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Defining the k^{th} Powers of the Dirac-Delta Distribution for Negative Integers

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Abstract—In [1], Koh and Kuan defined the powers of the Dirac-delta distribution for positive integers. Here we extend their definition for negative integers. Also, we give meaning to the distribution δ_+^{-k}.

Keywords—Distribution, Dirac-delta function, Neutrix, Neutrix limit, Neutrix composition of distribution.

1. INTRODUCTION

In the theory of distributions, no meaning can be given to expressions of the form $H(\delta^{(s)}(x))$, δ^k, $\ln\delta$, δ_+^k, and the like. However, in physics, one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interaction [2, p. 141]. Bremermann used the Cauchy representations of distributions to define $\sqrt{\delta_+}$, δ_+, and $\ln\delta_+$ (see [3]). Unfortunately, his definition does not carry over to $\sqrt{\delta}$, $\ln\delta$, and δ_+^k. In [4], Antosik gave the results $\sqrt{\delta}=0$, $\sqrt{\delta^2+1}=1+\delta$, $\log(1+\delta)=0$, $\sin\delta=0$, $\cos\delta=1$, and $(\delta+1)^{-1}=1$. Recently, Koh and Kuan [1] used the fixed δ-sequence to give meaning to the distributions δ^k and $(\delta')^k$ for $k\in(0,1)$ and $k=2,3,\dots$. Later they redefined $\delta^k(x)$ as the boundary value of $\delta^k(z-i\epsilon)$ as $\epsilon\to 0^+$[5]. Fisher and Kou considered the more general form $(\delta^{(s)}(x))^k$ in [6].

2. THE NEUTRIX COMPOSITION OF DISTRIBUTIONS

The technique of neglecting appropriately defined infinite quantities was devised by Hadamard [7] and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. In fact, his method can be regarded as a particular application of the neutrix calculus developed by van der Corput (see [7]). This is a general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been exploited in the context of distributions by Fisher in connection with the problem of distributional multiplication, convolution, and composition (see [8–10]).

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To define the composition of two distributions, we shall first let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$, for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

For our purpose, we additionally assume that $\rho(x)$ is decreasing on the open interval $(0,1)$. We define $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$. It follows that $\delta_n(x)$ is a regular sequence of infinitely differentiable functions converging to Dirac-delta function $\delta(x)$. Let f be an arbitrary distribution and define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle.$$

Then $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to f.

Let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ the space of distributions.

DEFINITION. *Let F and f be distributions in $\mathcal{D}'$. We say that the distribution $F(f(x))$, the neutrix composition of F and f, exists and is equal to $h(x)$ on the interval (a,b) if the double neutrix limit*

$$N - \lim_{n\to\infty} \left[N - \lim_{m\to\infty} \int_{-\infty}^{\infty} F_n(f_m(x))\, \phi(x)\, dx \right] = \langle h(x), \phi(x) \rangle,$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a,b), where $F_n(x) = (F * \delta_n)(x)$, $f_m(x) = (f * \delta_m)(x)$, and N is the neutrix having domain $N' = \{1, 2, \dots, n, \dots\}$, range the real numbers with negligible functions which are finite linear sums of the functions $n^\lambda \ln^{r-1} n$, $\ln^r n$ ($\lambda > 0, r = 1, 2, \dots$), and all functions which converge to zero in the usual sense as n tends to infinity (see [7,8,10]).

3. RESULTS

Koh and Kuan [1] define δ^k for the fixed δ-sequence as follows:

$$\delta^k(x) = \begin{cases} 0, & \text{if } k \text{ even}, \\ C_k \delta^{(k-1)}(x), & \text{if } k \text{ odd}, \end{cases}$$

for $k = 1, 2, \dots$, where $C_k = [2^{k-1}((k-1)/2)! k^{k/2} \pi^{(k-1)/2}]^{-1}$ (see [1]). We extend this result to the negative integers as follows.

THEOREM 1. *The distribution δ^{-k} exists on the real line and*

$$\delta^{-k}(x) = 0,$$

for $k = 1, 2, \dots$.

PROOF. Writing $x^{-k} = ((-1)^{k-1}/(k-1)!)(\ln x)^{(k)}$, we have

$$\left(x^{-k}\right)_n = x^{-k} * \delta_n(x) = \frac{(-1)^{k-1}}{(k-1)!} \int_{-1/n}^{1/n} \ln|t-x| \delta_n^{(k)}(t)\, dt.$$

Then for $\phi \in \mathcal{D}$,

$$\begin{aligned} \left\langle \left[(\delta_m(x))^{-k} \right]_n, \phi(x) \right\rangle = {} & \frac{(-1)^{k-1}}{(k-1)!} \int_{|x|\geq 1/m} \phi(x) \int_{-1/n}^{1/n} \ln|t-\delta_m(x)|\, \delta_n^{(k)}(t)\, dt\, dx \\ & + \frac{(-1)^{k-1}}{(k-1)!} \int_{|x|<1/m} \phi(x) \int_{-1/n}^{1/n} \ln|t-\delta_m(x)|\, \delta_n^{(k)}(t)\, dt\, dx. \end{aligned}$$

The function $\delta_m(x)$ is equal to zero on the set $\{x : |x| \geq 1/m\}$. Thus,

$$\lim_{m\to\infty} \int_{|x|\geq 1/m} \phi(x) \int_{-1/n}^{1/n} \ln|t-\delta_m(x)|\, \delta_n^{(k)}(t)\, dt\, dx = \int_{-\infty}^{\infty} \phi(x)\, dx \int_{-1/n}^{1/n} \ln|t|\delta_n^{(k)}(t)\, dt$$

and making the substitution $nt = u$, we have

$$N-\lim_{n\to\infty} \left[\lim_{m\to\infty} \int_{|x|\geq 1/m} \phi(x) \int_{-1/n}^{1/n} \ln|t-\delta_m(x)|\delta_n^{(k)}(t)\, dt\, dx \right]$$
$$= \int_{-\infty}^{\infty} \phi(x)\, dx \left[N-\lim_{n\to\infty} n^k \int_{-1}^{1} [\ln|u| - \ln n]\rho^{(k)}(u)\, du \right] = 0.$$

Next

$$\begin{aligned} \int_{-1/n}^{1/n} \left|\ln|t-\delta_m(x)|\, \delta_n^{(k)}(t)\right| dt &\leq n^{k+1} \sup_t \left\{ \left|\rho^{(k)}(t)\right| \right\} \int_{-1/n}^{1/n} |\ln|t-\delta_m(x)||\, dt \\ \leq n^{k+1} \sup_t \left\{ \left|\rho^{(k)}(t)\right| \right\} &\left\{ \left| \left(\frac{1}{n} - \delta_m(x)\right) \ln\left|\frac{1}{n} - \delta_m(x)\right| - \left(\frac{1}{n} + \delta_m(x)\right) \ln \right| \right. \\ &\left. \times \frac{1}{n} + \delta_m(x) \left| + \frac{2}{n} \right| \right\} \leq n^k \sup_t \left\{ \left|\rho^{(k)}(t)\right| \right\} (2\ln n + 3), \end{aligned} \tag{1}$$

for $(1/m)\rho^{-1}(1/mn) \leq |x| < 1/m$.

If $|x| < (1/m)\rho^{-1}(1/mn)$, we have $1/n < \delta_m(x) \leq m\rho(0)$ and

$$\ln(1+\rho(0)m) = \sup_t \left\{ \sup_{|x|<(1/m)\rho^{-1}(1/mn)} |\ln|t-\delta_m(x)|| \right\}.$$

Thus,

$$\left| \int_{|x|<1/m} \phi(x) \int_{-1/n}^{1/n} \ln|t-\delta_m(x)|\, \delta_n^{(k)}(t)\, dt\, dx \right|$$
$$= \left| \int_{|x|<(1/m)\rho^{-1}(1/mn)} \phi(x) \int_{-1/n}^{1/n} \ln|t-\delta_m(x)|\, \delta_n^{(k)}(t)\, dt\, dx \right.$$
$$\left. + \int_{(1/m)\rho^{-1}(1/mn)\leq|x|<1/m} \phi(x) \int_{-1/n}^{1/n} \ln|t-\delta_m(x)|\, \delta_n^{(k)}(t)\, dt\, dx \right| \leq 4n^k \sup_t$$
$$\left\{ \left|\rho^{(k)}(t)\right| \right\} \sup|\phi(x)| \left\{ \frac{1}{m}\rho^{-1}\left(\frac{1}{mn}\right) \ln(1+\rho(0)m) + \frac{1}{m}\left[1-\rho^{-1}\left(\frac{1}{mn}\right)\right] (2\ln n + 3) \right\} \to 0,$$

as $m \to \infty$.

It follows from what we have just proved that

$$N-\lim_{n\to\infty} \left[N - \lim_{m\to\infty} \left\langle \left[(\delta_m(x))^{-k}\right]_n, \phi(x) \right\rangle \right] = 0,$$

for all $\phi \in \mathcal{D}$. This completes the proof of the theorem.

In [11], the definition of the distribution δ_+^k was given by

$$\delta_+^k = \frac{(-1)^{(k-1)}}{(k-1)!} C_{k,\rho} \delta^{(k-1)}(x),$$

for $k = 1, 2, \dots$, where $C_{k,\rho} = \int_{-1}^{1} [\rho(y)]^k y^{k-1}\,dy$. In particular,

$$\delta_+^k = 0, \qquad \text{for even } k.$$

Similarly, we now give the definition of the k^{th} power of δ_+ for negative integers.

THEOREM 2. *The distribution δ_+^{-k} on the real line is defined by*

$$\delta_+^{-k} = 0,$$

for all $k = 1, 2, \dots$.

PROOF. Put

$$\left(x_+^{-k}\right)_n = x_+^{-k} * \delta_n(x) = \begin{cases} \dfrac{(-1)^k}{(k-1)!} \displaystyle\int_{-1/n}^{x} \ln(x-t)\delta_n^{(k)}(t)\,dt, & x < \dfrac{1}{n}, \\ \dfrac{(-1)^k}{(k-1)!} \displaystyle\int_{-1/n}^{1/n} \ln(x-t)\delta_n^{(k)}(t)\,dt, & x \geq \dfrac{1}{n}, \\ 0, & x \leq -\dfrac{1}{n}, \end{cases}$$

$$\left[(\delta_m(x))_+^{-k}\right]_n = \begin{cases} \dfrac{(-1)^k}{(k-1)!} \displaystyle\int_{-1/n}^{m\rho(mx)} \ln(\delta_m(x)-t)\delta_n^{(k)}(t)\,dt, & \delta_m(x) < \dfrac{1}{n}, \\ \dfrac{(-1)^k}{(k-1)!} \displaystyle\int_{-1/n}^{1/n} \ln\left(\delta_m(x)-t\right)\delta_n^{(k)}(t)\,dt, & \delta_m(x) \geq \dfrac{1}{n}, \\ 0, & \delta_m(x) \leq -\dfrac{1}{n}. \end{cases}$$

For $\phi \in \mathcal{D}$, we have

$$\begin{aligned} \left\langle \left[(\delta_m(x))_+^{-k}\right]_n, \phi(x) \right\rangle = \frac{(-1)^k}{(k-1)!} \Bigg\{ &\int_{|x|\geq 1/m} \phi(x) \int_{-1/n}^{m\rho(mx)} \ln(\delta_m(x)-t)\delta_n^{(k)}(t)\,dt\,dx \\ &+ \int_{(1/m)\rho^{-1}(1/mn)<|x|<1/m} \phi(x) \int_{-1/n}^{m\rho(mx)} \ln(\delta_m(x)-t)\delta_n^{(k)}(t)\,dt\,dx \\ &+ \int_{|x|\leq (1/m)\rho^{-1}(1/mn)} \phi(x) \int_{-1/n}^{1/n} \ln\left(\delta_m(x)-t\right)\delta_n^{(k)}(t)\,dt\,dx \Bigg\}. \end{aligned} \tag{2}$$

If $|x| \geq 1/m$, then $\delta_m(x) = 0$. Thus,

$$\lim_{m\to\infty} \int_{|x|\geq 1/m} \phi(x) \int_{-1/n}^{m\rho(mx)} \ln(\delta_m(x)-t)\delta_n^{(k)}(t)\,dt\,dx = \int_{-\infty}^{\infty} \phi(x)\,dx \int_{-1/n}^{0} \ln(-t)\delta_n^{(k)}(t)\,dt,$$

and by making the substitution $nt = u$,

$$\begin{aligned} N-\lim_{n\to\infty} &\left[\lim_{m\to\infty} \int_{|x|\geq 1/m} \phi(x) \int_{-1/n}^{m\rho(mx)} \ln(\delta_m(x)-t)\delta_n^{(k)}(t)\,dt\,dx \right] \\ &= \int_{-\infty}^{\infty} \phi(x)\,dx \left[N-\lim_{n\to\infty} (-n)^k \int_0^1 [\ln(-u) - \ln n]\rho^{(k)}(u)\,du \right] = 0. \end{aligned} \tag{3}$$

Now, as in equation (1), we have

$$\begin{aligned} \int_{-1/n}^{m\rho(mx)} \left| \ln(\delta_m(x)-t)\delta_n^{(k)}(t) \right| dt &\leq n^{k+1} \sup_t \left\{ \left| \rho^{(k)}(t) \right| \right\} \int_{-1/n}^{m\rho(mx)} \left| \ln\left(\delta_m(x)-t\right) \right| dt \\ &\leq 2n^k \sup_t \left\{ \left| \rho^{(k)}(t) \right| \right\} (\ln 2 + \ln n), \end{aligned}$$

for $(1/m)\rho^{-1}(1/mn) < |x| < 1/m$. Thus,

$$\begin{aligned}\left|\int_{(1/m)\rho^{-1}(1/mn)<|x|<1/m} \phi(x) \int_{-1/n}^{m\rho(mx)} \ln(\delta_m(x) - t)\delta_n^{(k)}(t)\,dt\,dx\right| \\ \leq 4n^k \sup_t \left\{\left|\rho^{(k)}(t)\right|\right\} (\ln 2 + \ln n) \sup|\phi(x)| \left[\frac{1}{m} - \frac{1}{m}\rho^{-1}\left(\frac{1}{mn}\right)\right] \to 0,\end{aligned} \tag{4}$$

as $m \to \infty$. Similarly, if $|x| \leq (1/m)\rho^{-1}(1/mn)$, then $1/n \leq \delta_m(x) \leq \rho(0)m$, and so

$$\begin{aligned}&\int_{-1/n}^{1/n} \left|\ln(\delta_m(x) - t)\delta_n^{(k)}(t)\right| dt \\ &\qquad \leq n^k \sup_t \left\{\left|\rho^{(k)}(t)\right|\right\} \left\{\rho(0)mn \left|\ln\left|\frac{\rho(0)mn - 1}{\rho(0)mn + 1}\right|\right| + 2|\ln(\rho(0)m + 1)| + 2\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\left|\int_{|x|\leq(1/m)\rho^{-1}(1/mn)} \phi(x) \int_{-1/n}^{1/n} \ln(\delta_m(x) - t)\delta_n^{(k)}(t)\,dt\,dx\right| \leq 2n^k \sup_t \left\{\left|\rho^{(k)}(t)\right|\right\} \sup \\ |\phi(x)|\rho^{-1}\left(\frac{1}{mn}\right) \left\{\rho(0)n \left|\ln\left|\frac{\rho(0)mn - 1}{\rho(0)mn + 1}\right|\right| + \frac{2}{m}|\ln(\rho(0)m + 1)| + \frac{2}{m}\right\} \to 0,\end{aligned} \tag{5}$$

as $m \to \infty$.

It follows from equations (2)–(5) that

$$N - \lim_{n\to\infty} \left[N - \lim_{m\to\infty} \left\langle \left[(\delta_m(x))_+^{-k}\right]_n, \phi(x)\right\rangle\right] = 0,$$

for all $\phi \in \mathcal{D}$. This completes the proof of the theorem.

REMARK. The existence of δ^k and δ^{-k} as a power of δ has never been proved, although the symbols continue to appear in print. In this paper, we have given a class of distributions δ^{-k} and δ_+^{-k} which are the neutrix limit of the k^{th} power of a delta sequence for negative integers.

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