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Rings whose pure-injective right modules are direct sums of lifting modules

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ABSTRACT

It is shown that every pure-injective right module over a ring R is a direct sum of lifting modules if and only if R is a ring of finite representation type and right local type. In particular, we deduce that every left and every right pure-injective R-module is a direct sum of lifting modules if and only if R is (both sided) serial artinian. Several examples are given to show that this condition is not left–right symmetric.

1. Introduction

Let R be an associative ring with identity. A module M is called *extending* or *CS* if every submodule of M is essential in a direct summand of M. Dually, M is called *lifting* if every submodule N of M lies above a direct summand of M. I.e., there exists a direct sum decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq N$ and $N \cap M_2$ superfluous in M_2. Extending modules generalize (quasi-)injective, semisimple, and uniform modules while lifting modules extend semisimple and hollow modules. These modules have been extensively studied in the last years (see, for instance, [2,6] for a detailed account on them).

It is known that every Σ-extending module M (i.e., any direct sum of copies of M is extending) is a direct sum of indecomposable Σ-quasi-injective modules [12] and that a ring R is both sided artinian and serial with $J^2 = 0$ if and only if every right (left) R-module is extending [8] (see also [13]). At first sight, these facts seem to suggest that the theory of extending modules runs parallel to that of

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injective ones. However, this is far from being true. For instance, the modules M such that $M^{(\aleph_0)}$ is extending do not need to be Σ-extending, nor a direct sum of indecomposable direct summands (see [7]). And indeed, the above results relay on deep cardinality arguments inspired on the work of Osofsky [16,17] (see also [14]).

It has been recently proved by Er in [11] that the following properties are equivalent for a ring R:

1. $(*)_r$ Every right R-module is a direct sum of extending modules.
2. R has finite type and every (finitely generated) indecomposable right R-module has simple socle (i.e., R is of right colocal type).

Moreover, in this case, R is (both sided) artinian and right serial, and every right R-module is a direct sum of uniform modules. As a consequence, Er deduces that every right and every left R-module is a direct sum of extending modules iff the ring R is (both sided) serial artinian. The key part in the proof of this result is to show that a ring in which every right module is a direct sum of extending modules is right pure-semisimple (i.e., every right module is pure-injective). This fact is proved again by means of Set Theoretical arguments.

On the other hand, several results in the structure of lifting modules (see e.g. [2,18]) suggest that the above structure theorem might also have a counterpart in terms of lifting modules. The aim of this paper is to extend the results in [11] by obtaining similar characterizations for rings whose right modules are direct sums of lifting modules. Namely, we prove:

Theorem 2.1. *The following are equivalent for a ring R:*

1. *Every right R-module is a direct sum of lifting modules.*
2. *Every pure-injective right R-module is a direct sum of lifting modules.*
3. *R is of finite type and right local type.*

We recall that a ring R is said to be *of finite representation type* (*finite type*, for short) when there exists a finite set of indecomposable right R-modules such that any other right module is isomorphic to a direct sum of copies of them. In this case, R is left and right artinian and there also exists a finite set of indecomposable left R-modules such that any other left module is isomorphic to a direct sum of copies of them. And a ring R is *of right local type* when every indecomposable right R-module is local.

In order to prove our theorem, we extend to the framework of pure-injectivity a result in [9] which asserts that every injective right R-module is a direct sum of lifting modules if and only if R is right noetherian and every indecomposable injective right R-module is hollow. Let us note that our theorem highlights the role played by pure-injectivity in the above characterizations.

As a consequence of our result, we are able to extend the characterization given in [11] by proving:

Corollary 2.3. *Let R be a ring. The following are equivalent*:

1. *R is (both sided) serial artinian.*
2. *Every left and every right R-module is a direct sum of lifting modules.*
3. *Every left and every right pure-injective R-module is a direct sum of lifting modules.*
4. *Every left and every right R-module is a direct sum of extending modules.*

Note that any two-sided artinian serial ring R is of finite type, and every left and every right R-module is a direct sum of uniserial modules (see e.g. [1, Theorem 32.3]). And every uniserial module is trivially lifting. We also give examples showing that the rings satisfying the hypotheses of our main theorem do not need to be of left local type and thus, our characterization is not left–right symmetric.

Throughout this paper, all rings will be associative rings with an identity element and all modules will be unitary right modules. We will denote by Mod-R the category of all right R-modules, and by Ab, the category of abelian groups. $E(M)$ will denote the injective envelope of a module M and $PE(M)$, its pure-injective envelope. A module M is called indecomposable if it cannot be written as the direct sum $M = M_1 \oplus M_2$ of two non-zero submodules. We will use the notation $N \ll M$ to stress that N is a superfluous submodule of M. We refer to [1,2,6,14,20] for any undefined notion used along the text.

2. Results

We begin this section by recalling some basic properties of purity and functor categories which will be used in the proof of our main result. A Grothendieck category $\mathcal{C}$ is called *locally finitely presented* if there exists a generator set of $\mathcal{C}$ consisting of finitely presented objects, where an object $C \in \mathcal{C}$ is *finitely presented* if the functor $\mathrm{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \to Ab$ commutes with direct limits. Every locally finitely presented Grothendieck category $\mathcal{C}$ has enough injective objects and every object $C \in \mathcal{C}$ can be essentially embedded in an injective object $E(C)$, called the injective envelope of C (see e.g. [22]). Baer criterium for injectivity can be adapted to this framework [22, Proposition V.2.9] and therefore, in order to show that an object $C \in \mathcal{C}$ is injective, it is enough to prove that C is injective with respect to the finitely presented objects belonging to the above generator set of $\mathcal{C}$.

A short exact sequence

$$0 \to N \to M \to M/N \to 0$$

in Mod-R is called *pure* if

$$0 \to N \otimes_R X \to M \otimes_R X \to (M/N) \otimes_R X \to 0$$

remains exact in Ab for any left R-module X. This is equivalent to asserting that the functor

$$\mathrm{Hom}_R(L, -) : \text{Mod-}R \to Ab$$

preserves exactness, for every finitely presented module $L \in$ Mod-R. And a module $E \in$ Mod-R is called *pure-injective* if it is injective with respect to pure-exact sequences.

It is well known (see e.g. [3,21]) that there exists a locally finitely presented Grothendieck category $\mathcal{C}$ (usually called the functor category of Mod-R) and a fully faithful additive functor

$$T : \text{Mod-}R \to \mathcal{C}$$

satisfying the following properties:

1. The functor T admits a right adjoint functor $H : \mathcal{C} \to$ Mod-R.
2. A short exact sequence

$$\Sigma \equiv 0 \to X \to Y \to Z \to 0$$

   in Mod-R is pure if and only if the sequence $T(\Sigma)$ is exact (and pure) in $\mathcal{C}$.
3. T carries finitely generated objects in Mod-R to finitely generated objects in $\mathcal{C}$.
4. The image of Mod-R under the functor T is the full subcategory of $\mathcal{C}$ consisting of the all FP-injective objects in $\mathcal{C}$. I.e., those objects $C \in \mathcal{C}$ such that $\mathrm{Ext}^1_{\mathcal{C}}(X, C) = 0$ for every finitely presented object $X \in \mathcal{C}$.
5. A module $M \in$ Mod-R is pure-injective if and only if $T(M)$ is an injective object of $\mathcal{C}$.

6. Every module $M \in$ Mod-R admits a pure embedding in a pure-injective object $PE(M) \in$ Mod-R such that the image of this embedding under T is the injective envelope of $T(M)$ in $\mathcal{C}$. Thus, $PE(M)$ is uniquely determined up to isomorphisms and it is called the *pure-injective envelope* of M.

In particular, it follows that every direct sum of pure-injective modules in Mod-R is pure-injective if and only if every injective object in $\mathcal{C}$ is Σ-injective. And this last condition is equivalent to claim that $\mathcal{C}$ is locally noetherian (see e.g. [22]). As pure subobjects of Σ-pure-injective objects are direct summands of them, we deduce that $\mathcal{C}$ is locally noetherian if and only if every object in Mod-R is pure-injective. In this case, the ring R is called *right pure-semisimple*.

We can now state our main result.

Theorem 2.1. *The following are equivalent for a ring R:*

1. *Every right R-module is a direct sum of lifting modules.*
2. *Every pure-injective right R-module is a direct sum of lifting modules.*
3. *R has finite type and right local type.*

Moreover, in this case R is (both sided) artinian and left serial, and every right R-module is a direct sum of local modules.

Proof. $(1) \Rightarrow (2)$ and $(3) \Rightarrow (1)$ are clear. Let us prove $(2) \Rightarrow (3)$.

Let us assume that every pure-injective right R-module is a direct sum of lifting modules. We claim that the ring R is right pure-semisimple. In order to prove our claim, let $T : \text{Mod-}R \to \mathcal{C}$ be the above embedding. We know that R is right pure-semisimple if and only if the category $\mathcal{C}$ is locally noetherian. On the other hand, it is well known that this is the case if and only if every direct sum of copies of injective objects in $\mathcal{C}$ is again injective (see [22, Proposition V.4.3]). But, as $\mathcal{C}$ is locally finitely generated, the set of isomorphism classes of the injective envelopes of simple objects in $\mathcal{C}$ cogenerates the category and therefore, we deduce that $\mathcal{C}$ is locally noetherian if and only if any direct sum of injective envelopes of simple objects in $\mathcal{C}$ is injective.

Let us assume on the contrary that $\mathcal{C}$ is not locally noetherian. Then there must exist a direct sum of injective envelopes of simple objects in $\mathcal{C}$, say $\bigoplus_I E(S_i)$, which is not injective. By Baer's criterium for locally finitely presented Grothendieck categories [22, Proposition V.2.9], there is a finitely presented object $C \in \mathcal{C}$, a subobject C' of C and a morphism $g : C' \to \bigoplus_{i\in I} E(S_i)$ which does not extend to C. As finite direct sums of injective objects in $\mathcal{C}$ are injective, this means that there does not exist any finite subset $I' \subseteq I$ such that $\text{Im}(g) \subseteq \bigoplus_{I'} E(S_i)$. Therefore, if we call $\pi_i : \bigoplus_I E(S_i) \to E(S_i)$ the canonical projection, then $\pi_i \circ g \neq 0$ for infinitely many indexes $i \in I$. Let us choose an infinite countable subset $\bar{I} \subseteq I$ such that $\pi_i \circ g \neq 0$ for all $i \in \bar{I}$ and let $\pi : \bigoplus_I E(S_i) \to \bigoplus_{\bar{I}} E(S_i)$ be the canonical projection. Call now $g' = \pi \circ g$.

We claim that for any infinite subset $\bar{I}' \subseteq \bar{I}$, the map $q \circ g' : C' \to \bigoplus_{\bar{I}'} E(S_i)$ does not extend to C, where $q : \bigoplus_{\bar{I}} E(S_i) \to \bigoplus_{\bar{I}'} E(S_i)$ is the canonical projection. Let us note that, in particular, this implies that $\bigoplus_{\bar{I}} E(S_i)$ does not have any infinite injective subsum. Assume otherwise that $f : C \to \bigoplus_{\bar{I}'} E(S_i)$ is an extension. Then, as C is finitely presented, $\text{Im}(f)$ embeds in a finite subsum, say $\bigoplus_{\bar{I}''} E(S_i)$, of $\bigoplus_{\bar{I}'} E(S_i)$. And therefore, $\text{Im}(g)$ also embeds in $\bigoplus_{\bar{I}''} E(S_i)$. But this means that $\pi_i \circ q \circ g' = \pi_i \circ g = 0$ for all $i \in \bar{I}' \setminus \bar{I}''$ and this is not possible since $\pi_i \circ g \neq 0$ for all $i \in \bar{I}$. A contradiction that proves our claim.

Let us write $\bigoplus_I E(S_i) = \bigoplus_{\mathbb{N}} E(S_n)$. As each $E(S_n)$ is an injective object in $\mathcal{C}$, there exists an indecomposable pure-injective object $Q_n \in$ Mod-R such that $T(Q_n) = E(S_n)$. Thus, $\bigoplus_{\mathbb{N}} Q_n$ is a direct sum of indecomposable pure-injective modules in Mod-R with the property that no infinite direct subsum of it is pure-injective.

On the other hand, we are assuming that any pure-injective right R-module is a direct sum of lifting modules. Therefore, there exists a family $\{M_j\}_{j\in J}$ of lifting modules such that $PE(\bigoplus_{\mathbb{N}} Q_n) = \bigoplus_{j\in J} M_j$. Applying now the functor T, we get that $E(\bigoplus_{\mathbb{N}} E(S_n)) = \bigoplus_{j\in J} T(M_j)$.

We claim that there exists a $j \in J$ such that $T(M_j)$ does not have finitely generated socle. Assume on the contrary that $\mathrm{Soc}(T(M_j))$ is finitely generated for every $j \in J$. As the socle of $\bigoplus_{\mathbb{N}} E(S_n)$ is not finitely generated, this means that there exists an infinite subset $J' \subseteq J$ such that $\mathrm{Soc}(T(M_j)) \neq 0$ for every $j \in J'$. We can choose, for any $j \in J'$, an $n_j \in \mathbb{N}$ such that S_{n_j} embeds in $T(M_j)$. Thus, $E(S_{n_j})$ also embeds in $T(M_j)$. And therefore, $\bigoplus_{j \in J'} E(S_{n_j})$ is a direct summand of $\bigoplus_{j \in J} T(M_j) = E(\bigoplus_{\mathbb{N}} E(S_n))$. This means that $\bigoplus_{j \in J'} E(S_{n_j})$ is injective, contradicting the fact that no infinite subsum of $E(\bigoplus_{\mathbb{N}} E(S_n))$ is injective. This proves our claim.

Therefore, there exists a $j \in J$ such that $T(M_j)$ has infinitely generated socle. As direct summands of lifting modules are lifting, we may assume without loss of generality that the socle of $T(M_j)$ is essential in $T(M_j)$ and that $T(M_j) = E(\bigoplus_{\mathbb{N}} E(S_n))$. But then, as $PE(\bigoplus_{\mathbb{N}} Q_n)$ is lifting, there exist direct summands Q, Q' of $PE(\bigoplus_{\mathbb{N}} Q_n)$ such that $PE(\bigoplus_{\mathbb{N}} Q_n) = Q \oplus Q'$, $Q \subseteq \bigoplus_{\mathbb{N}} Q_n$ and $Q' \cap (\bigoplus_{\mathbb{N}} Q_n) \ll Q'$. Let us denote by $\beta : Q \oplus Q' \to Q'$ the canonical projection.

We claim that $T(Q)$ is essential in $T(M_j)$. Assume on the contrary that there exists an $n_0 \in \mathbb{N}$ such that $E(S_{n_0}) \cap T(Q) = 0$. Then we have that $T(\beta)(E(S_{n_0})) \subseteq T(Q')$ and therefore $T(\beta)(E(S_{n_0}))$ is a direct summand of $T(Q')$. And this means that Q_{n_0} is a direct summand of Q'. But $\beta(Q_{n_0}) \subseteq Q' \cap B$. So $\beta(Q_{n_0})$ is superfluous in Q' and it cannot be a direct summand of Q' unless it is zero. This shows that $T(Q)$ is essential in $T(M_j)$ and thus, $T(Q) = \bigoplus_{\mathbb{N}} E(S_n)$. And we deduce that $\bigoplus_{\mathbb{N}} E(S_n)$ is injective, a contradiction that proves our original claim that $\mathcal{C}$ is locally noetherian.

We have shown that $\mathcal{C}$ is locally noetherian and, equivalently, Mod-R is pure-semisimple. Therefore, any right R-module is a direct sum of indecomposable modules. As any right pure-semisimple ring is right artinian, we deduce that any indecomposable projective right R-module is cyclic and there exists a finite number of isomorphism classes of them. Let us call

$$t_0 = \max\{l(P) \mid P \text{ is an indecomposable projective in Mod-}R\}$$

where $l(P)$ denotes the composition length of P. And let us now show that any indecomposable right R-module M has composition length bounded by t_0. As R is right artinian, every indecomposable module M has a projective cover, say $p : P_M \to M$. Moreover, as M is lifting, P_M is indecomposable. This means that $l(M) \leqslant l(P_M)$.

On the other hand, it was proved by Prest in [19] (see also [20, Theorem 8.26] and [24]) that any right pure-semisimple ring has a finite number of isomorphism classes of indecomposable right modules of length bounded by n, for each $n \in \mathbb{N}$. As we have shown that any indecomposable right R-module M has length bounded by t_0, we deduce that there are only finitely many isomorphism classes of indecomposable right R-modules and thus, the ring R is of finite type. Note that, as any indecomposable right module is lifting, R is of right local type.

Finally, assume that R satisfies any of the above equivalent conditions. Then R is left and right artinian since it is of finite type. Hence every right R-module is a direct sum of local modules by [10, Corollary 2]. On the other hand, R is left serial by [15, Theorem 2.12 and Corollary 3.4]. The proof is now complete. □

Remark 2.2. We stress that, in the proof of (2) ⇒ (3) in the above theorem, we are only using the fact that the pure-injective envelope of every countable direct sum of indecomposable pure-injective right modules is a direct sum of lifting modules. Therefore, the above statements are also equivalent to this weaker condition.

In particular, we get:

Corollary 2.3. *Let R be any ring. The following are equivalent*:

1. *R is (both sided) serial artinian.*
2. *Every left and every right R-module is a direct sum of lifting modules.*
3. *Every left and every right pure-injective R-module is a direct sum of lifting modules.*
4. *Every left and every right R-module is a direct sum of extending modules.*

Proof. By Theorem 2.1 and [11, Corollary 2]. □

We are going to close this note with several examples which show the limits of our results. We first give an example of an artinian left serial ring which does not satisfy condition (1) of Theorem 2.1:

Example 2.4. (See [15, Example 3.17].) Let R be a local artinian ring with radical W such that $W^2 = 0$, $Q = R/W$ is commutative, $dim({}_Q W) = 1$, and $dim(W_Q) = 3$. Then R is left serial but not right serial. Let $W = w_1 R \oplus w_2 R \oplus w_3 R$. By [5, Lemmas 4.1 and 4.2], the right R-module $X_5 = (R_R \oplus R_R)/((w_1, 0)R + (0, w_1)R + (w_2, w_3)R)$ is an indecomposable and 2-generated right R-module of length 5 and it is not local. It is proved in [15, Example 3.17] that X_5 is not $\oplus$-supplemented. Then X_5 cannot be a direct sum of lifting modules by [15, Theorem 2.12].

We now give two examples which show that condition (1) in Theorem 2.1 is not left–right symmetric. They also show that a ring R satisfying condition (1) in Theorem 2.1 does not need to be right serial.

Example 2.5. Set $R = \left[\begin{smallmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} \end{smallmatrix}\right]$, which is a 5-dimensional hereditary $\mathbb{R}$-algebra. Although R is left serial, it is not right serial. So, by Theorem 2.1, R does not satisfy the left version of condition (1) in Theorem 2.1. Let $e_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $e_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. It is easy to see that both indecomposable projective right R-modules $e_1 R$ and $e_2 R$ are lifting modules. Let $H_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}_{\mathbb{R}}, \mathbb{R}_{\mathbb{R}}) \cong \mathbb{C}_{\mathbb{C}}$. Then the right R-module $\left[\begin{smallmatrix} \mathbb{C} & 0 \\ H & \mathbb{R} \end{smallmatrix}\right]$, which is the minimal injective cogenerator (see [23, Corollary 10.3]), has length 3. The two indecomposable injective right R-modules are $U_1 = \left[\begin{smallmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $U_2 = \left[\begin{smallmatrix} 0 & 0 \\ H & \mathbb{R} \end{smallmatrix}\right]$ where U_1 is simple and U_2 is a uniserial right R-module of length 2. Hence both U_1 and U_2 are also lifting modules. It is known that R is of finite type with 4 indecomposable right R-modules. Hence, $e_1 R$, $e_2 R$, U_1 and U_2 are the only 4 indecomposable right R-modules and every right R-module is a direct sum of these 4 modules. Therefore, R satisfies the condition (1) of Theorem 2.1. Note that R is not a ring of left local type.

Example 2.6. (See [15, Example 3.16].) Let R be a local artinian ring with radical W such that $W^2 = 0$, $Q = R/W$ is commutative, $dim({}_Q W) = 1$ and $dim(W_Q) = 2$. Then R is left serial but not right serial. Therefore, by Theorem 2.1, there exists a left R-module which cannot be written as a direct sum of lifting left R-modules. Note that every right R-module is a direct sum of indecomposable modules. As it is sated in [15, Example 3.16], by [4, Proposition 3], there are three isomorphism classes of indecomposable right R-modules, namely, $A_1 = R/W$ (the simple module), $A_2 = R/uR$ (the injective module), and $A_3 = R_R$, where $W = uR \oplus vR$ and each of the A_i's is a lifting module. Note that, again, R is not a ring of left local type.

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