Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 649 – 657
© TÜBİTAK
doi:10.3906/mat-1210-15

Some rings for which the cosingular submodule of every module is a direct summand

Derya KESKİN TÜTÜNCÜ[1], **Nil ORHAN ERTAŞ**[2,*], **Patrick F. SMITH**[3], **Rachid TRIBAK**[4]
[1]Department of Mathematics, Hacettepe University, Beytepe, Ankara, Turkey
[2]Department of Mathematics, Karabük University, Karabük, Turkey
[3]Department of Mathematics, Glasgow University, Glasgow, Scotland
[4]Regional Center for Career Education and Training (CRMEF)-Tangier, Tangier, Morocco

Received: 05.10.2012 • **Accepted:** 25.11.2013 • **Published Online:** 25.04.2014 • **Printed:** 23.05.2014

Abstract: The submodule $\overline{Z}(M) = \cap\{N \mid M/N$ is small in its injective hull$\}$ was introduced by Talebi and Vanaja in 2002. A ring R is said to have property (P) if $\overline{Z}(M)$ is a direct summand of M for every R-module M. It is shown that a commutative perfect ring R has (P) if and only if R is semisimple. An example is given to show that this characterization is not true for noncommutative rings. We prove that if R is a commutative ring such that the class $\{M \in Mod-R \mid \overline{Z}_R(M) = 0\}$ is closed under factor modules, then R has (P) if and only if the ring R is von Neumann regular.

Key words: von Neumann regular ring, perfect ring, (non)cosingular submodule

1. Introduction

Throughout this paper all rings have identity and all modules are unital right modules. Let R be a ring and M an R-module. A submodule L of M is called a *small submodule* (notation $L \ll M$) if $M \neq L + N$ for any proper submodule N of M. The module M is said to be *small* if it is a small submodule of some R-module; equivalently, M is small in its injective hull. In [13], Talebi and Vanaja introduced the submodule $\overline{Z}(M) = \cap\{U \leq M \mid M/U$ is small$\}$. If $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$), then the module M is called *cosingular* (*noncosingular*).

If for every R-module M, $\overline{Z}(M)$ is a direct summand of M, we will say that R has property (P). The aim of this paper is to shed some light on the structure of rings having (P). Note that the rings satisfying the dual of our condition (P), namely those whose singular submodules $Z(M)$ are direct summands, have been studied in [2] and [3] extensively.

In Section 2 we present some properties of rings having (P). It is shown that the class of rings having (P) is closed under finite products. We also prove that if R is a commutative ring such that the class of cosingular modules is closed under factor modules, then R has (P) if and only if the ring R is von Neumann regular.

Section 3 deals with the structure of perfect rings having (P). We show that a commutative perfect ring R has (P) if and only if R is semisimple. An example is given to show that this characterization is not true for noncommutative rings.

*Correspondence: orhannil@yahoo.com
2010 *AMS Mathematics Subject Classification:* 16D10, 16D99.

2. Some properties of rings having (P)

Proposition 2.1 *The following are equivalent for a module M:*

(i) $\overline{Z}(M)$ *is a direct summand of M;*

(ii) M *is a direct sum of a cosingular summodule and a noncosingular submodule.*

In this case $\overline{Z}(M)$ is the largest noncosingular submodule of M.

Proof (i) $\Rightarrow$ (ii) Let N be a submodule of M such that $M = \overline{Z}(M) \oplus N$. By [13, Proposition 2.1(7)], N is cosingular. Since $\overline{Z}(M) = \overline{Z}(\overline{Z}(M)) \oplus \overline{Z}(N)$ (by [13, Proposition 2.1(4)]), we have $\overline{Z}(M) = \overline{Z}(\overline{Z}(M))$. Hence, $\overline{Z}(M)$ is noncosingular. This proves the result.

(ii) $\Rightarrow$ (i) Let N be a cosingular submodule of M and let K be a noncosingular submodule of M such that $M = N \oplus K$. By [13, Proposition 2.1(4)], $\overline{Z}(M) = \overline{Z}(N) \oplus \overline{Z}(K)$. Thus, $\overline{Z}(M) = K$ is a direct summand of M.

For the last statement: if L is a noncosingular submodule of M, then $L = \overline{Z}(L) \subseteq \overline{Z}(M)$. □

Example 2.2 By applying the last result and some results of [13], we can get some examples of rings having property (P).

(1) By [13, Proposition 2.5], if R is a cosemisimple ring, then every R-module is noncosingular. Therefore, R has property (P).

(2) If R is a ring such that every cosingular R-module is projective, then R has property (P) by [13, Theorem 3.8(4)].

Proposition 2.3 *For any ring R the following conditions are equivalent:*

(1) *R has (P);*

(2) *Every R-module is a direct sum of a noncosingular module and a cosingular module;*

(3) (a) *If N is a noncosingular submodule of a module M such that M/N is cosingular, then N is a direct summand of M, and*

(b) *The preradical $\overline{Z}$ is idempotent.*

Proof (1) $\Leftrightarrow$ (2) By Proposition 2.1.

(1) $\Rightarrow$ (3)(a) By (1), $\overline{Z}(M) \oplus L = M$ for some submodule $L \leq M$. Since $\overline{Z}(M/N) = 0$, $\overline{Z}(M) \subseteq N$ by [13, Proposition 2.1(7)]. Then $N = \overline{Z}(M) \oplus (L \cap N)$ and $M = N + L$. As $M/\overline{Z}(M) \cong L$, we have $\overline{Z}(L) = 0$. Hence, $\overline{Z}(N \cap L) = 0$. On the other hand, since $L \cap N$ is a direct summand of N, $L \cap N$ is noncosingular. It follows that $\overline{Z}(N \cap L) = N \cap L = 0$. Thus, $M = N \oplus L$.

(1) $\Rightarrow$ (3)(b) By Proposition 2.1.

(3) $\Rightarrow$ (1) Let M be any R-module. By [13, Proposition 2.1], we have $\overline{Z}(M/\overline{Z}(M)) = 0$. Moreover, we have $\overline{Z}(M) = \overline{Z}^2(M)$ by (b). Therefore, $\overline{Z}(M)$ is a direct summand of M by (a). □

Corollary 2.4 *Consider the following conditions:*

(i) *For any $N \leq M \in Mod-R$, we have $\overline{Z}(N) = N \cap \overline{Z}(M)$;*

(ii) *The class* $\{M \in Mod-R \mid \overline{Z}(M) = M\}$ *is closed under submodules.*

Then (i) $\Rightarrow$ *(ii) and if* R *has* (P)*, then (ii)* $\Rightarrow$ *(i).*

Proof By Proposition 2.3 and [4, Proposition 6.9(1)]. □

Corollary 2.5 *Consider the following conditions for a ring* R*:*

(i) R *has* (P)*;*

(ii) $\operatorname{Ext}(S, M) = 0$ *for every cosingular module* S *and noncosingular module* M*.*

Then (i) implies (ii). If the preradical $\overline{Z}$ *is idempotent, then (ii) implies (i).*

Note that (ii) does not imply (i) in the above corollary. Consider the ring $\mathbb{Z}$. By Lemma 4.12 of [8], a $\mathbb{Z}$-module M is noncosingular if and only if it is injective. So condition (ii) is satisfied. But the ring $\mathbb{Z}$ does not satisfy (P) (see Proposition 2.6).

Proposition 2.6 *Let* R *be a Dedekind domain. The following are equivalent:*

(i) R *has* (P)*;*

(ii) R *is a field.*

Proof (i) $\Rightarrow$ (ii) Let M be any module. By [15, Bemerkung 1.7 and Satz 2.10], there exists an R-module N such that $M \leq N$ and $M = \overline{Z}^2(N)$. By assumption, we also have that $N = \overline{Z}(N) \oplus K$ for some submodule K of N. Then $\overline{Z}(N) = \overline{Z}^2(N) \oplus \overline{Z}(K) = M \oplus \overline{Z}(K) = M$. Thus, M is noncosingular. By [8, Lemma 4.12], M is also injective. It follows that R is semisimple. Thus, R is a field.

(ii) $\Rightarrow$ (i) This is clear. □

Lemma 2.7 *Let* $R = R_1 \oplus R_2$ *where* R_i $(i = 1, 2)$ *are nonzero 2-sided ideals of* R*. Let* M *be an* R*-module. Then:*

(1) $M = MR_1 \oplus MR_2$ *and* MR_i $(i = 1, 2)$ *can be regarded as an* R_i*-module such that the submodules of* MR_i *are the same whether it is regarded as an* R_i*-module or as an* R*-module.*

(2)(a) *If* E *is an injective* R*-module, then* ER_i *is an injective* R_i*-module.*

(b) *If* E_i *is an injective* R_i*-module, then* E_i *is an injective* R*-module for the following multiplication:* $x_i(r_1 + r_2) = x_i r_i$*, where* $r_j \in R_j$ $(j = 1, 2)$ *and* $x_i \in E_i$*.*

(3)(a) *Let* N_i *be a submodule of the* R*-module* MR_i*. Then* MR_i/N_i *is a small* R_i*-module if and only if* MR_i/N_i *is a small* R*-module.*

(b) *We have* $\overline{Z}_{R_i}(MR_i) = \overline{Z}_R(MR_i)$ *for* $i = 1, 2$*.*

(4) *If* $\{M \in Mod-R \mid \overline{Z}_R(M) = 0\}$ *is closed under homomorphic images, then so is* $\{M \in Mod-R_i \mid \overline{Z}_{R_i}(M) = 0\}$*.*

Proof (1) This is obvious.

(2) (a) Let X_i be an R_i-module with $ER_i \subseteq X_i$. Clearly X_i is an R-module and ER_i is an injective R-module. Thus, ER_i is a direct summand of X_i, and so ER_i is an injective R_i-module.

(b) Let X be an R-module with $E_i \subseteq X$. Then $E_iR_i \subseteq XR_i$. Hence, $E_i \subseteq XR_i$. By hypothesis, E_i is a direct summand of XR_i. Since XR_i is a direct summand of X, E_i is a direct summand of X. It follows that E_i is an injective R-module.

(3) (a) Assume that MR_i/N_i is a small R_i-module. Thus there is an injective R_i-module E_i containing MR_i/N_i such that $MR_i/N_i \ll E_i$. By (2), E_i is an injective R-module. Thus, MR_i/N_i is a small R-module.

Conversely, suppose that MR_i/N_i is a small R-module. Thus there is an injective R-module E containing MR_i/N_i such that $MR_i/N_i \ll E$. Therefore, $MR_i/N_i \ll ER_i$. Since ER_i is an injective R_i-module, MR_i/N_i is a small R_i-module.

(b) By (a).

(4) This follows from (3)(b). □

Proposition 2.8 *Let $R = R_1 \oplus R_2$ be a ring decomposition. Then R has property (P) if and only if R_1 and R_2 both have property (P).*

Proof Let M be an R-module. By assumption, we have $M = MR_1 \oplus MR_2$ such that MR_i is an R_i-module for $i = 1, 2$. Note that $\overline{Z}_{R_i}(MR_i) = \overline{Z}_R(MR_i)$ for $i = 1, 2$ (see Lemma 2.7). Then $\overline{Z}_R(M) = \overline{Z}_R(MR_1) \oplus \overline{Z}_R(MR_2) = \overline{Z}_{R_1}(MR_1) \oplus \overline{Z}_{R_2}(MR_2)$. Since R_i has property (P), then $\overline{Z}_{R_i}(MR_i)$ is a direct summand of MR_i for $i = 1, 2$. Hence, R has property (P). Conversely, consider an R_i-module M_i. Then M_i can be regarded as an R-module for the following multiplication: $x_i(r_1 + r_2) = x_ir_i$, where $r_j \in R_j$ $(j = 1, 2)$ and $x_i \in M_i$ and the submodules of M_i are the same over R and over R_i $(i = 1, 2)$. Hence, $\overline{Z}_{R_i}(M_i) = \overline{Z}_{R_i}(M_iR_i) = \overline{Z}_R(M_iR_i) = \overline{Z}_R(M_i)$ by Lemma 2.7. Thus, if R has property (P), then R_1 and R_2 have property (P). □

Proposition 2.9 *Let R be a commutative ring having property (P). Then $R = R_1 \oplus R_2$ such that R_1 is a von Neumann regular ring and R_2 is a ring having property (P) with $\overline{Z}(R_2) = 0$.*

Proof By Proposition 2.1, $R = R_1 \oplus R_2$ such that $\overline{Z}(R_1) = R_1$ and $\overline{Z}(R_2) = 0$. By Proposition 2.8, R_1 and R_2 both have property (P). By [13, Corollary 2.6], R_1 is a cosemisimple ring. But R_1 is commutative. Then R_1 is a von Neumann regular ring. This completes the proof. □

In the sequel, let $\underline{C}_R = \{M_R \mid \overline{Z}(M) = 0\}$ denote the class of cosingular R-modules.

Lemma 2.10 *If the class $\underline{C}_R$ is closed under homomorphic images, then $\overline{Z}(R) \neq 0$.*

Proof Assume that $\overline{Z}(R) = 0$. Then $\overline{Z}(R^{(I)}) = 0$ for every index set I by [13, Proposition 2.1(4)]. By hypothesis, every module is cosingular, a contradiction (see [13, Proposition 2.8]). □

Proposition 2.11 *Let $R = R_1 \oplus R_2$ be a ring decomposition. Assume that $\underline{C}_R$ is closed under homomorphic images. Then $\overline{Z}_R(R_i) \neq 0$.*

Proof Suppose that $\overline{Z}_R(R_1) = 0$. By Lemma 2.7, we have $\overline{Z}_{R_1}(R_1) = 0$. Since $\underline{C}_R$ is closed under homomorphic images, $\underline{C}_{R_1}$ is closed under homomorphic images (see Lemma 2.7). By Lemma 2.10, $\overline{Z}_{R_1}(R_1) \neq 0$, a contradiction. □

Theorem 2.12 *Let R be a commutative ring such that $\underline{C}_R$ is closed under homomorphic images. The following are equivalent:*

(1) *R has (P);*

(2) *$R_R = R_1 \oplus R_2$ such that $R_1 \in \underline{C}_R$ and $\overline{Z}(R_2) = R_2$;*

(3) *R is von Neumann regular.*

Proof (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) By Proposition 2.11, we have $R_1 = 0$. Hence, $\overline{Z}(R) = R$. Thus R is cosemisimple by [13, Corollary 2.6]. Since R is commutative, R is von Neumann regular.

(3) $\Rightarrow$ (1) By [13, Proposition 2.5], every module is noncosingular. Thus R has (P). □

3. When perfect rings have property (P)

Lemma 3.1 *Let R be a right perfect ring with (P). Then the class $\underline{C}_R$ is closed under homomorphic images.*

Proof Since R is right perfect, every R-module is amply supplemented. Let M be a cosingular module and let $N \leq M$. By [13, Theorem 3.5], $\overline{Z}^2(M/N) = (\overline{Z}^2(M)+N)/N$. But $\overline{Z}^2(M/N) = \overline{Z}(M/N)$ and $\overline{Z}^2(M) = \overline{Z}(M)$ by Proposition 2.3. Then $\overline{Z}(M/N) = (\overline{Z}(M) + N)/N = 0$. □

Note that $r(J)$ and $l(J)$ will denote the right and left annihilator of the Jacobson radical J of a ring R, respectively.

Proposition 3.2 *Let R be a right perfect ring with Jacobson radical J. Then for any R-module M, we have $\overline{Z}(M) = Mr(J)$.*

Proof Let M be any module. By [4, 6.14], [13, Proposition 2.1(3)], and Lemma 3.1, we have $\overline{Z}(M) = M\overline{Z}(R_R)$. Therefore $\overline{Z}(M) = Mr(J)$ by [12, Proposition 2.6]. □

Theorem 3.3 *Let R be a right perfect ring such that $r(J) = l(J)$. The following are equivalent:*

(1) *R has (P);*

(2) *$r(J)$ is injective;*

(3) *R is semisimple.*

Proof Note that since $r(J) = l(J)$, we have $r(J) = Soc(R_R) = Soc({}_RR)$ by [1, Proposition 15.17].

(1) $\Rightarrow$ (2) By Proposition 3.2, $\overline{Z}(R_R) = Soc(R_R)$. Since R has (P), $Soc(R_R)$ is a noncosingular direct summand of R_R. Thus, $Soc(R_R) = \oplus_{i=1}^{n} S_i$ for some simple right ideals S_i $(1 \leq i \leq n)$ of R. By [13, Proposition 2.1(4)], every S_i $(1 \leq i \leq n)$ is noncosingular. So every S_i $(1 \leq i \leq n)$ is injective. Hence, $r(J) = Soc(R_R)$ is injective.

(2) $\Rightarrow$ (3) Suppose that R is not semisimple. By (2), there is a nonzero right ideal I of R such that $R = Soc(R_R) \oplus I$. By [1, Theorem 28.4], $Soc(I) \neq 0$, a contradiction.

(3) $\Rightarrow$ (1) This is clear. □

Corollary 3.4 *Let R be a commutative perfect ring. Then R has (P) if and only if R is semisimple.*

Proof This follows from Theorem 3.3. □

Proposition 3.5 *Let R be a right perfect ring having (P). Then R has a simple injective module.*

Proof Assume that R has no simple injective modules. Let M be any R-module. By Proposition 2.3, we have $\overline{Z}(M) = \overline{Z}^2(M)$. By [13, Theorem 3.8(3)], we have $\overline{Z}^2(M) = 0$. Thus, $\overline{Z}(M) = 0$ for every module M, a contradiction (see [13, Proposition 2.8]). □

Lemma 3.6 *Let R be a local ring with maximal right ideal m such that R/m is a nonsmall module. Then R has (P) if and only if R is a division ring.*

Proof $(\Rightarrow)$ The module R/m is injective. Thus, every simple R-module is injective. Therefore, R is cosemisimple, so $J = m = 0$. Hence, R is a division ring.
$(\Leftarrow)$ Clear. □

Corollary 3.7 *Let R be a right perfect local ring with maximal right ideal m. Then R has (P) if and only if R is a division ring.*

Proof By Proposition 3.5, the module R/m is not small. Then the rest is clear by Lemma 3.6. □

Proposition 3.8 *Let R be a right perfect ring. If R has (P), then $R_R = (\oplus_{i=1}^n L_i) \oplus (\oplus_{i=1}^m K_i)$ is a direct sum of local submodules such that $\overline{Z}(L_i) = L_i$ $(1 \leq i \leq n)$ and $\overline{Z}(K_i) = 0$ $(1 \leq i \leq m)$.*

Proof By [14, 42.6], $R_R = (\oplus_{i=1}^n L_i) \oplus (\oplus_{i=1}^m K_i)$ is a direct sum of local submodules such that $\overline{Z}(L_i) = L_i$ and $\overline{Z}(K_i) \neq K_i$. Since $\overline{Z}(K_i) \neq K_i$, $\overline{Z}(K_i) \ll K_i$. Thus, $\overline{Z}^2(K_i) = 0$. Proposition 2.3 shows that $\overline{Z}(K_i) = 0$. □

Proposition 3.9 *Let R be a right perfect ring such that $R_R = (\oplus_{i=1}^n L_i) \oplus (\oplus_{i=1}^m K_i)$ is a direct sum of local submodules with $\overline{Z}(L_i) = L_i$ and each K_i $(1 \leq i \leq m)$ is simple small. Then R has (P).*

Proof Let M be any R-module. It is well known that M is a homomorphic image of a free R-module. So $M = M_1 + M_2$ such that M_1 is a homomorphic image of a noncosingular module by [13, Proposition 2.4] and M_2 is a homomorphic image of a direct sum of K_is. By [13, Proposition 2.4], M_1 is noncosingular and by [11, Lemma 9], M_2 is small and hence cosingular. Since $M/M_1 \cong M_2/(M_1 \cap M_2)$ is small, we have $\overline{Z}(M/M_1) = 0$. By [13, Proposition 2.1(1)], we have $\overline{Z}(M) \subseteq M_1$. But $M_1 = \overline{Z}(M_1) \subseteq \overline{Z}(M)$. Then $\overline{Z}(M) = M_1$. Therefore, $M = \overline{Z}(M) + M_2$ with M_2 semisimple. Let N be a submodule of M_2 such that $M_2 = (\overline{Z}(M) \cap M_2) \oplus N$. Thus, $M = \overline{Z}(M) \oplus N$. □

The following example gives a ring satisfying the conditions of Proposition 3.9 and shows that a right perfect ring having (P) need not be semisimple.

Example 3.10 Let R be a left and right hereditary Artinian serial ring with $J^2 = 0$ (e.g., we can take the ring of all upper triangular 2×2 matrices with entries in a field K) (see [5, Example 13.6]). By [5, 13.5], every right ideal is a direct sum of an injective module and a semisimple module. By [14, 42.6], $R_R = (\oplus_{i=1}^{n} L_i) \oplus (\oplus_{i=1}^{m} K_i)$ is a direct sum of local submodules such that L_i are injective and K_i are simple. Without loss of generality we can assume that all K_i $(1 \leq i \leq m)$ are small. Since R is hereditary, every injective module is noncosingular. By Proposition 3.9, the ring R has (P).

Proposition 3.11 (1) *Let R be a ring with (P) such that every nonzero injective R-module is not cosingular. Then every injective R-module is noncosingular.*

(2) *Let R be a right Artinian ring. If R has (P), then every injective R-module is noncosingular.*

Proof (1) Let M be an injective R-module. Then $M/\overline{Z}(M)$ is injective cosingular. By hypothesis, $M = \overline{Z}(M)$.

(2) By [13, Corollary 2.10] and (1). □

Recall that a ring R is called a *right H-ring* if every injective right R-module is lifting.

Theorem 3.12 *Let R be a right H-ring. Consider the following conditions:*

(1) *R is semisimple;*

(2) *For every module M and every submodule A of M, we have $\overline{Z}(A) = A \cap \overline{Z}(M)$;*

(3) *R has (P);*

(4) *For every R-module M, $\overline{Z}(M) = Mr(J)$ is injective;*

(5) *The class of injective modules coincides with the class of noncosingular modules;*

(6) *Every injective module is noncosingular.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (3) and if R is a QF-ring, then (6) $\Rightarrow$ (1).

Proof (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) Let M be a module. Then M has a decomposition $M = M_1 \oplus M_2$ with M_1 injective and $\overline{Z}(M_2) = 0$ by [4, 28.10]. Since R is right Noetherian, $M_1 = \oplus_{i\in I} L_i$ with $(L_i)_{i\in I}$ indecomposable injective submodules. Since each L_i is lifting, each L_i is local. If $\overline{Z}(L_i) \neq L_i$, then $\overline{Z}^2(L_i) = 0$. But $\overline{Z}^2(L_i) = \overline{Z}(L_i) \cap \overline{Z}(M) = \overline{Z}(L_i)$ by (2). Thus, $\overline{Z}(L_i) = 0$. The result follows from [13, Proposition 2.1(4)].

(3) $\Rightarrow$ (4) Let M be an R-module. By [4, 28.10], $M = N \oplus K$ such that N is injective and K is a small module. Therefore, $\overline{Z}(M) = \overline{Z}(N) \oplus \overline{Z}(K)$. But $\overline{Z}(N) = N$ (see Proposition 3.11) and $\overline{Z}(K) = 0$. It follows that $\overline{Z}(M) = N$ is injective. Moreover, $\overline{Z}(M) = Mr(J)$ by Proposition 3.2.

(4) $\Rightarrow$ (5) Let E be a noncosingular module. Then $E = \overline{Z}(E)$ is injective. The result follows from Proposition 3.11.

(5) $\Rightarrow$ (6) This is clear.

(6) $\Rightarrow$ (3) Let M be any module. Then $M = K \oplus L$ such that K is injective and L is a small module. By [13, Proposition 2.1], $\overline{Z}(M) = \overline{Z}(K) \oplus \overline{Z}(L)$. But $\overline{Z}(L) = 0$. Then $\overline{Z}(M) = K$ is a direct summand of M.

(6) $\Rightarrow$ (1) Assume that R is a QF-ring. Since (6) implies (3), R has (P). The result follows by Theorem 3.3 and [7, Corollary 15.7]. □

Note that the QF condition in implication (6) $\Rightarrow$ (1) of Theorem 3.12 is not superfluous. As an example we can take the ring of all upper triangular 2×2 matrices with entries in a field K. This ring is not QF since

$r(J) \neq l(J)$, where J is the Jacobson radical. On the other hand, by Example 3.10, the ring has (P) but is not semisimple. This ring is also an H-ring by [9, Corollary 2.5].

4. Examples

Proposition 4.1 *Let R be a ring with Jacobson radical J such that R/J is a simple Artinian ring, $J \neq 0$ and $J^2 = 0$. Then:*

(1) *For every module M, $\overline{Z}(M) = MJ = Rad(M)$;*

(2) *The class $\underline{C}_R = \{M_R \mid \overline{Z}(M) = 0\}$ is closed under homomorphic images;*

(3) *The ring R does not have (P).*

Proof (1) Up to isomorphism, R has a unique simple right module U. Since J is a nonzero right R/J-module, there exists a submodule $V \leq J_R$ such that $U \cong V$. As V is small in R, U is a small module. Let M be a nonzero R-module. We want to show that $\overline{Z}(M) = \mathrm{Rad}(M)$.

Now let $N \leq M$ with M/N small. Then $M/N \subseteq EJ$, where $E = E(M/N)$. Thus, $(M/N)J = 0$ and hence $MJ \subseteq N$. Therefore, $MJ \subseteq \overline{Z}(M)$.

Now assume that $M \neq MJ$. Then M/MJ is a direct sum of isomorphic copies of U. Note that M/MJ is an R/J-module. By assumption and [11, Lemma 9], M/MJ is small. Hence, $\overline{Z}(M) \subseteq MJ$. Therefore, $\overline{Z}(M) = MJ$.

(2) By (1).

(3) Assume that R has (P). By (1), $\mathrm{Rad}(M)$ is a direct summand of M for every module M. Since $J(R) \ll R_R$, we have $\overline{Z}(R_R) = 0$. Moreover, it follows from (2) that every module is cosingular, a contradiction. □

Let R be a ring. R is called a right *Goldie* ring if R_R is finite dimensional and satisfies the ascending chain condition on right annihilator ideals. R is called a right *primitive* ring if there exists a simple right R-module U with $ann_R(U) = 0$.

The proof of the following last Proposition has the same techniques as the proof of [10, Proposition 12].

Proposition 4.2 *Let R be a prime right Goldie ring that is not right primitive. Then every cyclic right R-module is small. In particular, every cyclic module is cosingular.*

Proof Let $M = xR$ and $E = E(M)$, the injective hull of M. We want to show that $M \ll E$. Let $E = M + T$ with $T \leq E$. Assume $x \in M \backslash T$. Then E/T is nonzero and cyclic. Hence, there exists a maximal submodule K/T of E/T. Now K is a maximal submodule of E and the module $U = E/K$ is simple. By hypothesis, $I = ann_R(U) \neq 0$. Since R is prime, I_R is essential in R_R. By [6, Proposition 5.9], I contains a regular element, namely a nonzero divisor c. Now $E = Ec \subseteq EI \subseteq E$ implies that $EI = E$. Thus $E = K$, a contradiction. Hence, $x \in T$ and so $E = T$. □

Acknowledgment

This paper was written while the third author was visiting the Hacettepe University Mathematics Department in 2011. All authors wish to thank the Scientific and Technological Research Council of Turkey (TÜBİTAK) for financial support under the program 2221. We would also like to thank the referees for carefully reading this paper and for the numerous valuable comments on the paper.

References

[1] Anderson FW, Fuller KR. Rings and Categories of Modules. New York, NY, USA: Springer, 1974.

[2] Cateforis VC, Sandomierski FL. The singular submodule splits off. J Algebra 1968; 10: 149–165.

[3] Cateforis VC, Sandomierski FL. On commutative rings over which the singular submodule is a direct summand for every module. Pac J Math 1969; 31: 289–292.

[4] Clark J, Lomp C, Vanaja N, Wisbauer R. Lifting Modules: Frontiers in Mathematics. Boston, MA, USA: Birkhäuser, 2006.

[5] Dung NV, Huynh DV, Smith PF, Wisbauer R. Extending Modules: Pitman Research Notes in Mathematics Series. Harlow, UK: Longman, 1994.

[6] Goodearl KR, Warfield RB. An Introduction to Noncommutative Noetherian Rings. London Mathematical Society Student Texts 16. Cambridge, UK: Cambridge University Press, 1989.

[7] Lam TY. Lectures on Modules and Rings. Vol. 189 of Graduate Texts in Mathematics. New York, NY, USA: Springer. 1998.

[8] Orhan-Ertas N, Tribak R. Two generalizations of lifting modules. International Journal of Algebra 2009; 3(13): 599–612.

[9] Oshiro K, Wisbauer R. Modules with every subgenerated module lifting. Osaka J Math 1995; 32: 513–519.

[10] Özcan A.Ç. On a class of modules. Publ Math Debrecen 2000; 57(3–4): 269–275.

[11] Rayar M. On small and cosmall modules. Acta Math Acad Sci Hungar 1982; 39(4): 389–392.

[12] Tribak R, Keskin Tütüncü D. On $\overline{Z}_M$-semiperfect modules. East-West J Math 2006; 8(2): 193–203.

[13] Talebi Y, Vanaja N. The torsion theory cogenerated by M-small modules. Comm Algebra 2002; 30(3): 1449–1460.

[14] Wisbauer R. Foundations of Module and Ring Theory. Philadelphia, PA, USA: Gordon and Breach, 1991.

[15] Zöschinger H. Kosinguläre und Kleine Moduln. Comm Algebra 2005; 33: 3389–3404 (in German).