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Dicompleteness and real dicompactness of ditopological texture spaces

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A R T I C L E   I N F O

Dedicated to the memory of Melvin Henriksen

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A B S T R A C T

The authors consider interrelations between the completeness of certain initial di-uniformities and the real dicompactness of completely biregular bi-T_2 nearly plain ditopological spaces. Completions and real dicompactifications of almost plain spaces are also considered.

1. Introduction

This paper continues the study of dicompact and real dicompact completely biregular bi-T_2 spaces begun in [24–26], and in particular relates this work with the completeness of certain compatible di-uniformities. Our source of inspiration is again [2, Chapter 3] on bitopological notions of compactness and real compactness, and other papers in this area such as [13]. Indeed, there is a close relationship between the bitopological and ditopological theory restricted to plain textures, as exemplified by the isomorphism $\mathfrak{K}: \mathbf{pCReg_{w2}} \to \mathbf{ifPCbiR_2}$ between the construct of pairwise completely regular weakly pairwise Hausdorff bitopological spaces in the sense of J.C. Kelly [16] and pairwise continuous functions, and the category of completely biregular bi-T_2 plain ditopological texture spaces and ω-preserving bicontinuous functions. See also [22] for a discussion of this functor in a wider context.

Unlike the classical case [15], the family $\mathrm{BDF}(S)$ of real bicontinuous difunctions [12,24], and the family $\mathrm{BA}(S)$ of real ω-preserving bicontinuous point functions [24], are not rings but T-lattices [2]. Such parts of the theory of T-lattices developed in [2] as are required here have already been given in [24–26], and will not be repeated. A theory of di-uniformities on textures has been developed in [18], and the relationship with classical uniformities and quasi-uniformities discussed in [19]. In this paper we shall find it convenient to use the representation in terms of dicovers. In a closely related direction [27]

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gives an analogue of quasi-proximities for textures. The subject of completeness and total boundedness for di-uniformities is discussed in [21], and a completion of a di-uniformity on a plain texture is constructed in [20]. We refer the reader to [10] for a discussion of normal dicovers that will be needed at one point in the text, to [11] for results on compactness and stability, and to [23] for a discussion on the representation of real difunctions. We note that there is open access to [11,12,21,23,27] from www.mat.hacettepe.edu.tr/hjms.

The standard references for textures and ditopological texture spaces are [4–9]. However, although we cite these papers at various points in the text, most if not all of these concepts and results have been repeated in the papers mentioned earlier. Our standard reference for notions and results from category theory is [1], while the reader may consult [14] for terms from lattice theory not mentioned here.

Since our main aim in considering completeness is to link this with our earlier work on real dicompactness, we continue to work within the same framework. Hence, in Section 2, we begin by extending the notion of a nearly plain extension of an almost plain ditopological texture space introduced in [26] to a uniform nearly plain extension, and in particular to a nearly plain dicompletion. This section contains several results that are important in their own right, and which will also be needed later on. It cumulates in the establishment of a separated nearly plain dicompletion reflector.

Section 3 gives the promised link between real dicompactness and dicompleteness of a suitable di-uniformity. It shows that the Stone–Čech and Hewitt reflectors given earlier are particular instances of the dicompletion reflector, and concludes with some results on the existence of minimal compatible di-uniformities that generalize and improve results in [2].

2. Nearly plain dicomplete extensions

We begin by extending the notion of nearly plain extension of almost plain ditopological texture spaces, discussed in [26], to nearly plain dicomplete extensions.

First we recall that if $(S,\mathcal{S})$ is a texture and $U \subseteq S$ has the property that $\mathcal{U} = \mathcal{S}_U = \{A \cap U \mid A \in \mathcal{S}\}$ is a texturing of U, then $(U,\mathcal{U})$ is called an induced subtexture of $(S,\mathcal{S})$. Moreover, if $(S,\mathcal{S},\tau,\kappa)$ is a ditopological texture space, an induced subtexture $(U,\mathcal{S}_U)$ equipped with the induced ditopology (τ_U,κ_U) is referred to as an induced subspace.

Finally, if $(S,\mathcal{S})$ is nearly plain, then $(U,\mathcal{S}_U,\tau_U,\kappa_U)$ is said to be a dense subspace of $(S,\mathcal{S},\tau,\kappa)$ if

1. $\varphi_p^S(U) \subseteq U_p$, and
2. U is dense in S under the joint topology of (τ,κ).

Now let $(S,\mathcal{S})$ be a texture and υ a dicovering uniformity on $(S,\mathcal{S})$. If $(U,\mathcal{S}_U)$ is an induced subtexture of $(S,\mathcal{S})$, we may define the *induced dicovering uniformity* υ_U *on* $(U,\mathcal{S}_U)$ as the covering uniformity on $(U,\mathcal{S}_U)$ with base $\{\mathcal{C}|_U \mid \mathcal{C} \in \upsilon\}$, where $\mathcal{C}|_U = \{(A\cap U, B\cap U) \mid (A,B) \in \mathcal{C}\}$, and then $(U,\mathcal{S}_U,\upsilon_U)$ will be called a *uniform induced subspace*.

In case $(S,\mathcal{S})$ is nearly plain, $(U,\mathcal{S}_U,\upsilon_U)$ will be called a *dense uniform subspace* if the above density condition holds for the uniform ditopologies.

Now we may give:

Definition 2.1. The diuniform nearly plain texture space $(V,\mathcal{V},\nu)$ will be called a *nearly plain uniform extension* of the (necessarily almost plain) uniform texture space $(S,\mathcal{S},\upsilon)$ if $(S,\mathcal{S},\upsilon)$ is uniformly di-isomorphic to a dense uniform subspace of $(V,\mathcal{V},\nu)$.

In case $(V,\mathcal{V},\nu)$ is dicomplete it is called a *nearly plain dicompletion* of $(S,\mathcal{S},\upsilon)$.

Here, by a uniform di-isomorphism we mean a bijective difunction which, together with its inverse, is uniformly bicontinuous.

Naturally, the notion of nearly plain dicompletion includes that of plain dicompletion as a special case. Clearly, a dicomplete nearly plain uniform texture space is a nearly plain dicompletion of itself.

The concept of canonical nearly plain dicompletion can also be defined in the obvious way. Namely, $(V,\mathcal{V},\nu)$ is called a *canonical nearly plain dicompletion* of $(S,\mathcal{S},\upsilon)$ if $(S_p,\mathcal{S}_p,\upsilon_p)$ is regarded as an induced uniform subspace of $(V,\mathcal{V},\nu)$, and hence of $(V_p,\mathcal{V}_p,\nu_p)$.

First we give the following important lemma.

Lemma 2.2. *Let $(S,\mathcal{S})$ be an almost plain texture with associated plain texture $(S_p,\mathcal{S}_p)$ and canonical isomorphism $(f_\epsilon,F_\epsilon) : (S_p,\mathcal{S}_p) \to (S,\mathcal{S})$. Let υ be a dicovering uniformity on $(S,\mathcal{S})$, and denote by υ_p the induced dicovering uniformity on $(S_p,\mathcal{S}_p)$. Then $(f_\epsilon,F_\epsilon) : (S_p,\mathcal{S}_p,\upsilon_p) \to (S,\mathcal{S},\upsilon)$ is a uniform di-isomorphism.*

Proof. As in the proof of [26, Lemma 2.10] we have $f_\epsilon^{\leftarrow} A = F_\epsilon^{\leftarrow} A = A \cap S_p$ for each $A \in \mathcal{S}$, so $(f_\epsilon,F_\epsilon)^{-1}(\mathcal{C}) = \mathcal{C}|_{S_p}$ and we see that (f_ϵ,F_ϵ) is uniformly bicontinuous. On the other hand (f_ϵ,F_ϵ) is known to be bijective by [26, Lemma 3.5(3)], so its inverse is $(F_\epsilon^{\leftarrow}, f_\epsilon^{\leftarrow})$. Now for $A \in \mathcal{S}$,

$$\left(F_\epsilon^{\leftarrow}\right)^{\leftarrow} A \cap S_p = \left(\left(F_\epsilon^{\leftarrow}\right)^{\leftarrow}\right)^{\to} A \cap S_p = F_\epsilon^{\to} A \cap S_p.$$

Hence, by [7, Corollary 2.33(2)] we have $A \cap S_p = f_\epsilon^{\leftarrow}(F_\epsilon^{\to} A \cap S_p) = (F_\epsilon^{\to} A \cap S_p) \cap S_p$ since (f_ϵ, F_ϵ) is injective. Hence $A \cap S_p = ((F_\epsilon^{\leftarrow})^{\leftarrow} A \cap S_p) \cap S_p$, whence $A = (F_\epsilon^{\leftarrow})^{\leftarrow} A \cap S_p$ by the definition of almost plain texture. Likewise, $A = (f_\epsilon^{\leftarrow})^{\leftarrow} A \cap S_p$, and we deduce that $(F_\epsilon^{\leftarrow}, f_\epsilon^{\leftarrow})^{-1}(\mathcal{C}|S_p) = \mathcal{C}$. Thus the inverse of (f_ϵ, F_ϵ) is also uniformly bicontinuous. □

Lemma 2.3. *Let $(V, \mathcal{V}, \nu)$ be an almost plain dicovering uniform texture space. Then $(V, \mathcal{V}, \nu)$ is dicomplete if and only if the plain space $(V_p, \mathcal{V}_p, \nu_p)$ is dicomplete.*

Proof. We sketch the proof of necessity, leaving that of sufficiency to the interested reader.

Let $\mathcal{F}_p \times \mathcal{G}_p$ be a regular Cauchy difilter in $(V_p, \mathcal{V}_p, \nu_p)$ and define $\mathcal{F} = \{A \in \mathcal{V} \mid A \cap V_p \in \mathcal{F}_p\}$, $\mathcal{G} = \{A \in \mathcal{V} \mid A \cap V_p \in \mathcal{G}_p\}$. Clearly $\mathcal{F} \times \mathcal{G}$ is a regular Cauchy difilter in $(V, \mathcal{V}, \nu)$, hence is diconvergent. Thus there exists $v_1, v_2 \in V$ with $P_{v_2} \not\subseteq Q_{v_1}$ for which $\mathcal{F} \to v_1$ and $\mathcal{G} \to v_2$. Now we may choose $v \in V_p$ with $P_{v_2} \not\subseteq Q_v$ and $P_v \not\subseteq Q_{v_1}$, and it is straightforward to verify that $\mathcal{F}_p \to v$ and $\mathcal{G}_p \to v$. □

Proposition 2.4. *A uniform di-isomorphism between almost plain uniform texture spaces preserves dicompleteness.*

Proof. A uniform di-isomorphism induces a textural isomorphism between the corresponding plain spaces, and this clearly preserves dicompleteness. The proof is now completed by applying Lemma 2.3. □

Applying the above results to the nearly plain case shows that, up to a uniform di-isomorphism, there is no loss of generality in considering plain dicompletions in place of nearly plain dicompletions.

Theorem 2.5. *Every almost plain di-uniform space $(S, \mathcal{S}, \upsilon)$ has a plain dicompletion.*

Proof. In [20] a dicompletion of an arbitrary plain di-uniform texture space was constructed. Applying this construction to the plain space $(S_p, \mathcal{S}_p, \upsilon_p)$ gives a dicomplete dicovering uniform space $(\widetilde{S}_p, \widetilde{\mathcal{S}}_p, \widetilde{\upsilon}_p)$, called the prime dicompletion of $(S_p, \mathcal{S}_p, \upsilon_p)$, and a uniformly bicontinuous difunction (e, E) between $(S_p, \mathcal{S}_p, \upsilon_p)$ and a dense subspace of $(\widetilde{S}_p, \widetilde{\mathcal{S}}_p, \widetilde{\upsilon}_p)$. It is easy to verify that the inverse of (e, E) on this dense subspace is also uniformly bicontinuous, and that the notion of density used in [20] coincides with that in Definition 2.1. By Lemma 2.2 the composition $(e, E) \circ (F_\epsilon^{\leftarrow}, f_\epsilon^{\leftarrow})$ gives us a uniform di-isomorphism of $(S, \mathcal{S}, \upsilon)$ with a dense subspace of the dicomplete plain space $(\widetilde{S}_p, \widetilde{\mathcal{S}}_p, \widetilde{\upsilon}_p)$, whence this is a plain dicompletion of $(S, \mathcal{S}, \upsilon)$, as required. □

Corollary 2.6. *Every almost plain separated di-uniform space $(S, \mathcal{S}, \upsilon)$ has a separated plain dicompletion.*

Proof. In place of the prime dicompletion in the proof of Theorem 2.5 it is sufficient to take the separated prime dicompletion [20], which is just its T_0 quotient [3]. □

Proposition 2.7. *Let $(S, \mathcal{S}, \upsilon)$ be a nearly plain dicovering uniform texture space, $(U, \mathcal{S}_U, \upsilon_U)$ a uniform dense subspace of S_p, $(V, \mathcal{V}, \nu)$ a dicomplete separated almost plain dicovering uniform texture space and $\varphi : U \to V$ a uniformly bi-continuous ω-preserving point function. Then there exists a uniformly bicontinuous extension $\widehat{\varphi}$ of φ to S which is ω-preserving and satisfies $\widehat{\varphi}(S) \subseteq V_p$. Moreover, if φ_1, φ_2 are any two such extensions of φ then for $u, s, v \in S$,*

$$u\,\omega\, s,\ s\,\omega\, v \quad \Rightarrow \quad Q_{\varphi_1(u)} \subseteq Q_{\varphi_2(s)} \subseteq Q_{\varphi_1(v)}. \tag{2.1}$$

Proof. First we extend φ to S_p. For $s \in S_p$, define

$$\mathcal{F}_s = \left\{ B \in \mathcal{V} \;\middle|\; \exists G_j \in \tau_\upsilon \text{ with } G_j \not\subseteq Q_s,\ j = 1, \ldots, n,\ \left(\bigcap_{j=1}^{n} G_j \right) \cap U \subseteq \varphi^{\leftarrow} B \right\},$$

$$\mathcal{G}_s = \left\{ B \in \mathcal{V} \;\middle|\; \exists K_j \in \kappa_\upsilon \text{ with } P_s \not\subseteq K_j,\ j = 1, \ldots, n,\ \varphi^{\leftarrow} B \subseteq \left(\bigcup_{j=1}^{n} K_j \right) \cap U \right\}.$$

Clearly $\mathcal{F}_s \times \mathcal{G}_s$ is a difilter on $(V, \mathcal{V}, \nu)$, and it is regular because U is dense in S for the joint topology of υ. We show it is ν-Cauchy. To this end take $\mathcal{D} \in \nu$. Since φ is uniformly bicontinuous, $\varphi^{-1}(\mathcal{D}) \in \upsilon_U$ so we have $\mathcal{C} \in \upsilon$ with $\mathcal{C}|_U \prec \varphi^{-1}(\mathcal{D})$. As $\eta^*(s) \times \mu^*(s)$ is diconvergent in S_p it is υ_p-Cauchy by [21, Proposition 3.2], hence we have $(C_1, C_2) \in \mathcal{C} \cap (\eta^*(s) \times \mu^*(s))$. Now we have $D_1 \mathcal{D} D_2$ with $C_1 \cap U \subseteq \varphi^{\leftarrow} D_1$, $\varphi^{\leftarrow} D_2 \subseteq C_2 \cap U$, and we deduce $(D_1, D_2) \in \mathcal{D} \cap (\mathcal{F}_s \times \mathcal{G}_s) \neq \emptyset$, as required.

Since $(V, \mathcal{V}, \nu)$ is dicomplete, $\mathcal{F}_s \times \mathcal{G}_s$ is diconvergent, say $\mathcal{F}_s \to v_1$, $\mathcal{G}_s \to v_2$ with $v_1\, \omega\, v_2$. Now we have $v \in V_p$ with $v_1\, \omega\, v$, $v\, \omega\, v_2$, and clearly $\mathcal{F}_s \to v$, $\mathcal{G}_s \to v$ by [21, Lemma 2.7]. This point $v \in V_p$ is unique. Indeed, if $\mathcal{F}_s \to v'$, $\mathcal{G}_s \to v'$ for $v' \neq v$ in V_p, we may assume without loss of generality that $P_v \not\subseteq P_{v'}$, whence $Q_v \not\subseteq Q_{v'}$ as v' is a plain point. Since the uniform ditopology is completely biregular and T_0, hence bi-T_2, we have $H \in \tau_\nu$, $K \in \kappa_\nu$ with $H \subseteq K$, $P_v \nsubseteq K$ and $H \nsubseteq Q_{v'}$. This leads at once to $H \in \mathcal{F}_s$, $K \in \mathcal{G}_s$, which contradicts the fact that $\mathcal{F}_s \times \mathcal{G}_s$ is regular.

We set $\widehat{\varphi}_p(s) = v$, so defining a function $\widehat{\varphi}_p : S_p \to V$ satisfying $\widehat{\varphi}_p(S_p) \subseteq V_p$. To show that $\widehat{\varphi}_p$ extends φ, take $u \in U$. Since $U \subseteq S_p$, u is a plain point, so certainly $\varphi(u) \in V_p$ since φ is ω-preserving. It is straightforward to verify that $\mathcal{F}_u \to \varphi(u)$, $\mathcal{G}_u \to \varphi(u)$, whence we have $\widehat{\varphi}_p(u) = \varphi(u)$, as required.

The proof that $\widehat{\varphi}_p$ is ω-preserving is left to the interested reader, so we establish uniform bicontinuity. Hence, take $\mathcal{D} \in \nu$ and $\mathcal{C} \in \upsilon$ with $\mathcal{C}|_U \prec \varphi^{-1}(\mathcal{D})$, as above. Without loss of generality we may assume $\mathcal{C}$ is open, co-closed and that $\mathcal{D}$ is closed, co-open. We claim that $\mathcal{C} \prec \widehat{\varphi}_p^{-1}(\mathcal{D})$. Take $C_1 \,\mathcal{C}\, C_2$ and $D_1 \,\mathcal{D}\, D_2$ with $C_1 \cap U \subseteq \varphi^{\leftarrow} D_1$, $\varphi^{\leftarrow} D_2 \subseteq C_2 \cap U$ and suppose that $C_1 \nsubseteq \widehat{\varphi}_p^{\leftarrow} D_1$. Then for some $s \in S_p$, $C_1 \nsubseteq Q_s$ and $P_s \nsubseteq \widehat{\varphi}_p^{\leftarrow} D_1$. Now since $C_1 \in \tau_\upsilon$ we obtain $D_1 \in \mathcal{F}_s$, while since $\mathcal{G}_s \to \widehat{\varphi}_p(s)$, $P_{\widehat{\varphi}_p(s)} \nsubseteq D_1 \in \kappa_\nu$ gives $D_1 \in \mathcal{G}_s$, which contradicts the regularity of $\mathcal{F}_s \times \mathcal{G}_s$. Hence $C_1 \subseteq \widehat{\varphi}_p^{\leftarrow} D_1$, and likewise $\widehat{\varphi}_p^{\leftarrow} D_2 \subseteq C_2$, as required.

We now extend $\widehat{\varphi}$ to S by composing it with $\varphi_p : S \to S_p$, which is ω-preserving and uniformly bicontinuous since clearly $\varphi_p^{\leftarrow} \mathcal{C}|_{S_p} = \mathcal{C}$ for any $\mathcal{C} \in \upsilon$. Hence, $\widehat{\varphi} = \widehat{\varphi}_p \circ \varphi_p$ is an extension of φ to S with the required properties.

Finally (2.1) follows from the density of U and the fact that the ditopology on $(V, \mathcal{V}, \nu)$ is bi-T_2, and we omit the details. □

Generally (2.1) does not give the uniqueness of the extension in Proposition 2.7, but it clearly does if $(S, \mathcal{S})$ is plain, for then we may take $u = s = v$.

Proposition 2.8. *Any two nearly plain separated dicompletions of an almost plain separated uniform texture space are uniformly di-isomorphic under a uniform di-isomorphism that can be taken to preserve S_p.*

Proof. By Lemma 2.2 there is no loss of generality in assuming that the given dicompletions $(V_1, \mathcal{V}_1, \nu_1)$, $(V_2, \mathcal{V}_2, \nu_2)$ of $(S, \mathcal{S}, \upsilon)$ are plain, or in assuming that they are canonical plain dicompletions, so that $(S_p, \mathcal{S}_p, \upsilon_p)$ is a dense induced uniform subspace of both of them. Denote by φ_k, $k = 1, 2$, the identity on S_p regarded as mapping $S_p \subseteq V_k$ into V_l, $l \neq k$. Then certainly φ_k, $k = 1, 2$, satisfies the conditions of Proposition 2.7, so has an extension $\widehat{\varphi}_k : V_k \to V_l$ which is an ω-preserving uniformly bicontinuous point function. To complete the proof it is sufficient to show that $\widehat{\varphi}_l$ is the inverse of $\widehat{\varphi}_k$. Suppose that $w = \widehat{\varphi}_l(\widehat{\varphi}_k(v)) \neq v$ for some $v \in V_k$.

Consider the case $P_w \nsubseteq P_v$. Then $Q_w \nsubseteq Q_v$ as the texture is plain, so as the uniform ditopology on $(V_k, \mathcal{V}_k, \nu_k)$ is bi-T_2 we have an open set H and a closed set K with $H \subseteq K$, $v \in H$ and $w \notin K$. Since $\widehat{\varphi}_l$ and $\widehat{\varphi}_k$ are cocontinuous

$$F = \widehat{\varphi}_k^{\leftarrow}\big(\widehat{\varphi}_l^{\leftarrow} K\big)$$

is a closed set in V_k and clearly $P_v \nsubseteq F$. Hence $v \in H \setminus F \neq \emptyset$, whence by the density of S_p in V_k we have $s \in S_p \cap (H \setminus F)$. But $\widehat{\varphi}_k(s) = \varphi_k(s) = s$, $\widehat{\varphi}_l(s) = \varphi_l(s) = s$, which easily leads to the contradiction $s \in F$. Likewise, a contradiction is obtained for the case $P_v \nsubseteq P_w$, and the proof is complete. □

We now present a second corollary of Proposition 2.7.

Corollary 2.9. *With $(S, \mathcal{S}, \upsilon)$ almost plain, $(U, \mathcal{S}_U, \upsilon_U)$, $(V, \mathcal{V}, \nu)$ and φ as in Proposition* 2.7, *there exists a unique uniformly bicontinuous difunction $(h, H) : (S, \mathcal{S}, \upsilon) \to (V, \mathcal{V}, \nu)$ which extends φ in the sense that $(h, H) \circ (e, E) = (f_\varphi, F_\varphi)$, where $(e, E) : (U, \mathcal{S}_U) \to (S, \mathcal{S})$ is the inclusion difunction.*

Proof. Again consider the extension $\widehat{\varphi}_p$ to S_p, and the corresponding difunction $(f_{\widehat{\varphi}_p}, F_{\widehat{\varphi}_p}) : (S_p, \mathcal{S}_p, \upsilon_p) \to (V, \mathcal{V}, \nu)$, which is clearly uniformly bicontinuous. Recalling from Lemma 2.2 that $(F_\epsilon^{\leftarrow}, f_\epsilon^{\leftarrow}) : (S, \mathcal{S}, \upsilon) \to (S_p, \mathcal{S}_p, \upsilon_p)$ is a uniform di-isomorphism, we may set $(h, H) = (f_{\widehat{\varphi}_p}, F_{\widehat{\varphi}_p}) \circ (F_\epsilon^{\leftarrow}, f_\epsilon^{\leftarrow})$. It is easy to check that $(h, H) \circ (e, E) = (f_\varphi, F_\varphi)$, so we have an extension with the required properties.

To prove uniqueness, let (h_k, H_k), $k = 1, 2$ be extensions of φ with the given properties. By [7, Proposition 2.27] it is sufficient to show that $h_1 = h_2$, and hence to show that $h_1^{\leftarrow} A = h_2^{\leftarrow} A$ for all $A \in \mathcal{V}$. Suppose that $h_1^{\leftarrow} A \nsubseteq h_2^{\leftarrow} A$ for some $A \in \mathcal{V}$. Since $h_1^{\leftarrow} A = H_1^{\leftarrow} A$ we have $s \in S$, $t_1, t_2 \in V$ with $\overline{P}_{(s,t_1)} \nsubseteq H_1$, $A \nsubseteq Q_{t_1}$ and $h_2 \nsubseteq \overline{Q}_{(s,t_2)}$, $P_{t_2} \nsubseteq A$. We deduce $Q_{t_2} \nsubseteq Q_{t_1}$, so by the bi-$T_2$ property for $(V, \mathcal{V}, \tau_\nu, \kappa_\nu)$ we have $G \in \tau_\nu$, $K \in \kappa_\nu$ with $G \subseteq K$, $G \nsubseteq Q_{t_1}$ and $P_{t_2} \nsubseteq K$.

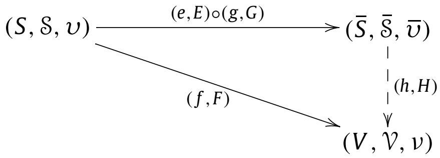
It follows now that $H_1^{\leftarrow} G \nsubseteq h_2^{\leftarrow} K$, while the given properties of (h_1, H_1), (h_2, H_2) imply that $H_1^{\leftarrow} G \in \tau_\upsilon$, $h_2^{\leftarrow} K \in \kappa_\upsilon$. The density of U now gives $u \in U$ with $u \in H_1^{\leftarrow} K \setminus h_2^{\leftarrow} K$, and arguing as above we obtain $v_1, v_2 \in V$ with $\overline{P}_{(u,v_1)} \nsubseteq H_1$, $h_2 \nsubseteq \overline{Q}_{(u,v_2)}$ and $P_{v_2} \nsubseteq P_{v_1}$. On the other hand (h_1, H_1), (h_2, H_2) are represented by φ on U, which leads to $h_k \nsubseteq \overline{Q}_{(u,\varphi(u))}$, $\overline{P}_{(u,\varphi(u))} \nsubseteq H_k$, $k = 1, 2$, and applying the condition *DF2* for difunctions leads to the contradiction $P_{v_2} \subseteq P_{v_1}$. □

We will denote by **dfDiU** the category of di-uniform texture spaces and uniformly bicontinuous difunctions, and by **dfDiU$_0$** the full subcategory of separated di-uniform spaces. A restriction to plain, nearly plain or almost plain textures will be indicated by adding **P**, **Np** or **Ap**, as the case may be, while a restriction to dicomplete di-uniformities will be indicated by **Dc**.

Theorem 2.10. **dfNpDcDiU$_0$** *is a reflective subcategory of* **dfApDiU$_0$**.

Proof. It is certainly a subcategory, so we need only prove it is reflective. For $(S, \mathcal{S}, \upsilon) \in \mathrm{Ob}\,\mathbf{dfApDiU_0}$, choose a nearly plain separated dicompletion $(\bar{S}, \bar{\mathcal{S}}, \bar{\upsilon})$ and let (g, G) be a uniform di-isomorphism between $(S, \mathcal{S}, \upsilon)$ and a dense uniform subspace $(U, \bar{\mathcal{S}}_U, \bar{\upsilon}_U)$ of $(\bar{S}, \bar{\mathcal{S}}, \bar{\upsilon})$, which without loss of generality we assume is plain. Denoting by (e, E) the inclusion difunction $(e, E) : (U, \bar{\mathcal{S}}_U, \bar{\upsilon}_U) \to (\bar{S}, \bar{\mathcal{S}}, \bar{\upsilon})$, we show that $(e, E) \circ (g, G) : (S, \mathcal{S}, \upsilon) \to (\bar{S}, \bar{\mathcal{S}}, \bar{\upsilon})$ is an **dfNpDcDiU$_0$**-reflection arrow for $(S, \mathcal{S}, \upsilon)$.

To this end, take $(V, \mathcal{V}, \nu) \in \mathrm{Ob}\,\mathbf{dfNpDcDiU_0}$ and a **dfApDiU$_0$**-morphism $(f, F) : (S, \mathcal{S}, \upsilon) \to (V, \mathcal{V}, \nu)$. We require a unique **dfNpDcDiU$_0$**-morphism $(h, H) : (\bar{S}, \bar{\mathcal{S}}, \bar{\upsilon}) \to (V, \mathcal{V}, \nu)$ making the following diagram commutative:



The difunction $(f, F) \circ (G^{\leftarrow}, g^{\leftarrow}) : (U, \bar{\mathcal{S}}_U, \bar{\upsilon}_U) \to (V, \mathcal{V}, \nu)$ is (uniquely) representable by an ω-preserving uniformly bicontinuous point function φ which by Proposition 2.7 has an extension $\widehat{\varphi} : \bar{S} \to V$. Clearly $(f_{\widehat{\varphi}}, F_{\widehat{\varphi}}) \circ (e, E) = (f_\varphi, F_\varphi) = (f, F) \circ (G^{\leftarrow}, g^{\leftarrow})$, whence

$$(f_{\widehat{\varphi}}, F_{\widehat{\varphi}}) \circ (e, E) \circ (g, G) = (f, F).$$

Hence, the **dfNpDcDiU$_0$**-morphism $(h, H) = (f_{\widehat{\varphi}}, F_{\widehat{\varphi}})$ makes the diagram commutative. On the other hand, for any **dfNpDcDiU$_0$**-morphism (h, H) making the diagram commutative we have $(h, H) \circ (e, E) = (f, F) \circ (G^{\leftarrow}, g^{\leftarrow}) = (f_\varphi, F_\varphi)$. Hence both (h, H) and $(f_{\widehat{\varphi}}, F_{\widehat{\varphi}})$ extend φ, so they are equal by Corollary 2.9. □

Clearly the above result extends [20, Theorem 3.5], which concerns plain textures and separated prime dicompletions.

As mentioned in the introduction, as with our earlier papers [24–26] this paper too is motivated by the work on bitopological spaces in [2]. The functor $\mathfrak{K}$ introduced in [25], and shown to be an isomorphism between **pCReg$_{\mathbf{w2}}$** and **ifPCbiR$_2$** [25, Theorem 3.7], sets on a firm foundation the bijection between bireal compact spaces in the sense of [2], and real dicompact plain ditopological texture spaces.

The functor $\mathfrak{K}$ is discussed in a more general setting in [22], and we now show how it may be extended to take into account quasi-uniformities and uniform continuity on the one hand, and plain di-uniform spaces and uniformly bicontinuous ω-preserving functions on the other. Since it is shown in [22] that $\mathfrak{K}$ is only defined for weakly pairwise T_0 bitopological spaces, we restrict our attention to quasi-uniformities $\mathcal{Q}$ on a set X that are *separated* in the sense that the bitopological space

$$(X, u, v) = (X, \mathcal{T}_{\mathcal{Q}}, \mathcal{T}_{\mathcal{Q}^{-1}})$$

is weakly pairwise T_0. Here, $\mathcal{Q}^{-1}$ denotes the conjugate of $\mathcal{Q}$. We will find it convenient to regard $\mathcal{Q}$ as a dual covering quasi-uniformity, see for example [19, Proposition 3.6]. In [19, Corollary 3.12] a bijection was set up between the dual covering quasi-uniformities on X and dicovering uniformities on the discrete texture $(X, \mathcal{P}(X))$. This bijection has the disadvantage that a quasi-uniformity that is separated in the above sense need not correspond to a dicovering uniformity on $(X, \mathcal{P}(X))$ that is separated in the sense of [18], so we replace $(X, \mathcal{P}(X))$ by $(X, \mathcal{K}_{uv})$, the texture involved in the definition of $\mathfrak{K}$. Here we recall [22,25] that $\mathcal{K}_{uv}$ is the smallest plain texturing of X containing $u \cup v^c$. Somewhat similarly to [19, Proposition 3.10] we have:

Proposition 2.11. *Let $U = \{(A_j, B_j) \mid j \in J\}$ be an open dual cover of (X, u, v). Then*

$$u^*(U) = \{(A_j, X \setminus B_j) \mid j \in J\}$$

is a dicover of $(X, \mathcal{K}_{uv})$. Moreover, if U satisfies $A_j \cap B_j \neq \emptyset\ \forall j \in J$ then $u^*(U)$ is anchored.

Proof. Since U is open we certainly have $A_j \in u \subseteq \mathcal{K}_{uv}$, $X \setminus B_j \in v^c \subseteq \mathcal{K}_{uv}$, so to prove $u^*(U)$ is a dicover it is sufficient to show that $\mathcal{P} = \{(P_x, Q_x) \mid x \in X\} \prec u^*(U)$ for each $x \in X$. Since U is a dual cover we have $j \in J$ with $x \in A_j \cap B_j$, and now $P_x \subseteq A_j$, $X \setminus B_j \subseteq Q_x$, which gives $\mathcal{P} \prec u^*(U)$.

The remainder of the proof is as for [19, Proposition 3.10], and is omitted. □

It is well known that a dual covering quasi-uniformity $\mathcal{Q}$ has a base $\mathcal{B}$ of open dual covers U satisfying $A\,U\,B \Rightarrow A \cap B \neq \emptyset$ for each $U \in \mathcal{B}$, so we may define $u^*(\mathcal{Q})$ by

$$u^*(\mathcal{Q}) = \{\mathcal{C} \mid \mathcal{C} \text{ is a dicover of } (X, \mathcal{K}_{uv}) \text{ and there exists } U \in \mathcal{B} \text{ with } u^*(U) \prec \mathcal{C}\}.$$

Then:

Theorem 2.12.

(1) $u^*(\mathcal{Q})$ *is a separated dicovering uniformity on* $(X, \mathcal{K}_{uv})$.
(2) *The mapping* u^* *is a bijection between the separated dual covering quasi-uniformities* $\mathcal{Q}$ *on* X *and the separated dicovering uniformities on* $(X, \mathcal{K}_{uv})$.

Proof. (1) It is straightforward to show that $u^*(\mathcal{Q})$ is a dicovering uniformity on $(X, \mathcal{K}_{uv})$, and that

$$\tau_{u^*(\mathcal{Q})} = u = \mathcal{T}_{\mathcal{Q}}, \qquad \kappa_{u^*(\mathcal{Q}^{-1})} = v^c = (\mathcal{T}_{\mathcal{Q}^{-1}})^c.$$

Hence the ditopology $(\tau_{u^*(\mathcal{Q})}, \kappa_{u^*(\mathcal{Q}^{-1})})$ is T_0, which means that $u^*(\mathcal{Q})$ is separated.

(2) Given an open, co-closed dicover $\mathcal{U} = \{(A_j, B_j) \mid j \in J\}$ of $(X, \mathcal{K}_{uv}, u, v^c)$, we obtain an open dual cover $^*u(\mathcal{U})$ of (X, u, v) by setting $^*u(\mathcal{U}) = \{(A_j, X \setminus B_j) \mid j \in J\}$, and if $\mathcal{U}$ is anchored then $^*u(\mathcal{U})$ satisfies $A\, {}^*u(\mathcal{U})\, B \Rightarrow A \cap B \neq \emptyset$. Hence, corresponding to a dicovering uniformity υ on $(X, \mathcal{K}_{uv})$ we have the compatible dual covering quasi-uniformity $^*u(\upsilon)$ on (X, u, v) generated by the dual covers $^*u(\mathcal{U})$ for $\mathcal{U} \in \upsilon$ open, co-closed and anchored.

It is straightforward to verify that $^*u(u^*(\mathcal{Q})) = \mathcal{Q}$ and $u^*(^*u(\upsilon)) = \upsilon$, whence u^* is bijective. □

To set up our functor from the category $\mathbf{QUni_0}$ of separated quasi-uniform spaces and uniformly continuous functions, to the category $\mathbf{ifPDiU_0}$ of separated di-uniform plain texture spaces and uniformly bicontinuous ω-preserving functions, we require the following:

Lemma 2.13. *Let* $(X_1, \mathcal{Q}_1)$, $(X_2, \mathcal{Q}_2)$ *be separated quasi-uniform spaces and suppose that the function* $f : (X_1, \mathcal{Q}_1) \to (X_2, \mathcal{Q}_2)$ *is uniformly continuous. Then* $f : (X_1, \mathcal{K}_{u_1 v_1}, u^*(\mathcal{Q}_1)) \to (X_2, \mathcal{K}_{u_2 v_2}, u^*(\mathcal{Q}_2))$ *is* ω*-preserving and uniformly bicontinuous.*

Proof. If $f : (X_1, \mathcal{Q}_1) \to (X_2, \mathcal{Q}_2)$ is uniformly continuous then $f : (X_1, u_1, v_1) \to (X_2, u_2, v_2)$ is pairwise continuous, whence $f : (X_1, \mathcal{K}_{u_1 v_1}) \to (X_2, \mathcal{K}_{u_2 v_2})$ is ω-preserving by the discussion preceding [22, Theorem 3.7]. □

Now let us note the following:

Lemma 2.14. *For a uniform texture space* $(S, \mathcal{S}, \upsilon)$*, a plain uniform texture space* $(T, \mathcal{T}, \nu)$ *and an* ω*-preserving point function* $\varphi : S \to T$*,* φ *is uniformly bicontinuous if and only if* $\mathcal{C} \in \nu \Rightarrow \varphi^{-1}\mathcal{C} \in \upsilon$.

Proof. This result, with $(S, \mathcal{S})$ plain, was given in [20, Lemma 2.8]. Only minor changes to the proof are needed to establish the general result. □

Using Lemma 2.14, the uniform continuity of $f : (X_1, \mathcal{Q}_1) \to (X_2, \mathcal{Q}_2)$ gives the uniform bicontinuity of $f : (X_1, \mathcal{K}_{u_1 v_1}, u^*(\mathcal{Q}_1)) \to (X_2, \mathcal{K}_{u_2 v_2}, u^*(\mathcal{Q}_2))$.

Now we define the mapping $\mathfrak{K}_u : \mathbf{QUni_0} \to \mathbf{ifPDiU_0}$ by

$$\mathfrak{K}_u\big((X_1, \mathcal{Q}_1) \xrightarrow{f} (X_2, \mathcal{Q}_2)\big) = \big(X_1, \mathcal{K}_{u_1 v_1}, u^*(\mathcal{Q}_1)\big) \xrightarrow{f} \big(X_2, \mathcal{K}_{u_2 v_2}, u^*(\mathcal{Q}_2)\big).$$

This mapping is well defined by Theorem 2.12(1) and Lemma 2.13, and is clearly a functor. By Theorem 2.12(2) it is bijective on objects, and it is clearly faithful. Finally it is easy to verify that the converse of Lemma 2.13 is also true, whence $\mathfrak{K}_u$ is full. By [1, Remark 3.28(2)] we therefore have:

Theorem 2.15. $\mathfrak{K}_u : \mathbf{QUni_0} \to \mathbf{ifPDiU_0}$ *is an isomorphism.* □

We may regard $\mathbf{QUni_0}$ as a concrete category over $\mathbf{Bitop_{w0}}$ via the functor $\mathfrak{U}$,

$$\mathfrak{U}\big((X_1, \mathcal{Q}_1) \xrightarrow{f} (X_2, \mathcal{Q}_2)\big) = (X_1, \mathcal{T}_{\mathcal{Q}_1}, \mathcal{T}_{\mathcal{Q}_1^{-1}}) \xrightarrow{f} (X_2, \mathcal{T}_{\mathcal{Q}_2}, \mathcal{T}_{\mathcal{Q}_2^{-1}}),$$

which forgets the quasi-uniformities but remembers the uniform bitopology they generate. Likewise $\mathbf{ifPDiU_0}$ is a concrete category over $\mathbf{ifPDitop_0}$ via the forgetful functor $\mathfrak{V}$,

$$\mathfrak{V}\big((X_1, \mathcal{X}_1, \upsilon_1) \xrightarrow{f} (X_2, \mathcal{X}_2, \upsilon_2)\big) = (X_1, \mathcal{X}_1, \tau_{\upsilon_1}, \kappa_{\upsilon_1}) \xrightarrow{f} (X_2, \mathcal{X}_2, \tau_{\upsilon_2}, \kappa_{\upsilon_2}).$$

It is easy to verify that the following diagram is commutative.

$$\begin{array}{ccc}
\mathbf{QUni_0} & \xrightarrow{\mathfrak{K}_u} & \mathbf{ifPDiU_0} \\
\downarrow{\scriptstyle \mathfrak{U}} & & \downarrow{\scriptstyle \mathfrak{V}} \\
\mathbf{Bitop_{w0}} & \xrightarrow{\mathfrak{K}} & \mathbf{ifPDitop_0}
\end{array}$$

Since $\mathfrak{K}$ is an isomorphism by [22, Theorem 3.7] it is faithful, so $\mathbf{QUni_0}$ may also be regarded as being concrete over $\mathbf{ifPDitop_0}$, whence $\mathfrak{K}_u$ is a concrete isomorphism over $\mathbf{ifPDitop_0}$. Likewise, it is a concrete isomorphism over $\mathbf{Bitop_{w0}}$. Naturally, in the above $\mathbf{Bitop_{w0}}$ may be replaced by $\mathbf{pCReg_{w2}}$ and $\mathbf{ifPDitop_0}$ by $\mathbf{ifPCbiR_2}$.

Finally, we note the following:

Proposition 2.16. *The functor $\mathfrak{K}_u$ preserves completeness.*

Proof. Suppose first that $(X, \mathcal{Q})$ is complete and let $\mathcal{F} \times \mathcal{G}$ be a regular Cauchy difilter on $(X, \mathcal{K}_{uv}, u^*(\mathcal{Q}))$. Then $\mathcal{F}$ is a base for a filter $\mathcal{B}_u$ on X, $\mathcal{G}^c$ a base for a filter $\mathcal{B}_v$ on X, and clearly $\mathcal{B} = \mathcal{B}_u \times \mathcal{B}_v$ is a regular bifilter on X that is Cauchy for $\mathcal{Q}$ in the sense of [2, Definition 1.7.4]. By hypothesis $\mathcal{B}$ converges to some $x \in X$, and it follows that $\mathcal{F} \times \mathcal{G}$ diconverges to x since $(X, \mathcal{K}_{uv})$ is plain. Hence, $(X, \mathcal{K}_{uv}, u^*(\mathcal{Q}))$ is dicomplete.

Similarly it may be shown that if $(X, \mathcal{K}_{uv}, u^*(\mathcal{Q}))$ is dicomplete then $(X, \mathcal{Q})$ is complete, and we omit the details. □

This discussion makes it clear how the notion of real dicompactness and its relation to dicompleteness, which will be the subject of the next section, represent an extension of the corresponding bitopological notions and results [2, Chapter 3] to more general spaces.

3. Real dicompactness and dicompleteness

Now we will look at the relations between real dicompactness and the dicompleteness of a certain di-uniformity. We begin by establishing a dicovering uniformity on $(\mathbb{R}, \mathcal{R})$.

Lemma 3.1. *For $\epsilon > 0$ let $\mu_\epsilon = \{(Q_{x+\epsilon}, P_{x-\epsilon}) \mid x \in \mathbb{R}\}$. Then $\{\mu_\epsilon \mid \epsilon > 0\}$ is a base of anchored dicovers for a dicomplete dicovering uniformity on $(\mathbb{R}, \mathcal{R})$ whose uniform ditopology is $(\tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$.*

Proof. For $x \in X$ we have $x \in Q_{x+\epsilon} \setminus P_{x-\epsilon}$, so μ_ϵ is a dicover by [22, Lemma 3.4], and it is anchored since $(\mathbb{R}, \mathcal{R})$ is plain. Clearly $\mu_\epsilon \prec_{(\star)} \mu_{3\epsilon}$ and $\mu_{\epsilon\wedge\delta} \prec (\mu_\epsilon) \wedge (\mu_\delta)$, so the family $\{\mu_\epsilon \mid \epsilon > 0\}$ is a base for a dicovering uniformity $\upsilon_{\mathbb{R}}$ on $(\mathbb{R}, \mathcal{R})$. It is trivial to verify that the uniform ditopology of $(\mathbb{R}, \mathcal{R}, \upsilon_{\mathbb{R}})$ is $(\tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$, so we outline the proof of dicompleteness.

Let $\mathcal{F} \times \mathcal{G}$ be a regular Cauchy difilter on $(\mathbb{R}, \mathcal{R})$. Then for each $n > 0$ we have $x_n \in \mathbb{R}$ with $(Q_{x_n+1/n}, P_{x_n-1/n}) \in \mathcal{F} \times \mathcal{G}$. We claim the sequence (x_n) is Cauchy in $\mathbb{R}$ for the usual metric topology. Take $\epsilon > 0$ and an integer N satisfying $N > 2/\epsilon$. For $m, n \geqslant N$ we have $Q_{x_n+1/n} \in \mathcal{F}$, $P_{x_m-1/m} \in \mathcal{G}$, whence $Q_{x_n+1/n} \not\subseteq P_{x_m-1/m}$ by regularity, and we obtain $x_m - x_n < 1/m + 1/n < 2/N < \epsilon$. Likewise $x_n - x_m < \epsilon$, and our claim is justified. Since $\mathbb{R}$ is complete we have $x_n \to x \in \mathbb{R}$, and it is trivial to verify that $\mathcal{F} \times \mathcal{G}$ is diconvergent to x (see [22]). □

We shall refer to the di-uniformity $\upsilon_{\mathbb{R}}$ as the *standard di-uniformity* on $(\mathbb{R}, \mathcal{R})$. The interested reader may easily verify that this is the image under $\mathfrak{K}_u$ of the standard quasi-uniformity on $(\mathbb{R}, s, t)$ considered in [2].

Now let $(S, \mathcal{S}, \tau, \kappa)$ be a completely biregular ditopological texture space and $B \subseteq \mathrm{BA}(S)$ bigenerating. Then by [18, Definition 5.23]

$$\left\{ \left(\bigwedge_{k=1}^{n} (f_{\varphi_k}, F_{\varphi_k})^{-1} (\mu_{\epsilon_k})^{\Delta} \right)^{\Delta} \,\middle|\, \varphi_k \in B,\ \epsilon_k > 0,\ 1 \leqslant k \leqslant n,\ n \in \mathbb{N}^+ \right\}$$

is a base for the initial covering di-uniformity υ_B on $(S, \mathcal{S})$ generated by the space $(\mathbb{R}, \mathcal{R}, \upsilon_{\mathbb{R}})$ and the difunctions (f_φ, F_φ), $\varphi \in B$. For convenience we will write $(f_\varphi, F_\varphi)^{-1}(\mu_\epsilon)$ as $\varphi^{-1}(\mu_\epsilon)$, where

$$\varphi^{-1}(\mu_\epsilon) = \left\{ \left(\varphi^{\leftarrow} Q_{x+\epsilon}, \varphi^{\leftarrow} P_{x-\epsilon} \right) \,\middle|\, x \in \mathbb{R} \right\}. \tag{3.1}$$

Lemma 3.2. *Setting $\varphi_\delta = \{(\varphi^{\leftarrow} Q_{\varphi(s)+\delta}, \varphi^{\leftarrow} P_{\varphi(s)-\delta}) \mid s \in S\}$, $\delta > 0$, we have*

(1) $\varphi_\delta \prec \varphi^{-1}(\mu_\delta)^{\Delta}$,
(2) $\varphi^{-1}(\mu_\delta)^{\Delta} \prec \varphi_{2\delta}$.

Proof. For $s \in S$ suppose that $\varphi^{\leftarrow} Q_{\varphi(s)+\delta} \not\subseteq \mathrm{St}(\varphi^{-1}(\mu_\delta), P_s)$. Then for some $t \in S$ we have $\varphi^{\leftarrow} Q_{\varphi(s)+\delta} \not\subseteq Q_t$ and $P_t \not\subseteq \mathrm{St}(\varphi^{-1}(\mu_\delta), P_s)$. From the first result we deduce $\varphi(t) < \varphi(s) + \delta$, and from the second, $P_s \not\subseteq \varphi^{\leftarrow} P_{x-\delta} \Rightarrow P_t \not\subseteq \varphi^{\leftarrow} Q_{x+\delta}$ for all $x \in \mathbb{R}$, whence $\varphi(s) > x - \delta \Rightarrow \varphi(t) \geqslant x + \delta$, or equivalently $\varphi(t) < x + \delta \Rightarrow \varphi(s) \leqslant x - \delta$. Now, taking $x = \varphi(s)$, we obtain the contradiction $\varphi(s) \leqslant \varphi(s) - \delta$.

This shows $\varphi^{\leftarrow} Q_{\varphi(s)+\delta} \subseteq \mathrm{St}(\varphi^{-1}(\mu_\delta), P_s)$, and $\mathrm{CSt}(\varphi^{-1}(\mu_\delta), Q_s) \subseteq \varphi^{\leftarrow} P_{\varphi(s)-\delta}$ can be proved in the same way, so (1) is established.

For (2), suppose $\mathrm{St}(\varphi^{-1}(\mu_\delta), P_s) \not\subseteq \varphi^{\leftarrow} Q_{\varphi(s)+2\delta}$. Then we have $t \in S$ with $\mathrm{St}(\varphi^{-1}(\mu_\delta), P_s) \not\subseteq Q_t$ and $P_t \not\subseteq \varphi^{\leftarrow} Q_{\varphi(s)+2\delta}$. Now $\varphi^{\leftarrow} Q_{x+\delta} \not\subseteq Q_t$, $P_s \not\subseteq \varphi^{\leftarrow} P_{x-\delta}$ for some $x \in \mathbb{R}$, whence $\varphi(t) < x+\delta$ and $\varphi(s) > x-\delta$. However, this gives $\varphi(t) < \varphi(s)+2\delta$, which contradicts $P_t \not\subseteq \varphi^{\leftarrow} Q_{\varphi(s)+2\delta}$. Likewise it can be shown that $\varphi^{\leftarrow} P_{\varphi(s)-2\delta} \subseteq \mathrm{CSt}(\varphi^{-1}(\mu_\delta), Q_s)$. □

We will require the following generalization of a well-known classical result.

Lemma 3.3. *For $\varphi, \psi \in B$, $r \in \mathbb{R}$, the functions $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi + \mathbf{r}$, φ_* and φ^* are υ_B–$\upsilon_{\mathbb{R}}$ uniformly bicontinuous.*

Proof. Using a similar argument to that used in the proof of Lemma 3.2(2), it is not difficult to establish that

$$(\varphi_\delta \wedge \psi_\delta)^\Delta \prec (\varphi \vee \psi)_{2\delta}.$$

Since $(\varphi^{-1}(\mu_{\delta/2})^\Delta \wedge \psi^{-1}(\mu_{\delta/2})^\Delta)^\Delta \prec (\varphi_\delta \vee \psi_\delta)^\Delta$ by Lemma 3.2(2), we deduce that $(\varphi \vee \psi)_{2\delta} \in \upsilon_B$. Finally, $(\varphi \vee \psi)_{2\delta} \prec (\varphi \vee \psi)^{-1}(\mu_{2\delta})^\Delta$ by Lemma 3.2(1), so $(\varphi \vee \psi)^{-1}(\mu_{2\delta})^\Delta \in \upsilon_B$ and we have established that $\varphi \vee \psi$ is υ_B–$\upsilon_{\mathbb{R}}$ uniformly bicontinuous.

The uniform bicontinuity of $\varphi \wedge \psi$ follows likewise from

$$(\varphi_\delta \wedge \psi_\delta)^\Delta \prec (\varphi \wedge \psi)_{2\delta},$$

and that of $\varphi + \mathbf{r}$, φ_* and φ^* follow trivially from $(\varphi + \mathbf{r})^{\leftarrow} B_x = \varphi^{\leftarrow} B_{x-r}$, $\varphi_*^{\leftarrow} B_x = \varphi^{\leftarrow} B_x$ and $\varphi^{*\leftarrow} B_x = \varphi^{\leftarrow} B_x$ for $B_x = P_x, Q_x$, $x \in \mathbb{R}$ (see [24, Lemma 3.3]). □

Corollary 3.4. *$\upsilon_B = \upsilon_{\langle B\rangle} = \upsilon_{\langle B\rangle^*}$, where $\langle B\rangle$ is the smallest sub-T-lattice of* BA(S) *containing B and $\langle B\rangle^*$ is the smallest such T-lattice closed under the operations $_*$ and *.*

Lemma 3.5. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a completely biregular ditopological texture space and $B \subseteq$ BA(S) bigenerating. Then the initial di-uniformity υ_B is compatible with (τ, κ).*

Proof. Since $\varphi \in B$ is υ_B-$\upsilon_{\mathbb{R}}$ uniformly bicontinuous it is τ_{υ_B}-$\tau_{\upsilon_{\mathbb{R}}}$ continuous. Hence, for $r \in \mathbb{R}$, $\varphi^{\leftarrow} Q_r \in \tau_{\upsilon_B}$ as $Q_r \in \tau_{\mathbb{R}}$. But $\{\varphi^{\leftarrow} Q_r \mid \varphi \in B, r \in \mathbb{R}\}$ is a subbase for τ since B is bigenerating [25, Definition 2.2], so $\tau \subseteq \tau_{\upsilon_B}$.

Conversely, take $G \in \tau_{\upsilon_B}$ and $s \in S$ with $G \not\subseteq Q_s$. By [18, Definition 4.5] we have $\mathcal{C} \in \upsilon_B$ with $\mathrm{St}(\mathcal{C}, P_s) \subseteq G$, and then we have $\varphi_k \in B$, $\epsilon_k > 0$, $k = 1, 2, \dots, n$ with $\mathcal{D} = (\bigwedge_{k=1}^n \varphi_k^{-1}(\mu_{\epsilon_k})^\Delta)^\Delta \prec \mathcal{C}$. Since $\varphi_k \in B$ is τ-$\tau_{\mathbb{R}}$ continuous, $\varphi_k^{-1}(\mu_{\epsilon_k})$ is clearly open with respect to τ for $1 \leqslant k \leqslant n$, whence $\mathcal{D}$ is also open and hence $\mathrm{St}(\mathcal{D}, P_s) \in \tau$. Finally $P_s \subseteq \mathrm{St}(\mathcal{D}, P_s) \subseteq G$, and we deduce $G \in \tau$ by [8, Theorem 3.2(1 iii)].

This establishes $\tau = \tau_{\upsilon_B}$, and the proof of $\kappa = \kappa_{\upsilon_B}$ is dual and is omitted. □

We are now going to relate the dicompleteness of $(S, \mathcal{S}, \upsilon_B)$ with the B-real dicompactness of $(S, \mathcal{S}, \tau, \kappa)$. The following lemma will be crucial.

Lemma 3.6. *Let $(S, \mathcal{S}, \tau, \kappa)$ be completely biregular, $B \subseteq$ BA(S) a bigenerating sub-T-lattice and $\mathcal{F} \times \mathcal{G}$ a regular Cauchy difilter. Then the ρ_b-regular bi-ideal $(L_{\mathcal{F}} \cap \langle B\rangle, M_{\mathcal{G}} \cap \langle B\rangle)$ in $\langle B\rangle$ is finite.*

Proof. Since $\mathcal{F} \times \mathcal{G}$ is Cauchy, and $\upsilon_B = \upsilon_{\langle B\rangle}$ by Corollary 3.4, for $\varphi \in \langle B\rangle$ and $\delta > 0$ there exists $s \in S$ with

$$\varphi^{\leftarrow} Q_{\varphi(s)+\delta} \in \mathcal{F}, \qquad \varphi^{\leftarrow} P_{\varphi(s)-\delta} \in \mathcal{G} \tag{3.2}$$

by Lemma 3.2. We claim that for $\alpha = \varphi(s) - \delta$, $\beta = \varphi(s) + \delta$ we have $[\boldsymbol{\alpha}] \leqslant [\varphi] \leqslant [\boldsymbol{\beta}]$ in $\langle B\rangle/(L_{\mathcal{F}} \cap \langle B\rangle, M_{\mathcal{G}} \cap \langle B\rangle)$ for all $\varphi \in \langle B\rangle$, which will establish that $(L_{\mathcal{F}} \cap \langle B\rangle, M_{\mathcal{G}} \cap \langle B\rangle)$ is finite.

Suppose that $[\boldsymbol{\alpha}] \nleqslant [\varphi]$ for some $\varphi \in B$. Then by [2] (see also [25]) there are two cases to consider.

Case 1. There exists $r \in \mathbb{R}$ with $\varphi - \mathbf{r} \in L_{\mathcal{F}} \cap \langle B\rangle$ and $r < \alpha$. Choose $\epsilon > 0$ with $r + \epsilon < \alpha$. Then $(\varphi - \mathbf{r})^{\leftarrow} P_\epsilon \in \mathcal{F}$, and as $\mathcal{F} \times \mathcal{G}$ is regular, $(\varphi - \mathbf{r})^{\leftarrow} P_\epsilon \notin \mathcal{G}$, so by (3.2), $(\varphi - \mathbf{r})^{\leftarrow} P_\epsilon \not\subseteq \varphi^{\leftarrow} P_{\varphi(s)-\delta}$. Bearing in mind that $(\varphi - \mathbf{r})^{\leftarrow} P_\epsilon = \varphi^{\leftarrow} P_{r+\epsilon}$ we obtain $r + \epsilon > \varphi(s) - \delta$, which contradicts $r + \epsilon < \alpha$.

Case 2. $r \leqslant \alpha$ and $\varphi - \mathbf{r} \notin M_{\mathcal{G}} \cap \langle B\rangle$. Since $\varphi - \mathbf{r} \in \langle B\rangle$ we have $\varphi - \mathbf{r} \notin M_{\mathcal{G}}$, so there exists $\epsilon > 0$ with $(\varphi - \mathbf{r})^{\leftarrow} Q_{-\epsilon} \notin \mathcal{G}$. Using (3.2) as above now leads to the contradiction $r > r - \epsilon > \varphi(s) - \delta = \alpha$.

This gives $[\boldsymbol{\alpha}] \leqslant [\varphi]$, and the proof of $[\varphi] \leqslant [\boldsymbol{\beta}]$ is dual and is omitted. □

Proposition 3.7. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a completely biregular ditopological texture space and $B \subseteq \mathrm{BA}(S)$ bigenerating.*

(1) *If $\mathcal{F} \times \mathcal{G}$ is a regular difilter on $(S, \mathcal{S})$ and C denotes $\langle B\rangle$ or $\langle B\rangle^*$ then $\mathcal{F} \times \mathcal{G}$ is υ_B-Cauchy if and only if $(L_{\mathcal{F}} \cap C, M_{\mathcal{G}} \cap C)$ is a real bi-ideal in the T-lattice C.*
(2) *If (L, M) is a real $*$-bi-ideal in the T-lattice $\langle B\rangle^*$ then the difilter $\mathcal{F}_L \times \mathcal{G}_M$ is υ_B-Cauchy.*

Proof. (1) *Necessity*. Let $\mathcal{F} \times \mathcal{G}$ be a regular Cauchy difilter. We recall from [24] that

$$L_{\mathcal{F}} = \{\varphi \in \mathrm{BA}(S) \mid \varphi^{\leftarrow} P_r \in \mathcal{F}\ \forall r > 0\},$$
$$M_{\mathcal{G}} = \{\varphi \in \mathrm{BA}(S) \mid \varphi^{\leftarrow} Q_{-r} \in \mathcal{G}\ \forall r > 0\}.$$

By [2, Lemma 3.1.2] it is straightforward to show that the bi-ideal $(L_{\mathcal{F}} \cap C, M_{\mathcal{G}} \cap C)$ is ρ_b-regular, and we omit the details. Take a maximal ρ_b-regular bi-ideal (L, M) in C satisfying $(L_{\mathcal{F}} \cap C, M_{\mathcal{G}} \cap C) \prec (L, M)$. Since $(L_{\mathcal{F}} \cap C, M_{\mathcal{G}} \cap C)$ is finite by Lemma 3.6, (L, M) is real by [2, Proposition 3.1.7, Corollary 2]. Hence, there exists $p \in H_C$ with $(L, M) = (L^p, M^p)$. We now show that $(L^p, M^p) \prec (L_{\mathcal{F}} \cap C, M_{\mathcal{G}} \cap C)$, whence $(L_{\mathcal{F}} \cap C, M_{\mathcal{G}} \cap C) = (L, M)$ is real, as required.

Suppose that $L^p \not\subseteq L_{\mathcal{F}} \cap C$ and take $\varphi \in L^p$, $\varphi \notin L_{\mathcal{F}} \cap C$. Then $\varphi \notin L_{\mathcal{F}}$, so there exists $\delta > 0$ with $\varphi^{\leftarrow} P_{2\delta} \notin \mathcal{F}$. As $\mathcal{F} \times \mathcal{G}$ is Cauchy there exists $s \in S$ with $\varphi^{\leftarrow} Q_{\varphi(s)+\delta} \not\subseteq \varphi^{\leftarrow} P_{2\delta}$ by (3.2), so $\varphi(s) > \delta$. Also by (3.2),

$$\begin{aligned}
\varphi^{\leftarrow} P_{\varphi(s)-\delta} \in \mathcal{G} \quad &\Rightarrow \quad \big(\varphi + \boldsymbol{\delta} - \boldsymbol{\varphi}(\mathbf{s})\big)^{\leftarrow} Q_{-r} \in \mathcal{G}, \quad \forall r > 0 \\
&\Rightarrow \quad \varphi + \boldsymbol{\delta} - \boldsymbol{\varphi}(\mathbf{s}) \in M_{\mathcal{G}} \cap C \subseteq M = M^p \\
&\Rightarrow \quad p(\varphi) + \delta - \varphi(s) \geqslant 0.
\end{aligned}$$

This now gives $p(\varphi) > 0$, which contradicts $\varphi \in L^p$. Hence $L^p \subseteq L_{\mathcal{F}} \cap C$, and a similar argument gives $M^p \subseteq M_{\mathcal{G}} \cap C$.

Sufficiency. Let $\mathcal{F} \times \mathcal{G}$ be a regular difilter with $(L_{\mathcal{F}} \cap C, M_{\mathcal{G}} \cap C)$ real in C. Take $\varphi \in C$. Then there exists a (unique) $\alpha \in \mathbb{R}$ with $\varphi - \boldsymbol{\alpha} \in L_{\mathcal{F}} \cap C \cap M_{\mathcal{G}} \cap C$. Take $\delta > 0$ and $r \in \mathbb{R}$ with $0 < r < \delta/2$. Then

$$(\varphi - \boldsymbol{\alpha})^{\leftarrow} P_r \not\subseteq (\varphi - \boldsymbol{\alpha})^{\leftarrow} Q_{-r}$$

since $\mathcal{F} \times \mathcal{G}$ is regular. If we take $s \in S$ with $(\varphi - \boldsymbol{\alpha})^{\leftarrow} P_r \not\subseteq Q_s$, $P_s \not\subseteq (\varphi - \boldsymbol{\alpha})^{\leftarrow} Q_{-r}$ it is trivial to verify that

$$(\varphi - \boldsymbol{\alpha})^{\leftarrow} P_r \subseteq \varphi^{\leftarrow} Q_{\varphi(s)+\delta} \quad \text{and} \quad (\varphi - \boldsymbol{\alpha})^{\leftarrow} Q_{-r} \supseteq \varphi^{\leftarrow} P_{\varphi(s)-\delta},$$

which gives $(\varphi^{\leftarrow} Q_{\varphi(s)+\delta}, \varphi^{\leftarrow} P_{\varphi(s)-\delta}) \in \mathcal{F} \times \mathcal{G}$. In view of Lemma 3.2(1) this shows that $\mathcal{F} \times \mathcal{G}$ is $\upsilon_B = \upsilon_C$-Cauchy.

(2) Let (L, M) be a real $*$-bi-ideal in $\langle B\rangle^*$. We recall from [24] that

$$\mathcal{F}_L = \{A \in \mathcal{S} \mid \exists \varphi \in L,\ r > 0 \text{ with } \varphi^{\leftarrow} P_r \subseteq A\},$$
$$\mathcal{G}_M = \{A \in \mathcal{S} \mid \exists \varphi \in M,\ r > 0 \text{ with } A \subseteq \varphi^{\leftarrow} Q_{-r}\}.$$

Since (L, M) is a ρ_b-regular $*$-bi-ideal, $\mathcal{F}_L \times \mathcal{G}_M$ is regular by [24, Proposition 3.5(2)], whence $(L_{\mathcal{F}_L} \cap \langle B\rangle^*, M_{\mathcal{G}_M} \cap \langle B\rangle^*)$ is a ρ_b-regular bi-ideal in $\langle B\rangle^*$. However it is easy to see that $(L, M) \preccurlyeq (L_{\mathcal{F}_L}, M_{\mathcal{G}_M})$, and since a real bi-ideal is maximal ρ_b-regular we have $(L, M) = (L_{\mathcal{F}_L}, M_{\mathcal{G}_M})$. Applying (1) for $C = \langle B\rangle^*$ to the real bi-ideal $(L_{\mathcal{F}_L}, M_{\mathcal{G}_M})$, we deduce that $\mathcal{F}_L \times \mathcal{G}_M$ is υ_B-Cauchy. □

Theorem 3.8. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a completely biregular, almost plain bi-T_2 ditopological $*$-space. Then $(S, \mathcal{S}, \tau, \kappa)$ is B-real dicompact if and only if the di-uniform texture space $(S, \mathcal{S}, \upsilon_B)$ is dicomplete.*

Proof. *Necessity*. Let $(S, \mathcal{S}, \tau, \kappa)$ be B-real dicompact and $\mathcal{F} \times \mathcal{G}$ a regular υ_B-Cauchy difilter. By Proposition 3.7(1), $(L_{\mathcal{F}}, M_{\mathcal{G}})$ is a real bi-ideal in $\langle B\rangle$ and hence difixed by some $s \in S_p$. For $\varphi \in \langle B\rangle$ we have $\varphi - \boldsymbol{\varphi}(\mathbf{s}) \in L(s) \cap M(s)$, whence for $\delta > 0$ we have $\varphi^{\leftarrow} Q_{\varphi(s)+\delta} \in \mathcal{F}$, $\varphi^{\leftarrow} P_{\varphi(s)-\delta} \in \mathcal{G}$. It is easy to deduce that $\mathcal{F} \to s$, $\mathcal{G} \to s$, while $P_s \not\subseteq Q_s$ so $\mathcal{F} \times \mathcal{G}$ is diconvergent [21, Definition 2.6]. Hence, $(S, \mathcal{S}, \upsilon_B)$ is dicomplete.

Sufficiency. Let $(S, \mathcal{S}, \upsilon_B)$ be dicomplete and (L, M) a real bi-ideal in $\langle B\rangle^*$. Since $(S, \mathcal{S}, \tau, \kappa)$ is a $*$-space, (L, M) is a real $*$-bi-ideal in $\langle B\rangle^* = \langle B\rangle$. By Proposition 3.7(2), $(\mathcal{F}_L, \mathcal{G}_M)$ is a regular υ_B-Cauchy difilter, hence diconvergent in $(S, \mathcal{S}, \tau_{\upsilon_B}, \kappa_{\upsilon_B})$. Let $\mathcal{F}_L \to s$, $\mathcal{G}_M \to s'$ with $P_{s'} \not\subseteq Q_s$. Since $(S, \mathcal{S})$ is almost plain there exists $a \in S_p$ with $P_{s'} \not\subseteq Q_a$, $P_a \not\subseteq Q_s$. We will show that (L, M) is difixed by a.

Suppose that $L(a) \not\subseteq L$. Then we have $\varphi \in L(a)$ with $\varphi \notin L$. Since (L, M) is real there exists a (unique) $\alpha \in \mathbb{R}$ with $\varphi - \boldsymbol{\alpha} \in L \cap M$, and $\varphi \not\leqslant \varphi - \boldsymbol{\alpha}$ since $\varphi \notin L$, so $\alpha > 0$. Choose $\delta > 0$ with $2\delta < \alpha$. Now $P_a \subseteq \varphi^{\leftarrow} Q_{\varphi(a)+\delta} \in \tau_{\upsilon_B} = \tau$ by Lemma 3.5, and $\varphi^{\leftarrow} Q_{\varphi(a)+\delta} \not\subseteq Q_s$, so $\varphi^{\leftarrow} Q_{\varphi(a)+\delta} \in \mathcal{F}_L$ as $\mathcal{F}_L \to s$. However, $\varphi^{\leftarrow} Q_{\varphi(a)+\delta} \subseteq (\varphi - \boldsymbol{\alpha})^{\leftarrow} Q_{-\delta} \in \mathcal{G}_M$ gives $\varphi^{\leftarrow} Q_{\varphi(a)+\delta} \in \mathcal{G}_M$, which contradicts the fact that $(\mathcal{F}_L, \mathcal{G}_M)$ is regular. This gives $L(a) \subseteq L$, and dually $M(a) \subseteq M$. But $(L(a), M(a))$ is real and hence maximal ρ_b-regular, so $(L, M) = (L(a), M(a))$ and we have established that (L, M) is difixed by a as required. Hence, $(S, \mathcal{S}, \tau, \kappa)$ is B-real dicompact. □

Example 3.9. Consider the real ditopological texture space $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ and the bigenerating subset $B = \{\iota_{\mathbb{R}}, \mathbf{0}\}$ of $\mathrm{BA}(\mathbb{R})$, as in [25, Example 2.6]. For $\epsilon > 0$ we clearly have

$$\mu_\epsilon \prec \mu_\epsilon = \iota^{-1}(\mu_\epsilon), \qquad \mu_\epsilon \prec \{(\mathbb{R},\mathbb{R}), (\mathbb{R},\emptyset), (\emptyset,\emptyset)\} = \mathbf{0}^{-1}(\mu_\epsilon),$$

so $\upsilon_B = \upsilon_{\mathbb{R}}$. Since $(\mathbb{R}, \mathcal{R}, \upsilon_{\mathbb{R}})$ is dicomplete by Lemma 3.1, Theorem 3.8 gives us a second proof that $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is B-real dicompact, hence real dicompact by [25, Corollary 2.8].

By considering the T-lattice $\mathrm{BA}^*(S)$ of bounded functions in $\mathrm{BA}(S)$, and recalling from [25, Proposition 2.20] that $\mathrm{BA}^*(S)$-real dicompactness coincides with dicompactness, we may give at once the following consequence of Theorem 3.8:

Corollary 3.10. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a completely biregular, almost plain bi-T_2 ditopological $*$-space. Then $(S, \mathcal{S}, \tau, \kappa)$ is dicompact if and only if the di-uniform texture space $(S, \mathcal{S}, \upsilon_{\mathrm{BA}^*(S)})$ is dicomplete.*

Now let B be a bigenerating sub-T-lattice of $\mathrm{BA}(S_p)$ and consider $H_B \subseteq \mathbb{R}^B$. We may define a dicovering uniformity on H_B as the restriction to H_B of the initial dicovering uniformity on $\mathbb{R}^B$ given by the projection functions $\rho_\varphi : \mathbb{R}^B \to \mathbb{R}$ and the dicovering uniformity $\upsilon_{\mathbb{R}}$ on $\mathbb{R}$. We will denote this dicovering uniformity on H_B by υ_{ρ_B}. Then:

Corollary 3.11. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a completely biregular ditopological almost plain space and $B \subseteq \mathrm{BA}(S_p)$ a bigenerating sub-T-lattice. Then $(H_B, \mathcal{H}_B, \upsilon_{\rho_B})$ is a separated plain dicompletion of $(S, \mathcal{S}, \upsilon_B)$.*

Proof. It is easy to verify that the embedding $\xi_B : S_p \to H_B$ is a uniformly bicontinuous bijection between $(S, \mathcal{S}, \upsilon)$ and the restriction of $(H_B, \mathcal{H}_B, \upsilon_{\rho_B})$ to $\xi_B(S_p)$, and we omit the details. Since H_B is jointly closed in $\mathbb{R}^B$ by [25, Proposition 2.17(2)] it is real dicompact by [25, Theorem 2.19], and an examination of the proof of that theorem shows that it is actually ρ_B-real dicompact. Hence, by Theorem 3.8, $(H_B, \mathcal{H}_B, \upsilon_{\rho_B})$ is dicomplete. □

This result enables us to regard the Hewitt reflector [26, Proposition 3.3] and the Stone–Čech reflector [26, Proposition 3.18] as particular instances of the dicompletion reflector of Theorem 2.10 for the appropriate categories.

Let υ be a compatible dicovering uniformity on $(S, \mathcal{S}, \tau, \kappa)$ or $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$. In the remainder of this section we consider the family $U(\upsilon)$ of real ω-preserving υ–$\upsilon_{\mathbb{R}}$ uniformly bicontinuous point functions on S or S_p, as the case may be.

Proposition 3.12. *If B is a bigenerating sub-T-lattice of $\mathrm{BA}(S_p)$ then*

$$B \subseteq U(\upsilon_B) \subseteq \nu\big(\mathrm{BA}(H_A)\big).$$

In particular, for $B \in \mathcal{B}$, the set of bigenerating finitely ρ_b-prime-complete sub-T-lattices of $\mathrm{BA}(S_p)$, we have $B = U(\upsilon_B)$.

Proof. By definition, every element of B is υ_B–$\upsilon_{\mathbb{R}}$ uniformly bicontinuous, so we have $B \subseteq U(\upsilon_B)$.

Let us recall from [26] that the mapping $\nu : \mathrm{BA}(H_B) \to \mathrm{BA}(S_p)$ is given by $\nu(\varphi) = \varphi \circ \xi_B$, $\varphi \in \mathrm{BA}(H_B)$, where $\xi : S_p \to H_B$ is given by $\xi(s) = \widehat{s}|_B$, and $\widehat{s} : \mathrm{BA}(S_p) \to \mathbb{R}$ by $\widehat{s}(\varphi) = \varphi(s)$, $s \in S_p$.

Now take $\varphi \in U(\upsilon_B)$. Then $\varphi : S_p \to \mathbb{R}$ is υ_B–$\upsilon_{\mathbb{R}}$ uniformly bicontinuous, and recalling that ξ_B is a uniformly bicontinuous bijection between S_p and $\xi(S_p)$ we see that $\varphi \circ \xi_B^{-1}$ is a uniformly bicontinuous mapping from the restriction of $(H_B, \mathcal{H}_B, \upsilon_{\rho_B})$ on $\xi(S_p)$ to $(\mathbb{R}, \mathcal{R}, \upsilon_{\mathbb{R}})$. But $\xi(S_p)$ is jointly dense in H_B by [25, Proposition 2.17(1)], and $\upsilon_{\mathbb{R}}$ is dicomplete and separated, so by Proposition 2.7, $\varphi \circ \xi_B^{-1}$ has a uniformly bicontinuous extension $\widehat{\varphi \circ \xi_B^{-1}}$ to H_B. In particular this function is bicontinuous for the uniform ditopologies, so $\widehat{\varphi \circ \xi_B^{-1}} \in \mathrm{BA}(H_B)$. Finally, for $s \in S_p$,

$$\nu\big(\widehat{\varphi \circ \xi_B^{-1}}\big)(s) = \widehat{\varphi \circ \xi_B^{-1}}\big(\xi_B(s)\big) = \big(\varphi \circ \xi_B^{-1}\big)\big(\xi_B(s)\big) = \varphi(s),$$

whence $\nu(\widehat{\varphi \circ \xi_B^{-1}}) = \varphi$, that is $\varphi \in \nu(\mathrm{BA}(H_B))$, as required.

By [26, Corollary 3.13], $\nu(\mathrm{BA}(H_B))$ is a finite ρ_b-prime-completion of B, hence for $B \in \mathcal{B}$ we have $\nu(\mathrm{BA}(H_B)) = B$, and so $B = U(\upsilon_B)$. □

For our next result we will require the following lemma. Although this is the textural counterpart of a known bitopological result we sketch the proof for the sake of completeness.

Lemma 3.13. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a bi-R_1 dicompact plain space. Then $(S, \mathcal{S}, \tau, \kappa)$ is completely biregular and for any compatible dicovering uniformity υ we have $U(\upsilon) = \mathrm{BA}(S)$.*

Proof. The complete biregularity of $(S, \mathcal{S}, \tau, \kappa)$ follows from [11, Proposition 4.3] and [9, Corollary 5.24], and clearly $U(\upsilon) \subseteq \mathrm{BA}(S)$. To prove the opposite inclusion take $\varphi \in \mathrm{BA}(S)$ and $\delta > 0$. By compatibility, for each $s \in S$ we have $\mathcal{C}_s \in \upsilon$ with $\mathrm{St}(\mathcal{C}_s, P_s) \subseteq \varphi^{\leftarrow} Q_{\varphi(s)+\delta}$, $\varphi^{\leftarrow} P_{\varphi(s)-\delta} \subseteq \mathrm{CSt}(\mathcal{C}_s, Q_s)$, and we take $\mathcal{D}_s \in \upsilon$ open, co-closed with $\mathcal{D}_s \prec_{(\star)} \mathcal{C}_s$. Now $\mathcal{D}^{\Delta}$ is an open, co-closed dicover of $(S, \mathcal{S}, \tau, \kappa)$, so by [11, Theorem 4.8(3)] there exists $s_1, s_2, \ldots, s_n \in S$ for which $\{\mathrm{St}(\mathcal{D}_{s_k}, P_{s_k}), \mathrm{CSt}(\mathcal{D}_{s_k}, Q_{s_k}) \mid 1 \leqslant k \leqslant n\}$ is a dicover of $(S, \mathcal{S})$. It is straightforward to verify

$$\mathcal{D}_{s_1} \wedge \mathcal{D}_{s_2} \wedge \cdots \wedge \mathcal{D}_{s_n} \prec \varphi_\delta,$$

whence $\varphi \in U(\upsilon)$ as required. □

Proposition 3.14. *For a bigenerating sub-T-lattice B of $\mathrm{BA}^*(S_p)$, $U(\upsilon_B) \in \mathcal{B}^*$.*

Proof. In this case $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$ is dicompact, so by Lemma 3.13 we have $U(\upsilon_{\rho_B}) = \mathrm{BA}(H_B)$. Hence, $U(\upsilon_B) = \nu(\mathrm{BA}(H_B)) \in \mathcal{B}^*$. □

As a consequence of this proposition we see that there is a one to one correspondence between $\mathcal{B}^*$ and the set of totally bounded dicovering uniformities compatible with $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$. On the other hand, it is possible to have a bigenerating sub-T-lattice of $\mathrm{BA}(S_p)$ for which $U(\upsilon_B) \notin \mathcal{B}$, as the following example shows.

Example 3.15. We continue to develop Example 3.9. Hence, for the real texture $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ we take $B = \{\iota, \mathbf{0}\}$, which gives $\upsilon_{\langle B\rangle} = \upsilon_B = \upsilon_{\mathbb{R}}$. Since $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is real dicompact and B-real dicompact we see from [25, Proposition 2.11] that $\mathrm{BA}(\mathbb{R})$ is a finite ρ_b-refinement of $\langle B\rangle$, and indeed an examination of the proof of this proposition reveals that $\mathrm{BA}(\mathbb{R})$ is actually a finite ρ_b-prime-refinement of $\langle B\rangle$, and hence of $U(\upsilon_B) = U(\upsilon_{\mathbb{R}})$. If we can show that $U(\upsilon_{\mathbb{R}}) \neq \mathrm{BA}(\mathbb{R})$ it will follow that $U(\upsilon_B) = U(\upsilon_{\langle B\rangle}) = U(\upsilon_{\mathbb{R}}) \notin \mathcal{B}$. To this end take

$$\varphi(x) = \mathrm{e}^x \in \mathrm{BA}(\mathbb{R})$$

and suppose that $\varphi \in U(\upsilon_{\mathbb{R}})$. Then, given $\delta > 0$ there exists $\epsilon > 0$ with $\mu_\epsilon \prec \varphi^{-1}(\mu_\delta)$ since we are dealing with a plain texture. Hence, given $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ with $Q_{x+\epsilon} \subseteq \varphi^{\leftarrow} Q_{y+\delta}$, $P_{x-\epsilon} \subseteq \varphi^{\leftarrow} P_{y-\delta}$, which gives $0 < \mathrm{e}^{\epsilon/2} - \mathrm{e}^{-\epsilon/2} < 2\delta \mathrm{e}^{-x}$. This gives a contradiction for large enough x, so $\varphi \notin U(\upsilon_{\mathbb{R}})$, as required.

Using an obvious notation we know from the general theory above that $U^*(\upsilon_{\mathbb{R}}) \in \mathcal{B}^*$. Actually, $U^*(\upsilon_{\mathbb{R}}) = \mathrm{BA}^*(\mathbb{R})$ as the interested reader may easily verify.

For our final theorem we wish to extend the families $\mathcal{B}$ and $\mathcal{B}^*$ to $\mathrm{BA}(S)$. We begin by taking a compatible dicovering uniformity υ on $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ and extending it to $(S, \mathcal{S}, \tau, \kappa)$. To do this, note that given $A \in \mathcal{S}_p$ we have a unique set $\widehat{A} \in \mathcal{S}$ with $\widehat{A} \cap S_p = A$, hence for a dicover $\mathcal{C}$ on S_p we may define $\widehat{\mathcal{C}} = \{(\widehat{A}, \widehat{B}) \mid A \,\mathcal{C}\, B\}$. Then:

Lemma 3.16. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a nearly plain ditopological texture space. Then $\widehat{\upsilon} = \{\widehat{\mathcal{C}} \mid \mathcal{C} \in \upsilon\}$ is the unique compatible dicovering uniformity on $(S, \mathcal{S}, \tau, \kappa)$ whose restriction to $(S_p, \mathcal{S}_p)$ is υ.*

Proof. The verification of the various properties of $\widehat{\upsilon}$ is quite straightforward, if somewhat tedious, and we omit the details, proving only that if $\mathcal{C}$ is anchored then so is $\widehat{\mathcal{C}}$. By [18, Definition 2.1(1)] we must first show that $\mathcal{P} \prec \widehat{\mathcal{C}}$, so take $s \in S$. Since $(S, \mathcal{S})$ is nearly plain we have $\varphi_p(s) \in S_p$ with $Q_s = Q_{\varphi_p(s)}$, and since $\mathcal{C}$ is anchored we have $A \,\mathcal{C}\, B$ with $P^p_{\varphi_p(s)} \subseteq A$ and $B \subseteq Q^p_{\varphi_p(s)}$. But now $P_s \subseteq P_{\varphi_p(s)} \subseteq \widehat{A}$ and $\widehat{B} \subseteq Q_{\varphi_p(s)} = Q_s$, which establishes $\mathcal{P} \prec \widehat{\mathcal{C}}$.

Secondly, for $\widehat{A} \,\widehat{\mathcal{C}}\, \widehat{B}$ we have $A \,\mathcal{C}\, B$, so by [18, Definition 2.1(2)] we have $s \in S_p$ satisfying conditions (a) and (b) of that definition. It is straightforward to verify that $\widehat{A} \,\widehat{\mathcal{C}}\, \widehat{B}$ also satisfies (a) and (b) for the same point $s \in S_p \subseteq S$, and the proof is complete. □

Now let us note that from Lemma 2.14 we trivially have $\varphi \in U(\widehat{\upsilon}) \iff \varphi|_{S_p} \in U(\upsilon)$. If we recall from Proposition 2.7 that every element of $U(\upsilon)$ can be (non-uniquely) extended to an element of $U(\widehat{\upsilon})$ we see that $U(\upsilon) = \{\varphi|_{S_p} \mid \varphi \in U(\widehat{\upsilon})\}$. This suggests the following definition:

Definition 3.17. For a bigenerating set $B \subseteq \mathrm{BA}(S_p)$ we set $\widehat{B} = \{\varphi \in \mathrm{BA}(S) \mid \varphi|_{S_p} \in B\}$ and $\widehat{\mathcal{B}} = \{\widehat{B} \mid B \in \mathcal{B}\}$. Likewise, $\widehat{\mathcal{B}}^*$ denotes the above family for $B \subseteq \mathrm{BA}^*(S_p)$.

Lemma 3.18. *With $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{B}}^*$ as above,*

(1) $\widehat{B} \in \widehat{\mathcal{B}} \Rightarrow \widehat{B} = U(\upsilon_{\widehat{B}})$.
(2) *If $C \subseteq \mathrm{BA}^*(S)$ is bigenerating then $U(\upsilon_C) \in \widehat{\mathcal{B}}^*$.*

Proof. (1) $\widehat{B} \in \widehat{\mathcal{B}} \Rightarrow B \in \mathcal{B} \Rightarrow B = U(\upsilon_B)$ by Proposition 3.12. Hence $\widehat{B} = U(\widehat{\upsilon_B}) = U(\upsilon_{\widehat{B}})$, as required.

(2) For $C \subseteq \mathrm{BA}^*(S)$ we have $C|_{S_p} \subseteq \mathrm{BA}^*(S_p)$ and so $U(\upsilon_{C|_{S_p}}) \in \mathcal{B}^*$ by Proposition 3.14. Hence, $U(\upsilon_C) \in \widehat{\mathcal{B}}^*$. □

Remark 3.19. The sets $\widehat{B} \in \widehat{\mathcal{B}}$ have the property that for $r \in \mathbb{R}$ with $r > 0$ we have $\varphi \in \widehat{B} \Rightarrow r\varphi \in \widehat{B}$. Here, $(r\varphi)(s) = r(\varphi(s))$, $s \in S$. To see this we only have to verify that $\varphi \in U(\upsilon_{\widehat{B}}) \Rightarrow r\varphi \in U(\upsilon_{\widehat{B}})$, and this follows easily from the evident equality $(r\varphi)^{\leftarrow} B_x = \varphi^{\leftarrow} B_{x/r}$, where $B_x = P_x$ or Q_x as usual.

Before stating our final theorem we require the following definition:

Definition 3.20. Let B be a bigenerating sub T-lattice of $\mathrm{BA}(S)$.

1. A pair (F, G), $F \in \mathcal{S} \setminus \{\emptyset\}$, $G \in \mathcal{S} \setminus \{S\}$, is called *completely B-excluding* if for some $\vartheta \in B$, $-1 \leqslant \vartheta \leqslant 1$, we have $F \subseteq \vartheta^{\leftarrow} P_{-1}$ and $\vartheta^{\leftarrow} Q_1 \subseteq G$. Moreover, (F, G) will be called *closed, co-open* if $F \in \kappa$ and $G \in \tau$.
2. A pair (H, K), $H \in \mathcal{S} \setminus \{S\}$, $K \in \mathcal{S} \setminus \{\emptyset\}$, is called *completely B-co-excluding* if for some $\vartheta \in B$, $-1 \leqslant \vartheta \leqslant 1$, we have $H \cup \vartheta^{\leftarrow} P_{-1} = S$ and $K \cap \vartheta^{\leftarrow} Q_1 = \emptyset$. Moreover, (H, K) will be called *open, co-closed* if $H \in \tau$ and $K \in \kappa$.

Theorem 3.21. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a completely biregular nearly plain ditopological space.*

(a_1) *Let B be a minimal element of $\widehat{\mathcal{B}}$. Then for every completely B-coexcluding pair (H, K), either every $\mathcal{S}$-cofilter $\mathcal{G}$ with $H \in \mathcal{G}$ has a cluster point in S_p or every $\mathcal{S}$-filter $\mathcal{K}$ with $K \in \mathcal{F}$ has a cluster point in S_p.*

(a_2) *Let B be a minimal element of $\widehat{\mathcal{B}}$. Then for every completely B-excluding pair (F, G), either every $\mathcal{S}$-cofilter $\mathcal{G}$ with $P \notin \mathcal{G}$ has a cluster point in S_p or every $\mathcal{S}$-filter $\mathcal{K}$ with $F \notin \mathcal{K}$ has a cluster point in S_p.*

(b) *Let $B \in \widehat{\mathcal{B}}^*$ and suppose that for every closed, co-open completely B-excluding pair (F, G), either F is compact or G is cocompact. Then B is the smallest element of $\widehat{\mathcal{B}}$.*

Proof. (a_1) Suppose that for some completely B-excluding pair (H, K), there is a filter $\mathcal{K}$ with $K \in \mathcal{K}$ that has no cluster point in S_p and a cofilter $\mathcal{G}$ with $H \in \mathcal{G}$ that has no cluster point in S_p. We claim that $\mathcal{K} \times \mathcal{G}$ is a (not necessarily regular) υ_B-Cauchy difilter.

Since an arbitrary point $a \in S_p$ is not a cluster point of $\mathcal{K}$, there exists $K(a) \in \kappa$ with $K(a) \in \mathcal{K}$, $P_a \nsubseteq K(a)$, likewise there exists $H(a) \in \tau$ with $H(a) \in \mathcal{G}$, $H(a) \nsubseteq Q_a$. Hence we have $\mathcal{E}_a \in \upsilon_B$ with $\mathrm{St}(\mathcal{E}_a, P_a) \subseteq H(a)$ and $K(a) \subseteq \mathrm{CSt}(\mathcal{E}_a, Q_a)$. Now take the restriction υ_B^p of υ_B to S_p, let $\mathcal{E} \in \upsilon_B^p$ be arbitrary, and consider $\mathcal{E} \wedge \mathcal{E}_a|_{S_p} \in \upsilon_B^p$. This is a normal dicover, so as in the proof of [10, Proposition 3.5] we have open, co-closed $\mathcal{D}_n \in \upsilon_B^p$, $n \in \mathbb{N}$, with $\mathcal{D}_0 \prec \mathcal{E} \wedge \mathcal{E}_a|_{S_p}$ and $\mathcal{D}_{n+1} \prec_{(\star)} \mathcal{D}_n$ for all $n \in \mathbb{N}$, and a corresponding admissible pseudo-dimetric $(\overline{\rho}, \underline{\rho})$ on S_p satisfying $\mathcal{O}_1 \prec \mathcal{D}_1 \prec \mathcal{D}_0 \prec \mathcal{E} \wedge \mathcal{E}_a|_{S_p}$.

Define $\varphi = \varphi_a^{\mathcal{E}}$ by $\varphi(s) = \overline{\rho}(a, s)$, $s \in S_p$. Clearly $\varphi : S_p \to \mathbb{R}$ is ω-preserving. Take $\delta > 0$. If $n \geqslant 1$ satisfies $3^{-n} < \delta$ it is easy to show that $\mathcal{O}_n \prec \varphi^{-1}(\mu_\delta)$, while for a sufficiently large m we have $\mathcal{D}_m \prec \mathcal{O}_n$, so $\varphi^{-1}(\mu_\delta) \in \upsilon_B^p$ as $\mathcal{D}_m \in \upsilon_B^p$. Applying Proposition 2.7, and noting that $(S, \mathcal{S})$ satisfies $S^\flat = S$, we have an extension $\widehat{\varphi} : S \to \mathbb{R}$ of φ with $\widehat{\varphi}^{-1}(\mu_\delta) \in \upsilon_B$. Hence, $\widehat{\varphi} = \widehat{\varphi_a^{\mathcal{E}}} \in U(\upsilon_B) = B$ since $B \in \mathcal{B}$. Also, working within S_p,

$$\varphi^{\leftarrow} Q_{1/3} \subseteq N_1(a) \subseteq \mathrm{St}\big(\mathcal{D}_1, P_a^p\big) \subseteq \mathrm{St}\big(\mathcal{E}, P_a^p\big)$$

from which we deduce $\widehat{\varphi}^{\leftarrow} Q_{\widehat{\varphi}(a)+1/3} = \widehat{\varphi}^{\leftarrow} Q_{1/3} \subseteq \mathrm{St}(\mathcal{E}, P_a)$.

Now we show that these functions generate τ. Firstly,

$$\widehat{\varphi}^{\leftarrow} Q_{\widehat{\varphi}(a)+1/3} = \widehat{\varphi_a^{\mathcal{E}}}^{\leftarrow} Q_{\widehat{\varphi_a^{\mathcal{E}}}(a)+1/3} \in \tau_{\upsilon_B} = \tau.$$

On the other hand, if $G \in \tau$, $G \nsubseteq Q_s$ for $s \in S$, we have $a \in S_p$ with $G \nsubseteq Q_a$ and $P_a \nsubseteq Q_s$. Taking $\mathcal{E} \in \upsilon_B$ with $\mathrm{St}(\mathcal{E}, P_a) \subseteq G$ gives $P_s \subseteq P_a \subseteq \mathrm{St}(\mathcal{E}, P_a) \subseteq G$, whence the family $\{\widehat{\varphi_a^{\mathcal{E}}} \mid a \in S_p,\ \mathcal{E} \in \upsilon_B\}$ does indeed generate the topology τ. In exactly the same way, defining $\psi = \psi_a^{\mathcal{E}}$ by $\psi(s) = \underline{\rho}(a, s)$, $s \in S_p$, we have an extension $\widehat{\psi}$ of ψ to S which belongs to B, and the family $\{\widehat{\psi_a^{\mathcal{E}}} \mid a \in S_p,\ \mathcal{E} \in \upsilon_B\}$ generates the cotopology κ. It follows that the set

$$C = \big\{\widehat{\varphi_a^{\mathcal{E}}} \wedge (1/3) \,\big|\, a \in S_p,\ \mathcal{E} \in \upsilon_B\big\} \cup \{\mathbf{0}\} \cup \big\{\widehat{\psi_a^{\mathcal{E}}} \wedge (1/3) \,\big|\, a \in S_p,\ \mathcal{E} \in \upsilon_B\big\}$$

is bigenerating and satisfies $U(\upsilon_C) \subseteq B$. However, $C \subseteq \mathrm{BA}^*(S)$ so $U(\upsilon_C) \in \widehat{\mathcal{B}}^* \subseteq \widehat{\mathcal{B}}$ by Lemma 3.18(2), and since B is minimal in $\widehat{\mathcal{B}}$ this gives

$$B = U(\upsilon_C). \tag{3.3}$$

Now from $\mathcal{O}_1 \prec \mathcal{E}_a$ we obtain, for any $\delta > 0$,

$$S = \widehat{\varphi_a^{\mathcal{E}}} \wedge (1/3)^{\leftarrow} Q_{1/3+\delta}, \qquad \widehat{\varphi_a^{\mathcal{E}}} \wedge (1/3)^{\leftarrow} P_{1/3-\delta} \subseteq H(a),$$
$$K(a) \subseteq \widehat{\psi_a^{\mathcal{E}}} \wedge (1/3)^{\leftarrow} Q_{1/3+\delta}, \qquad \widehat{\psi_a^{\mathcal{E}}} \wedge (1/3)^{\leftarrow} P_{1/3-\delta} = \emptyset.$$

Hence, in view of (3.3), we see that $\mathcal{K}\times\mathcal{G}$ is indeed υ_B-Cauchy. Now let us take $\vartheta\in B$ with $-1\leqslant\vartheta\leqslant 1$, $H\cup\vartheta^{\leftarrow}P_{-1}=S$ and $K\cap\vartheta^{\leftarrow}Q_1=\emptyset$. Since $(\mathcal{K}\times\mathcal{G})\cap\vartheta_{1/3}\neq\emptyset$ we have $s\in S$ with $\vartheta^{\leftarrow}Q_{\vartheta(s)+1/3}\in\mathcal{K}$ and $\vartheta^{\leftarrow}P_{\vartheta(s)-1/3}\in\mathcal{G}$. Now $K\in\mathcal{K}$, $H\in\mathcal{G}$ gives

$$\vartheta^{\leftarrow}Q_{\vartheta(s)+1/3}\cap K\neq\emptyset \quad\text{and}\quad \vartheta^{\leftarrow}P_{\vartheta(s)-1/3}\cup H\neq S,$$

which leads to a contradiction. Hence, (a$_1$) is proved.

(a$_2$) Let $B\in\widehat{\mathcal{B}}$ be minimal and (F,G) a completely B-excluding pair for which there exists an $\mathcal{S}$-cofilter $\mathcal{G}$ with $F\notin\mathcal{G}$ that has no cluster point in S_p and an $\mathcal{S}$-filter $\mathcal{K}$ with $G\notin\mathcal{K}$ that has no cluster point in S_p. Proceeding exactly as in the proof of (a$_1$) we may show that $\mathcal{K}\times\mathcal{G}$ is υ_B-Cauchy, whence for $\vartheta\in B$ with $F\subseteq\vartheta^{\leftarrow}P_{-1}$ and $\vartheta^{\leftarrow}Q_1\subseteq G$ we have $s\in S$ with $\vartheta^{\leftarrow}Q_{\vartheta(s)+1/3}\in\mathcal{K}$ and $\vartheta^{\leftarrow}P_{\vartheta(s)-1/3}\in\mathcal{G}$. This time, $F\notin\mathcal{G}$ and $G\notin\mathcal{K}$ give

$$F\nsubseteq\vartheta^{\leftarrow}P_{\vartheta(s)-1/3} \quad\text{and}\quad \vartheta^{\leftarrow}Q_{\vartheta(s)+1/3}\nsubseteq G,$$

which lead to a contradiction as before, so proving (a$_2$).

(b) Suppose that $B\in\widehat{\mathcal{B}}^*$ has the property stated in (b), and that B' is any other element of $\widehat{\mathcal{B}}^*$. We wish to show $B\subseteq B'$, so take $\varphi\in B$. Since B' certainly contains all constant real functions on S we may assume φ is not constant, whence $\alpha=\inf\varphi(S)$, $\beta=\sup\varphi(S)$ satisfy $\alpha<\beta$. Take $\delta>0$, choose n with $2/n<\min\{\delta,(\beta-\alpha)/2\}$, and integers p,q satisfying

$$(p-1)/n\leqslant\alpha<p/n \quad\text{and}\quad q/n<\beta\leqslant(q+1)/n.$$

Clearly $(\varphi^{\leftarrow}P_{i/n},\varphi^{\leftarrow}Q_{(i+1)/n})$, $p\leqslant i\leqslant q-1$ are closed, co-open completely B-excluding pairs, whence by hypothesis either the first set is compact or the second cocompact. Suppose, for some given i, that $\varphi^{\leftarrow}P_{i/n}$ is compact. Take $s\in S_p$ with $\varphi^{\leftarrow}P_{i/n}\nsubseteq Q_s$. Then $\varphi^{\leftarrow}Q_{(i+1)/n}\nsubseteq Q_s$, so since B' is a bigenerating sub-T-lattice we have $\psi_s\in B'$ and $x\in\mathbb{R}$ with $P_s\subseteq\psi_s^{\leftarrow}Q_x\subseteq Q_{(i+1)/n}$. Now $\psi_s^{\leftarrow}Q_x\nsubseteq Q_s$ so $\psi(s)<x$ and we may choose $\epsilon>0$ with $\psi(s)+\epsilon<x$. Taking into account Remark 3.19, we may now choose $r,k\in\mathbb{R}$ with $r>0$ so that $\varphi_s=((r\varphi+k)\vee(-\mathbf{1}))\wedge\mathbf{1}\in B'$ satisfies

$$\psi_s^{\leftarrow}Q_{\psi(s)+\epsilon}\subseteq\varphi_s^{\leftarrow}P_{-1} \quad\text{and}\quad \varphi_s^{\leftarrow}Q_1\subseteq\varphi^{\leftarrow}Q_{(i+1)/n}.$$

Using the compactness of $\varphi^{\leftarrow}P_{i/n}$ now leads to a function $\vartheta_i\in B'$ satisfying $\varphi^{\leftarrow}P_{i/n}\subseteq\vartheta_i^{\leftarrow}P_{-1}$, $\vartheta_i^{\leftarrow}Q_1\subseteq\varphi^{\leftarrow}Q_{(i+1)/n}$, and a dual argument gives a function ϑ_i with the same properties if $\varphi^{\leftarrow}Q_{(i+1)/n}$ is cocompact.

In order to prove $\varphi\in U(\upsilon_{B'})$ it will be sufficient, by Lemma 3.2(1), to show that $\varphi_\delta\in\upsilon_{B'}$, and by Lemma 3.2(2) this will follow from $(\bigwedge_{i=p}^{q-1}(\vartheta_i)_{1/2})^{\Delta}\prec\varphi_\delta$. Take $s\in S$ and suppose that

$$\mathrm{St}\Bigl(\bigwedge_{i=p}^{q-1}(\vartheta_i)_{1/2},P_s\Bigr)\nsubseteq\varphi^{\leftarrow}Q_{\varphi(s)+\delta}.$$

If we take $t\in S$ with $\mathrm{St}(\bigwedge_{i=p}^{q-1}(\vartheta_i)_{1/2},P_s)\nsubseteq Q_t$, $P_t\nsubseteq\varphi^{\leftarrow}Q_{\varphi(s)+\delta}$ there exist $s_i\in S$, $p\leqslant i\leqslant q-1$, with $\bigcap_{i=p}^{q-1}\vartheta^{\leftarrow}Q_{\vartheta(s_i)+1/2}\nsubseteq Q_t$ and $P_s\nsubseteq\bigcup_{i=p}^{q-1}\vartheta_i^{\leftarrow}P_{\vartheta_i(s_i)-1/2}$. Now there exists j with $j/n<\varphi(s)\leqslant(j+1)/n$, from which we obtain $\vartheta_{j+1}(s)=-1$, and since $\varphi(t)\geqslant\varphi(s)+\delta>\varphi(s)+2/n$ we have $\varphi(t)>(j+2)/n$ from which we obtain $\vartheta_{j+1}(t)=1$. However, $\vartheta_{j+1}(t)\leqslant\vartheta_{j+1}(s_{j+1})+1/2$ and $\vartheta_{j+1}(s)\geqslant\vartheta_{j+1}(s_{j+1})-1/2$, which leads to an immediate contradiction.

A similar argument for the dual inclusion also holds, so we deduce $\varphi\in U(\upsilon_{B'})$, that is $B\subseteq B'$ as required. □

The following result uses the notions of stability and costability. Stability for bitopological spaces was introduced by R. Kopperman [17]. We recall that a stable (costable) ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ is one for which every $F\in\kappa\setminus\{S\}$ ($G\in\tau\setminus\{\emptyset\}$) is compact (respectively cocompact).

Corollary 3.22. *A stable, costable bi-T_2 ditopological texture space has a unique compatible totally bounded di-uniformity.*

Proof. We know from [11, Theorem 3.7] that an R_1 costable space is regular, while by [11, Theorem 3.5] a regular stable space is normal, so using [9, Corollary 5.24] we see that a bi-R_1 stable, costable space is completely biregular. Hence, Theorem 3.21 applies. Now for any $B\in\mathcal{B}$ and a completely B-excluding pair (F,G) we have $F\in\kappa$, and as clearly $F\subseteq G$ and $G\neq S$ we also have $F\neq S$ so F is compact by stability. Likewise, G is cocompact by costability. Hence B is the smallest element of $\mathcal{B}$, that is $\mathcal{B}$ has a single element which generates the unique compatible totally bounded di-uniformity. □

Clearly, the same result holds for a completely biregular bi-T_2 space that is either stable or costable. An example of a space satisfying the conditions of the corollary is $(\mathbb{R},\mathcal{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ restricted to the set $\{r\in\mathbb{R}\mid r\geqslant 0\}$.

Bearing in mind that a dicompact space is stable and costable, and that every compatible di-uniformity is totally bounded, Corollary 3.22 gives at once:

Corollary 3.23. *Every dicompact bi-T_2 ditopological texture space has a unique compatible di-uniformity.*

Remark 3.24. An examination of the proof of (b) shows that it would be enough to replace the assumption that F is compact by the weaker requirement that every open cover of G has a finite subfamily covering F, which would be equivalent to saying that every cofilter $\mathcal{G}$ not containing F has a cluster point in $G^{\flat}$. Likewise, the assumption that G be cocompact could be weakened in the same way. Clearly, even these weaker requirements are in general stronger than the conclusion of (a_2).

Remark 3.25. For the plain case, the necessary condition (a_1) occurs, along with the sufficient condition (b), in bitopological form in [2, Theorem 3.3.4]. The reason (a_1) is interesting there is that, for the single topology case, it coincides with (b). Generally, however, (a_1) would seem to be of less relevance than (a_2). For instance, it is easy to see that in the subspace of $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ mentioned above, (a_1) holds only by virtue of the fact that in this space there can be no completely B-co-excluding pairs.

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