DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **21**, Number **4**, August **2008**

Website: http://aimSciences.org

pp. **1185–1198**

LONG-TIME BEHAVIOUR OF WAVE EQUATIONS WITH NONLINEAR INTERIOR DAMPING

A. Kh. Khanmamedov

Department of Mathematics, Faculty of Science
Hacettepe University, Beytepe 06532, Ankara, Turkey

Abstract. We prove the existence of attractors for higher dimensional wave equations with nonlinear interior damping which grows faster than polynomials at infinity.

1. **Introduction.** The main purpose of the paper is to investigate the long-time behaviour of the following wave equation:

$$u_{tt} - \Delta u + g(u_t) + f(u) = 0 \qquad \text{in } \ (0, \ +\infty) \times \Omega \tag{1.1}$$

with boundary condition

$$u = 0 \qquad \text{on } \ (0, \ +\infty) \times \partial\Omega \tag{1.2}$$

and with initial data condition

$$u(0, \cdot) = u_0 \ , \qquad u_t(0, \cdot) = u_1 \qquad \text{in } \ \Omega, \tag{1.3}$$

where $\Omega \subset R^n$ is a bounded domain with a sufficiently smooth boundary.

The problem of the existence of attractors for autonomous wave equations with linear interior damping was investigated in [1], [2], [5], [9], [12], [17] and references therein. In [3], [6], [15] the existence of attractors for wave equations with nonlinear interior damping was established assuming a large value for the damping. In [13] attractors were studied without assuming large value for the damping in the two-dimensional case. For the three-dimensional case the large damping restriction has been removed in [16] and later the result of [16] is improved in [10].

In all articles mentioned above attractors were studied under a polynomial growth condition at infinity on the damping term. In [7] the attractors were investigated for one-dimensional wave equations without upper restriction on the growth of the damping term, where the embedding $H^1(\Omega) \subset C(\overline{\Omega})$ available in the one-dimensional case is critically used.

In this paper, we study the existence of attractors for higher dimensional wave equations with nonlinear interior damping which grows faster than polynomials at infinity. The paper organized as follows: In the next section we state our main result, in Section 3 we establish the asymptotic compactness property of solutions of (1.1)-(1.3), and finally in Section 4 we prove the existence of a global $(\mathcal{H}, \mathcal{H})_{\mathfrak{B}}-$attractor for the problem (1.1)-(1.3).

2000 *Mathematics Subject Classification.* 35B41, 35L70;
Key words and phrases. Attractors, wave equations.

2. **Statement of the main result.** We begin with the assumptions on the functions f and g.

Assumption 2.1.

- $g \in C^1(R),\ g(0) = 0,\quad g$ is odd function and $g'(\cdot) \geq \alpha > 0,$ (2.1)
- $|g'(s)| \leq c(1 + g^2(s)), \qquad \forall\, s \in R,$ (2.2)
- $\liminf\limits_{|s|\to\infty} \dfrac{g(ks)}{g(s)} > 1 \quad$ for some $k > 1,$ (2.3)
- $|g(s-t)| \leq \beta\,|g(s) - g(t)|\,, \quad \forall\, s, t \in R, \quad$ for some $\beta > 0,$ (2.4)
- $f \in C^1(R), \quad |\ f'(s)| \leq c, \qquad \liminf\limits_{|s|\to\infty} \dfrac{f(s)}{s} > -\lambda_1,$ (2.5)

where $\lambda_1$is the first eigenvalue of $-\Delta$ with zero Dirichlet data.

Remark 2.1. The class of functions which satisfy conditions (2.1)-(2.4) is quite large. It is easy to see that if a function satisfying (2.1)-(2.2) also has non-decreasing derivative on R_+, then it satisfies (2.3) and (2.4). For example, $g_1(s) = |s|^p\, s + s$ $(p \geq 0)$ and $g_2(s) = \begin{cases} e^s - 1, & \text{if } s \geq 0 \\ -e^{-s} + 1, & \text{if } s < 0 \end{cases}$ are such functions.

Applying Galerkin's method one can prove the following existence theorem.

Theorem 2.1. *Assume that Assumption 2.1 holds and*

$$u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \qquad u_1 \in H_0^1(\Omega) \cap C(\overline{\Omega}).$$

Then for every $T > 0$ there exists a unique strong solution $u(t,x)$ of problem (1.1)-(1.3) on $[0,T] \times \Omega$, that is $u \in W^{2,\infty}(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ and

$$u_{tt} - \Delta u + g(u_t) + f(u) = 0, \qquad a.e.\ on\ \ [0,T] \times \Omega.$$

Remark 2.2. In the case when g and f satisfy certain polynomial growth conditions Theorem 2.1 was proved in [8] for initial data (u_0, u_1) such that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $g(u_1) \in L^2(\Omega)$. In [4], the existence of strong solutions of wave equations with nonlinear damping satisfying some polynomial growth conditions was studied for a larger class of initial data.

Let us define a generalized solution:

Definition 2.1. A function $u \in C([0,\ T];\ H_0^1(\Omega)) \cap C^1([0,\ T];\ L^2(\Omega))$ possessing the properties $u(0,\cdot) = u_0$ and $u_t(0,\cdot) = u_1$ is said to be generalized (weak) solution to problem (1.1)-(1.3) on $[0,\ T] \times \Omega$, iff there exists a sequence of strong solutions $\{u^n(t,\ x)\}$ to problem (1.1)-(1.3) with initial data $(u_0^n,\ u_1^n)$ instead of $(u_0,\ u_1)$ such that

$$\lim_{n\to\infty} \left(\|u - u^n\|_{C([0,T];\ H_0^1(\Omega))} + \|u_t - u_t^n\|_{C([0,T];\ L^2(\Omega))} \right) = 0.$$

Using density argument and Theorem 2.1 we have the following theorem.

Theorem 2.2. *Assume that Assumption 2.1 holds and $(u_0, u_1) \in \mathcal{H} := H_0^1(\Omega) \times L^2(\Omega)$. Then for every $T > 0$ the problem (1.1)-(1.3) has a unique generalized solution on $[0,T] \times \Omega$, which satisfies*

$$E(u(t), u_t(t)) + \int\limits_{\tau}^{t} \int\limits_{\Omega} g(u_t(s,x))u_t(s,x)dxds + \int\limits_{\Omega} F(u(t,x))dx$$

$$\leq E(u(\tau), u_t(\tau)) + \int_\Omega F(u(\tau, x))dx, \qquad 0 \leq \tau \leq t < T. \tag{2.6}$$

Moreover if $v(t,x)$ *is a generalized solution to (1.1)-(1.3) on* $[0,T] \times \Omega$ *with initial data* (v_0, v_1) *and* $\max\{\|(u_0,u_1)\|_{\mathcal{H}}, \|(v_0,v_1)\|_{\mathcal{H}}\} \leq R$*, then there exists* $C = C(R) > 0$ *such that*

$$E(u(t) - v(t), u_t(t) - v_t(t)) \leq CE(u_0 - v_0, u_1 - v_1), \qquad \forall\, t \in [0,T],$$

where $F(u) = \int_0^u f(v)dv$, $E(u,v) = \frac{1}{2}\left(\|\nabla u\|^2 + \|v\|^2\right)$ *and* $\|\cdot\|$ *is a norm in* $L^2(\Omega)$.

Thus under Assumption 2.1, problem (1.1)-(1.3) generates a continuous semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{H}$ by the formula $S(t)(u_0,u_1) = (u(t), u_t(t))$, where $u(t,x)$ is a generalized solution with initial data (u_0, u_1).

Now let us introduce the following family of sets:

$$\mathfrak{B} = \Bigg\{B : B \text{ is a bounded subset of } \mathcal{H} \text{ and for any } \varepsilon > 0, \text{ there exists } m = m(\varepsilon, B) > 0 \text{ such that } \sup_{(u,v)\in B} \int_{\{x: x\in\Omega,\ |u(x)|>m\}} |\nabla u(x)|^2 dx \leq \varepsilon \Bigg\}.$$

Definition 2.2. We say that a set $\mathcal{A} \in \mathfrak{B}$ is a global $(\mathcal{H},\mathcal{H})_{\mathfrak{B}}-$attractor for the semigroup $\{S(t)\}_{t\geq 0}$ iff

- $\mathcal{A}$ is compact in $\mathcal{H}$;
- $\mathcal{A}$ is invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}, \forall\, t \geq 0$;
- $\lim\limits_{t\to\infty} \sup\limits_{v\in B} \inf\limits_{u\in\mathcal{A}} \|S(t)v - u\|_{\mathcal{H}} = 0$ for each $B \in \mathfrak{B}$.

Our main result is:

Theorem 2.3. *Under Assumption 2.1 the semigroup* $\{S(t)\}_{t\geq 0}$ *generated by the problem (1.1)-(1.3) possesses a global* $(\mathcal{H},\mathcal{H})_{\mathfrak{B}}-$*attractor.*

Remark 2.3. By the definition we see that the projection of an element of $\mathfrak{B}$ on $H_0^1(\Omega)$ is a bounded set which has a "compactness" (or regularity) property in some sense. We will use this property to prove asymptotic compactness (see proof of Lemma 3.2).

Remark 2.4. By the definition it follows that a global $(\mathcal{H},\mathcal{H})_{\mathfrak{B}}-$attractor is maximal as invariant set belonging to $\mathfrak{B}$ and minimal as closed attractor attracting every element of $\mathfrak{B}$. Since every bounded subset of $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$ and $W^{1,\,2+\varepsilon}(\Omega) \times L^2(\Omega)$ belongs to $\mathfrak{B}$, a global $(\mathcal{H},\mathcal{H})_{\mathfrak{B}}-$attractor attracts each bounded subset of $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$ and $W^{1,\,2+\varepsilon}(\Omega) \times L^2(\Omega)$ in the topology of $\mathcal{H}$, where $\varepsilon > 0$.

3. **Asymptotic compactness.**

Let $u(t,x)$ be a strong solution of (1.1)-(1.3) with initial data (u_0, u_1). We use decomposition used in [5], [6] and [8]. So we decompose $u(t,x)$ as a sum $w(t,x) + v(t,x)$, where

$$\begin{cases} w_{tt} - \Delta w + g(w_t) + f(u) = 0 & \text{in } (0, +\infty) \times \Omega \\ w = 0 & \text{on } (0, +\infty) \times \partial\Omega \\ w(0,\cdot) = 0\ , \qquad w_t(0,\cdot) = 0 & \text{in } \Omega \end{cases} \tag{3.1}$$

and

$$\begin{cases} v_{tt} - \Delta v + g(v_t + w_t) - g(w_t) = 0 & \text{in } (0, +\infty) \times \Omega \\ v = 0 & \text{on } (0, +\infty) \times \partial\Omega \\ v(0,\cdot) = u_0 , \qquad v_t(0,\cdot) = u_1 & \text{in } \Omega \end{cases} \tag{3.2}$$

Lemma 3.1. *Let Assumption 2.1 holds. Then for every $(u_0, u_1) \in \mathcal{H}$, there exists a unique strong solution $w \in W^{2,\infty}(0,\infty; L^2(\Omega)) \cap W^{1,\infty}(0,\infty; H_0^1(\Omega)) \cap L^\infty(0,\infty; H^2(\Omega) \cap H_0^1(\Omega))$ of (3.1) such that*

$$\|\Delta w(t)\| + \|\nabla w_t(t)\| + \|w_{tt}(t)\| \le c(\|(u_0, u_1)\|_{\mathcal{H}}), \quad \forall t \ge 0, \tag{3.3}$$

where $c(\cdot) : R_+ \to R_+$ is a nondecreasing function.

Proof. Since uniqueness of the strong solution is trivial, we will prove the existence of a solution which satisfies (3.3). Let $\{\varphi_i\}_{i=1}^\infty$ be eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, i.e.

$$\begin{cases} -\Delta\varphi_i = \lambda_i\varphi_i, \text{ in } \Omega, \\ \varphi_i \mid_\Omega = 0, \end{cases} , \; i = 1, 2, \dots .$$

Since $\Omega \subset R^n$ is a bounded domain with sufficiently smooth boundary, by standard elliptic theory we have $\varphi_i \in C^\infty(\overline{\Omega})$, $i = 1, 2, \dots$. Set $w^m(t) = \sum_{j=1}^m a_{mj}(t)\varphi_j$ and consider the following system of ordinary differential equations:

$$\begin{aligned} &\frac{d^2}{dt^2} \langle w^m(t), \varphi_j \rangle + \langle \nabla w^m(t), \nabla\varphi_j \rangle + \left\langle g(\frac{d}{dt} w^m(t)), \varphi_j \right\rangle \\ &+ \langle f(u(t)), \varphi_j \rangle = 0, \quad j = \overline{1, m} \end{aligned} \tag{3.4}$$

with initial conditions

$$a_{mj}(0) = 0, \qquad a'_{mj}(0) = 0, \qquad j = \overline{1, m} \tag{3.5}$$

where $\langle w, \varphi \rangle = \int\limits_\Omega w(x)\varphi(x)dx$ and $u(t,x)$ is the generalized solution of (1.1)-(1.3) with initial data (u_0, u_1). Existence theory of ordinary differential equations implies that there exists a solution of (3.4)-(3.5) on $[0, T_m)$. Multiplying both sides of (3.4) by $2\lambda_j \frac{d}{dt} a_{mj}(t)$, summing from 1 to m and integrating over $[0, t] \subset [0, T_m)$ we obtain

$$\begin{aligned} &\|\nabla w_t^m(t)\|^2 + \|\Delta w^m(t)\|^2 + 2\int\limits_0^t \int\limits_\Omega g'(w_t^m(s,x)) \left|\nabla w_t^m(s,x)\right|^2 dxds \\ = &- 2\sum_{j=1}^n \int\limits_0^t \int\limits_\Omega f'(u(s,x)) u_{x_j}(s,x) w_{tx_j}^m(s,x) dxds, \end{aligned}$$

which together with (2.1), (2.5) and (2.6) yields

$$\|\nabla w_t^m(t)\|^2 + \|\Delta w^m(t)\|^2 \le c_1(\|(u_0, u_1)\|_{\mathcal{H}}) T_m, \quad 0 \le t < T_m.$$

Hence $w^m(t, \cdot)$ can be extended to an interval $[0, T]$ and

$$\|\nabla w_t^m(t)\|^2 + \|\Delta w^m(t)\|^2 \le c_1(\|(u_0, u_1)\|_{\mathcal{H}}) T, \; 0 \le t \le T. \tag{3.6}$$

Differentiating both sides of (3.4), multiplying by $2\frac{d^2}{dt^2}a_{mj}(t)$, summing from 1 to m and integrating over $[0,t]$ we obtain

$$\|w_{tt}^m(t)\|^2 + \|\nabla w_t^m(t)\|^2 + 2\int_0^t\int_\Omega g'(w_t^m(s,x))\,|w_{tt}^m(s,x)|^2\,dxds$$

$$= \|w_{tt}^m(0)\|^2 - 2\int_0^t\int_\Omega f'(u(s,x))u_t(s,x)w_{tt}^m(s,x)dxds,$$

which again together with (2.1), (2.5) and (2.6) yields

$$\|w_{tt}^m(t)\|^2 + \|\nabla w_t^m(t)\|^2 \le c_2(\|(u_0,\ u_1)\|_{\mathcal{H}}), \qquad \forall\, t\ge 0. \tag{3.7}$$

On the other hand multiplying both sides of (3.4) by $\frac{d}{dt}a_{mj}(t)$, summing from 1 to m and integrating over $[0,T]$ we find

$$\int_0^T\int_\Omega g(w_t^m(s,x))w_t^m(s,x)dxds \le c_3(\|(u_0,\ u_1)\|_{\mathcal{H}})T$$

and consequently

$$\int_0^T\int_\Omega N(g(w_t^m(s,x)))dxds \le c_3(\|(u_0,\ u_1)\|_{\mathcal{H}})T, \tag{3.8}$$

where $N(x) = \int_0^x g^{-1}(y)dy$. Taking into account (3.6), (3.7) and applying [11, Theorem 14.4, p. 131] to (3.8) we can say that there exists a subsequence $\{m_k\}$ such that

$$\begin{cases} w^{m_k} \to w \quad \text{weakly star in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \\ w_t^{m_k} \to w_t \quad \text{weakly star in } L^\infty(0,T;H_0^1(\Omega)) \\ w_{tt}^{m_k} \to w_{tt} \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \\ \int_0^T\int_\Omega g(w_t^{m_k})\psi dxds \to \int_0^T\int_\Omega g(w_t)\psi dxds, \quad \forall\, \psi\in L^\infty((0,T)\times\Omega) \end{cases} \tag{3.9}$$

Now passing to limit in (3.4) we obtain

$$\langle w_{tt}(t),\varphi_j\rangle + \langle \Delta w(t),\varphi_j\rangle + \langle g(w_t(t)),\varphi_j\rangle$$
$$+ \langle f(u(t)),\varphi_j\rangle = 0, \quad \text{a.e. on } [0,T], \quad j=1,2...$$

from which we find that $g(w_t) \in L^\infty(0,T;L^2(\Omega))$ and $w\in W^{2,\infty}(0,T;L_2(\Omega))\cap W^{1,\infty}(0,T;H_0^1(\Omega))\cap L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ satisfies $(3.1)_1$ a.e. on $[0,T]\times\Omega$.

Now let us prove inequality (3.3). Passing to limit in (3.7) we have

$$\|w_{tt}(t)\|^2 + \|\nabla w_t(t)\|^2 \le c_2(\|(u_0,\ u_1)\|_{\mathcal{H}}), \quad \text{for a.a. } t\ge 0. \tag{3.10}$$

Denoting $z^m(t) = w^m(t+\Delta t) - w^m(t)$, from (3.4) we obtain

$$\langle z_{tt}^m(t),\varphi_j\rangle + \langle \nabla z^m(t),\nabla\varphi_j\rangle + \langle g(w_t^m(t+\Delta t)) - g(w_t^m(t)),\varphi_j\rangle$$
$$+ \langle f(u(t+\Delta t)) - f(u(t)),\varphi_j\rangle = 0, \qquad j=\overline{1,m} \tag{3.11}$$

Multiplying both sides of (3.11) by $\frac{d}{dt}(\ a_{mj}(t+\Delta t)-a_{mj}(t))$, summing from 1 to m and integrating over $[0,T]$ we find

$$\int_0^T\int_\Omega (g(w_t^m(t+\Delta t))-g(w_t^m(t)))(w_t^m(t+\Delta t))-w_t^m(t))dxdt$$
$$\leq l(\|z_t^m(0)\|^2+\|\nabla z^m(0)\|^2+\int_0^T\|u(t+\Delta t)-u(t)\|^2\,dt)$$

which together with (2.1), (2.5), (2.6) and (3.7) yields

$$\frac{1}{(\Delta t)^2}\int_0^T\int_\Omega (g(w_t^{m_k}(t+\Delta t))-g(w_t^{m_k}(t)))(w_t^{m_k}(t+\Delta t)-w_t^{m_k}(t))dxdt\leq$$
$$\leq c_4(\|(u_0,\ u_1)\|_{\mathcal{H}}),\qquad \text{for } \Delta t>0,$$

where $l>0$ and $c_4(\cdot):R_+\to R_+$ is a nondecreasing function. Taking into account (3.9) and applying Fatou's lemma from last inequality we obtain

$$\frac{1}{(\Delta t)^2}\int_0^T\int_\Omega (g(w_t(t+\Delta t))-g(w_t(t)))(w_t(t+\Delta t)-w_t(t))dxdt$$
$$\leq c_4(\|(u_0,\ u_1)\|_{\mathcal{H}}),\qquad \text{for } \Delta t>0,$$

and consequently

$$\int_0^T\int_\Omega g'(w_t(t,x))\,|w_{tt}(t,x)|^2\,dxds\leq c_4(\|(u_0,\ u_1)\|_{\mathcal{H}}),\quad \forall\, T\geq 0. \tag{3.12}$$

Now differentiating $(3.1)_1$ with respect to t we obtain

$$\frac{\partial}{\partial t}w_{tt}-\Delta w_t+\frac{\partial}{\partial t}g(w_t)+f'(u)u_t=0. \tag{3.13}$$

By (2.2) and (3.12) we find $\frac{\partial}{\partial t}g(w_t)=g'(w_t)w_{tt}\in L^1((0,T)\times\Omega)$, which together with $g(w_t)\in L^\infty(0,T;L^2(\Omega))$ yields $g(w_t)\in C(0,T;L^1(\Omega))$. Applying [14, Lemma 8.1, p.275] we have $g(w_t)\in C_s(0,T;L^2(\Omega))$. On the other hand since $\Delta w_t\in L^\infty(0,T;H^{-1}(\Omega))$ and $f'(u)u_t\in L^\infty(0,T;L^2(\Omega))$ from (3.13) we find $\frac{\partial}{\partial t}w_{tt}\in L^1(0,T;L^1(\Omega)+H^{-1}(\Omega))$, which together with $w_{tt}\in L^\infty(0,T;L^2(\Omega))$ yields $w_{tt}\in C(0,T;L^1(\Omega)+H^{-1}(\Omega))$. Applying again [14, Lemma 8.1, p.275] we have $w_{tt}\in C_s(0,T;L^2(\Omega))$.

Denote

$$g_n(z)=\begin{cases} g(-n), & z<-n,\\ g(z), & -n\leq z\leq n,\\ g(n), & z>n\end{cases}.$$

Since $g_n(w_t) \in L^\infty(0,T;L^\infty(\Omega) \cap H_0^1(\Omega))$ by (3.13) have

$$\int_0^t \left\langle \frac{\partial}{\partial t} w_{tt}(s), g_n(w_t(s)) \right\rangle ds + \int_0^t \langle \nabla w_t(s), \nabla g_n(w_t(s)) \rangle \, ds$$
$$+ \int_0^t \left\langle \frac{\partial}{\partial t} g(w_t(s)), g_n(w_t(s)) \right\rangle ds + \int_0^t \langle f'(u)u_t, g_n(w_t) \rangle \, ds = 0. \tag{3.14}$$

Let us estimate each integral in (3.14):

$$\int_0^t \left\langle \frac{\partial}{\partial t} w_{tt}(s), g_n(w_t(s)) \right\rangle ds = \langle w_{tt}(t), g_n(w_t(t)) \rangle - \int_0^t \left\langle w_{tt}(s), \frac{\partial}{\partial t} g_n(w_t(s)) \right\rangle ds$$
$$= \langle w_{tt}(t), g_n(w_t(t)) \rangle - \int_0^t \int_{\{x: x\in\Omega,\ |w_t(s,x)|\le n\}} g_n'(w_t(s,x)) \left| w_{tt}(s,x) \right|^2 dxds$$
$$\ge - \left\| w_{tt}(t) \right\| \left\| g(w_t(t)) \right\| - \int_0^t \int_\Omega g'(w_t(s,x)) \left| w_{tt}(s,x) \right|^2 dxds. \tag{3.15}$$

$$\int_0^t \langle \nabla w_t(s), \nabla g_n(w_t(s)) \rangle \, ds$$
$$= \int_0^t \int_{\{x: x\in\Omega,\ |w_t(s,x)|\le n\}} g_n'(w_t(s,x)) \left| \nabla w_t(s,x) \right|^2 dxds \ge 0. \tag{3.16}$$

$$\int_0^t \left\langle \frac{\partial}{\partial t} g(w_t(s)), g_n(w_t(s)) \right\rangle ds = \langle g(w_t(t)), g_n(w_t(t)) \rangle$$
$$- \int_0^t \left\langle g_n(w_t(s)), \frac{\partial}{\partial t} g_n(w_t(s)) \right\rangle ds = \langle g(w_t(t)), g_n(w_t(t)) \rangle - \frac{1}{2} \left\| g_n(w_t(t)) \right\|^2$$
$$\ge \frac{1}{2} \left\| g_n(w_t(t)) \right\|^2 . \tag{3.17}$$

Using (2.5) and Young inequality (see [11]) we have

$$\int_0^t \langle f'(u)u_t, g_n(w_t) \rangle \, ds \ge -c \int_0^t \int_\Omega \left| u_t(s,x) \right| \left| g_n(w_t(s,x)) \right| dxds$$
$$\ge - c \int_0^t \int_\Omega \left| u_t(s,x) \right| \left| g(w_t(s,x)) \right| dxds \ge -c \int_0^t \int_\Omega g(u_t(s,x)) u_t(s,x) dxds$$
$$- c \int_0^t \int_\Omega g(w_t(s,x)) w_t(s,x) dxds. \tag{3.18}$$

Taking into account (3.15)-(3.18) in (3.14) we find

$$\frac{1}{2}\left\|g_n(w_t(t))\right\|^2 \le \|w_{tt}(t)\|\,\|g(w_t(t))\| + \int_0^t\int_\Omega g'(w_t(s,x))\left|w_{tt}(s,x)\right|^2 dxds$$
$$+ c\left[\int_0^t\int_\Omega g(u_t(s,x))u_t(s,x)dxds + \int_0^t\int_\Omega g(w_t(s,x))w_t(s,x)dxds\right]$$

which together with (2.6), (3.10) and (3.12) implies

$$\|g(w_t(t))\| \le c_5(\|(u_0,\ u_1)\|_{\mathcal{H}}) + \int_0^t\int_\Omega g(w_t(s,x))w_t(s,x)dxds, \qquad t \ge 0, \tag{3.19}$$

where $c_5(\cdot) : R_+ \to R_+$ is a nondecreasing function.

Denote by $K(t)$ a solution operator of (3.1), i.e. $(w(t), w_t(t)) = K(t)(u_0, u_1)$. Let $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H_0^1(\Omega) \cap C(\overline{\Omega}))$, $(u(t), u_t(t)) = S(t)(u_0, u_1)$, $(w(t), w_t(t)) = K(t)(u_0, u_1)$ and $v(t,) = u(t,) - w(t,)$. As shown above $w \in W^{2,\infty}(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$. Then by Theorem 1.1, $v \in W^{2,\infty}(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ satisfies $(3.2)_1$ a.e. on $(0, T) \times \Omega$ for every $T > 0$. Multiplying $(3.2)_1$ by v_t, integrating over $(s, t) \times \Omega$ and taking into account (2.4) we obtain

$$E(v(t), v_t(t)) + \frac{1}{\beta}\int_s^t\int_\Omega g(v_t(s,x))v_t(s,x)dxds \le E(v(s), v_t(s)), \quad \forall\, t \ge s \ge 0, \tag{3.20}$$

and using Young inequality (see [11])

$$\begin{aligned}
&\int_0^t\int_\Omega g(w_t(s,x))w_t(s,x)dxds\\
\le&\beta\int_0^t\int_\Omega (g(u_t(s,x)) - g(v_t(s,x)))(u_t(s,x) - v_t(s,x))dxds\\
\le&3\beta\left[\int_0^t\int_\Omega g(u_t(s,x))u_t(s,x)dxds + \int_0^t\int_\Omega g(v_t(s,x))v_t(s,x)dxds\right]\\
\le&c_6(\|(u_0,\ u_1)\|_{\mathcal{H}}), \qquad \forall\, t > 0.
\end{aligned} \tag{3.21}$$

where $c_6(\cdot) : R_+ \to R_+$ is a nondecreasing function. Using density argument it is easy to see that (3.21) also holds for every $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

By (3.19) and (3.21) we have

$$\|g(w_t(t))\| \le c_7(\|(u_0,\ u_1)\|_{\mathcal{H}}), \qquad t \ge 0,$$

where $c_7(\cdot) : R_+ \to R_+$ is a nondecreasing function. Taking into account the last inequality together with (2.6) and (3.10) in equation $(3.1)_1$ we obtain (3.3). □

Lemma 3.2. *Assume that Assumption 2.1 holds and $B \in \mathfrak{B}$. Then for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $T_0 = T_0(\varepsilon, B) > 0$ such that*

$$\|S(T)\theta - K(T)\theta\|_{\mathcal{H}} \le \varepsilon, \qquad \forall\, T \ge T_0 \quad \textit{and}\ \forall\, \theta \in O_\delta(B), \tag{3.22}$$

where $O_\delta(B)$ is $\delta-$neighbourhood of B in $\mathcal{H}$.

Proof. Let $\theta = (u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H_0^1(\Omega) \cap C(\overline{\Omega}))$, $(u(t), u_t(t)) = S(t)(u_0, u_1)$ and $(w(t), w_t(t)) = K(t)(u_0, u_1)$. Then as mentioned in proof of Lemma 3.1, the function

$$v = u - w \in W^{2,\infty}(0,T; L^2(\Omega)) \cap W^{1,\infty}(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega))$$

satisfies $(3.2)_1$ a.e. on $(0,T) \times \Omega$ for every $T > 0$.

Set $\widehat{u}_0(x) = \begin{cases} u_0(x) + m, & u_0(x) < -m \\ 0, & |u_0(x)| \le m \\ u_0(x) - m, & u_0(x) > m \end{cases}$.

Multiplying $(3.2)_1$ by $\frac{1}{t+1}(v(t,x) - \widehat{u}_0(x))$ and integrating over $(0,T) \times \Omega$ we have

$$\begin{aligned} &\int_0^T \frac{1}{t+1} \|\nabla v(t)\|^2 dt - \int_0^T \frac{1}{t+1} \langle \nabla v(t), \nabla \widehat{u}_0 \rangle dt - \int_0^T \frac{1}{t+1} \|v_t(t)\|^2 dt \\ &\le \int_0^T \int_\Omega g(w_t(t,x)) \frac{1}{t+1} \int_0^t v_t(s,x) ds dx dt - \int_0^T \int_\Omega g(u_t(t,x)) \frac{1}{t+1} \int_0^t v_t(s,x) ds dx dt \\ &\quad + \int_0^T \int_\Omega g(w_t(t,x)) \frac{1}{t+1} (u_0(x) - \widehat{u}_0(x)) dx dt \\ &\quad - \int_0^T \int_\Omega g(u_t(t,x)) \frac{1}{t+1} (u_0(x) - \widehat{u}_0(x)) dx dt \\ &\quad + \frac{1}{2} \|u_0 - \widehat{u}_0\|^2 - \frac{1}{T+1} \langle v_t(T), v(T) - \widehat{u}_0 \rangle + \langle u_1, u_0 - \widehat{u}_0 \rangle . \end{aligned} \tag{3.23}$$

Now let us estimate first four terms on the right side of (3.23).

Using Young and Jensen inequalities (see [11]) we find

$$\begin{aligned} &\left| \int_0^T \int_\Omega g(w_t(t,x)) \frac{1}{t+1} \int_0^t v_t(s,x) ds dx dt \right| \le \int_0^T \int_\Omega N(\lambda g(w_t(t,x))) dx dt \\ &+ \int_0^T \int_\Omega M \left(\frac{1}{\lambda(t+1)} \int_0^t v_t(s,x) ds \right) dx dt \le \int_0^T \int_\Omega N(\lambda g(w_t(t,x))) dx dt \\ &+ \int_0^T \int_\Omega \frac{t}{\lambda(t+1)} M \left(\frac{1}{t} \int_0^t v_t(s,x) ds \right) dx dt \le \int_0^T \int_\Omega N(\lambda g(w_t(t,x))) dx dt \\ &+ \int_0^T \int_\Omega \frac{1}{\lambda(t+1)} \int_0^t M(v_t(s,x)) ds dx dt, \qquad \forall \lambda > 1, \end{aligned} \tag{3.24}$$

where $M(z) = \int_0^z g(x) dx$ and $N(z) = \int_0^z g^{-1}(x) dx$. By (2.1) and (2.3) there exists $l(k) > 1$ such that

$$g(kx) \ge l(k) g(x), \quad \forall x \in R,$$

and consequently

$$g(k^n x) \geq (l(k))^n g(x), \quad \forall\, x \in R.$$

The last inequality together with (2.1) yields that

$$\begin{aligned}
&\int_0^T \int_\Omega N((l(k))^n g(w_t(t,x)))dxdt \\
\leq &(l(k))^n \int_0^T \int_\Omega g(w_t(t,x)) g^{-1}((l(k))^n g(w_t(t,x)))dxdt \\
\leq &(l(k))^n k^n \int_0^T \int_\Omega g(w_t(t,x)) w_t(t,x)dxdt. \qquad (3.25)
\end{aligned}$$

Setting $\lambda = (l(k))^n$, by (3.24)-(3.25) we obtain

$$\begin{aligned}
&\left| \int_0^T \int_\Omega g(w_t(t,x)) \frac{1}{t+1} \int_0^t v_t(s,x)dsdxdt \right| \\
\leq &(l(k))^n k^n \int_0^T \int_\Omega g(w_t(t,x)) w_t(t,x)dxdt \\
&+ \frac{\ln(T+1)}{(l(k))^n} \int_0^T \int_\Omega g(v_t(t,x)) v_t(t,x)dxdt,
\end{aligned}$$

which together with (3.21) implies

$$\begin{aligned}
&\left| \int_0^T \int_\Omega g(w_t(t,x)) \frac{1}{t+1} \int_0^t v_t(s,x)dsdxdt \right| \\
&\leq \left((l(k))^n k^n + \frac{\ln(T+1)}{(l(k))^n} \right) c_6(\|(u_0,\ u_1)\|_{\mathcal{H}}), \quad \forall\, T \geq 0,\ n = 1, 2, \ldots . \qquad (3.26)
\end{aligned}$$

By the same way we find

$$\begin{aligned}
&\left| \int_0^T \int_\Omega g(u_t(t,x)) \frac{1}{t+1} \int_0^t v_t(s,x)dsdxdt \right| \\
&\leq \left((l(k))^n k^n + \frac{\ln(T+1)}{(l(k))^n} \right) c_8(\|(u_0,\ u_1)\|_{\mathcal{H}}), \ \forall\, T \geq 0, n = 1, 2, \ldots , \qquad (3.27)
\end{aligned}$$

where $c_8(\cdot): R_+ \to R_+$ is a nondecreasing function. By definition of $\widehat{u}_0(x)$, we have

$$\left|\int_0^T\int_\Omega g(w_t(t,x))\frac{1}{t+1}(u_0(x)-\widehat{u}_0(x))dxdt\right| \le m\int_0^T\int_\Omega \frac{1}{t+1}\left|g(w_t(t,x))\right| dxdt$$

$$\le m\int_0^T \int_{\{x:x\in\Omega,\ |w_t(t,x)|\le\rho\}} \frac{1}{t+1}\left|g(w_t(t,x))\right| dxdt$$

$$+ m\int_0^T \int_{\{x:x\in\Omega,\ |w_t(t,x)|>\rho\}} \frac{1}{t+1}\left|g(w_t(t,x))\right| dxdt$$

$$\le mg(\rho)\ln(T+1)mes\Omega + \frac{m}{\rho}\int_0^T\int_\Omega \frac{1}{t+1}g(w_t(t,x))w_t(t,x)dxdt$$

$$\le mg(\rho)\ln(T+1)mes\Omega + \frac{m}{\rho}c_6(\|(u_0,\ u_1)\|_{\mathcal{H}}), \quad \forall\, T\ge 0,\ \forall\, \rho>0. \tag{3.28}$$

Similarly we have

$$\left|\int_0^T\int_\Omega g(u_t(t,x))\frac{1}{t+1}(u_0(x)-\widehat{u}_0(x))dxdt\right|$$

$$\le mg(\rho)\ln(T+1)mes\Omega + \frac{m}{\rho}c_8(\|(u_0,\ u_1)\|_{\mathcal{H}}), \quad \forall\, T\ge 0,\ \forall\, \rho>0. \tag{3.29}$$

Taking into account (3.26)-(3.29) in (3.23) we find

$$\frac{1}{2}\int_0^T \frac{1}{t+1}\left\|\nabla v(t)\right\|^2 dt \le \frac{1}{2}\ln(T+1)\left\|\nabla\widehat{u}_0\right\|^2 + \int_0^T \frac{1}{t+1}\left\|v_t(t)\right\|^2 dt$$

$$+\left((l(k))^n k^n + \frac{\ln(T+1)}{(l(k))^n} + \frac{m}{\rho}\right)\left(c_6(\|(u_0,\ u_1)\|_{\mathcal{H}}) + c_8(\|(u_0,\ u_1)\|_{\mathcal{H}})\right)$$

$$+ 2mg(\rho)\ln(T+1)mes\Omega + \frac{1}{T+1}\left\|v_t(T)\right\|\left\|v(T)-\widehat{u}_0\right\| + \frac{1}{2}\left\|u_0-\widehat{u}_0\right\|^2$$

$$+ \left\|u_1\right\|\left\|u_0-\widehat{u}_0\right\|, \quad \forall\, T\ge 0,\ \forall\, \rho>0,\ \forall\, m>0,\quad n=1,2,...,$$

which together with (2.1) and (3.20) yields

$$\ln(T+1)E(v(T),v_t(T)) \le \int_0^T \frac{1}{t+1}E(v(t),v_t(t))dt$$

$$\le \frac{1}{2}\ln(T+1)\int_{\{x:x\in\Omega,\ |u_0(x)|>m\}} \left|\nabla u_0\right|^2 dx + 2mg(\rho)\ln(T+1)mes\Omega$$

$$+ c_9(\|(u_0,\ u_1)\|_{\mathcal{H}})\left((l(k))^n k^n + \frac{\ln(T+1)}{(l(k))^n} + \frac{m}{\rho}\right)$$

$$+ c_9(\|(u_0,\ u_1)\|_{\mathcal{H}}), \quad \forall\, T>0,\ \forall\, \rho>0,\ \forall\, m>0,\quad n=1,2,\dots,$$

where $c_9(\cdot): R_+ \to R_+$ is a nondecreasing function. Choosing m and n large enough then ρ small enough, from the last inequality we obtain (3.22) for large T. □

Now we can prove the asymptotic compactness property of solutions, which is included in the following theorem:

Theorem 3.1. *Assume Assumption 2.1 holds and $B \in \mathfrak{B}$. Then any sequence of the form $\{S(t_n)\theta_n\}_{n=1}^{\infty}$, $t_n \to \infty$, $\theta_n \in O_{\varepsilon_n}(B)$, $\varepsilon_n \searrow 0$, has a convergent subsequence in $\mathcal{H}$.*

Proof. Since by Lemma 3.1 the sequence $\{K(t_n)\theta_n\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{H}$, there exists a subsequence $\{K(t_{n_m})\theta_{n_m}\}_{m=1}^{\infty}$ which converges in $\mathcal{H}$. So for any $\varepsilon > 0$ there exists $N_1 = N_1(\varepsilon)$, such that

$$\|K(t_{n_m})\theta_{n_m} - K(t_{n_k})\theta_{n_k}\|_{\mathcal{H}} \le \frac{\varepsilon}{3}, \quad \forall\, m, k \ge N_1. \tag{3.30}$$

On the other hand by Lemma 3.2, there exists $N_2 = N_2(\varepsilon)$, such that

$$\|S(t_{n_m})\theta_{n_m} - K(t_{n_m})\theta_{n_m}\|_{\mathcal{H}} \le \frac{\varepsilon}{3}, \quad \forall\, m \ge N_2,$$

which together with (3.30) gives that

$$\|S(t_{n_m})\theta_{n_m} - S(t_{n_k})\theta_{n_k}\|_{\mathcal{H}} \le \varepsilon, \quad \forall\, m, k \ge \max\{N_1,\ N_2\}.$$

In other words $\{S(t_{n_m})\theta_{n_m}\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}$ and consequently converges in $\mathcal{H}$. □

From this theorem immediately the following corollary follows.

Corollary 1. *Under Assumption 2.1 for every $B \in \mathfrak{B}$, the sets $\omega(B) = \bigcap_{t\ge 0} \overline{\bigcup_{\tau \ge t} S(\tau)B}$ and $\widehat{\omega}(B) = \bigcap_{\varepsilon>0} \bigcap_{t\ge 0} \overline{\bigcup_{\tau\ge t} S(\tau)O_\varepsilon(B)}$ are nonempty invariant compacts which attract B.*

4. **Proof of Theorem 2.3.** Since $\theta \in \mathfrak{B}$, for every $\theta \in \mathcal{H}$, by Corollary 3.1 the set $\omega(\theta) = \bigcap_{t\ge 0} \overline{\bigcup_{\tau\ge t} S(\tau)\theta}$ is nonempty invariant compact which attracts a semitrajectory beginning from θ. Set

$$Z = \{\varphi \in \mathcal{H},\ S(t)\varphi = \varphi,\ \forall\, t \ge 0\}.$$

It is easy to see that under condition (2.5) the set Z belongs to $\mathfrak{B}$.

Let $\theta = (u_0, u_1) \in \mathcal{H}$ and $(\varphi_0, \varphi_1) \in \omega(\theta)$. By the definition of $\omega(\theta)$ there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$ and

$$S(t_n)(u_0, u_1) \to (\varphi_0, \varphi_1) \text{ strongly in } \mathcal{H}. \tag{4.1}$$

On the other hand by (2.6) the Lyapunov function $L(u(t), u_t(t)) := E(u(t), u_t(t)) + \int_\Omega F(u(t,x))dx$ is nonincreasing and lower bounded on any semitrajectory $\bigcup_{t\ge 0}(u(t), u_t(t))$. So there exists a constant l, such that

$$\lim_{t\to\infty} L(S(t)(u_0, u_1)) = l,$$

which together with (4.1) yields

$$L(\varphi_0, \varphi_1) = l, \qquad \forall\, (\varphi_0, \varphi_1) \in \omega(\theta). \tag{4.2}$$

Set $(\varphi(t), \varphi_t(t)) = S(t)(\varphi_0, \varphi_1)$. Since $(\varphi(t), \varphi_t(t)) \in \omega(\theta)$, by (2.1), (2.6) and (4.2) we have $\varphi(t) \equiv \varphi_0$, $\varphi_1 = 0$ and consequently $(\varphi_0, \varphi_1) \in Z$. Thus $\omega(\theta) \subset Z$ for every $\theta \in \mathcal{H}$.

Now let us show that $\widehat{\omega}(Z) = \bigcap_{\varepsilon>0} \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} S(\tau)O_\varepsilon(Z)}$ is a global $(\mathcal{H},\mathcal{H})_{\mathfrak{B}}-$ attractor. By Corollary 3.1 the set $\widehat{\omega}(Z)$ is invariant and compact. Hence it is sufficient to prove that $\widehat{\omega}(Z)$ attracts every element of $\mathfrak{B}$. Assume it is not true. Then there exist $\varepsilon_0 > 0$, $B \in \mathfrak{B}$, $\{v_n\} \subset B$, $t_n \to \infty$ such that

$$\inf_{u\in\widehat{\omega}(Z)} \|S(t_n)v_n - u\| \geq \varepsilon_0, \quad n = 1, 2, ... \; . \tag{4.3}$$

By Theorem 3.1 and Corollary 3.1 for any $\delta > 0$ there exists $T(\delta) > 0$ such that

$$S(t)B \subset O_\delta(\omega(B)), \qquad \forall\, t \geq T(\delta). \tag{4.4}$$

On the other hand as shown above for any $\varepsilon > 0$ and $\varphi \in \omega(B)$ there exists $t(\varepsilon, \varphi) > 0$ such that

$$S(t)\varphi \in O_\varepsilon(\omega(\varphi)) \subset O_\varepsilon(Z), \quad \forall\, t \geq t(\varepsilon, \varphi). \tag{4.5}$$

By the continuity of $S(t)$ and (4.5) there exists $\delta(t(\varepsilon,\varphi), \varphi) > 0$ such that

$$S(t(\varepsilon,\varphi))O_{\delta(t(\varepsilon,\varphi),\varphi)}(\varphi) \in O_\varepsilon(Z).$$

Since $\omega(B)$ is compact and $\omega(B) \subset \bigcup_{\varphi\in\omega(B)} O_{\delta(t(\varepsilon,\varphi),\varphi)}(\varphi)$, there exist $\delta_\varepsilon > 0$ and a finite number $O_{\delta_i^\varepsilon}(\varphi_i^\varepsilon)$, t_i^ε $(i = \overline{1, k_\varepsilon})$ such that

$$\begin{cases} O_{\delta_\varepsilon}(\omega(B)) \subset \bigcup_{i=1}^{k_\varepsilon} O_{\delta_i^\varepsilon}(\varphi_i^\varepsilon) \\ S(t_i^\varepsilon)O_{\delta_i^\varepsilon}(\varphi_i^\varepsilon) \in O_\varepsilon(Z), \quad i = \overline{1, k_\varepsilon} \end{cases} .$$

Now let $\varepsilon_m \searrow 0$. Then by the argument above done for every $m \in \mathbb{N}$, there exist $\delta_m > 0$, $k_m \in \mathbb{N}$, $\{\varphi_i^m\}_{i=1}^{k_m} \subset \omega(B)$, $\delta_i^m > 0$ $(i = \overline{1, k_m})$ and $t_i^m > 0$ $(i = \overline{1, k_m})$ such that

$$\begin{cases} O_{\delta_m}(\omega(B)) \subset \bigcup_{i=1}^{k_m} O_{\delta_i^m}(\varphi_i^m) \\ S(t_i^m)O_{\delta_i^m}(\varphi_i^m) \in O_{\varepsilon_m}(Z), \quad i = \overline{1, k_m} \end{cases} . \tag{4.6}$$

Since $t_n \to \infty$, for every $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ such that

$$t_{n_m} \geq m + T(\delta_m) + \max_{1\leq i\leq k_m} t_i^m.$$

On the other hand by (4.4) and (4.6) for every v_{n_m} there exists $t_{i_m}^m$ $(i_m \in \{1, 2, ..., k_m\})$ such that $S(T(\delta_m) + t_{i_m}^m)v_{n_m} \in O_{\varepsilon_m}(Z)$. So setting $\tau_m = t_{n_m} - T(\delta_m) - t_{i_m}^m$ and $w_m = S(T(\delta_m) + t_{i_m}^m)v_{n_m}$ we have $\tau_m \to \infty$ as $m \to \infty$ and $w_m \subset O_{\varepsilon_m}(Z)$ for every $m \in \mathbb{N}$. Consequently by Theorem 3.1 and Corollary 3.1 there exists a subsequence $\{m_\nu\}$ such that

$$\lim_{\nu\to\infty} \inf_{u\in\widehat{\omega}(Z)} \|S(\tau_{m_\nu})w_{m_\nu} - u\| = 0,$$

or

$$\lim_{\nu\to\infty} \inf_{u\in\widehat{\omega}(Z)} \left\|S(t_{n_{m_\nu}})v_{n_{m_\nu}} - u\right\| = 0.$$

The last equality contradicts (4.3). Thus $\widehat{\omega}(Z)$ is a global $(\mathcal{H},\mathcal{H})_{\mathfrak{B}}-$ attractor.

REFERENCES

[1] J. Arrietta, A. Carvalho and J. Hale, *A damped hyperbolic equations with critical exponents*, Commun. PDE, **17** (1992), 841–866.

[2] A. V. Babin and M. I. Vishik, "Attractors of Evolution Equations," 1^{st} edition, North-Holland, Amsterdam, 1992.

[3] I. Chueshov and I. Lasiecka, *Attractors for second order evolution equations with a nonlinear damping*, J. Dynamics and Differential Equations, **16** (2004), 469–512.

[4] I. Chueshov and I. Lasiecka, "Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping," Memoirs of AMS, 2007.

[5] E. Feireisl and E. Zuazua, *Global attractors for semilinear wave equations with locally distributed nonlinear damping and critical exponent*, Commun. PDE., **18** (1993), 1538–1555.

[6] E. Feireisl, *Attractors for wave equations with nonlinear dissipation and critical exponent*, C. R. Acad. Sci. Paris, **315** (1992), 551–555.

[7] E. Feireisl, *Finite dimensional asymptotic behavior of some semilinear damped hyperbolic problems*, J. Dynamics and Differential Equations, **6** (1994), 23–35.

[8] E. Feireisl, *Global attractors for semilinear damped wave equations with supercritical exponent*, J. Differential Equations, **116** (1995), 431–447.

[9] J. Hale, "Asymptotic Behavior of Dissipative Systems," 1^{st} edition, AMS, Providence, 1988.

[10] A. Kh. Khanmamedov, *Global attractors for wave equations with nonlinear interior damping and critical exponents*, J. Differential Equations, **230** (2006), 702–719.

[11] M. Krasnoselskii and Y. Rutickii, "Convex Functions and Orlicz Spaces," 1^{st} edition, P. Noordhoff Ltd., Groningen, 1961.

[12] O. A. Ladyzhenskaya, *On the determination of minimal global attractors for the Navier-Stokes equations and other partial differential equations*, UMN, **42** (1987), 25–60 [English translation; Russian Math. Surveys, **42** (1987), 27–73].

[13] I. Lasiecka and A. R. Ruzmaikina, *Finite dimensionality and regularity of attractors for 2-D semilinear wave equation with nonlinear dissipation*, J. Math. Anal. Appl., **270** (2002), 16–50.

[14] J. L. Lions and E. Magenes, "Non-homogeneous Boundary Value Problems and Applications," 1^{st} edition, Springer-Verlag Berlin Heidelberg, New York, vol. **I**, 1972.

[15] G. Raugel, *Une equation des ondes avec amortissment non lineaire dans le cas critique en dimensions trois*, C. R. Acad.Sci. Paris, **314** (1992), 177–182.

[16] C. Sun, M. Yang and C. Zhong, *Global attractors for the wave equation with nonlinear damping*, J. Differential Equations, **227** (2006), 427–443.

[17] R. Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," 1^{st} edition, Springer-Verlag, New York, 1988.

Received October 2007; revised March 2008.

E-mail address: `azer@hacettepe.edu.tr`