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Research Article

Remarks on Separation of Convex Sets, Fixed-Point Theorem, and Applications in Theory of Linear Operators

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Some properties of the linear continuous operator and separation of convex subsets are investigated in this paper and a dual space for a subspace of a reflexive Banach space with a strictly convex norm is constructed. Here also an existence theorem and fixed-point theorem for general mappings are obtained. Moreover, certain remarks on the problem of existence of invariant subspaces of a linear continuous operator are given.

1. Introduction

In this paper, the separation of convex sets in a real reflexive Banach space are investigated, existence of a fixed-point theorem for a general mapping acting in a Banach space and the obtained results are applied to study certain properties of continuous linear operators. Furthermore, here is proved the solvability theorem for an inclusion with sufficiently general mapping. Fixed-point theorems obtained here are some generalizations of results obtained earlier in [1, 2] (see, also [3]).

It is known that (see [4–6]) sufficiently general results about the separation of convex sets are available for the case when the space considered is a finite-dimensional Euclidean space. But, if X is infinite-dimensional, it is not possible to prove such results since the geometrical characteristics of an infinite-dimensional space essentially differ from those of a finite-dimensional space. Here we prove results about the separation of convex sets in an infinite-dimensional space which resemble the results in the finite-dimensional case, provided that the space has a geometry satisfying some complementary conditions. These results concern the separation of convex sets in a reflexive Banach space which, together with its dual space, has a strictly convex norm (it is known that [7–10] in a reflexive Banach space, such equivalent norm can be defined to consider that the space in this norm and his dual space in the respective norm are strictly convex spaces). Moreover, the obtained results are used to prove some fixed-point theorems for sufficiently general mappings. It should be noted that to investigated the existence of fixed-points, sufficiently many works are dedicated (see, e.g., [1–3, 11–14], etc. and references therein).

Here we investigate certain properties of continuous linear operators acting in a reflexive Banach space, and obtain conditions under which such operator has an eigenvector (clearly this implies that the operator has an invariant subspace). It should be noted that many works are devoted to the problem of type of the existence of an invariant subspace of the linear operator (see, e.g., [15–18], etc.) and one of the essential results is obtained in [16] (see, also [17]). In these papers, the connection of the considered linear operator with a completely continuous operator played a basic role as in [16] (see, also [17, 18]).

In particular, here is obtained the following assertion. Let X and Y be Banach spaces, let $\mathbb{B}(X;Y)$ be the Banach space of linear bounded operators acting from X into Y (in particular, if $Y = X$, then $\mathbb{B}(X;Y) \equiv \mathbb{B}(X)$, as usual). Let $B_r^X(0) \equiv \{x \in X \mid \|x\|_X \leq r\}$ and let X_0 be a subspace of X, let $x_0 \in X_0$ be an element, then let $B_r^{X_0}(x_0) \equiv B_r^X(0) \cap X_0 + \{x_0\}$ be a closed ball of X_0.

Theorem 1.1. *Let X be a reflexive Banach space with strictly convex norm together with its dual space. Then the operator $A \in \mathbb{B}(X)$ possesses an eigenvector if and only if there exist numbers $r, \mu \neq 0$, a subspace X_0 of X and an element $x_0 \in X_0$ with $\|x_0\|_X > r > 0$ such that*

$$\mu A : B_r^{X_0}(x_0) \longrightarrow B_r^X(x_0), \qquad \mu A\left(B_r^{X_0}(x_0)\right) \cap X_0 \neq \varnothing \tag{1.1}$$

holds where $B_r^{X_0}(x_0)$ is a closed ball of X_0.

Further, we conduct a result about existence of an invariant subspace of a linear bounded operator without using a completely continuous operator.

2. Remarks on the separation of convex sets in a Banach space

We will cite the following known results (see [6, 12, 19, 20]) on the separation of convex sets.

Theorem 2.1. *Let $\Re^n$ $(n \geq 2)$ be n-dimensional Euclidian space and K_0, K_1 are nonempty convex sets in $\Re^n$. In order that there exists a hyperplane separating K_0 and K_1 properly, it is necessary and sufficient that the relative interiors $\operatorname{ri} K_0$ and $\operatorname{ri} K_1$ have no point in common, that is, $\operatorname{ri} K_0 \cap \operatorname{ri} K_1 = \varnothing$. In other words, K_0 and K_1 are properly separated if and only if there exists a vector $x_0 \in \Re^n$ such that*

$$\begin{aligned} \inf\{\langle x, x_0\rangle \mid x \in K_0\} &\geq \sup\{\langle x, x_0\rangle \mid x \in K_1\}, \\ \sup\{\langle x, x_0\rangle \mid x \in K_0\} &> \inf\{\langle x, x_0\rangle \mid x \in K_1\}. \end{aligned} \tag{2.1}$$

Further, in order that there exists a hyperplane separating these sets strongly, it is necessary and sufficient that there exists a vector $x_0 \in \Re^n$ such that

$$\inf\{\langle x, x_0\rangle \mid x \in K_0\} > \sup\{\langle x, x_0\rangle \mid x \in K_1\} \tag{2.2}$$

or

$$\inf\{\,|x_1 - x_2|\mid x_1 \in K_0,\ x_2 \in K_1\} > 0. \tag{2.3}$$

In other words, $0 \notin \operatorname{cl}(K_0 - K_1)$ *(i.e.,* 0 *is not in the closure of the set* $K_0 - K_1$*).*

The general result on the separation of convex sets in an infinite-dimensional space X has the following known formulation.

Theorem 2.2. *Let* K_0 *and* K_1 *be disjoint convex subsets of a linear space* X*, and let* K_0 *have an internal point. Then there exists a nonzero linear functional* f *which separates* K_0 *and* K_1*.*

In a linear topological space, any two disjoint convex sets, one of which has an interior point, can be separated by a nonzero continuous linear functional, that is, $K_0 \cap K_1 = \varnothing$, $\int K_0 \neq \varnothing$*, and there exists an element* $x_0^* \in X^*$ *such that*

$$\inf\{\langle x, x_0^*\rangle \mid x \in K_0\} \geq \sup\{\langle x, x_0^*\rangle \mid x \in K_1\}. \tag{2.4}$$

Moreover, if $K_0, K_1 \subset X$ *are open convex subsets in* X*, then they are strictly separated.*

If K_0 *and* K_1 *are disjoint closed convex subsets of a locally convex linear topological space* X*, and if* K_0 *is compact, then there exist constants* c *and* ε, $\varepsilon > 0$*, and a non-zero continuous linear functional* $x_0^* \in X^*$ *on* X*, such that*

$$\begin{gathered}\inf\{\langle x, x_0^*\rangle \mid x \in K_0\} \geq c > c - \varepsilon \geq \sup\{\langle x, x_0^*\rangle \mid x \in K_1\},\\ (\{\langle x, x_0^*\rangle \mid \forall x \in K_0\} \geq c > c - \varepsilon \geq \{\langle x, x_0^*\rangle \mid \forall x \in K_1\}).\end{gathered} \tag{2.5}$$

Now let X, Y be real Banach spaces and let X^*, Y^* be their dual spaces. Here and hereafter we will denote by X and Y reflexive Banach spaces with strictly convex norm together with their dual spaces X^*, Y^*. A Banach space X is called strictly convex [12, 21] if and only if $\|tx + (1-t)y\|_X < 1$ provided that $\|x\|_X = \|y\|_X = 1$, $x \neq y$, and $0 < t < 1$, consequently any point from the unit sphere $S_1^X(0)$ is an extremal point.

We begin by proving a result on the dual space of a subspace of a reflexive Banach space. We recall that a subset X_0 of a Banach space X is called a subspace of X if it is a linear closed subspace in X.

Proposition 2.3. *Let* X *and its dual space* X^* *be strictly convex reflexive Banach spaces, and let* $X_0 \subset X$ *be a subspace of* X*. Then the dual space of a subspace* $X_0 \subset X$ *is equivalent to a subspace of* X^* *which is determined by the subspace* X_0*, that is,* X^* *has a subspace* $X_0^* \subset X^*$ *defined by the unit sphere of the subspace* X_0 *and* $X_0^* \equiv (X_0)^*$*. Consequently,* X_0 *and its dual space* $(X_0)^*$ *are strictly convex reflexive Banach spaces under the norms induced by the norms of* X *and* X^**, respectively.*

Proof. It is known from [5, 22, 23] that the dual space of the subspace $X_0 \subset X$ is equivalent to a factor (quotient) space of the form $X^*/X_0^{\perp}$, where $X_0^{\perp} \subset X^*$ is the annihilator of $X_0 \subset X$:

$$X_0^{\perp} \equiv \{x^* \in X^* \mid \langle x, x^*\rangle = 0,\ \forall x \in X_0\}. \tag{2.6}$$

(Here the expression $\langle\cdot,\cdot\rangle$ denotes the dual form for the pair (X,X^*), or an inner product if X is a Hilbert space.) It is also known from [23] that the subspace $X_0 \subset X$ is a reflexive Banach space under the norm induced from X and that its dual space $(X_0)^*$ is also reflexive. Moreover, if X is a strictly convex reflexive Banach space, then so is X_0. In addition, by [22], if X is a strictly convex reflexive Banach space, then an arbitrary element of the unit sphere is an extremal point and the dual space $(X_0)^*$ is equivalent to a subspace of X^*. It remains, therefore, to identify this subspace.

In order to construct a dual subspace to X_0, we will consider the duality mapping $\mathfrak{I}: X \to X^*$ for the pair $(X;X^*)$, that is, $X \overset{\mathfrak{I}}{\longleftrightarrow} X^*$ (see, [8, 9, 20, 21, 24] and the references therein). In the case under consideration, the duality mapping is bijective and together with its inverse mapping is strictly monotone, surjective, odd, demicontinuous, bounded and coercive. Hence we have $\langle x,x^*\rangle \equiv \langle x^*,x\rangle$ for any $x \in X$, $x^* \in X^*$, and in particular for any $x \in X$ we have $x \leftrightarrow x^* = \mathfrak{I}(x)$, that is, it is an equivalence relation [5, 8, 22]. It follows from this that it will be enough to consider these mappings on the unit spheres of X and X^*.

We will denote the unit spheres of X and X^* by S_1^X and $S_1^{X^*}$, respectively. Then we have $\mathfrak{I}(S_1^X) \equiv S_1^{X^*}$. In addition, the following relations hold:

$$\begin{gathered}(\forall x)\big(x \in S_1^X \Longleftrightarrow \mathfrak{I}(x) = x^* \in S_1^{X^*}\big),\\ (\forall x)\big(x \in S_1^X \Longleftrightarrow \langle x,\mathfrak{I}(x)\rangle = \langle x,x^*\rangle = \|x\|_X \cdot \|x^*\|_{X^*} = 1\cdot 1\big),\end{gathered} \tag{2.7}$$

and conversely

$$(\forall x^*)\big(x^* \in S_1^{X^*} \Longleftrightarrow \langle x^*,\mathfrak{I}^{-1}(x^*)\rangle = \langle x^*,x\rangle = \|x^*\|_{X^*} \cdot \|x\|_X = 1\cdot 1\big), \tag{2.8}$$

since the duality mapping is a homeomorphism, by virtue of the conditions of the proposition (see, [8, 21] and the references therein). Moreover, the following relation holds:

$$\forall x \in X,\ \exists\, \tilde{x} \in S_1^X \quad \text{so that } x = \tilde{x}\|x\|_X. \tag{2.9}$$

Hence we have that the unit sphere S_1^X defines the whole space X in the sense that $X \equiv S_1^X \times \mathfrak{R}_+$; $\mathfrak{R}_+ \equiv \{\tau \in \mathfrak{R} : \tau \ge 0\}$.

Hence, if $X_0 \subset X$ is a subspace of X, then X_0 can be defined through a subset of the unit sphere of the form $S_1^{X_0} \equiv S_1^X \cap X_0 \equiv S_1^X(0) \cap X_0$. Here, $S_1^{X_0}$ denotes the unit sphere of X_0 with the norm induced from X. Thus, the space X_0 is a strictly convex reflexive Banach space. Consequently, there exists an equivalent norm such that X_0, together with its dual space, is a strictly convex reflexive Banach space. Under the induced topology—which we obtain by virtue of the duality mapping $\mathfrak{I}$ from X onto X^*—the sphere $S_1^{X_0}$ will be transformed onto a subset which can be expressed in the form

$$\widetilde{S}_1^* \equiv \{x^* \in S_1^{X^*} \mid \langle \mathfrak{I}^{-1}(x^*),x^*\rangle = \|x\|_X \cdot \|x^*\|_{X^*} = 1,\ \mathfrak{I}^{-1}(x^*) = x \in S_1^{X_0}\}, \tag{2.10}$$

because we have

$$\mathfrak{I}^{-1}(\widetilde{S}_1^*) \equiv S_1^{X_0}, \quad \forall x^* \in \widetilde{S}_1^* \subset S_1^{X^*} \Longleftrightarrow x = \mathfrak{I}^{-1}(x^*) \in S_1^{X_0}. \tag{2.11}$$

It is known that if X and X^* are strictly convex reflexive spaces, then the duality mapping $\Im : X \rightleftarrows X^* : \Im^{-1}$ is the Gateaux-differential of a strictly convex functional $\digamma$ and $\Im^{-1}$ is the Gateaux-differential of a strictly convex functional $\digamma^*$, that is, the duality mapping $X \overset{\Im}{\longleftrightarrow} X^*$ is a positively homogeneous potential operator with strictly convex potential. In addition, there is a strongly monotone increasing continuous function $\Phi : \Re_+ \to \Re_+$, $\Phi(0) = 0$, $\Phi(\tau) \nearrow +\infty$ when $\tau \nearrow +\infty$ such that $\Im(\tau x) = \Phi(\tau)x^*$ for any $x \in S_1^X$ and $x^* \in S_1^{X^*}$, where $\langle x, x^* \rangle \equiv 1$ and $\tau \in \Re_+$ [8]. Consequently, $\Im(B_1^{X_0})$ is a convex subset X^* (see also [5]). Thus we obtain that $\widetilde{S}_1^*$ defined by (2.10) is the unit sphere of the subspace $(X_0)^*$ from X^*, which we can denote by X_0^* (i.e., $(X_0)^* \equiv X_0^*$) that also is equivalent to $X^*/X_0^\perp$.

In other words we have obtained that X_0^* is equivalent to the dual space of the subspace X_0 of X, and so a subspace X_0 of a reflexive Banach space X is a reflexive Banach space under the induced topology under the conditions of the proposition. □

Note 2.4. It should be noted that the validity of results of this type also follows from results obtained by Phelps in [25] concerning the uniqueness of the extension of a linear functional to the whole of a Banach space.

Remark 2.5. We note that the annihilator of X_0^*, which is a subspace ${}^\perp X_0^*$ of X, is orthogonal to X_0, that is,

$$ {}^\perp X_0^* \equiv \{ y \in X \mid \|x + \lambda y\| \geq \|x\|, \ \forall x \in X_0, \ \forall \lambda \in [-1, 1] \}. \tag{2.12} $$

In other words, the subspace ${}^\perp X_0^*$ of X is generated by a subset $S_1^{{}^\perp X_0^*}$ of the sphere S_1^X which has the form

$$ S_1^{{}^\perp X_0^*} \equiv \{ y \in S_1^X \mid \|x + \lambda y\| \geq 1, \ \forall x \in S_1^{X_0}, \ \lambda = \pm 1 \}. \tag{2.13} $$

We will now show that if X is a reflexive Banach space which, together with its dual space X^* has a strictly convex norm, we may prove (under certain general conditions) certain generalizations of the results on separation of convex sets.

Theorem 2.6. *Let K_0 and K_1 be disjoint bounded convex subsets of a reflexive Banach space X which, together with its dual space X^*, has a strictly convex norm, and let K_0 have an internal point relative to the subspace $X_0 \subset X$,* $\operatorname{codim}_X X_0 \geq 1$. *Then there exists a nonzero linear continuous functional $x_0^* \in X^*$ which properly separates K_0 and K_1. That is,*

$$ \begin{aligned} &\inf \{ \langle x, x_0^* \rangle \mid x \in K_0 \} \geq \sup \{ \langle x, x_0^* \rangle \mid x \in K_1 \}, \\ &\sup \{ \langle x, x_0^* \rangle \mid x \in K_0 \} > \inf \{ \langle x, x_0^* \rangle \mid x \in K_1 \}. \end{aligned} \tag{2.14} $$

Proof. It is easy to see that $K_0 \subset X_0$, and that it has a nonempty interior relative to X_0. We will consider all possible cases with respect to the position of the sets K_0 and K_1, which are as follows:

(1) $K_1 \cap X_0 \equiv K_{10} \neq \varnothing$ (particular case, $K_1 \subset X_0$, i.e., $K_1 \equiv K_{10}$);

(2) $K_1 \cap X_0 \equiv \varnothing$.

First we will consider the subcase of case 1 for which $K_1 \subset X_0$, that is, $K_1 \equiv K_{10}$. We can study separation in this case with the help of Proposition 2.3 because we can see the subspace X_0 as a space X_0 by virtue of Proposition 2.3. Then, by using Theorem 2.2, we obtain the existence of a linear functional $x_0^* \in X_0^*$ which separates K_0 and K_{10}, and using the Hahn-Banach theorem we obtain an extension of this functional x_0^* to X (X^*) which is equal to x_0^* on X_0 because X (X^*) is a strictly convex Banach space.

Now assume that $K_1 \neq K_{10}$. Then in a similar way, we obtain the existence of a continuous linear functional $x_0^* \in X_0^*$ which separates the sets K_0 and K_{10} relative to the subspace X_0, that is,

$$\begin{gathered} \inf\{\langle x, x_0^*\rangle \mid x \in K_0\} \geq \sup\{\langle x, x_0^*\rangle \mid x \in K_{10}\}, \\ \sup\{\langle x, x_0^*\rangle \mid x \in K_0\} > \inf\{\langle x, x_0^*\rangle \mid x \in K_{10}\} \end{gathered} \tag{2.15}$$

hold. From this, we obtain that there exists $x_1 \in X_0$ such that

$$\sup\{\langle x, x_0^*\rangle \mid x \in K_{10}\} = \langle x_1, x_0^*\rangle = c_0, \tag{2.16}$$

since K_1 is a convex bounded set. Clearly, such an assertion is valid for any functional $x_0^* \in X_0^*$ which separates the sets K_0 and K_{10}. Since the linear functional x_0^* is defined on the subspace X_0, by the Hahn-Banach theorem, we can extend it to a continuous linear functional on the whole space X. Therefore, at least for the point $x_1 \in X$ determined by (2.16), we have a corresponding support hyperplane on this point separating the sets K_0 and K_1. In other words, if we will consider hyperplanes $\{L(x_0^*)\}$ which contain the hyperplane generated by the functional x_0^* relative to X_0, then there exists L_1 in $\{L(x_0^*)\}$ which separates the sets K_0 and K_1. If this is not so, then there would exist a point $\tilde{x}$ of K_{10} such that the relation $\langle \tilde{x}, x_0^*\rangle \leq c_0$ is not fulfilled, that is, $\tilde{x}$ is contained in the other half-space relative to the hyperplane generated by the functional x_0^*. This contradiction shows that the assertion of the theorem is valid in case 1.

Now we will consider case 2. Since the set K_1 is convex, there exist a subspace $\widehat{X} \subset X$, $\operatorname{codim}_X \widehat{X} = 1$, such that $K_0 \subset X_0 \subset \widehat{X}$ and the half-spaces $X_{\widehat{X}}^{\pm}$ generated by it are such that either $K_1 \subset \operatorname{cl} X_{\widehat{X}}^{+}$ or $K_1 \subset \operatorname{cl} X_{\widehat{X}}^{-}$. Indeed, if we assume that such a subspace does not exist, then we will obtain a contradiction with the condition that K_1 is a convex set [7]. We should note that the "induction" method (or applying Zorn's lemma) can be used for the proof of this proposition in the sense that we can choose a sequence of expanding subspaces in X which contain the subspace X_0 (as in [19, 2]). More exactly, if $X_1 \subset X$ is a subspace such that $X_0 \subset X_1$, $\operatorname{codim}_{X_1} X_0 = 1$, then it is not difficult to see that if $K_1 \cap X_1 = K_{11} \neq \varnothing$ then at least either $K_{11} \subset \operatorname{cl}(X_1)_{X_0}^{+}$ or $K_{11} \subset \operatorname{cl}(X_1)_{X_0}^{-}$ (since K_1 is a convex set). □

A subset K of X is called open relative to a subspace X_0 of X if for any element $x \in K$, there exists a neighborhood $U(x)$ from X such that $U(x) \cap X_0 \subset K$, and a subset K of X is called closed relative to the subspace X_0 of X if the complement $C_{X_0}K$ is open set relative to X_0. Consequently, if X_0 is a subspace of a Banach space X, if a set is closed relative to subspace X_0, it is closed with respect to X.

Theorem 2.7. *Let X be a space as in Theorem 2.6, and let K_0 and K_1 be disjoint bounded open convex sets relative to subspaces X_0 and X_1 of X, respectively, that is, $K_0 \subset X_0$ and*

$K_1 \subset X_1$ ($\operatorname{codim}_X X_0 \geq 1$, $\operatorname{codim}_X X_1 \geq 1$). *Then K_0 and K_1 are strictly separated, that is, there exists an element $x_0^* \in X^*$ such that*

$$\{\langle x, x_0^* \rangle \mid \forall x \in K_1\} > \{\langle x, x_0^* \rangle \mid \forall x \in K_0\}. \tag{2.17}$$

Proof. We will consider all possible cases separately, as in the proof of Theorem 2.6. These cases have the following form:

(1) $K_0 \subset X_0$ and $K_1 \subset X_0$, that is, $X_0 \equiv X_1$ (the subspaces or hyperplanes X_0, X_1 are the same);

(2) $K_0 \subset X_0$ and $K_1 \cap X_0 = \varnothing$;

(3) $K_0 \subset X_0$ and $K_1 \cap X_0 = K_{10} \neq \varnothing$.

Case 1 follows from Proposition 2.3 and Theorem 2.2, therefore we will consider the remaining cases.

Separation of the sets considered in the remaining cases follows from Theorem 2.6. So, we must show that this separation is strict. Thus we assume that the sets K_0 and K_1 are open relative to the subspaces X_0 and X_1 of X, respectively, and we will consider case 2. For the proof in this case, we will use the theorem of Kakutani and Tukey [23]. We obtain with the help of these results that there exist two convex sets K_{00} and K_{11} such that $K_{00} \cap K_{11} = \varnothing$, $K_0 \subset K_{00}$, $K_1 \subset K_{11}$, and $K_{00}, K_{11} \subset X$. Then if we choose a set K_{00} such that K_{00} is a bounded open convex set of X, for example as $K_{11} = K_1$, then we can use a well-known result (Theorem 2.2). From here the statement of the theorem follows.

For the proof of case 3, one may use the proof of Theorem 2.6 and cases 1, 2. Thus we obtain the validity of Theorem 2.7. □

Note 2.8. The above theorems remain correct if we replace one of the subspaces X_0 and X_1 with a closed hyperplane. In this case, for example if $X_1 \equiv L$ is a closed hyperplane and $K_1 \subset L$, then $K_1 - x_0$ with $X_1 - x_0$ satisfies the condition of the theorem.

3. Some fixed-point theorems

Let X, Y and their dual spaces X^*, Y^* be strictly convex reflexive Banach spaces. We will consider a general mapping f acting from X into Y and investigate when the image of a certain set under this mapping contains zero. It is clear that this result is equivalent to the existence theorem for inclusion $y \in f(x)$. Moreover, if $Y = X$, we will investigate when this mapping f has a fixed point in some set from X. Here we will consider variants of the fixed-point theorems of the type proved earlier in [1]. Other results of this type may be proved analogously as in the papers mentioned above.

Specifically, let $f : D(f) \subseteq X \to Y$ be a bounded mapping (i.e., if $G \subseteq D(f)$ is the bounded subset of X, then $f(G)$ is a bounded subset of Y) which may be multivalued or discontinuous, and let B_1^Y and S_1^Y be the unit ball and unit sphere from Y, respectively. We will consider the following conditions. Let $G \subseteq D(f)$ be a bounded subset and

(i) there exists a subspace Y_0 of Y with $\operatorname{codim}_Y Y_0 \geq 1$ such that $f(G) \cap Y_0 \equiv f_{Y_0}(G)$ is an nonvoid open (or closed) convex set relative to the subspace Y_0;

(ii) for any $y^* \in S_1^{Y_0^*} \equiv S_1^{Y^*} \cap Y_0^*$, there exists $x \in G$ satisfying the inequality

$$\langle f_{Y_0}(x), y^* \rangle \cap \overline{\mathfrak{R}^+} \neq \varnothing, \quad \overline{\mathfrak{R}^+} \equiv \{\tau : \tau \geq 0\}, \tag{3.1}$$

and also

(i_1) there exists a subspace Y_0 of Y with $\operatorname{codim}_Y Y_0 \geq 1$ such that $f_{Y_0}(G)$ is a convex set with nonvoid internal relative to the subspace Y_0;

(ii_1) for any $y^* \in S_1^{Y_0^*}$, there exists $x \in G$ satisfying the inequality $\langle f_{Y_0}(x), y^* \rangle \cap (\overline{\mathfrak{R}^+} \setminus \{0\}) \neq \varnothing$, for a dual form of the pair (Y_0, Y_0^*).

Theorem 3.1. *Let $f : D(f) \subseteq X \to Y$ be a bounded mapping, and let Y and its dual space Y^* be reflexive Banach spaces with a strictly convex norm. Assume that on a bounded subset $G \subseteq D(f)$, f satisfies conditions (i), (ii) or conditions (i_1), (ii_1).*

Then there exists $x_0 \in G$ such that $0 \in f(x_0)$, that is, $0 \in f(G)$.

Proof. Let $f(G)$ be an open (or closed) convex set relative to the subspace Y_1. For the proof, it is sufficient to note that here we can use the separation theorem from the previous section. For this, we will consider the sets $f_{Y_0}(G)$ and $\{0\}$, and prove the result by *reductio ad absurdum*. Then it is enough to note that all the conditions of Theorem 2.6 (or of Theorem 2.2) are fulfilled relative to the pair (Y_0, Y_0^*). Consequently, we obtain the correctness of Theorem 3.1 with the aid of Theorem 2.7. □

The next corollary immediately follows from Theorem 3.1.

Corollary 3.2 (fixed-point theorem). *Let the mapping $f : D(f) \subseteq X \to X$ be a bounded mapping and let the space X be such as the space Y in Theorem 3.1. Assume that on a subset $G \subseteq D(f)$, the mapping f_0 defined by $f_0(x) \equiv x - f(x)$ for any $x \in G$ satisfies conditions (i), (ii) or (i_1), (ii_1) in the case when $Y \equiv X$ and $Y_0 \equiv X_0$, respectively.*

Then there exists $x_0 \in G$ such that $x_0 \in f(x_0)$, that is, the mapping f possesses a fixed point in the subset G.

For the proof, it is sufficient to note that under the conditions of the corollary, the mapping f_0 satisfies the conditions of Theorem 3.1. Consequently $0 \in f_0(G)$.

In particular, if the set $G \subseteq D(f)$ is a closed ball $B_r^X(x_0)$ centered at a point $x_0 \in X_0$ and having radius $r > 0$ for a subspace X_0 of X, then we can formulate this corollary in the following form (other results of such type exist in [3]). This may be proved using the duality mapping $\mathfrak{I} : X \overset{\mathfrak{I}}{\longleftrightarrow} X^*$. It is known from [8, 11, 23] that if a Banach space X is as above, then there exists a duality mapping $\mathfrak{I} : X \overset{\mathfrak{I}}{\longleftrightarrow} X^*$ which is a demicontinuous strictly monotone operator together with its inverse mapping.

Corollary 3.3. *Let $f : D(f) \subseteq X \to X$ and X be as in Corollary 3.2, and let $B_r^{X_1}(x_0) \subseteq D(f)$ be some ball with a point x_0 of X_1. Assume that $f(B_r^{X_1}(x_0)) \subseteq B_r^X(x_0)$ and that the mapping f_0 is such that $f_{0X_1}(B_r^{X_1}(x_0))$ is an open (or closed) convex set relative to the subspace X_1 from X, where $f_0(x) \equiv x - f(x)$ for any $x \in B_r^{X_1}(x_0)$, $f_{0X_1}(x) \equiv f_0(x) \cap X_1$, $B_r^{X_1}(x_0) \equiv B_r^X(x_0) \cap X_1$ and $\operatorname{codim}_X X_1 \geq 1$. Then f possesses a fixed point in $B_r^X(x_0)$, that is, there exists $\widetilde{x} \in B_r^X(x_0)$ such that $\widetilde{x} \in f(\widetilde{x})$.*

For the proof, it is enough to show that the necessary inequality is true for any $\overline{x} \in S_1^{X_1}$, which has the form

$$\begin{aligned}\langle f_{0X_0}(x_0 + r\overline{x}), \Im(\overline{x})\rangle &\equiv \langle x_0 + r\overline{x} - f(x_0 + r\overline{x}), \Im(\overline{x})\rangle \\ &= \langle r\overline{x}, \Im(\overline{x})\rangle - \langle f(x_0 + r\overline{x}) - x_0, \Im(\overline{x})\rangle \\ &\geq r - \|f(x_0 + r\overline{x}) - x_0\| \geq 0.\end{aligned} \tag{3.2}$$

Let X, Y be Banach spaces as above, and let $f : D(f) \subseteq X \to Y$ be a mapping which may be multivalued or discontinuous. Let B_1^Y and S_1^Y be the unit ball and unit sphere from Y, respectively. We will conduct results on the solvability of inclusion $y \in f(x)$ and a fixed-point theorem that is used in the following sections.

4. About completeness of the image of a set under a linear mapping

In beginning, we will prove the following result.

Lemma 4.1. *Let X and Y satisfy the above conditions and $A \in \mathbb{B}(X, Y)$. Then the image of each closed bounded convex subset of X under operator A will be a closed bounded convex subset of Y.*

Proof. It is known from [12, 19] that in the conditions of the lemma, the operator A is weakly compact. Let $K \subset X$ be a bounded closed convex set and $A(K) = M \subset Y$. It is easy to see that M is a bounded convex set of Y. So it remains to show that M is a closed set.

Let $\{y_m\} \subset M$ be a fundamental sequence (if the space is not separable, then we will consider a general sequence but here for simplicity we will not conduct this case). Then there exists $y_0 \in Y$ such that $\lim_{m\to\infty} y_m = y_0$.

We will consider an inverse image of the sequence $\{y_m\} \subset M$ from K and denote it by $\{x_m\}$. It is clear that, generally, the inverse image is a set of the form $\{x_m + \ker A\} \subset X$. Therefore, we must consider the set $\{x_m + \ker A\} \cap K$. Then there exists a subsequence $\{x_{m_k}\} \subset \{x_m + \ker A\} \cap K$ such that $x_{m_k} \rightharpoonup x_0$ weakly in X for some $x_0 \in X$ by virtue of reflexivity of the space X and boundedness of the set $\{x_m + \ker A\} \cap K$. From here, follows that $x_{m_k} \rightharpoonup x_0 \in K$ weakly in X by virtue of completeness and convexity of set K [23, 19].

Thus the sequence $\{A(x_{m_k})\}$ converges weakly in Y, furthermore $A(x_{m_k}) \rightharpoonup A(x_0)$ weakly in Y because A is weakly compact. On the other hand, we have $A(x_{m_k}) = y_{m_k}$ and $y_{m_k} \to y_0 \in Y$ in Y by assumption. From here, it follows that $A(x_0) = y_0$, consequently $y_0 \in M$.

So we have shown that if $K \subset X$ is a bounded closed convex subset, then so is $A(K) = M$ in Y. □

Corollary 4.2. *Under the conditions of the previous lemma, an affine mapping with the mentioned linear operator satisfies the statement of this lemma.*

The proof is obvious.

5. On existence of an eigenvector of a linear bounded operator

Let X be a Banach space such as above, and let $A \in \mathbb{B}(X)$, X_0 be a closed subspace of X.

LEMMA 5.1. *Let $A \in \mathbb{B}(X)$, $A \neq 0$, and there exist a closed subspace X_0 of X and a closed ball $B_r^{X_0}(x_0) \subset X_0$, $0 \notin B_r^{X_0}(x_0)$ with a radius $r > 0$ and a center $x_0 \in X_0$ such that for a $\mu \neq 0$, the expressions $\mu A : B_r^{X_0}(x_0) \to B_r^X(x_0) \subset X$ holds, also $\mu A(B_r^{X_0}(x_0)) \cap X_0 \neq \varnothing$. Then the operator A has a nontrivial eigenvector in the ball $B_r^X(x_0)$, that is, there exists $x_1 \in B_r^X(x_0) \cap X_0$ and $\lambda_1 \in \sigma(A)$ such that $Ax_1 = \lambda_1 x_1$.*

Proof. We will consider a mapping $f : X \to X$ defined in the form

$$f(x) \equiv x - \mu Ax + x_0 - \mu Ax_0 = x - \left[\mu A(x + x_0) - x_0\right] \equiv x - A_1 x. \tag{5.1}$$

From the condition, it is easy to see that

$$\mu A : B_r^{X_0}(x_0) \longrightarrow B_r^X(x_0) \Longrightarrow A_1 : B_r^{X_0}(0) \longrightarrow B_r^X(0) \tag{5.2}$$

holds and A_1 is an affine mapping.

Further, $f(K)$ is a convex subset of X for any convex subset K from X. We will show that if $x_1, x_2 \in K \subseteq X$ are arbitrary elements and $\alpha \in R^1$, $0 \leq \alpha \leq 1$, then $\alpha f(x_1) + (1 - \alpha) f(x_2) \in f(K)$. In fact,

$$\begin{aligned} y \equiv \alpha f(x_1) + (1-\alpha) f(x_2) &= \alpha x_1 - \alpha A_1 x_1 + (1-\alpha)(x_2 - A_1 x_2) \\ &= \alpha x_1 + (1-\alpha)x_2 - \left[\alpha A_1 x_1 + (1-\alpha)A_1 x_2\right] = \alpha x_1 + (1-\alpha)x_2 + x_0 \\ &\quad - \mu A(\alpha x_1 + (1-\alpha)x_2) - \mu A x_0 = x - \left[\mu A(x + x_0) - x_0\right] = f(x), \end{aligned} \tag{5.3}$$

here $x = \alpha x_1 + (1-\alpha)x_2 \in K$. Consequently, $f(x) = y \in f(K)$ by virtue of convexity of K. Thus we have that $f(B_r^{X_0}(0))$ is a convex subset of X.

From boundedness of the operator A, it follows that the image $f(B_r^X(0))$ is a bounded subset of X, that is, the inequality

$$\|f(x)\|_X \leq \|x\|_X + \|x_0\|_X + \|\mu A(x + x_0)\|_X \leq C(|\mu|, \|A\|)(r + \|x_0\|_X) \tag{5.4}$$

holds for any $x \in B_r^X(0)$ where $C(|\mu|, \|A\|) > 0$ is a number. Thus, using Corollary 3.3, we obtain that $f(B_r^X(0))$ is a bounded closed convex set of X.

Hence, the mapping f on the ball $B_r^{X_0}(0)$ satisfies all conditions of Theorem 3.1 (in particular, A_1 satisfies all conditions of Corollary 3.3) by virtue of the conditions of Lemma 5.1 and

$$\begin{aligned} \langle f(x), \Im(x)\rangle &= \langle x, \Im(x)\rangle - \langle A_1 x, \Im(x)\rangle = \langle x, \Im(x)\rangle \\ &\quad - \langle \mu A(x_0 + x) - x_0, \Im(x)\rangle \geq \|x\|_X\left(\|x\|_X - \|\mu A(x_0 + x) - x_0\|_X\right) \geq 0 \end{aligned} \tag{5.5}$$

holds for any $x \in S_r^{X_0}(0)$, where $\Im : X \leftrightarrows X^*$ is a duality mapping which in this case is a homeomorphism.

Consequently, we obtain using Corollary 3.3 that there exists $\widetilde{x} \in B_r^{X_0}(0)$ such that $A_1\widetilde{x} = \widetilde{x}$, that is, $\mu A(x_0 + \widetilde{x}) = x_0 + \widetilde{x}$. The last equality shows that the obtained element $x_0 + \widetilde{x}$ is an eigenvector of the operator A with respect to the eigenvalue $\lambda = \mu^{-1}$. (Obviously, $\mu^{-1} \leq \|A\|_{X\to X}$.) □

We must note that when $X_0 \equiv X$ this lemma follows also from the Tychonov-Schauder fixed-point theorem as the operator A is weakly compact.

The following statement immediately follows from Lemma 5.1.

Corollary 5.2. *Let X be as in Lemma 5.1, $A \in \mathbb{B}(X)$, $A \neq 0$, and $\widetilde{x}_0$ let be a nonzero element of X_0. Then there exist numbers $\mu \neq 0$ and $r > 0$ such that the mapping $f_0 : f_0(x) \equiv \mu(Ax + \widetilde{x}_0)$, for all $x \in X$, possesses a fixed point in the closed ball $B_r^X(0) \subset X$.*

Proof. For the proof, we must show that there is a closed ball $B_r^X(0) \subset X$ with radius $r > 0$ such that the mapping $f_0(x)$ satisfies the inequality $\|f_0(x)\|_X \leq r$ for any $x \in B_r^{X_0}(0)$, that is, we must find a number $r(\mu) > 0$. Such number exists under the conditions of the corollary. In fact, we have

$$||f_0(x)||_X = ||\mu(Ax + \widetilde{x}_0)||_X \leq |\mu|\left(\|A\|_{X_0\longrightarrow X}r + ||\widetilde{x}_0||_X\right) \tag{5.6}$$

for any $x \in B_r^{X_0}(0)$. For fulfilment of the inequality, $\|f_0(x)\|_X \leq r$ is enough for

$$r\|A\|_{X_0\to X} + ||\widetilde{x}_0||_X \leq \frac{r}{|\mu|} \tag{5.7}$$

to hold, and we have

$$r \geq |\mu|||\widetilde{x}_0||_X\left(1 - |\mu|\|A\|_{X_0\to X}\right)^{-1} \quad \text{or} \quad |\mu|^{-1} > \|A\|_{X_0\to X}. \tag{5.8}$$

Hence the necessary ball is found. Further, since the mapping f_0 satisfies all conditions of Corollary 3.3, we can apply this result to the considered case. Then we obtain Corollary 5.2 using Corollary 3.3, in other words, there exists a point $x_1 \in B_r^X(0)$ such that $f_0(x_1) = x_1$, that is, $Ax_1 + \widetilde{x}_0 = \mu^{-1}x_1$. □

Lemma 5.3. *Let X and the operator $A \in \mathbb{B}(X)$ be such as in Lemma 5.1, furthermore A possesses a nontrivial eigenvector x_{λ_0} corresponding to the eigenvalue $\lambda_0 : |\lambda_0| \leq \|A\|$. Then there exist a subspace X_0 of X, a nonzero element $x_0 \in X_0$, and numbers μ, r such that $\mu \neq 0$, $0 < r < \|x_0\|_X$, and the following relation holds:*

$$\mu A : B_r^{X_0}(x_0) \longrightarrow B_r^X(x_0), \qquad \mu A\left(B_r^{X_0}(x_0)\right) \cap X_0 \neq \varnothing. \tag{5.9}$$

Proof. Let $x_0 = x_{\lambda_0}$. Then for the proof, it is sufficient to show that there exist a needed subspace X_0 of X and numbers $\mu \neq 0$, $r > 0$, which are found by the following way.

We assume the existence of a subspace X_0 such that $\|A\|_{X_0\to X} \leq |\lambda_0|$. It is clear that such subspace X_0 exists (we can choose X_0 as a subspace over the eigenvector x_0, at least). Let $r > 0$ be a number such that $r < \|x_0\|$, then we have

$$\begin{aligned} ||\mu Ax - x_0||_X &= ||\mu Ax - \lambda_0^{-1}Ax_0||_X \\ &\leq ||A(\mu x - \lambda_0^{-1}x_0)||_X \leq \left|\lambda_0\right|^{-1}\|A\|_{X_0\to X}||\mu\lambda_0 x - x_0||_X \end{aligned} \tag{5.10}$$

for any $x \in B_r^{X_0}(x_0)$, where $B_r^{X_0}(x_0) \subset X$ is a closed ball. From (5.10), it follows that μ must be such that

$$|\lambda_0|^{-1}\|A\|_{X_0\to X}\|\mu\lambda_0 x - x_0\|_X \le r \quad \text{or} \quad \|\mu\lambda_0 x - x_0\|_X \le r. \tag{5.11}$$

Then it is sufficient to choose μ as $\mu = \lambda_0^{-1}$, because in this case inequality (5.11) holds for for all $x \in B_r^{X_0}(x_0)$. The assertion follows from here. □

We obtain the following theorem from Lemmas 5.1 and 5.3.

Theorem 5.4. *Let X be a Banach space such as above. Then an operator $A \in \mathbb{B}(X)$ possesses a nontrivial invariant subspace (an eigenvector, at least) if and only if there exist numbers μ, r, a subspace X_0 of X, and an element $x_0 \in X_0$ such that $x_0 \neq 0$, $0 < r < \|x_0\|_X$, $\mu \neq 0$ and*

$$\mu A : B_r^{X_0}(x_0) \longrightarrow B_r^X(x_0), \qquad \mu A\big(B_r^{X_0}(x_0)\big) \cap X_0 \neq \varnothing \tag{5.12}$$

hold for the closed ball $B_r^{X_0}(x_0)$.

Remark 5.5. It is easy to see that Theorem 5.4 is correct for a linear compact operator in the case of an arbitrary Banach space.

6. Some remarks on existence of the invariant subspace

Let X be a Banach space such as above and $A \in \mathbb{B}(X)$, and let $\mathbb{B}_A(X)$ be a subset of $\mathbb{B}(X)$ of operators that are commuting with A. It is obvious that $\mathbb{B}_A(X) \neq \varnothing$.

Since $\mathbb{B}_A(X)$ contains an operator satisfying the conditions of Theorem 5.4, then A possesses an invariant subspace in X, we will consider the case when this is not known.

So let $x_0 \neq 0$ be an element of X, and let $B_r(x_0) \subset X$ be a closed ball such that $0 < r < \|x_0\|_X$. As known [16] (see, also [17]), there exist operators $A_\beta \in \mathbb{B}_A(X)$, $\beta \in I \subset R^1$ such that $A_\beta(B_r(x_0)) \cap B_r(x_0) \neq \varnothing$, which can be shown by the same way. From Section 4, it follows that $A_\beta(B_r(x_0)) \cap B_r(x_0)$ is closed for each $\beta \in I$. Let

$$\begin{aligned} \mathbb{B}_0 &\equiv \{A_\beta \in \mathbb{B}_A(X) \mid A_\beta(B_r(x_0)) \cap B_r(x_0) \neq \varnothing,\ \beta \in I\}, \\ V_{A_\beta} &\equiv \{x \in B_r(x_0) \mid A_\beta \in \mathbb{B}_0,\ A_\beta(x) \in B_r(x_0)\}, \quad \beta \in I. \end{aligned} \tag{6.1}$$

It is clear that if $A_\beta \in \mathbb{B}_0$, then $\mu A_\beta \in \mathbb{B}_0$ also for some numbers μ, moreover we can choose these numbers such that

$$\mu A_\beta(B_r(x_0)) \cap B_r(x_0) \supseteq A_\beta(B_r(x_0)) \cap B_r(x_0). \tag{6.2}$$

So we will select μ_β as

$$\begin{aligned} \mu_\beta : \mu_\beta A_\beta(B_r(x_0)) \cap B_r(x_0) \\ = \sup\{\mu A_\beta(B_r(x_0)) \cap B_r(x_0) \mid \mu A_\beta(B_r(x_0)) \cap B_r(x_0) \supseteq A_\beta(B_r(x_0)) \cap B_r(x_0)\}. \end{aligned} \tag{6.3}$$

Thus we assume that $x_0 \in X$, $x_0 \neq 0$. Further, we regard μ_β selected such that the relation (6.3) holds, therefore we choose only one of such operators and define it as A_β.

Under these assumptions, we have $\bigcup_{A_\beta \in \mathbb{B}_0} V_{A_\beta} = B_r(x_0)$, because otherwise as known (see [16, 17], etc.), operators from $\mathbb{B}_0$ have invariant subspace.

It is easy to see that if $A_1, A_2 \in \mathbb{B}_0$ then $\alpha_1 A_1 + \alpha_2 A_2 \in \mathbb{B}_0$ for some numbers $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, besides the operator $\alpha_1 A_1 + \alpha_2 A_2 \equiv \widetilde{A}$ such that there exists $\widetilde{x} \in B_r(x_0) \cap V_{\widetilde{A}}$ for which $\widetilde{x} \notin V_{A_1} \cup V_{A_2}$ holds. For example, if $V_{A_1} \cap V_{A_2} = \varnothing$ then some convex subset of a convex hull on $V_{A_1} \cup V_{A_2}$ will be contained in $V_{\widetilde{A}}$. This shows that with use of this method, we can construct operators from $\mathbb{B}_0$ by using the operators from $\mathbb{B}_0$ for which $V_{\widetilde{A}} \subseteq B_r(x_0)$.

Let $\{A_\beta \mid \beta \in I_0 \subset I\}$ be a minimal subset of $\mathbb{B}_0$ for which

$$\bigcup_{\beta \in I_0} V_{A_\beta} = B_r(x_0) \tag{6.4}$$

(the number I_0 of such operators for which (6.4) takes place may be finite).

Now, we define the following mapping:

$$f(x) \equiv \left\{ \bigcup A_\beta x \mid \beta \in I_0 \right\}, \quad x \in \bigcup V_{A_\beta}, \tag{6.5}$$

where $\bigcup A_\beta x$ is the union of an image of the operators A_β for which $x \in V_{A_\beta}$. Obviously, f is a multivalued mapping (generally speaking) and $f(B_r(x_0)) \subseteq B_r(x_0)$. Therefore, we will consider the mapping $f_1 : f_1(x) \equiv x - f(x)$ for any $x \in B_r(x_0)$, that is, for any $x \in \bigcup_{\beta \in I_0} V_{A_\beta}$.

So, we consider the following condition.

(1) *Assume that the mapping f defined in (6.5) is such that there exist a subspace X_0 of X and a closed ball $B_r^{X_0}(x_0)$ on which $f_1(B_r^{X_0}(x_0)) \cap X_0$ is a convex closed (or open) subset of X_0.*

Theorem 6.1. *Let X be a Banach space as above and $A \in \mathbb{B}(X)$ is such that there exist an element $x_0 \in X$, a number $r : \|x_0\|_X > r > 0$, and a subset $\{A_\beta \in \mathbb{B}_0 \subset \mathbb{B}_A(X) \mid \beta \in I_0\}$ for which the mapping f defined in (6.5) satisfies condition 1. Then the operator A possesses an invariant subspace.*

The proof of the theorem follows from Corollary 3.3 as all conditions of Corollary 3.3 hold in this case. In fact with using Corollary 3.3, we obtain that in the class $\mathbb{B}_A(X)$ there exists an operator which possesses an eigenvector in the ball $B_r(x_0)$. The theorem follows from here.

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