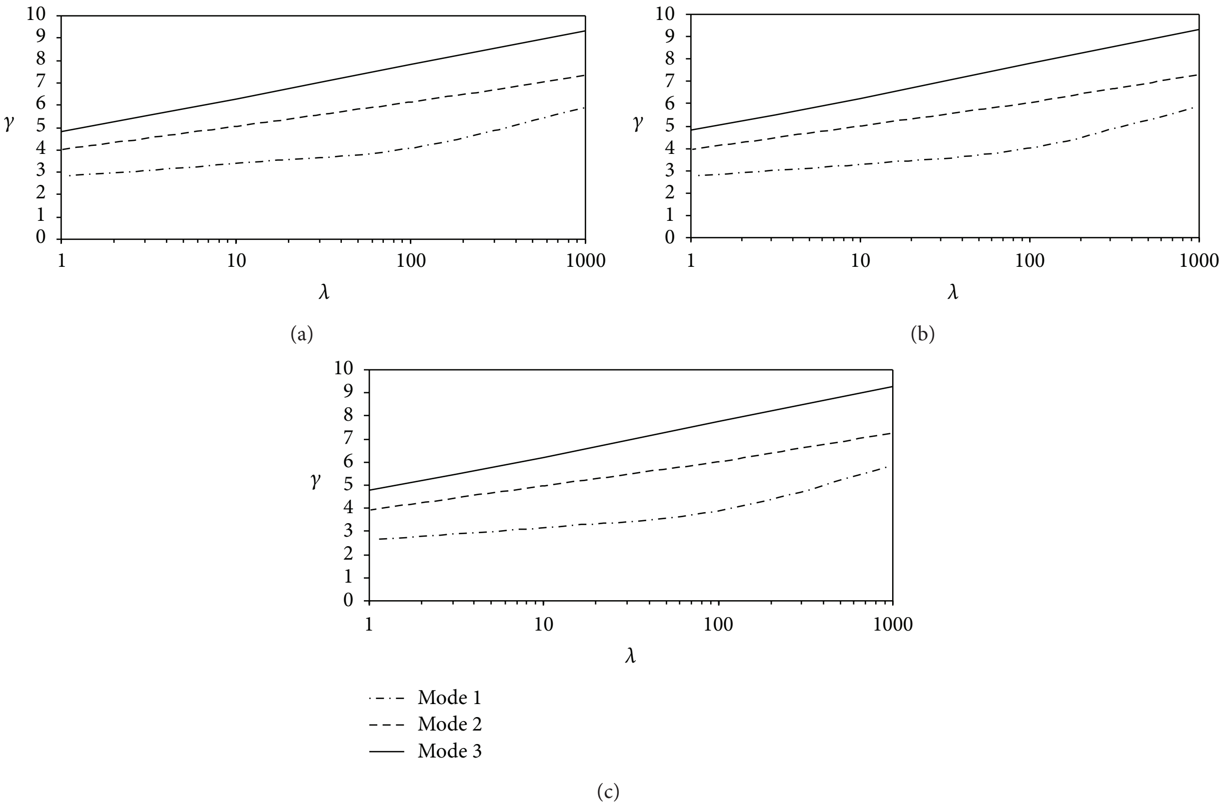

Figure 3: Variation of frequency factor, γ, with relative stiffness, λ, for a CS beam on elastic foundation: (a) for $N_r = 0.25$, (b) for $N_r = 0.5$, and (c) for $N_r = 0.75$.

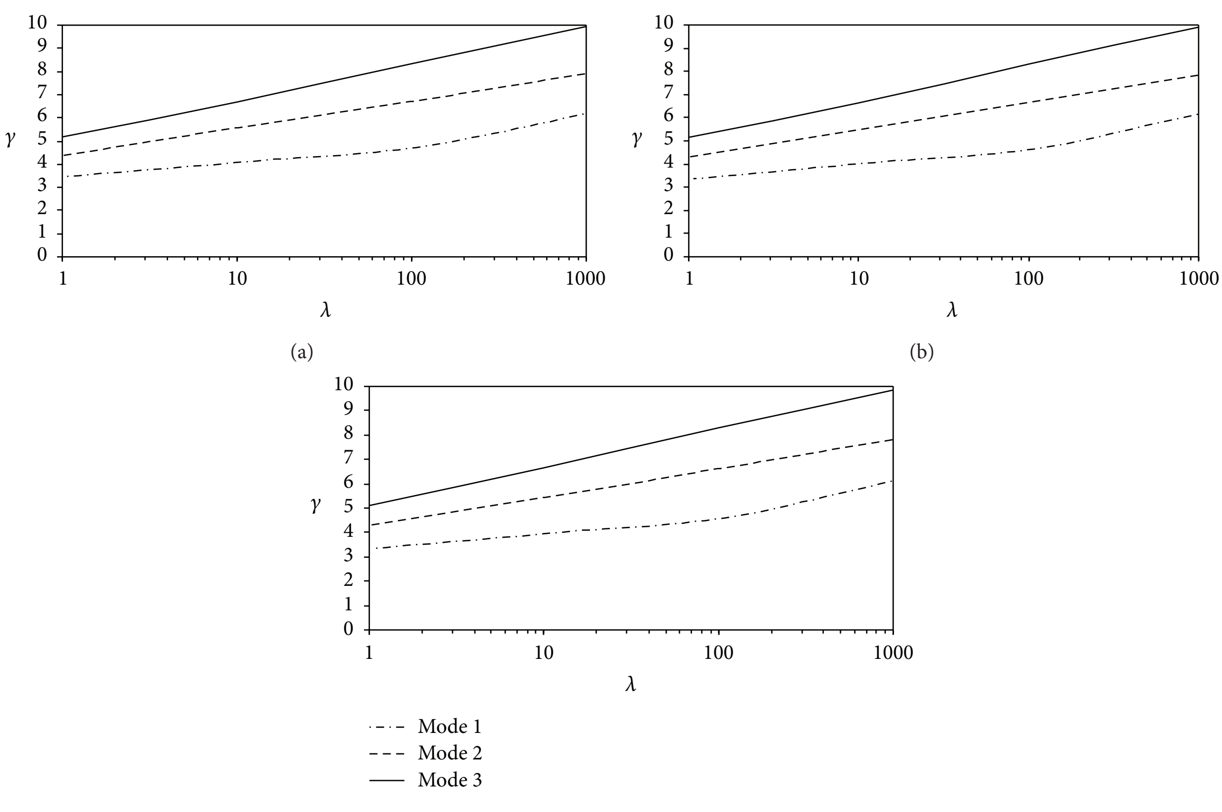

Figure 4: Variation of frequency factor, γ, with relative stiffness, λ, for a CC beam on elastic foundation: (a) for $N_r = 0.25$, (b) for $N_r = 0.5$, and (c) for $N_r = 0.75$.

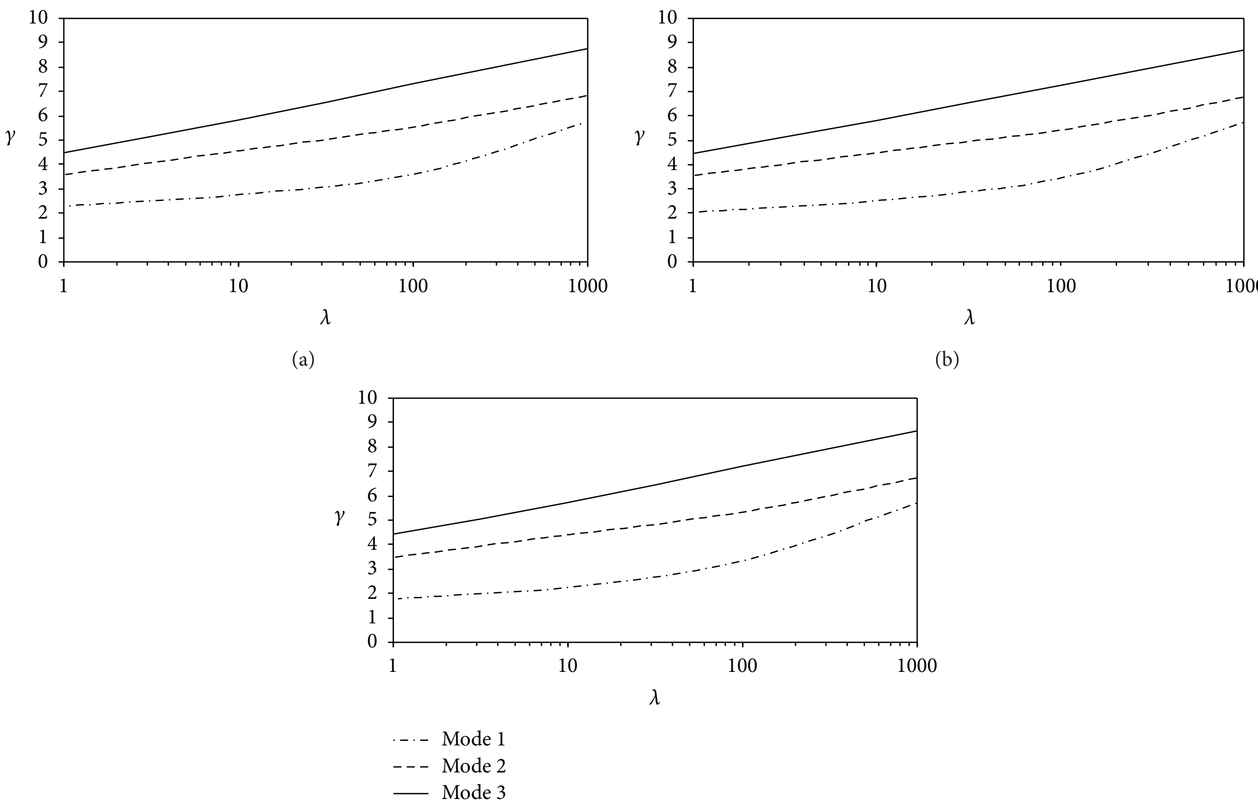

Figure 5: Variation of frequency factor, γ, with relative stiffness, λ, for an SS beam on elastic foundation: (a) for $N_r = 0.25$, (b) for $N_r = 0.5$, and (c) for $N_r = 0.75$.

Accordingly, this study provides the analytical results for free vibration analysis of beam on elastic foundation for three different axially loaded cases which are namely one end clamped, the other end simply supported (CS beam); both ends clamped (CC beam) and both ends simply supported (SS beam) cases.

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