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An analytic shooting-like approach for the solution of nonlinear boundary value problems

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ABSTRACT

In this paper a novel approach is presented for an analytic approximate solution of nonlinear differential equations with boundary conditions. By converting the nonlinear problem into an initial value form, a shooting-like procedure is introduced based on the powerful homotopy analysis technique. The proposed methodology is shown to work adequately for solving single or multiple solutions of some sample nonlinear boundary value problems.

1. Introduction

Finding approximate analytical solutions of nonlinear boundary value problems is extremely significant in engineering and physical sciences. The driving motivation behind this article is prompted by the recent publications [1,2], where the papers consider the approximate analytical solutions of some nonlinear boundary layer equations of fluid mechanics. In this connection, a *shooting-like* procedure which can be adopted for the solution of such equations is proposed here.

A commonly used numerical method for the solution of two-point boundary value problems is the shooting method. This well-known technique is an iterative algorithm which attempts to identify appropriate initial conditions for a related initial value problem that provides the solution to the original boundary value problem. Hence, solutions of boundary value problems with boundary conditions at distinct points are generally calculated by certain integration schemes in combination with a shooting procedure, see for instance [3,4]. The shooting method is implemented as a standard numerical procedure for the solution of two-point boundary-value problems (generally arising from the Navier–Stokes equations) in standard all-purpose mathematical software like MathCAD with its sbval function and Mathematica with its ndsolve function [5]. Estimation of the global discretization error in shooting methods for linear boundary value problems was discussed in [6]. In [7] using a spline approximation, the Falkner–Skan equation was solved through the use of the shooting technique for handling the problem when the conditions imposed are of boundary-value rather than an initial-value type for different values of its parameters. A nonlinear shooting method for two-point boundary value problems was proposed in [8]. Gebeily and Attili [9] introduced an iterative shooting method for a certain class of singular two-point boundary value problems. A shooting method for nonlinear heat transfer using automatic differentiation was implemented in [10]. Very recently, Attili and Syam [11] presented an efficient shooting method for solving two-point boundary value problems.

Together with certain widespread numerical techniques, another classical approach for approximate solutions of nonlinear boundary value problems is to pursue perturbation methods. However, the solutions obtained within perturbation

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techniques may not be uniform, restricting the applicability of results [12,13]. To overcome the limitations of the perturbative techniques Liao in [14] proposed a new analytic method for highly nonlinear problems, namely the homotopy analysis method. Unlike the perturbative and non-perturbative methods, this technique allows more than a uniformly valid analytic solution of nonlinear equations with no possible small parameters. In this method, according to the homotopy technique, a homotopy with an embedding parameter is constructed, and the embedding parameter is considered as a small parameter. Thus the original nonlinear problem is converted into an infinite number of linear problems without using the perturbation techniques. Further advantages of this new technique have been severally outlined in the literature as also briefly addressed in Section 2. After the introduction of the method, several problems of science and engineering have been revisited. For example, Liao successfully applied the method for the analytical solution of the Falkner–Skan equation [15]. Foundations and fundamentals of the method have been recently summarized in [16]. The homotopy analysis method keeps evolving steadily and the recent research clearly shows that it may replace the place of traditional perturbation or numerical methods in the near future, see for instance [17]. The idea of shooting was recently used in the homotopy analysis approach [18].

As opposed to the conventional shooting schemes implemented via different numerical methods, the objective of this paper is to propose an analytic approximate procedure for the two-point boundary value problems. The methodology that we develop relies essentially upon the recently fashionable and powerful homotopy analysis method. Within this aim, suitable auxiliary linear operators, convergence control parameters and initial guesses are suggested that generate an explicit analytic form of the solution of nonlinear two-point boundary value problems. The proposed homotopy analysis technique provides uniformly valid approximate series solutions. Three examples are discussed in this paper which provide samples of how simple shooting introduced here can be applied to the analysis of boundary value problems encountered in mechanical engineering.

The following strategy is adopted in the rest of the paper. In Section 2 the methodology of the homotopy analysis approach is presented. Application of the method to nonlinear boundary value problems is implemented in Section 3, in which analytic expressions are derived and compared with the numerical ones. Finally conclusions follow in Section 4.

2. The problem and proposed methodology

The basic idea of the shooting method for two-point boundary value problems is to reformulate the problem as a nonlinear parameter estimation problem. The new problem requires the solution of a related initial value problem with initial conditions chosen to approximate the boundary conditions at the other endpoint. If these boundary conditions are not satisfied to the desired accuracy, the process is repeated with a new set of initial conditions until the desired accuracy is achieved or an iteration limit is reached. To be more specific, we consider the two-point boundary value problems of the form

$$F[x, f(x), f'(x), f''(x), f'''(x)] = 0,$$
$$f(0) = \lambda_1, \qquad f'(0) = \lambda_2, \qquad f''(0) = \lambda_3, \qquad f(\infty) = 1, \qquad f'(\infty) = 1, \tag{2.1}$$

where either of the infinity boundary conditions is supplemented with only one unknown λ. It should be noticed that the unknown parameter λ can be scaled out of the boundary conditions for the sake of embedding it into the differential equation after some suitable transformations.

The usual algorithm known for the solution of (2.1) is the simple or single shooting method. While this method is effective for many problems, some specific deficiencies should be mentioned [19]. For instance, assuming (2.1) has a solution, there is no guarantee that the initial value problem replaced by the boundary value problem will have a solution on the interval of interest for all λ. Even if it does have a solution, the problem may be stiff. In such a case the solution at $x = \infty$ may be so inaccurate as to make the results of the Newton–Raphson step meaningless. When the accuracy of the solution at $x = \infty$ is known with sufficient accuracy, the local convergence of the Newton–Raphson step may prevent the iterations from converging to a solution of the original boundary value problem (2.1). This difficulty can be addressed by replacing the Newton–Raphson step with another iterative solver with improved convergence properties (e.g., modified Newton's method). Another difficulty is to truncate the infinity boundary condition at a finite value in numerical schemes, which may yield converged solutions not to the true solution of (2.1).

Liao in [14] proposed a new kind of analytic technique for nonlinear problems, namely the homotopy analysis method. This method is based on the homotopy and has several advantages. To underline, firstly its validity does not depend upon whether or not nonlinear equations under consideration contain small or large parameters, hence it can solve more of the strongly nonlinear equations than the perturbation techniques. Secondly, it provides us with a great freedom to select proper auxiliary linear operators and initial guesses so that uniformly valid approximations can be obtained. Thirdly, it gives a family of approximations which are convergent in a larger region. Fascinating examples are provided within the ref. [20].

Prior to an outline of the homotopy analysis method let us reformulate (2.1) into the form

$$N[u(t)] = 0, \tag{2.2}$$

with boundary conditions

$$B1[u, u', u''] = 0, \tag{2.3}$$

where N is a general nonlinear operator and $B1$ is a boundary operator. The important stage of the technique is that the boundary value problem (2.2) and (2.3) is transformed to an equivalent problem so that the conditions (2.3) involve an unknown parameter λ and are separated into

$$B[u, u', u''; \lambda] = 0, \qquad G[u, u', u''] = 0, \tag{2.4}$$

where the latter condition corresponds to the unused boundary condition at the other point that will later determine the value of λ.

Such a split of the original system (2.2)–(2.3) enables us to construct a homotopy

$$\begin{aligned} &(1-p)L(u-u_0) + phN = 0, \\ &B[u, u', u'; \lambda] = 0. \end{aligned} \tag{2.5}$$

In the homotopy system (2.5), $p \in [0, 1]$ is an embedding parameter, and h is the parameter to adjust the convergence of the homotopy series to be defined later. Moreover, L is an auxiliary linear differential operator whose proper shape depends on the particular example considered, while $u_0(t)$ is the initial guess for the solution.

At this stage λ might be treated in two different manners;

Case I. The unknown λ might be taken independent of the embedding parameter p and it is eventually computed from the unused boundary condition from (2.4), as implemented in [2] for instance. However, this process requires solution of a high-order algebraic equation involving λ, while it can be easily evaluated by a numerical scheme, but resulting in so many irrelevant roots to be dropped off.

Case II. In the second case, λ is taken as the function of p and determined sequentially during the calculation of homotopy variables; as a result an explicit analytic formula is constructed for λ. We discuss both of the methodologies in this paper, but only describe the second case in the following.

It is obvious from Eqs. (2.5) that for $p = 0$ we have the initial approximation $(u_0(t) = u(t, 0), \lambda_0 = \lambda(0))$ to the solution, and when $p = 1$ we have the exact solution $(u(t) = u(t, 1), \lambda = \lambda(1))$ to Eqs. (2.2)–(2.3). It can be deduced that the deformation process of p from zero to unity is just that of from $(u(t, 0), \lambda(0))$ to $(u(t, 1), \lambda(1))$. The zeroth-order deformations to homotopy (2.5) are thus basically the linear differential equation with the boundary conditions in (2.5) satisfied exactly and $N = 0$. Next, the kth-order deformation equations follow as

$$L(u_k - \kappa_k u_{k-1}) = -hN_k, \quad u_k(0) = 0 \tag{2.6}$$

and where $\kappa_k = 0$ for $k \le 1$ and $\kappa_k = 1$ otherwise. In addition to this, N_k and λ_k are defined by

$$N_k = \frac{1}{k!}\frac{\partial^k N}{\partial p^k}\bigg|_{p=0}, \qquad \lambda_k = \frac{1}{k!}\frac{\partial^k \lambda}{\partial p^k}\bigg|_{p=0}.$$

Taking into account Taylor series expansion of the solutions $u(t, p)$ and $\lambda(p)$ at $p = 0$ and later imposition of the expansion at $p = 1$ we obtain respectively

$$u(t) = u_0(t) + \sum_{k=1}^{\infty} u_k(t), \tag{2.7}$$

and

$$\lambda = \lambda_0 + \sum_{k=1}^{\infty} \lambda_k, \tag{2.8}$$

where u_k and λ_k are also defined by

$$u_k = \frac{1}{k!}\frac{\partial^k u}{\partial p^k}\bigg|_{p=0}, \qquad \lambda_k = \frac{1}{k!}\frac{\partial^k \lambda}{\partial p^k}\bigg|_{p=0}.$$

The Mth-order finite truncation of the series (2.7)–(2.8) yields the approximations to the solutions

$$\begin{aligned} u(t) &= \sum_{k=0}^{M} u_k(t), \\ \lambda &= \sum_{k=0}^{M} \lambda_k, \end{aligned} \tag{2.9}$$

which represent the approximate solution of (2.2) to the desired degree of accuracy.

It is worth indicating that up to this stage, the linear operator L, the initial approximation guess u_0, and the auxiliary parameter h have been chosen properly so that the series solutions (2.7)–(2.8) would be convergent. Meanwhile, owing to the unknown parameter λ called the shooting parameter, the above procedure resembles the shooting techniques as employed

Table 1
Illustrating the λ computed at the orders written for the problem (3.10).

α	$M = 1$	$M = 20$	$M = 40$	$M = 60$	Exact
−.20	0.29393	0.30365	0.30393	0.30415	0.30417
0.00	0.31622	0.33171	0.33191	0.33205	0.33206
0.20	0.29933	0.31298	0.31319	0.31334	0.31336
0.40	0.28000	0.29202	0.29225	0.29243	0.29244

during the conventional numerical schemes. Therefore, we call the above method the *shooting-like* analytic method in this paper. It is essential that the existence of unique or multiple solutions for the original boundary value problem (2.2)–(2.3) depends on the fact whether the unused boundary condition in (2.4) admits unique or multiple values for the formally introduced parameter λ in the boundary conditions. Setting the unused boundary condition at the Mth-order approximation (2.9) in the first case will produce the shooting parameter numerically, or in the second case setting the unused condition initially will yield the analytic expression for the shooting parameter.

3. Results and discussion

In this section we apply the above outlined homotopy algorithms to some selected physical two-point boundary value problems. Approximate analytical solutions obtained from the method have been compared with those obtained from the numerical computations.

The first example that is encountered here is the boundary layer flow over a flat plate which has a constant velocity opposite in direction to that of the uniform main stream. In terms of the Crocco variables [21] (which naturally reduce the infinity boundary condition to unity) and some further suitable transformations, the problem can be formulated as

$$\lambda^2 F(\eta)F''(\eta) + \frac{\eta+\alpha}{2} = 0, \qquad F(0) = 1, \qquad F'(0) = 0, \qquad F(1-\alpha) = 0, \tag{3.10}$$

where α is the ratio of the speed of the plate surface to the velocity of the free stream and λ corresponds to the skin friction coefficient, that appears due to the rescaling of the first boundary condition in (3.10). For clarity, we propose the following homotopy parameters for evaluation of the solutions $u(t, p)$ and $\lambda(p)$, that are to be substituted into Eqs. (2.5)

$$L = \frac{d^2}{d\eta^2}, \qquad F_0 = 1 - \frac{1}{(-1+\alpha)^2}\eta^2,$$
$$N_1 = \lambda_0^2 F_0 F_0'' + \frac{\eta+\alpha}{2},$$
$$N_k = \sum_{j=0}^{k-1} F1_j F2_{k-1-j}, \qquad F1_k = \sum_{j=0}^{k} \lambda_{k-j} F_j, \qquad F2_k = \sum_{j=0}^{k} \lambda_{k-j} F_j''.$$

As a result, the leading value of λ is calculated as

$$\lambda = |-1+\alpha| \frac{\sqrt{1+2\alpha}}{\sqrt{10}}.$$

Together with these and also taking into account α varying initially, and having assigned the convergence control parameter $h = -5$, the homotopy solutions in (2.9) of order $M = 1$, 5 and 20 are shown in Fig. 1 for the pair (α, λ) using case II. It is seen from the curves how the homotopy solutions converge. In addition to this, for specific values of α, the corresponding values of λ are compared using case II with the numerical solution in Table 1. It can be seen that as the order of the homotopy series solution in Eqs. (2.9) increases, a fast convergence takes place to bring our approximate solution into excellent agreement with the numerical solution; so, the approximation to the present order can be used to represent the exact solution. It is quite remarkable that the difference between our homotopy solution and the numerically calculated one decays quite fast as the order of iteration in the homotopy series (2.9) increases. This clearly implies the fact of convergence of the homotopy series solution to the true solution of system (3.10).

It is well-known that nonlinear boundary-value problems have multiple solutions in many cases, see for instance, the nonlinear problem arising in heat transfer [22], the strongly nonlinear Bratu equation [23] and the nonlinear reaction–diffusion model [24]. The second example that we consider is a model of convective flow in fluid-saturated porous medium which admits multiple (dual) solutions, see [25]. Under the boundary layer and Darcy–Boussinesq approximations and also further proper transformations the governing equations of the steady mixed convection flow past a plane of arbitrary shape can be reduced to the following form

$$2\lambda^2 F''(\eta) + F(\eta) - F^2(\eta) = 0, \qquad F(0) = 1+\beta, \qquad F'(0) = 1, \qquad F(\infty) = 1. \tag{3.11}$$

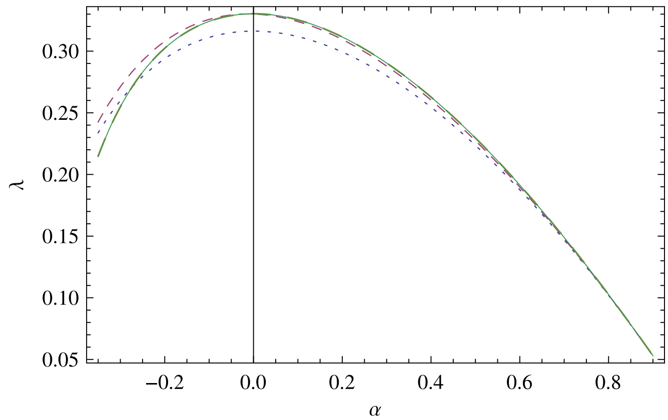

Fig. 1. Solution of Blasius equation (3.10), skin friction parameter λ versus α with $h = -5$: straight curve from the numerical solution and homotopy solutions are the thick-dashed curve from the 20th-order, dashed curve from the 5th-order and dotted curve is the leading-order approximation.

Table 2
Illustrating the values of λ computed at the orders written for the problem (3.11) for case II.

ϵ	$M = 1$	$M = 5$	$M = 10$	$M = 15$	Exact
−1.20	0.359487	0.379150	0.379492	0.379473	0.379473
−1.00	0.408248	0.408248	0.408248	0.408248	0.408248
−0.80	0.397048	0.386706	0.386446	0.386437	0.386437
−0.60	0.341822	0.329313	0.328692	0.328633	0.328633
−0.40	0.230940	0.242028	0.242272	0.242212	0.242212

Magyari et al. [26] showed that Eqs. (3.11) admit dual solutions for any given value of the parameter $b \in [\frac{-3}{2}, 0]$ which are

$$F = -\frac{1}{2} + \frac{3}{2}\tanh^2\left[\frac{\eta}{2\sqrt{2}} \pm \ln\left(\frac{\sqrt{3}+\sqrt{3+2\beta}}{\sqrt{3}-\sqrt{3+2\beta}}\right)\right], \tag{3.12}$$

so that the physical interest wall skin friction is obtained as follows

$$\lambda = \pm\beta\sqrt{\frac{2\beta+3}{6}}. \tag{3.13}$$

It should be remarked that λ appears in (3.11) due to the rescaling of the second boundary condition in (3.11).

For this problem, the corresponding auxiliary homotopy parameters are as listed

$$L = \frac{d^2}{d\eta^2} - 1, \qquad F_0 = 1 + (1+2\beta)e^{-\eta} - (1+\beta)e^{-2\eta},$$

$$N_k = F_{k-1} + 2\sum_{j=0}^{k-1}\sum_{i=0}^{j}\lambda_i\lambda_{j-i}F''_{k-1-j} - \sum_{j=0}^{k-1}F_jF_{k-1-j}.$$

The leading-order value of λ from the analysis is obtained as

$$\lambda = \pm\frac{\sqrt{-6-27\beta-16\beta^2}}{\sqrt{10}\sqrt{5+2\beta}}.$$

It is plausible that for $\beta = -1$, this homotopy formula yields $\lambda = \pm\frac{1}{\sqrt{6}}$, which are the exact values obtained from (3.13). The homotopy solution is next performed with the convergence control parameter $h = -1.65$. Fig. 2 demonstrates a comparison between the 5th-order homotopy series solution (2.9) and the numerically computed one from (3.13). Moreover, Fig. 3 shows a comparison between the homotopy solution for F and the exact solution corresponding to $\beta = -1$. As seen from the figures, the homotopy method generates results which get more accurate as the order of approximation increases. It is worth mentioning that the absolute residual error (evaluated as the integral of the difference) between the 15th-order homotopy solution and the exact solution is only 4.43055×10^{-8}.

Table 2 presents a list of values calculated at different orders of approximation, which clearly illustrate the convergence of homotopy series results (3.11) to the exact ones. Table 3 presents case 1 of the homotopy methodology as described in Section 2. It is seen that method II of determining λ is better than method I.

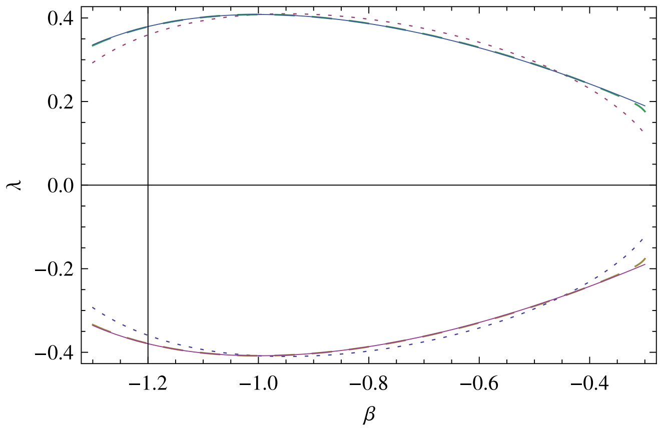

Fig. 2. Solution of Eq. (3.11), skin friction parameter λ versus α with $h = -1.65$: straight curve from the numerical solution of Eq. (3.13) and homotopy solutions are the thick-dashed curve from the 5th-order and dotted curve is the leading-order approximation.

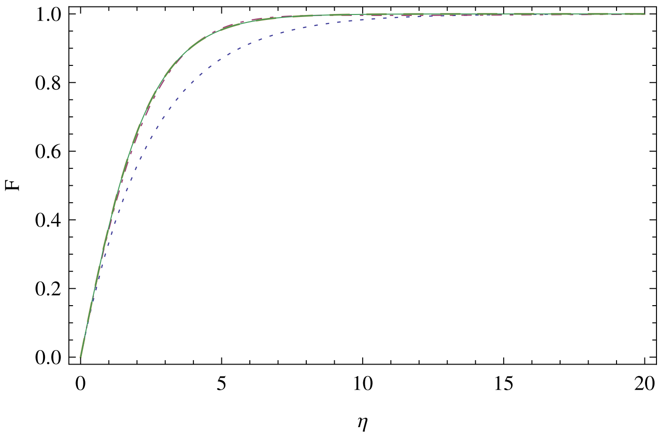

Fig. 3. Solution of Eq. (3.11) with $\beta = -1$ and $h = -1.65$: straight curve from the numerical solution of Eq. (3.12) and homotopy solutions are the thick-dashed curve from the 15th-order, dash–dotted curve from the 3th-order and dotted curve is the leading-order approximation.

Table 3
Illustrating the values of λ computed at the orders written for the problem (3.11) for case I.

ϵ	$M = 1$	$M = 5$	$M = 10$	$M = 15$	Exact
−1.20	0.359487	0.369419	0.371126	0.371397	0.379473
−1.00	0.408248	0.408248	0.408248	0.408248	0.408248
−0.80	0.397048	0.388123	0.386766	0.386521	0.386437
−0.60	0.341822	0.330334	0.328876	0.328673	0.328633
−0.40	0.230940	0.241217	0.242210	0.242193	0.242212

The final example that we consider here is a similar solution for the nano boundary layer with nonlinear Navier boundary condition, particularly the flow past a wedge, see [2]. The governing equations for the fluid flow are given by

$$f''' + \frac{2n-1}{3n-2}ff'' - \frac{n}{3n-2}(f'^2 - 1) = 0, \tag{3.14}$$

complemented with the boundary conditions

$$f(0) = 0, \qquad |f'(0)| - l|f''(0)|^n = 0, \qquad f'(\infty) = 1. \tag{3.15}$$

A prime denotes a derivative with respect to ζ and the parameters in Eqs. (3.14)–(3.15) are $n > 0$ is an arbitrary power parameter related to the wedge angle and the constant $l \geq 0$ is the slip length. Moreover, the second condition in (3.15) is the Navier slip boundary condition.

Liao [2] introduced $f'(0) = \alpha$ instead of the Navier boundary condition in Eq. (3.15) and pursued the usual homotopy method as outlined above. At the Mth-order approximation to the solution f, he then imposes $f''(0)$ which is related to α

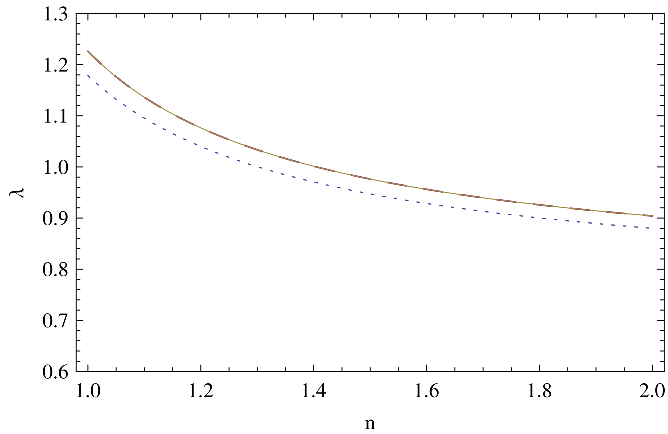

Fig. 4. Solution of Eq. (3.18), the parameter λ versus n with $l = 0$ and $h = -0.05$: straight curve from the numerical solution of equation and homotopy solutions are the dashed curve from the 5th-order and dotted curve is the leading-order approximation.

via the Navier boundary condition

$$|\alpha| = l|f''(0)|^n. \tag{3.16}$$

However, Eq. (3.16) is an algebraic equation which demands a numerical solution. We instead reformulate the system (3.14)–(3.15) in such a way that the shooting-like approach can be used. Keeping this in mind, let us use the new transformations

$$f = \frac{F}{\lambda}, \qquad \eta = \lambda\zeta, \tag{3.17}$$

where $\lambda = f''(0)$. Under these transformations, Eqs. (3.14)–(3.15) can be rewritten as

$$\lambda^2 F''' + \frac{2n-1}{3n-2}FF'' - \frac{n}{3n-2}(F'^2 - 1) = 0, \tag{3.18}$$
$$F(0) = 0, \qquad |F'(0)| = l|\lambda|^n, \qquad F''(0) = 1, \qquad F'(\infty) = 1.$$

For this physical two-point boundary value problem, the corresponding auxiliary homotopy parameters are as follows

$$L = \frac{d^3}{d\eta^3} + \frac{d^2}{d\eta^2}, \qquad F_0 = \frac{1}{2}(-2 + 3l) + \eta + (1 - 2l)e^{-\eta} + \frac{l}{2}e^{-2\eta},$$
$$N_k = \sum_{j=0}^{k-1}\sum_{i=0}^{j} \lambda_i\lambda_{j-i}F'''_{k-1-j} + \frac{2n-1}{3n-2}\sum_{j=0}^{k-1} F_jF''_{k-1-j} - \frac{n}{3n-2}\left(\sum_{j=0}^{k-1} F'_jF'_{k-1-j} - (1 - \kappa_k)\right).$$

The homotopy solution in this case is performed with the convergence control parameter $h = -0.05$. For instance, when $l = 0$, the leading-order λ as a function of n is obtained as

$$\lambda = \frac{\sqrt{-23 + 123n}}{6\sqrt{-4 + 6n}}. \tag{3.19}$$

Fig. 4 shows the variation of λ against n for $l = 0$ obtained from the homotopy series solution as well as the numerical solution.

Homotopy solutions corresponding to various combinations of n and l can be obtained from (3.18) using the shooting-like homotopy procedure. It can be readily deduced from Figs. 1–4 and Table 1–2 that the homotopy analysis solutions for the two-point boundary value problems considered are uniformly valid approximate solutions and hence they reliably represent the exact solutions. It is furthermore worthwhile to state that although specific problems are chosen as examples for the homotopy analysis solutions obtained here, the methodologies introduced can be applied to a vast variety of boundary value problems.

It should be further remarked here that the proposed mathematical software was validated for the problems with a single governing transport equation. However, although not implemented here, the methodology can also be applied for a problem that involves a system of governing transport equations, i.e. for a 3D flow like rotating/swirl flows (rotating disks etc.) [27].

4. Concluding remarks

In this paper the nonlinear two-point boundary value problems have been considered by means of the homotopy analysis technique. The governing equations have first been modified so that a *shooting-like* analytic approach based on the homotopy can be accessed with a proper proposal of auxiliary parameters involved.

The success of the method has later been tested by applying it to three physical problems taken from the literature. Even sufficiently low-order uniformly valid approximate analytic homotopy solutions whose graphs have been depicted here reveal excellent agreement with the numerical solutions. The advantages of the presented approaches have also been discussed.

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