DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **9**, Number **3**, June **2016**

doi:10.3934/dcdss.2016021

pp. **675–685**

1-DIMENSIONAL HARNACK ESTIMATES

Fatma Gamze Düzgün

Hacettepe University
06800, Beytepe, Ankara, Turkey

Ugo Gianazza

Dipartimento di Matematica "F. Casorati"
Università di Pavia
via Ferrata 1, 27100 Pavia, Italy

Vincenzo Vespri

Dipartimento di Matematica e Informatica "U. Dini"
Università di Firenze
viale Morgagni 67/A, 50134 Firenze, Italy

Dedicated to the memory of our friend Alfredo Lorenzi

Abstract. Let u be a non-negative super-solution to a 1-dimensional singular parabolic equation of p-Laplacian type ($1 < p < 2$). If u is bounded below on a time-segment $\{y\} \times (0, T]$ by a positive number M, then it has a power-like decay of order $\frac{p}{2-p}$ with respect to the space variable x in $\mathbb{R} \times [T/2, T]$. This fact, stated quantitatively in Proposition 1.2, is a "sidewise spreading of positivity" of solutions to such singular equations, and can be considered as a form of Harnack inequality. The proof of such an effect is based on geometrical ideas.

1. **Introduction.** Let $E = (\alpha, \beta)$ and define $E_{-\tau_o,T} = E \times (-\tau_o, T]$, for τ_o, $T > 0$. Consider the non-linear diffusion equation

$$u_t - (|u_x|^{p-2}u_x)_x = 0, \qquad 1 < p < 2. \tag{1.1}$$

A function

$$u \in C_{\mathrm{loc}}\big(-\tau_o, T; L^2_{\mathrm{loc}}(E)\big) \cap L^p_{\mathrm{loc}}\big(-\tau_o, T; W^{1,p}_{\mathrm{loc}}(E)\big) \tag{1.2}$$

is a local, weak super-solution to 1.1, if for every compact set $K \subset E$ and every sub-interval $[t_1, t_2] \subset (-\tau_o, T]$

$$\int_K u\varphi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \big[-u\varphi_t + |u_x|^{p-2}u_x\,\varphi_x\big]dxdt \ge 0 \tag{1.3}$$

for all non-negative test functions

$$\varphi \in W^{1,2}_{\mathrm{loc}}\big(-\tau_o, T; L^2(K)\big) \cap L^p_{\mathrm{loc}}\big(-\tau_o, T; W^{1,p}_o(K)\big).$$

This guarantees that all the integrals in 1.3 are convergent. These equations are termed singular since, for $1 < p < 2$, the modulus of ellipticity $|u_x|^{p-2} \to \infty$ as $|u_x| \to 0$.

2010 *Mathematics Subject Classification.* Primary: 35K65, 35B65; Secondary: 35B45.
Key words and phrases. Singular diffusion equations, p-laplacian, expansion of positivity.

Remark 1.1. Since we are working with *local* solutions, we consider the domain $E_{-\tau_o,T} = E\times(-\tau_o,T]$, instead of dealing with the more natural $E_T = E\times(0,T]$, in order to avoid problems with the initial conditions. The only role played by $\tau_o > 0$ is precisely to get rid of any difficulty at $t = 0$, and its precise value plays no role in the argument to follow.

Proposition 1.2. *Let u be a non-negative, local, weak super-solution to 1.1 in $E_{-\tau_o,T}$, in the sense of 1.2–1.3, satisfying*

$$u(y,t) > M \quad \forall t\in(0,\frac{T}{2}] \tag{1.4}$$

for some $y\in E$, and for some $M>0$. Let $\bar\rho \overset{def}{=} \left(\frac{2^{2-p}T}{M^{2-p}}\right)^{\frac{1}{p}}$, take $\rho\ge 4\bar\rho$, and assume that

$$B_\rho(\bar x)\subset B_{4\rho}(y)\subset E, \qquad \textit{where}\ \ \mathrm{dist}(\bar x,y)=2\rho.$$

There exists $\bar\sigma\in(0,1)$, that can be determined a priori, quantitatively only in terms of the data, and independent of M and T, such that

$$u(x,t)\ge\bar\sigma M\left(\frac{\bar\rho}{\rho}\right)^{\frac{p}{2-p}} \qquad \textit{for all}\ \ (x,t)\in B_{\frac{\rho}{4}}(\bar x)\times[\frac{T}{4},\frac{T}{2}] \tag{1.5}$$

Remark 1.3. Strictly speaking, it might not be possible to satisfy the assumption

$$\rho\ge 4\bar\rho \qquad \text{and} \qquad B_{4\rho}(y)\subset E,$$

if E were too small: nevertheless, we can always assume it without loss of generality. Indeed, if it were not satisfied, we would decompose the interval $(0,\frac{T}{2}]$ in smaller subintervals, each of width τ, such that the previous requirement is satisfied working with $\bar\rho$ replaced by

$$\widehat{\rho}=\left(\frac{2^{2-p}\tau}{M^{2-p}}\right)^{\frac{1}{p}}.$$

1.1. **Novelty and significance.** The measure theoretical information on the "positivity set" in $\{y\}\times(0,\frac{T}{2}]$ implies that such a positivity set actually "expands" sidewise in $\mathbb{R}\times[\frac{T}{4},\frac{T}{2}]$, with a power-like decay of order $\frac{p}{2-p}$ with respect to the space variable x. Although considered a sort of natural fact, to our knowledge this result has never been proven before; it is the analogue of the power-like decay of order $\frac{1}{p-2}$ with respect to the time variable t, known in the degenerate setting $p>2$ (see [2], [3, Chapter 4, Section 4], [7]). As the $t^{-\frac{1}{p-2}}$-decay is at the heart of the Harnack estimate for $p>2$, so Proposition 1.2 could be used to give a more streamlined proof of the Harnack inequality in the singular, super-critical range $\frac{2N}{N+1}<p<2$. This will be the object of future work, where we plan to address the general N-dimensional case.

The proof is based on geometrical ideas, originally introduced in two different contexts: the energy estimates of § 2 and the decay of § 3 rely on a method introduced in [8] in order to prove the Hölder continuity of solutions to an anisotropic elliptic equation, and further developed in [5, 6]; the change of variable used in the actual proof of Proposition 1.2 was used in [4].

1.2. **Further generalization.** Consider partial differential equations of the form

$$u_t - (\mathbf{A}(x,t,u,u_x))_x = 0 \qquad \text{weakly in } E_{-\tau_o,T}, \tag{1.6}$$

where the function $\mathbf{A} : E_{-\tau_o,T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is only assumed to be measurable and subject to the structure condition

$$\begin{cases} \mathbf{A}(x,t,u,u_x)\, u_x \ge C_o |u_x|^p \\ |\mathbf{A}(x,t,u,u_x)| \le C_1 |u_x|^{p-1} \end{cases} \qquad \text{a.e. in } \; E_{-\tau_o,T}, \tag{1.7}$$

where $1 < p < 2$, C_o and C_1 are given positive constants. It is not hard to show that Proposition 1.2 holds also for weak super-solutions to 1.6–1.7, since our proof is entirely based on the structural properties of 1.1, and the explicit dependence on u_x plays no role. However, to keep the exposition simple, we have limited ourselves to the prototype case.

2. **Energy estimates.** Let u be a non-negative, local, weak super-solution in $E_{-\tau_o,T}$, set

$$0 \le \mu_- = \inf_{E_{-\tau_o,T}} u,$$

and let $0 < \omega < +\infty$. Without loss of generality we may assume that $0 \in (\alpha, \beta)$. For ρ sufficiently small, so that $(-\rho,\rho) \subset (\alpha,\beta)$, let

$$\begin{aligned}
&B_\rho = (-\rho,\rho), \qquad Q = B_\rho \times (0,T],\\
&B_\rho(y) = (y-\rho, y+\rho), \qquad Q(y) = B_\rho(y) \times (0,T],\\
&a \in (0,1), \quad H \in (0,1] \quad \text{parameters that will be fixed in the following},\\
&A = \{(x,t) \in Q(y) : \; u(x,t) < \mu_- + (1-a)H\omega\},\\
&A(\tau) = \{x \in B_\rho(y) : \; u(x,\tau) < \mu_- + (1-a)H\omega\}, \quad 0 \le \tau \le T.
\end{aligned}$$

Proposition 2.1. *Let u be a non-negative, local, weak super-solution to 1.1 in $E_{-\tau_o,T}$, in the sense of 1.2–1.3. There exists a positive constant $\gamma = \gamma(p)$, such that for every cylinder $Q(y) = B_\rho(y) \times (0,T] \subset E_{-\tau_o,T}$, and every piecewise smooth, cutoff function ζ vanishing on $\partial B_\rho(y)$, such that $0 \le \zeta \le 1$, and $\zeta_t \le 0$,*

$$\begin{aligned}
&\int_{B_\rho(y)\cap\{u(x,0)<\mu_-+(1-a)H\omega\}} \Bigg[\frac{(u(x,0)-\mu_-+a\omega H)^{2-p}}{2-p}\\
&\quad - \frac{u(x,0)-\mu_-}{(\omega H)^{p-1}}\Bigg]\zeta^p(x,0)dx + \iint_A \frac{|u_x|^p}{(u-\mu_-+a\omega H)^p}\zeta^p\, dxdt \\
&\le \gamma \iint_A |\zeta_x|^p\, dxdt + \gamma \iint_A (u-\mu_-+a\omega H)^{2-p}\zeta^{p-1}|\zeta_t|\, dxdt.
\end{aligned} \tag{2.1}$$

Proof. Without loss of generality, we may assume $y = 0$. In the weak formulation of 1.1 take $\varphi = G(u)\zeta^p$ as test function, with

$$G(u) = \left[\frac{1}{(u-\mu_-+a\omega H)^{p-1}} - \frac{1}{(\omega H)^{p-1}}\right]_+,$$

and ζ a piecewise smooth, cutoff function vanishing on ∂B_ρ and on $B_\rho \times \{T\}$, such that $0 \le \zeta \le 1$, and $\zeta_t \le 0$. It is easy to see that we have

$$G'(u) = -\frac{p-1}{(u-\mu_-+a\omega H)^p}\,\chi_A.$$

Modulo a Steklov averaging process, we have

$$\iint_Q u_t G(u)\zeta^p\,dxdt + \iint_Q \zeta^p G'(u)|u_x|^p\,dxdt + p\iint_Q G(u)\,|u_x|^{p-2}\zeta^{p-1}u_x\cdot\zeta_x dxdt \ge 0,$$

$$\begin{aligned}(p-1)&\iint_A \frac{|u_x|^p}{(u-\mu_-+a\omega H)^p}\zeta^p\,dxdt\\ \le p&\iint_A \zeta^{p-1}\frac{|u_x|^{p-1}}{(u-\mu_-+a\omega H)^{p-1}}|\zeta_x|\,dxdt\\ &+\iint_A \frac{u_t}{(u-\mu_-+a\omega H)^{p-1}}\zeta^p\,dxdt-\iint_A\frac{u_t}{(\omega H)^{p-1}}\zeta^p\,dxdt,\end{aligned}$$

$$\begin{aligned}(p-1)&\iint_A \frac{|u_x|^p}{(u-\mu_-+a\omega H)^p}\zeta^p\,dxdt\\ \le p&\iint_A \zeta^{p-1}\frac{|u_x|^{p-1}}{(u-\mu_-+a\omega H)^{p-1}}|\zeta_x|\,dxdt\\ &+\iint_A \partial_t\left[\frac{(u-\mu_-+a\omega H)^{2-p}}{2-p}-\frac{u-\mu_-}{(\omega H)^{p-1}}\right]\zeta^p\,dxdt,\end{aligned}$$

$$\begin{aligned}(p-1)&\iint_A \frac{|u_x|^p}{(u-\mu_-+a\omega H)^p}\zeta^p\,dxdt\\ \le p&\iint_A \zeta^{p-1}\frac{|u_x|^{p-1}}{(u-\mu_-+a\omega H)^{p-1}}|\zeta_x|\,dxdt\\ &+\int_{A(T)}\left[\frac{(u(x,T)-\mu_-+a\omega H)^{2-p}}{2-p}-\frac{u(x,T)-\mu_-}{(\omega H)^{p-1}}\right]\zeta^p(x,T)\,dx\\ &-\int_{A(0)}\left[\frac{(u(x,0)-\mu_-+a\omega H)^{2-p}}{2-p}-\frac{u(x,0)-\mu_-}{(\omega H)^{p-1}}\right]\zeta^p(x,0)\,dx\\ &-p\iint_A\left[\frac{(u-\mu_-+a\omega H)^{2-p}}{2-p}-\frac{u-\mu_-}{(\omega H)^{p-1}}\right]\zeta^{p-1}\zeta_t\,dxdt.\end{aligned}$$

The second term on the right-hand side vanishes, as $\zeta(x,T)=0$. An application of Young's inequality yields

$$\begin{aligned}(p-1)&\iint_A \frac{|u_x|^p}{(u-\mu_-+a\omega H)^p}\zeta^p\,dxdt\\ &+\int_{B_\rho\cap\{u(x,0)<\mu_-+(1-a)H\omega\}}\left[\frac{(u(x,0)-\mu_-+a\omega H)^{2-p}}{2-p}\right.\\ &\left.-\frac{u(x,0)-\mu_-}{(\omega H)^{p-1}}\right]\zeta^p(x,0)\,dx\le\frac{p-1}{2}\iint_A\frac{|u_x|^p}{(u-\mu_-+a\omega H)^p}\zeta^p\,dxdt\\ &+\gamma\iint_A|\zeta_x|^p\,dxdt+p\iint_A\frac{(u-\mu_-+a\omega H)^{2-p}}{2-p}\zeta^{p-1}|\zeta_t|\,dxdt,\end{aligned}$$

where we have taken into account that $\zeta_t \le 0$. Therefore, we conclude

$$\int_{B_\rho\cap\{u(x,0)<\mu_-+(1-a)H\omega\}}\left[\frac{(u(x,0)-\mu_-+a\omega H)^{2-p}}{2-p}\right.$$
$$\left.-\frac{u(x,0)-\mu_-}{(\omega H)^{p-1}}\right]\zeta^p(x,0)\,dx+\frac{p-1}{2}\iint_A\frac{|u_x|^p}{(u-\mu_-+a\omega H)^p}\zeta^p\,dxdt$$
$$\le\gamma\iint_A|\zeta_x|^p\,dxdt+\gamma\iint_A(u-\mu_-+a\omega H)^{2-p}\zeta^{p-1}|\zeta_t|\,dxdt.$$

Notice that the first term on the left-hand side is non–negative. Indeed, since $1<p<2$, first of all we have

$$\frac{(u(x,0)-\mu_-+a\omega H)^{2-p}}{2-p}-\frac{u(x,0)-\mu_-}{(\omega H)^{p-1}}$$
$$\ge(u(x,0)-\mu_-+a\omega H)^{2-p}-\frac{u(x,0)-\mu_-}{(\omega H)^{p-1}}.$$

Now, if we let $v=u(x,0)-\mu_-$, we have

$$(u(x,0)-\mu_-+a\omega H)^{2-p}-\frac{u(x,0)-\mu_-}{(\omega H)^{p-1}}$$
$$=\frac{v}{(\omega H)^{p-1}}\left[\frac{\left(\frac{v}{\omega H}+a\right)^{2-p}}{\frac{v}{\omega H}}-1\right].$$

To conclude, it suffices to remark that for $0<s<1-a<1$ the function $f(s)=\frac{(s+a)^{2-p}}{s}$ is monotone decreasing, and $f(1-a)=\frac{1}{1-a}>1$. □

Remark 2.2. The constant γ deteriorates, as $p\to1$.

Remark 2.3. Even though in the next Section H basically plays no role, we chose to state the previous Proposition with an explicit dependence also on H for future applications. The same applies to ω: in the next Section it will play the role of the lower bound M for u on a proper set, and we could have directly used such a notation, as indicated below. However, we have in mind future applications, where ω will have a more general meaning.

3. **A decay lemma.**

Without loss of generality, we may assume $\mu_-=0$. Let $M=\omega$, $L\le\frac{M}{2}$, and suppose that

$$u(0,t)>M\quad\forall t\in(0,\frac{T}{2}].\tag{3.1}$$

Now, let s_o be an integer to be chosen, define

$$F_{s_o}=\{t\in(0,\frac{T}{2}]:\ \exists\,x\in B_{\frac{\rho}{2}},\ u(x,t)<\frac{L}{2^{s_o}}\}$$
$$F(t)=\{x\in B_{\frac{\rho}{2}}:\ u(x,t)<L(1-\frac{1}{2^{s_o}})\},\qquad t\in(0,\frac{T}{2}],$$

and notice that with the previous choices,

$$A=\{(x,t)\in B_\rho\times(0,T]:\ u(x,t)<L(1-\frac{1}{2^{s_o}})\}.$$

Lemma 3.1. *Let u be a non-negative, local, weak super-solution to 1.1 in $E_{-\tau_o,T}$, in the sense of 1.2–1.3. Let 3.1 hold and take*

$$L \le \min\{\frac{M}{2}, \left(\frac{T}{\rho^p}\right)^{\frac{1}{2-p}}\}.$$

Then, for any $\nu \in (0,1)$, there exists a positive integer s_o such that

$$|\{t \in (0, \frac{T}{2}] : \ \exists\, x \in B_{\frac{\rho}{2}},\ u(x,t) \le \frac{L}{2^{s_o}}\}| \le \nu |(0, \frac{T}{2}]|,$$

where $|G|$ denotes the N-dimensional Lebesgue measure of $G \subset \mathbb{R}^N$, with $N = 1$ or $N = 2$.

Proof. Take $t \in F_{s_o}$: by definition, there exists $\bar{x} \in B_{\frac{\rho}{2}}$ such that $u(\bar{x},t) < L/2^{s_o}$. On the other hand, by assumption $u(0,t) > 2L$, and therefore, $u(0,t)+(L/2^{s_o}) > L$. Hence

$$\ln_+ \frac{u(0,t) + \frac{L}{2^{s_o}}}{u(\bar{x},t) + \frac{L}{2^{s_o}}} > (s_o - 1)\ln 2,$$

and we obtain

$$\begin{aligned}
(s_o - 1)\ln 2 &\le \ln_+\left(\frac{L}{u(\bar{x},t) + \frac{L}{2^{s_o}}}\right) - \ln_+\left(\frac{L}{u(0,t) + \frac{L}{2^{s_o}}}\right)\\
&= \int_0^{\bar{x}} \frac{\partial}{\partial x}\left(\ln_+\left(\frac{L}{u(\xi,t) + \frac{L}{2^{s_o}}}\right)\right) d\xi\\
&\le \int_{-\frac{\rho}{2}}^{\frac{\rho}{2}} \left|\frac{\partial}{\partial x}\left(\ln_+\left(\frac{L}{u(x,t) + \frac{L}{2^{s_o}}}\right)\right)\right| dx\\
&= \int_{B_{\frac{\rho}{2}} \cap F(t)} \left|\frac{\partial}{\partial x}\left(\ln_+\left(\frac{L}{u(x,t) + \frac{L}{2^{s_o}}}\right)\right)\right| dx.
\end{aligned}$$

If we integrate with respect to time over the set F_{s_o}, we have

$$\begin{aligned}
(s_o - 1)|F_{s_o}|\ln 2 &\le \int_0^{\frac{T}{2}} \int_{B_{\frac{\rho}{2}} \cap F(t)} \left|\frac{\partial}{\partial x}\left(\ln_+\left(\frac{L}{u(x,t) + \frac{L}{2^{s_o}}}\right)\right)\right| dxdt\\
&\le \left[\int_0^{\frac{T}{2}} \int_{B_{\frac{\rho}{2}} \cap F(t)} \left|\frac{\partial}{\partial x}\left(\ln_+\left(\frac{L}{u(x,t) + \frac{L}{2^{s_o}}}\right)\right)\right|^p dxdt\right]^{\frac{1}{p}} |Q|^{\frac{p-1}{p}}\\
&\le \left[\iint_{Q \cap A} \frac{|u_x|^p}{(u + \frac{L}{2^{s_o}})^p} \zeta^p \, dxdt\right]^{\frac{1}{p}} |Q|^{\frac{p-1}{p}},
\end{aligned}$$

where ζ is as in Proposition 2.1, and is chosen such that $\zeta = \zeta_1(x)\zeta_2(t)$, where ζ_1 vanishes outside B_ρ and satisfies

$$0 \le \zeta_1 \le 1, \qquad \zeta_1 = 1 \ \text{ in } \ B_{\frac{\rho}{2}}, \qquad |\partial_x \zeta_1| \le \frac{\gamma_1}{\rho},$$

for an absolute constant γ_1 independent of ρ, and ζ_2 is monotone decreasing, and satisfies

$$0 \le \zeta_2 \le 1, \qquad \zeta_2 = 1 \ \text{ in } \ (0, \frac{T}{2}], \qquad \zeta_2 = 0 \ \text{ for } \ t \ge T, \qquad |\partial_t \zeta_2| \le \frac{\gamma_2}{T},$$

for an absolute constant γ_2 independent of T.

Apply estimates 2.1 with $a = \frac{1}{2^{s_o}}$, $H\omega = HM = L$. The requirement $H \le 1$ is satisfied, since $L \le \frac{M}{2}$. They yield

$$(s_o-1)|F_{s_o}| \le \gamma\, |Q|^{\frac{p-1}{p}} \left[\iint_A |\zeta_x|^p dxdt\right]^{\frac{1}{p}} + \gamma |Q|^{\frac{p-1}{p}} \left[\iint_A (u+\frac{L}{2^{s_o}})^{2-p}|\zeta_t| dxdt\right]^{\frac{1}{p}}.$$

By the choice of ζ we have

$$\begin{aligned}(s_o-1)|F_{s_o}| &\le \frac{\gamma}{\rho}|Q|^{\frac{p-1}{p}}|Q|^{\frac{1}{p}} + \gamma|Q|^{\frac{p-1}{p}}\left(\frac{L^{2-p}}{T}\right)^{\frac{1}{p}}|Q|^{\frac{1}{p}}\\ &\le \gamma\left[\frac{1}{\rho}+\left(\frac{L^{2-p}}{T}\right)^{\frac{1}{p}}\right]|Q|.\end{aligned}$$

If we require $L \le \left(\frac{T}{\rho^p}\right)^{\frac{1}{2-p}}$, and we substitute it back in the previous estimate, we have

$$(s_o-1)|F_{s_o}| \le \gamma_1|(0,\frac{T}{2}]|.$$

Therefore, if we want that $|F_{s_o}| \le \nu|(0,\frac{T}{2}]|$, it is enough to require that $s_o = \frac{\gamma_1}{\nu}+1$. □

The previous result can also be rewritten as

Lemma 3.2. *Let u be a non-negative, local, weak super-solution to 1.1 in $E_{-\tau_o,T}$, in the sense of 1.2–1.3. Let 3.1 hold. For any $\nu \in (0,1)$, there exists a positive integer s_o such that*

$$|\{t\in(0,\frac{T}{2}]:\ \exists x\in B_{\frac{\rho}{2}},\ u(x,t)\le\left(\frac{T}{\rho^p}\right)^{\frac{1}{2-p}}\frac{1}{2^{s_o}}\}| \le \nu|(0,\frac{T}{2}]|,$$

provided $\rho>0$ is so large that $\left(\frac{T}{\rho^p}\right)^{\frac{1}{2-p}} \le \frac{M}{2}$.

Now let $\bar\rho$ be such that

$$\left(\frac{T}{\bar\rho^p}\right)^{\frac{1}{2-p}} = \frac{M}{2} \qquad\Rightarrow\qquad \bar\rho = \left(\frac{2^{2-p}T}{M^{2-p}}\right)^{\frac{1}{p}}, \tag{3.2}$$

and assume that $B_{\bar\rho}\subset(\alpha,\beta)$. Then Lemmas 3.1–3.2 can be rephrased as

Lemma 3.3. *Let u be a non-negative, local, weak super-solution to 1.1 in $E_{-\tau_o,T}$, in the sense of 1.2–1.3. Let 3.1 hold. For any $\nu\in(0,1)$, there exists a positive integer s_o such that for any $\rho>\bar\rho$*

$$|\{t\in(0,\frac{T}{2}]:\ \exists x\in B_{\frac{\rho}{2}},\ u(x,t)\le\frac{M}{2^{s_o+1}}\left(\frac{\bar\rho}{\rho}\right)^{\frac{p}{2-p}}\}| \le \nu|(0,\frac{T}{2}]|,$$

provided that $B_\rho\subset(\alpha,\beta)$.

Remark 3.4. The previous corollary gives us the power-like decay, required in Proposition 1.2.

Let us now set $F^c_{s_o} \stackrel{\text{def}}{=} (0, \frac{T}{2}]\backslash F_{s_o}$. Then, if 3.1 holds, we conclude that for any $t \in F^c_{s_o}$ and for any $x \in B_{\frac{\rho}{2}}$ with $\rho > \bar\rho$

$$u(x,t) \geq \frac{M}{2^{s_o+1}} \left(\frac{\bar\rho}{\rho}\right)^{\frac{p}{2-p}}. \tag{3.3}$$

Let $c \geq 4$ denote a positive parameter, choose $\bar x \in (\alpha, \beta)$ such that $|\bar x| = 2c\bar\rho$, and consider $B_{c\bar\rho}(\bar x)$. Then, by 3.3

$$\forall\, x \in B_{c\frac{\bar\rho}{2}}(\bar x), \quad \forall\, t \in F^c_{s_o} \quad u(x,t) \geq \frac{M}{2^{s_o+1}} \left(\frac{2}{5c}\right)^{\frac{p}{2-p}}, \tag{3.4}$$

provided 3.1 holds, and $B_{c\bar\rho}(\bar x) \subset (\alpha, \beta)$.

4. **A DeGiorgi-Type lemma.** Assume that some information is available on the "initial data" relative to the cylinder $B_{2\rho}(y) \times (s, s + \theta\rho^p]$, say for example

$$u(x,s) \geq M \qquad \text{for a.e.} \;\; x \in B_{2\rho}(y) \tag{4.1}$$

for some $M > 0$. Then, the following Proposition is proved in [3, Chapter 3, Lemma 4.1].

Lemma 4.1. *Let u be a non-negative, local, weak super-solution to 1.1, and M be a positive number such that 4.1 holds. Then*

$$u \geq \tfrac{1}{2} M \qquad \textit{a.e. in} \;\; B_\rho(y) \times (s, s + \theta(4\rho)^p],$$

where

$$\theta = \delta\, M^{2-p} \tag{4.2}$$

for a constant $\delta \in (0,1)$ depending only upon p, and independent of M and ρ.

Remark 4.2. Lemma 4.1 is based on the energy estimates and Proposition 3.1 of [1], Chapter I, which continue to hold in a stable manner for $p \to 1$. These results are therefore valid for all $p \geq 1$, including a seamless transition from the singular range $p < 2$ to the degenerate range $p > 2$.

5. **Proof of Proposition 1.2.** Fix $y \in E$, define $\bar\rho$ as in 3.2, and choose a positive parameter $C \geq 4$, such that the cylindrical domain

$$B_{2^{\frac{p-2}{p}} C\bar\rho}(y) \times \big(0, \frac{T}{2}\big] \subset E_{-\tau_o, T}. \tag{5.1}$$

This is an assumption both on the size of the reference ball $B_{2^{\frac{p-2}{p}} C\bar\rho}(y)$ and on T; we can always assume it without loss of generality. Indeed, as we have already pointed out in Remark 1.3, if 5.1 were not satisfied, we would decompose the interval $(0, \frac{T}{2}]$ in smaller subintervals, each of width τ, such that 5.1 is satisfied working with $\bar\rho$ replaced by

$$\widehat\rho = \left(\frac{2^{2-p}\tau}{M^{2-p}}\right)^{\frac{1}{p}}.$$

The only role of C is in determining a sufficiently large reference domain

$$B_{2^{\frac{p-2}{p}} C\bar\rho}(y) \subset E,$$

which contains the smaller ball we will actually work with, and will play no other role; in particular the structural constants will not depend on C.

Now, introduce the change of variables and the new unknown function

$$z = 2^{\frac{2-p}{p}} \frac{x-y}{\bar{\rho}}, \qquad -e^{-\tau} = \frac{t - \frac{T}{2}}{\frac{T}{2}}, \qquad v(z,\tau) = \frac{1}{M} u(x,t) e^{\frac{\tau}{2-p}}. \tag{5.2}$$

This maps the cylinder in 5.1 into $B_C \times (0, \infty)$ and transforms 1.1 into

$$v_\tau - \frac{1}{2}(|v_z|^{p-2} v_z)_z = \frac{1}{2-p} v \quad \text{weakly in } B_C \times (0,\infty). \tag{5.3}$$

The only effect of the factor $\frac{1}{2}$ in front of $(|v_z|^{p-2}v_z)_z$ is to modify the constant γ in Proposition 2.1, and consequently s_o in Lemmas 3.1–3.3. By the previous change of variable, assumption 1.4 of Proposition 1.2 becomes

$$v(0,\tau) \geq e^{\frac{\tau}{2-p}} \quad \text{for all } \tau \in (0, +\infty). \tag{5.4}$$

Let $\tau_o > 0$ to be chosen and set

$$k = e^{\frac{\tau_o}{2-p}}.$$

With this symbolism, 5.4 implies

$$v(0,\tau) \geq k \quad \text{for all } \tau \in (\tau_o, +\infty). \tag{5.5}$$

Now consider the segment

$$I \stackrel{\text{def}}{=} \{0\} \times \big(\tau_o, \tau_o + k^{2-p}\big).$$

Let $\nu = \frac{1}{4}$ and s_o be the corresponding quantity introduced in Lemma 3.1. We can then apply Lemmas 3.1–3.3 with $T = k^{2-p}$, M substituted by k,

$$F_{s_o} = \{\tau \in (\tau_o, \tau_o + \frac{1}{2}k^{2-p}] : \ \exists\, z \in B_{\frac{\rho}{2}},\ v(z,\tau) < \frac{k}{2^{s_o+1}}\} \quad \text{for} \quad \rho > \rho_*,$$

with $\rho_* \stackrel{\text{def}}{=} 2^{\frac{2-p}{p}}$. Therefore, if $c \geq 4$ denotes a positive parameter, we choose $\bar{z} \in B_C$ such that $|\bar{z}| = 2c\rho_*$, and consider $B_{c\rho_*}(\bar{z})$, by 3.3

$$\forall\, z \in B_{c\frac{\rho_*}{2}}(\bar{z}), \quad \forall\, \tau \in F_{s_o}^c \quad v(z,\tau) \geq \frac{k}{2^{s_o+1}} \left(\frac{2}{5c}\right)^{\frac{p}{2-p}}, \tag{5.6}$$

provided $B_{c\rho_*}(\bar{z}) \subset B_C$. Summarising, there exists at least a time level τ_1 in the range

$$\tau_o < \tau_1 < \tau_o + \frac{1}{2}k^{2-p} \tag{5.7}$$

such that

$$\forall\, z \in B_{c\frac{\rho_*}{2}}(\bar{z}), \quad v(z,\tau_1) \geq \sigma_o e^{\frac{\tau_o}{2-p}} \quad \text{where} \quad \sigma_o = \frac{1}{2^{s_o+1}} \left(\frac{2}{5c}\right)^{\frac{p}{2-p}}.$$

Remark 5.1. Notice that σ_o is determined only in terms of the data and is independent of the parameter τ_o, which is still to be chosen.

5.1. **Returning to the original coordinates.** In terms of the original coordinates and the original function $u(x,t)$, this implies

$$u(\cdot,t_1)\ge \sigma_o M e^{-\frac{\tau_1-\tau_o}{2-p}} \stackrel{\text{def}}{=} M_o \qquad \text{in } \ B_{c\frac{\bar\rho}{2}}(\bar x)$$

where the time t_1 corresponding to τ_1 is computed from 5.2 and 5.7, and $\operatorname{dist}(\bar x,y)=2c\bar\rho$. Now, apply Lemma 4.1 with M replaced by M_o over the cylinder $B_{c\frac{\bar\rho}{2}}(\bar x)\times\big(t_1,t_1+\theta(c\bar\rho)^p\big]$. By choosing

$$\theta=\delta M_o^{2-p} \qquad \text{where} \qquad \delta=\delta(\text{data}),$$

the assumption 4.2 is satisfied, and Lemma 4.1 yields

$$\begin{aligned} u(\cdot,t)&\ge \tfrac12 M_o=\tfrac12\sigma_o M e^{-\frac{\tau_1-\tau_o}{2-p}} \\ &\ge \frac{1}{2^{s_o+2}}\left(\frac{2}{5c}\right)^{\frac{p}{2-p}} e^{-\frac{2}{2-p}e^{\tau_o}} M \end{aligned} \qquad \text{in} \quad B_{\frac{c\bar\rho}{4}}(\bar x) \tag{5.8}$$

for all times

$$t_1\le t\le t_1+\delta\frac{1}{2^{s_o(2-p)}}\left(\frac25\right)^p e^{-(\tau_1-\tau_o)}\frac{T}{2}. \tag{5.9}$$

Notice that 5.8 can be rewritten as

$$u(\cdot,t)\ge\bar\sigma\left(\frac{\bar\rho}{\rho}\right)^{\frac{p}{2-p}} M \text{ in } \quad B_{\frac{\rho}{4}}(\bar x), \tag{5.10}$$

with

$$\bar\sigma \stackrel{\text{def}}{=} \frac{1}{2^{s_o+2}}\left(\frac25\right)^{\frac{p}{2-p}} e^{-\frac{2}{2-p}e^{\tau_o}} \tag{5.11}$$

If the right hand side of 5.9 equals $\frac{T}{2}$, then 5.8 holds for all times in

$$\big(\frac{T}{2}-\varepsilon M^{2-p}(c\bar\rho)^p\,,\,\frac{T}{2}\big] \qquad \text{where} \qquad \varepsilon=\delta\sigma_o^{2-p}e^{-e^{\tau_o}}; \tag{5.12}$$

taking into account the expression for $\bar\rho$ and σ_o, we conclude that 5.8 holds for all times in the interval

$$\big(\frac{T}{2}-e^{-e^{\tau_o}}\frac{\delta}{2^{s_o(2-p)}}\left(\frac25\right)^p\frac{T}{2}\,,\,\frac{T}{2}\big]. \tag{5.13}$$

Thus, the conclusion of Proposition 1.2 holds, provided the upper time level in 5.9 equals $\frac{T}{2}$. The transformed τ_o level is still undetermined, and it will be so chosen as to verify such a requirement. Precisely, taking into account 5.2

$$\frac{T}{2}e^{-\tau_1}=-(t_1-\frac{T}{2})=\delta\frac{1}{2^{s_o(2-p)}}\left(\frac25\right)^p e^{-(\tau_1-\tau_o)}\frac{T}{2} \implies e^{\tau_o}=\left(\frac52\right)^p\frac{2^{s_o(2-p)}}{\delta}.$$

This determines quantitatively $\tau_o=\tau_o(\text{data})$, and inserting such a τ_o on the right-hand side of 5.11 and 5.13, yields a bound below that depends only on the data; 5.11 and 5.13 have been obtained relying on the bound below for u along the segment $\{y\}\times(0,\frac{T}{2}]$. However, the same argument on the bound along the shorter segment $\{y\}\times(0,s]$ for any $\frac{T}{4}\le s<\frac{T}{2}$ yields the same result with $\frac{T}{2}$ substituted by s: the proof of Proposition 1.2 is then completed.

Remark 5.2. In the proof of Proposition 1.2, the parameter c basically measures the relative size of ρ with respect to $\bar\rho$.

5.2. **A remark about the limit as $p \to 2$.** The change of variables 5.2 and the subsequent arguments, yield constants that deteriorate as $p \to 2$. This is no surprise, as the decay of solutions to linear parabolic equations is not power-like, but rather exponential-like, as in the fundamental solution of the heat equation.

Nevertheless, our estimates can be stabilised, in order to recover the correct exponential decay in the $p = 2$ limit. However, this would require a careful tracing of all the functional dependencies in our estimates, and we postpone it to a future work.

Acknowledgments. We are grateful to the anonymous referee for the comments and suggestions, which greatly improved the paper.

REFERENCES

[1] E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer–Verlag, New York, 1993.

[2] E. DiBenedetto, U. Gianazza and V. Vespri, Harnack estimates for quasi-linear degenerate parabolic differential equation, *Acta Mathematica*, **200** (2008), 181–209.

[3] E. DiBenedetto, U. Gianazza and V. Vespri, *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2012.

[4] E. DiBenedetto, U. Gianazza and V. Vespri, A New Approach to the Expansion of Positivity Set of Non-negative Solutions to Certain Singular Parabolic Partial Differential Equations, *Proc. Amer. Math. Soc.,* **138** (2010), 3521–3529.

[5] F. G. Düzgün, P. Marcellini and V. Vespri, An alternative approach to the Hoelder continuity of solutions to some elliptic equations, *Nonlinear Anal.*, **94** (2014), 133–141.

[6] F. G. Düzgün, P. Marcellini and V. Vespri, Space expansion for a solution of an anisotropic p-Laplacian equation by using a parabolic approach, *Riv. Mat. Univ. Parma*, **5** (2014), 93–111.

[7] T. Kuusi, Harnack estimates for weak supersolutions to nonlinear degenerate parabolic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, **7** (2008), 673–716.

[8] V. Liskevich and I. I. Skrypnik, Hölder continuity of solutions to an anisotropic elliptic equation, *Nonlinear Anal.*, **71** (2009), 1699–1708.

Received March 2015; revised July 2015.

E-mail address: gamzeduz@hacettepe.edu.tr
E-mail address: gianazza@imati.cnr.it
E-mail address: vespri@math.unifi.it