Complex and symplectic structures on panelled web 4-manifolds

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ABSTRACT

We analyze the symplectic and complex structures on the panelled web 4-manifolds of S. Akbulut and M. Kalafat [1]. In particular, we give infinite families of examples of almost complex but not symplectic and not complex 4-manifolds in the non-simply connected case.

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1. Introduction

Panelled web 4-manifolds are introduced in [1] which have a natural locally conformally flat (LCF) structure with negative scalar curvature. The underlying smooth manifolds have the nice property that they can be used to produce self-dual metrics with small signatures. See [2] for further information. In this paper we analyze complex and symplectic structures on these type of manifolds. Summarizing Theorems 2.3, 2.4, 2.5 and 2.6 we obtain the following result.

Theorem A. *The manifolds M_g^1, $M_{g,n}^2$, $M_{g,n}^3$ for $g, n > 0$ and M_n^4 for $n > 1$ are almost complex.*[1]

Furthermore as an outcome of the search for a complex structure we get,

Theorem 4.2. *The manifolds M_g^1, $M_{g,n}^2$, $M_{g,n}^3$, M_n^4 do not admit any complex structure for all $g, n > 0$.*

Finally, the search for a symplectic structure yields the following.

Theorem 3.2. *The manifolds M_g^1, $M_{g,n}^2$, $M_{g,n}^3$, M_n^4 do not admit any symplectic structure for all $g, n > 0$.*

It follows from the above theorems that the non-simply connected 4-manifolds we investigated all carry almost complex structures, but they are neither symplectic nor complex. The first kind of example of almost complex but noncomplex 4-manifolds was constructed by A. Van de Ven. In his paper [15] he proves that for a 4-manifold M, if the Chern numbers $p = c_1^2[M]$ and $q = c_2[M]$ belong to a restricted domain then there does not exist a complex structure. Combining this with

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[1] Throughout the paper we have the assumption $g > 0$. Panelled 4-manifolds are built upon lower dimensional hyperbolic manifolds. These hyperbolic manifolds are produced by some special Kleinian groups which results in this condition. See [11] for more information.
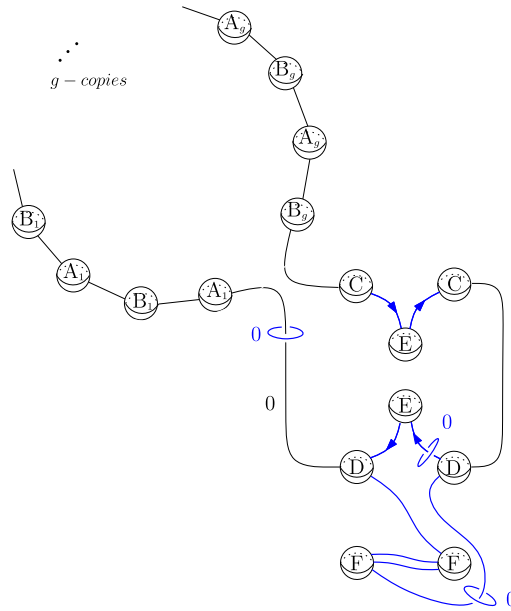
Fig. 1. The panelled web 4-manifolds M_g^1.

Milnor's result [13], namely if $p + q \equiv 0$ (mod 12) then an almost complex compact manifold can be constructed with these p, q; he constructs almost complex but noncomplex manifolds for special p and q's which are non-simply connected and reducible, i.e. of the form of a connected sum. Later on A. Howard [8] gave examples using the same method of simply connected almost complex but noncomplex 4-manifolds. Shing-Tung Yau produces the first parallelizable manifolds of this kind in [17] using Massey products, N. Brotherton provides more examples of parallelizable noncomplex almost complex manifolds in [4] using the same techniques. These are basically constructed by taking the product of a 3-manifold with a circle. They all depend on the fact that a compact complex 4-manifold with even first Betti number is Kähler, and the real Massey products on a compact Kähler manifold vanish. So they construct compact manifolds with first even Betti numbers and nonvanishing Massey products, so that there is no way for them to be complex. Here we also provide examples of 4-manifolds of this kind, namely almost complex, noncomplex and nonsymplectic manifolds. Ours have strictly negative (hence nontrivial) Euler characteristics, so that they are not parallelizable. In higher dimensions the existence of almost complex, noncomplex manifolds remains still as a conjecture. The reader may want to check [5] for symplectic, noncomplex examples.

In Sections 2 and 4 we analyze the complex structures and in Section 3 we investigate symplectic structures on the manifolds.

2. Almost complex structures

The following result is a guideline for us to understand almost complex structures on 4-manifolds, see [6] for references.

Lemma 2.1. *([7,16]) For a given 4-manifold X with signature $\tau(X)$ and Euler characteristic $\chi(X)$, for any element $h \in H_2(X; \mathbb{Z})$ satisfying the equation $h^2 = 3\tau(X) + 2\chi(X)$ and the congruence $h \equiv w_2$ (mod 2) there is an almost complex structure J on TX with $h = c_1(X; J)$.*

Also we will be using the following fact, see [6, p. 187] for details.

Lemma 2.2. *Let X be an oriented 4-manifold (not necessarily compact). If $H_1(X; \mathbb{Z})$ has no 2-torsion then X is spin iff Q_X is an even form.*

The first set of manifolds we want to work with are denoted by M_g^1, and can be seen through Fig. 1. This is an instance of a *handlebody diagram.* One can interpret the diagram as follows. Start with a blank page which stands for a copy of $\mathbb{R}^3$, a part of S^3 the three-dimensional sphere. The three-sphere bounds a 4-dimensional ball from one side. As you can notice, the spheres in the picture appears in pairs. Through each pair we attach a 4-dimensional 1-handle, i.e. a copy of $D^1 \times D^3$, where D^k is a k-dimensional (closed) ball. After this process, we attach 4-dimensional 2-handles, i.e. $D^2 \times D^2$ through the curves in the picture, each of which is a copy of a circle. Notice that a boundary piece of the 2-handle is $S^1 \times D^2$. After these two types of attachments, we need to attach the 3 and 4-handles, but these are uniquely attached so we do not

mention them. In [1] the reader can find out the motivation to work with these manifolds. There, LCF metrics on these manifolds are considered. Here, we are only interested in the underlying smooth manifolds. Alternatively, for construction one begins with $\Sigma_{g,2}$, the twice punctured genus g surface, then cross it with the interval $I = [0, 1]$ and then glue the boundary cylinders with each other with a flip. Finally make the so-called complex twist. This way one obtains the panelled web 3-manifold, and then by crossing this with S^1 and identifying its boundary one obtains the panelled web 4-manifold. A curious reader should check the reference to understand the relationship between the two constructions. These manifolds have the following characteristics.

$$\pi_1(M_g^1) = \langle a_1, b_1, \ldots, a_g, b_g, c, d, e \mid a_1^{-1}b_1^{-1}a_1b_1 \cdots a_g^{-1}b_g^{-1}a_gb_gcd^{-1} = ede^{-1}c^{-1} = df^3 = 1 \rangle,$$

$$\tau(M_g^1) = 0, \qquad Q_{M_g^1} = H, \qquad \chi(M_g^1) = -4g,$$

$$H_1(M_g^1; \mathbb{Z}) = \mathbb{Z}^{2g+2},$$

$$H_2(M_g^1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}.$$

Here, the fundamental group is generated by 1-handles, and lowercase letters are used to denote the corresponding generators. As an application of Lemma 2.2 they are all spin since they have even intersection form and there is no 2-torsion in $H_1(M_g^1; \mathbb{Z})$. We will see that the lemma is also applicable for the rest of the manifolds we will consider, so that they will be spin as well. Our first result is the following.

Theorem 2.3. *The manifolds M_g^1 are almost complex for all $g > 0$.*

Proof. We know $H_2(M_g^1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Say h_1 and h_2 are generators of $H_2(M_g^1; \mathbb{Z})$. Let $(a, b) = ah_1 + bh_2$ be an element of $H_2(M_g^1; \mathbb{Z})$. If (a, b) satisfies the above hypothesis we must have $(a, b) \cup (a, b) = 3\tau(M_g^1) + 2\chi(M_g^1) = 3(0) + 2(-4g) = -8g$. But since we know from the intersection form of M_g^1, $h_1^2 = h_2^2 = 0$ and $h_1h_2 = 1$ we get $(a, b) \cup (a, b) = a^2h_1 \cup h_1 + b^2h_2 \cup h_2 + 2abh_1 \cup h_2 = 2ab$. So we must have $2ab = -8g$, i.e. $ab = -4g$. Then for any fixed g, let $a = -2g$, $b = 2$ so that $h = -2gh_1 + 2h_2 \in H_2(M_g^1; \mathbb{Z})$. This gives an almost complex structure on TM_g^1 for any g. Note that the second condition in Wu's theorem, i.e. $h \equiv w_2 \pmod{2}$ is automatically satisfied for this choice of $h = (-2g, 2)$, since we know that M_g^1 is spin hence $w_2 = 0$ and we have $h = -2gh_1 + 2h_2 = 0 \pmod 2$. $\square$

In fact, the number of almost complex structures (up to homotopy) on each manifold of this family and upcoming families are finite. This is the consequence of that Lemma 2.1 is actually an if and only if statement, i.e. almost complex structures are in correspondence with (or parametrized by) the elements of $H_2(M; \mathbb{Z})$ satisfying the two conditions. One way to see this is that an almost complex structure turns the tangent bundle into a complex 2-bundle, and $U(2)$ bundles are determined by the first two Chern classes, the latter of which is equal to the Euler class that is fixed for the tangent bundle so only c_1 conditions are important. See pp. 29–31 of [6] for further information. For our family e.g. if g is a prime number, then $(\pm 2g, \mp 2)$ and $(\pm 2, \mp 2g)$ are the only possibilities for h. In general h^2 conditions, which is $ab = -4g$ for this family restricts the possibilities to the finite case and depends on the number of prime factors of g.

The second family of manifolds we want to work on are the manifolds $M_{g,n}^2$. See Fig. 27 in [1] to see the explicit handlebody diagram. As one compares to the first family, the number of CDE components is increased and the complex twist handle (the F component) in the handlebody diagram for M_g^1 is omitted. The fundamental group and the other topological invariants are computed as follows.

$$\pi_1(M_{g,n}^2) = \langle a_1, b_1, \ldots, a_g, b_g, c_1, d_1, e_1, \ldots, c_n, d_n, e_n \mid a_1^{-1}b_1^{-1}a_1b_1 \cdots a_g^{-1}b_g^{-1}a_gb_gc_1 \cdots c_nd_n^{-1} \cdots d_1^{-1}$$
$$= e_id_ie_i^{-1}c_i^{-1} = 1 \rangle,$$

$$\tau(M_{g,n}^2) = 0, \qquad Q_{M_{g,n}^2} = H, \qquad \chi(M_{g,n}^2) = 4 - 4g - 4n,$$

$$H_1(M_{g,n}^2; \mathbb{Z}) = \mathbb{Z}^{2g+2n},$$

$$H_2(M_{g,n}^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}.$$

From this set of constructions we obtain the following.

Theorem 2.4. *The manifolds $M_{g,n}^2$ are almost complex for all $g, n > 0$.*

Proof. We know $H_2(M_{g,n}^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Say h_1 and h_2 are generators of $H_2(M_{g,n}^2; \mathbb{Z})$. Let $(a, b) = ah_1 + bh_2$ be an element of $H_2(M_{g,n}^2; \mathbb{Z})$. If (a, b) satisfies the above hypothesis we must have $(a, b) \cup (a, b) = 3\tau(M_{g,n}^2) + 2\chi(M_{g,n}^2) = 3(0) + 2(4 - 4g - 4n) = 8 - 8g - 8n$. But since we know from the intersection form of the manifold, $h_1^2 = h_2^2 = 0$ and $h_1h_2 = 1$ we get $(a, b) \cup (a, b) = a^2h_1 \cup h_1 + b^2h_2 \cup h_2 + 2abh_1 \cup h_2 = 2ab$. So we must have $2ab = 8 - 8g - 8n$, i.e. $ab = 4 - 4g - 4n$. Then for any

fixed g, let $a = 2 - 2g - 2n$, $b = 2$ so that $h = (2 - 2g - 2n)h_1 + 2h_2 \in H_2(M_{g,n}^2; \mathbb{Z})$. This gives an almost complex structure on $TM_{g,n}^2$ for any g. Note that the second condition in Wu's theorem, i.e. $h \equiv w_2 \pmod{2}$ is automatically satisfied for this choice of $h = (2 - 2g - 2n, 2)$, since we know that $M_{g,n}^2$ is spin hence $w_2 = 0$ and we have $h = (2 - 2g - 2n)h_1 + 2h_2 = 0 \pmod 2$. □

In the third sequence of manifolds $M_{g,n}^3$ in [1] seen in Fig. 29, *CE*-components are taken away and some new building blocks, namely trivial I-bundles over punctured annuli $\Sigma_{0,3}$ are attached through the handles D_i. These manifolds have,

$$\pi_1(M_{g,n}^3) = \langle a_1, b_1, \ldots, a_g, b_g, k_1, l_1, m_1, \ldots, k_n, l_n, m_n \mid a_1^{-1}b_1^{-1}a_1b_1 \cdots a_g^{-1}b_g^{-1}a_gb_g c_1 \cdots c_n d_n^{-1} \cdots d_1^{-1} = 1 \rangle,$$

$$\tau(M_{g,n}^3) = 0, \qquad Q_{M_{g,n}^3} = (n+1)H, \qquad \chi(M_{g,n}^3) = 4 - 4g - 4n,$$

$$H_1(M_{g,n}^3; \mathbb{Z}) = \mathbb{Z}^{2g+3n},$$

$$H_2(M_{g,n}^3; \mathbb{Z}) = \mathbb{Z}^{2+2n}.$$

Our next result is the following.

Theorem 2.5. *The manifolds $M_{g,n}^3$ are almost complex for all $g, n > 0$.*

Proof. This time we have $H_2(M_{g,n}^3; \mathbb{Z}) = \mathbb{Z}^{2+2n}$, so that $H_2(M_{g,n}^3)$ is generated by $h_1, \ldots, h_{2+2n}$ with $h_i \cup h_i = 0$ for any i, and $h_i \cup h_{i+1} = h_{i+1} \cup h_i = 1$ for odd i's. So for instance we may take $h = 2h_1 + 2h_2 + 2h_3 - 2gh_4 + 2h_5 - 2nh_6$. With this choice of h, one can obtain $h \cup h = 8 - 8g - 8n = 3\tau(M_{g,n}^3) + 2\chi(M_{g,n}^3)$. Again since h is arranged to be even its mod 2 reduction is zero, which is the Stiefel–Whitney class of the spin manifold. □

In the last set of constructions of panelled web manifolds M_n^4 in [1] seen in Fig. 30, many copies of the new building blocks are attached to each other as a chain. By this way it is found that,

$$\pi_1(M_n^4) = \langle \cdots g_i, h_i, j_i, k_i, l_i, \ldots, m \mid k_ih_ik_i^{-1}g_i = l_i^{-1}j_il_ih_i = g_ih_ij_i = 1 \text{ for all } i \leqslant n \rangle,$$

$$\tau(M_n^4) = 0, \qquad Q_{M_n^4} = nH, \qquad \chi(M_n^4) = -2n,$$

$$H_1(M_n^4; \mathbb{Z}) = \mathbb{Z}^{2n+1},$$

$$H_2(M_n^4; \mathbb{Z}) = \mathbb{Z}^{2n}.$$

Our next result is the following.

Theorem 2.6. *The manifolds M_n^4 are almost complex for all $n > 1$ and do not admit any almost complex structure for $n = 1$.*

Proof. We split the proof into the following cases.

Case 1: Assume n is even; We have $H_2(M_n^4; \mathbb{Z}) = \mathbb{Z}^{2n}$, so that $H_2(M_n^4)$ is generated by $h_1, \ldots, h_{2n}$ with $h_i \cup h_j = 0$ for $j \neq i+1$ and $h_i \cup h_{i+1} = h_{i+1} \cup h_i = 1$ for odd i's. So for n even we may take $h = 2h_1 + 2h_2 + 2h_3 - 4h_4 + 2h_5 + 2h_6 + 2h_7 - 4h_8 + \cdots + 2h_{2n-3} + 2h_{2n-2} + 2h_{2n-1} - 4h_{2n}$ so that the coefficients take the values $(2, 2, 2, -4)$ repeatedly. With this choice of h, one can obtain $h \cup h = -4n = 3\tau(M_n^4) + 2\chi(M_n^4)$ and $h \pmod 2 = w_2 = 0$. So, the hypothesis of Wu's theorem is satisfied.

Case 2: Assume n is odd, $n \neq 1$ and $n = 2k + 1$ with k even (say $k = 2t$, $t \in \mathbb{Z}^+$); Again say $H_2(M_n^4)$ is generated by $h_1, \ldots, h_{2n}$ observe that $2n \geqslant 10$ since with this choice of n, $n \geqslant 5$. This time let $h = h_1 + \cdots + h_{2n-6} - 2h_{2n-5} + 2th_{2n-4} - 2h_{2n-3} + 2th_{2n-2} - 2h_{2n-1} + 2th_{2n}$ so that the first $(2n - 6)$ terms have coefficient 1, and the coefficients for the last 6 terms appear as $(-2, 2t, -2, 2t, -2, 2t)$. Then $h \cup h = 1 + 1 + \cdots + 1 - 4t - 4t - 4t - 4t - 4t - 4t = 2n - 6 - 24t = 2n - 6 - 3(8t) = 2n - 6 - 3(4k) = 2n - 6 - 3(2n - 2) = -4n = 3\tau(M_n^4) + 2\chi(M_n^4)$. Also, we know for all i, $h_i \neq 0 = w_2 \pmod 2$ since $h_i \cup h_{i+1} = 1 \pmod 2$ for odd i, but summing up $(2n - 6)$ of them and adding the last terms with coefficients a multiple of 2, guarantees the second condition in Wu's theorem.

Case 3: Assume n is odd, $n \neq 1$ and $n = 2k + 1$ with k odd (say $k = 2t + 1$, $t \in \mathbb{Z}$); Let $h = h_1 + \cdots + h_{2n-2} - 2h_{2n-1} + (6t + 4)h_{2n}$ so that the first $(2n - 2)$ terms have coefficient 1, and the coefficients for the last 2 terms are $(-2, 6t + 4)$. Then $h \cup h = 1 + 1 + \cdots + 1 - 12t - 8 - 12t - 8 = 2n - 2 - 24t - 16 = 2n - 2 - 12(2t + 1) - 4 = 2n - 2 - 12k - 4 = 2n - 2 - 6(2k + 1) + 2 = 2n - 2 - 6n + 2 = -4n = 3\tau(M_n^4) + 2\chi(M_n^4)$. Again we get $h = 0 \pmod 2$.

This completes the proof for all n such that $n \neq 1$. In fact for $n = 1$, M_n^4 are not almost complex since there cannot exist any generator of $H_2(M_n^4)$ satisfying the conditions of Wu's theorem in that case. □

3. Symplectic structures

Next we will investigate the symplectic structures on these manifolds. We follow the nice survey [9]. There is a classification scheme for symplectic 4-manifolds similar to that of the Kodaira classification for complex surfaces. In the symplectic case, first of all one needs to find a substitute for the canonical bundle of the complex surface. This is easily generalized. Since associated to a symplectic manifold (M, ω), there is the contractible space of compatible almost complex structures. Using any compatible almost complex structure J we define the symplectic Chern classes $c_k(M, \omega) := c_k(M, J)$ and hence the *symplectic canonical class* $-c_1(M, \omega)$ denoted sometimes by $-c_1(\omega)$ or K_ω which lies in $H^2(M; \mathbb{Z})$. Next for a minimal symplectic 4-manifold (M, ω) we define the *Kodaira dimension* in the following way. See [10,12].

$$\kappa(M, \omega) := \begin{cases} -\infty & \text{if } c_1(\omega) \cdot [\omega] > 0 \text{ or } c_1^2(\omega) < 0, \\ 0 & \text{if } c_1(\omega) \cdot [\omega] = 0 \text{ and } c_1^2(\omega) = 0, \\ 1 & \text{if } c_1(\omega) \cdot [\omega] < 0 \text{ and } c_1^2(\omega)^2 = 0, \\ 2 & \text{if } c_1(\omega) \cdot [\omega] < 0 \text{ and } c_1^2(\omega)^2 > 0. \end{cases}$$

We will also make use of the following theorem.

Theorem 3.1. *([9]) Let M be a closed oriented smooth 4-manifold and ω a symplectic form on M compatible with the orientation of M. If (M, ω) is symplectically minimal, then*

1. *The Kodaira dimension of (M, ω) is well-defined.*
2. *(M, ω) has Kodaira dimension $-\infty$ if and only if it is rational or ruled.*
3. *(M, ω) has Kodaira dimension 0 if and only if $c_1(\omega)$ is a torsion class.*

Notice that since all the manifolds M_g^1, $M_{g,n}^2$, $M_{g,n}^3$, M_n^4 have even intersection forms they cannot contain a sphere of self-intersection -1, hence they are all minimal. Now we are ready to prove the following.

Theorem 3.2. *None of the manifolds investigated above have symplectic structure for $g, n > 0$.*

Proof. Assume that the manifolds M_g^1, $M_{g,n}^2$, $M_{g,n}^3$, M_n^4 are symplectic, so that ω is a symplectic form on them. We have

$$c_1^2(\omega)\big[M_g^1\big] = (2\chi + 3\tau)\big(M_g^1\big) = -8g < 0,$$
$$c_1^2(\omega)\big[M_{g,n}^2\big] = 8 - 8g - 8n < 0,$$
$$c_1^2(\omega)\big[M_{g,n}^3\big] = 8 - 8g - 8n < 0 \quad \text{for } g, n \geqslant 1$$

and

$$c_1^2(\omega)\big[M_n^4\big] = 3\tau\big(M_n^4\big) + 2\chi\big(M_n^4\big) = -2n < 0 \quad \text{for } n \geqslant 1.$$

So all of the manifolds constructed must have Kodaira dimension $-\infty$, hence by the above theorem they must be rational or ruled if they are symplectic. If they were rational they should be $S^2 \times S^2$ or $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$ where k is any non-negative integer. This cannot happen since the manifolds we consider are all non-simply connected. If they were ruled then they should be of the form $S^2 \times \Sigma_g \# k\overline{\mathbb{CP}}_2$ where Σ_g is the surface of genus g and k is any integer. This also fails to occur since the fundamental groups of $M_g^1, M_{g,n}^2, M_{g,n}^3, M_n^4$ which are stated in the previous section are not equal to the fundamental group of $S^2 \times \Sigma_g \# k\overline{\mathbb{CP}}_2$. Therefore, they are neither rational nor ruled. So, they cannot be symplectic. $\quad\square$

4. Complex structures

In this section we will study the complex structures on our manifolds. The answer comes in the negative. For this study we will use the Enriques–Kodaira classification of surfaces. This classification is certainly an accumulation of works of many people, see [3] or [14] for references.

Lemma 4.1 *(Kodaira–Enriques classification). Every compact, complex and connected surface has a minimal model in exactly one of the following classes* (1) *to* (10).

Class of X	Kod	b_1	c_1^2	c_2
(1) *Minimal rational surfaces*	$-\infty$	0	8 *or* 9	4 *or* 3
(2) *Minimal surfaces of Kodaira's class VII*	$-\infty$	1	$\leqslant 0$	$\geqslant 0$
(3) *Ruled surfaces of genus* $\geqslant 0$	$-\infty$	$2g$	$8(1-g)$	$4(1-g)$
(4) *Enriques surfaces*	0	0	0	12
(5) *Bi-elliptic surfaces*	0	2	0	0
(6) *Kodaira surfaces*	0	3 *or* 1	0	0
(7) *K3 surfaces*	0	0	0	24
(8) *Tori*	0	4	0	0
(9) *Minimal properly elliptic surfaces*	1		0	$\geqslant 0$
(10) *Minimal surfaces of general type*	2	*even*	> 0	> 0

Next we will prove the following.

Theorem 4.2. *The manifolds M_g^1, $M_{g,n}^2$, $M_{g,n}^3$, M_n^4 do not admit any complex structure for all $g, n > 0$.*

Proof. For all the manifolds M_g^1, $M_{g,n}^2$, $M_{g,n}^3$, M_n^4 we have already cited their characteristics in Section 2, moreover we have $c_1^2(M_g^1) = -8g$, $c_1^2(M_{g,n}^2) = 8 - 8g - 8n$, $c_1^2(M_{g,n}^3) = 8 - 8g - 8n$, $c_1^2(M_n^4) = -2n$ which are all negative for $n, g \geqslant 1$. According to the Kodaira–Enriques classification this can happen only when they are either in the second category i.e. a minimal surface of Kodaira's class VII or in the third category i.e. a ruled surface $S^2 \times \Sigma_{g'} \# k\overline{\mathbb{CP}}_2$ of genus $g' \geqslant 0$. The first case cannot happen since minimal surfaces of Kodaira's class VII have $b_1 = 1$ which is not the case because $b_1(M_g^1) = 2g + 2 \neq 1$, $b_1(M_{g,n}^2) = 2g + 2n \neq 1$, $b_1(M_{g,n}^3) = 2g + 3n \neq 1$, $b_1(M_n^3) = 2n + 1 \neq 1$ for $n, g \geqslant 1$ (for the Betti numbers stated we refer to [1]). Finally, the ruled surface case is already ruled out as in the proof of Theorem 3.2. Hence, the manifolds M_g^1, $M_{g,n}^2$, $M_{g,n}^3$, M_n^4 are not complex. □

Acknowledgements

References

[1] S. Akbulut, M. Kalafat, Topology of non-simply connected Locally Conformally Flat (LCF) 4-manifolds, available at arXiv:0807.0837 [math.DG].
[2] H. Argüz, M. Kalafat, Y. Ozan, Self-dual metrics on non-simply connected 4-manifolds, preprint available at arXiv:1108.0433v1 [math.DG].
[3] W. Barth, K. Hulek, C. Peters, A.V. de Ven, Compact Complex Surfaces, 2nd edition, Springer-Verlag, Berlin, ISBN 3-540-00832-2, 2004, pp. xii+436.
[4] N. Brotherton, Some parallelizable four-manifolds not admitting a complex structure, Bull. Lond. Math. Soc. 10 (3) (1978) 303–304.
[5] M. Fernández, M. Gotay, A. Gray, Compact parallelizable four-dimensional symplectic and complex manifolds, Proc. Amer. Math. Soc. 103 (4) (1988) 1209–1212.
[6] R.E. Gompf, A.I. Stipsicz, 4-Manifolds and Kirby Calculus, American Mathematical Society, Providence, RI, 1999.
[7] F. Hirzebruch, H. Hopf, Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten, Math. Ann. 136 (1958) 156–172.
[8] Alan Howard, On the homotopy groups of an affine algebraic hypersurface, Ph.D. thesis, Brown Univ., 1965.
[9] Tian-Jun Li, The Kodaira dimension of symplectic 4-manifolds, in: Floer Homology, Gauge Theory, and Low-dimensional Topology, in: Clay Math. Proc., vol. 5, American Mathematical Society, Providence, RI, 2006, pp. 249–261.
[10] Tian-Jun Li, Symplectic 4-manifolds with Kodaira dimension zero, J. Differential Geom. 74 (2) (2006) 321–352.
[11] B. Maskit, Panelled web groups, in: Kleinian Groups and Related Topics, Oaxtepec, 1981, in: Lecture Notes in Math., vol. 971, Springer, Berlin–New York, 1983, pp. 79–108.
[12] D. McDuff, D. Salamon, A survey of symplectic 4-manifolds with $b^+ = 1$, Turkish J. Math. 20 (1) (1996) 47–60.
[13] J. Milnor, On the cobordism ring Ω^* and a complex analogue, I and II, Amer. J. Math. 82 (1960) 505–521.
[14] A.I. Stipsicz, The geography problem of 4-manifolds with various structures, Acta Math. Hungar. 87 (4) (2000) 267–278.
[15] A. Van de Ven, On the Chern numbers of certain complex and almost complex manifolds, Proc. Natl. Acad. Sci. USA 55 (1966) 1624–1627.
[16] Wen-Tsun Wu, Sur les classes caractéristiques des structures fibrées sphériques, in: Publ. Inst. Math. Univ. Strasbourg 11, in: Actualités Sci. Ind., vol. 1183, Hermann & Cie, Paris, 1952, pp. 5–89, 155–156.
[17] S.T. Yau, Parallelizable manifolds without complex structure, Topology 15 (1) (1976) 51–53.