A STUDY ON DIRECT SUMMAND SUBMODULES OVER NONCOMMUTATIVE RINGS

DEĞİŞMELİ OLMAYAN HALKALARDA DİK TOPLANAN ALT MODÜLLER ÜZERİNE BİR ÇALIŞMA

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To the memory of my beloved father Mustafa Altun.

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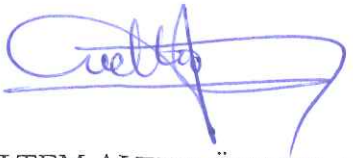

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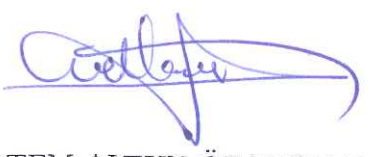

MELTEM ALTUN ÖZARSLAN

ABSTRACT

A STUDY ON DIRECT SUMMAND SUBMODULES OVER NONCOMMUTATIVE RINGS

Meltem ALTUN ÖZARSLAN

Doctor of Philosophy, Department of Mathematics

Supervisor: Prof. Dr. Ayşe Çiğdem ÖZCAN

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A natural problem to consider in ring and module theory is to investigate the cancellation property of a given object. This problem was first considered by Jónsson and Tarski for any algebraic system and then gave rise to many variations related to the cancellation theme such as substitution and internal cancellation. In the mid-30s of the last century, just before the cancellation problem was treated for any algebraic system by Jónsson and Tarski, a ground-breaking invention was made by von Neumann. He developed the theory of continuous geometries. One of the main ideas of this new structure was the construction of a dimension function whose range is a continuum of real numbers and this construction was based on the perspectivity relation. Throughout this work we discuss new concepts derived from cancellation and continuity. This dissertation consists of four chapters. In the first chapter, we recall the ring and module theoretical properties that play an important role within our framework like stable range conditions, the exchange property, and perspectivity. In the second chapter, we study the class of internally cancellable rings, i.e., the class of rings that satisfy internal cancellation property with respect to their one-sided ideals. By considering a condition, we obtain new characterizations of internally cancellable rings, unit regular

rings, and rings with stable range one. We also investigate internally cancellable rings with the summand sum property. In Chapter 3, we introduce the lifting of elements having (idempotent) stable range one from a quotient of a ring R modulo a two-sided ideal I by providing several examples and investigating the relations with other lifting properties, including lifting idempotents, lifting units, and lifting of von Neumann regular elements. In the case where the ring R is a left or a right duo ring, we show that stable range one elements lift modulo every two-sided ideal iff R is a ring with stable range one. Under a mild assumption, we further prove that the lifting of elements having idempotent stable range one implies the lifting of von Neumann regular elements. In the last chapter, we study the most recent variations of continuity and discreteness concepts, namely $C4$- and $D4$-modules, in terms of perspective direct summands by providing new characterizations and results. Endomorphism rings of $C4$-modules and extensions of right $C4$-rings are also investigated. Decompositions of $C4$-modules with restricted ACC on direct summands and $D4$-modules with restricted DCC on direct summands are obtained.

Keywords: Internal cancellation, perspectivity, stable range one, idempotent stable range one, lifting units, lifting idempotents, quasi-continuous and quasi-discrete modules, $C4$- and $D4$-modules.

ÖZET

DEĞİŞMELİ OLMAYAN HALKALARDA DİK TOPLANAN ALT MODÜLLER ÜZERİNE BİR ÇALIŞMA

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Halka ve modül teorisinde ilk akla gelebilecek problemlerden biri verilen nesnenin sadeleşme özelliğini incelemektir. Bu problem, ilk olarak Jónsson ve Tarski tarafından genel bir cebirsel yapı için ele alınmış ve sonrasında yerine koyma ve içsel sadeleşme gibi sadeleşme kavramı ile ilgili varyasyonların tanımlanmasına imkan vermiştir. 1930'ların ortalarında, sadeleşme problemi Jónsson ve Tarski tarafından genel bir cebirsel yapı için ele alınmadan hemen önce, von Neumann çığır açan bir buluşa imza attı ve sürekli geometrilerin teorisini geliştirdi. Sürekli geometrilerin önemli özelliklerinden biri görüntü kümesi $[0, 1]$ olan bir boyut fonksiyonuna sahip olmasıydı ve bu boyut fonksiyonunun inşaasında ise temel yapı taşı perspektiflik bağıntısıydı. Bu tez çalışmasında, sadeleşme ve süreklilik kavramlarından türeyen yeni nosyonları ele alacağız. Bu tez dört bölümden oluşmaktadır. Tezin birinci bölümünde, stable range 1, değişim özelliği ve perspektiflik gibi çalışmalarımızda önem arz eden halka ve modül teorik kavramlar tanıtılmıştır. Tezin ikinci bölümünde, içsel sadeleşebilen halkalar olarak adlandırılan tek yönlü ideallerine göre içsel sadeleşme özelliğini sağlayan halkalar ele alınmıştır. Daha sonra özel bir koşul tanıtılarak içsel sadeleşebilen halkalar, birimsel düzenli halkalar

ve stable range 1 özelliğine sahip halkalar için yeni karakterizasyonlar elde edilmiştir. Ayrıca, dik toplanan sağ ideallerinin toplamı da dik toplanan olan halka sınıfı, içsel sadeleşebilir halkalar ile birlikte ele alınmıştır. Tezin üçüncü bölümünde, bir R halkasının (eşkare) stable range 1 özelliğine sahip elemanlarının bir I idealine göre, R/I bölüm halkasından yükseltilmesi kavramı tanıtılmıştır. Bu kavramlar tanıtılırken çeşitli örnekler verilmiş ve bugüne kadar literatürde önemli yer edinmiş eşkare elemanların yükseltilmesi, birimsel elemanların yükseltilmesi ve von Neumann düzenli elemanların yükseltilmesi gibi özellikler ile bağlantıları incelenmiştir. Daha sonra bir R halkasının sol veya sağ duo olması koşulu altında stable range 1 özelliğine sahip elemanların her ideale göre yükseltilmesi ile o halkanın stable range 1 özelliğine sahip olmasının denk olduğu gösterilmiştir. Ayrıca, zayıf bir koşul altında (eşkare) stable range 1 özelliğine sahip elemanların yükseltilmesi kavramının von Neumann düzenli elemanların yükseltilmesi kavramını gerektirdiği gösterilmiştir. Tezin dördüncü ve son bölümünde, süreklilik ve ayrıklık kavramlarının en son varyasyonları $C4$- ve $D4$-modüller perspektif dik toplanan alt modüller açısından ele alınmış, yeni karakterizasyonlar ve sonuçlar elde edilmiştir. Bunula birlikte, $C4$-modüllerin endomorfizma halkaları ve $C4$-halkaların bazı genişlemeleri incelenmiştir. Son olarak yeni tanımlanan bir zincir koşulu ile $C4$-modüller ve $D4$-modüller için ayrışım teoremleri verilmiştir.

Anahtar Kelimeler: İçsel sadeleşme, perspektiflik, stable range 1, eşkare stable range 1, birimsel elemanların yükseltilmesi, eşkare elemanların yükseltilmesi, yarı-sürekli ve yarı-ayrık modüller, $C4$- ve $D4$-modüller.

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LIST OF SYMBOLS AND ACRONYMS

Symbols

R	An associative ring with identity
$\mathrm{J}(R)$	The Jacobson radical of a ring R
S_r	The right socle of a ring R
S_l	The left socle of a ring R
$r_R(a)$	$\{r \in R \mid ar = 0\}$, the right annihilator of $a \in R$
$l_R(a)$	$\{r \in R \mid ra = 0\}$, the left annihilator of $a \in R$
$\mathrm{U}(R)$	The set of all invertible elements in a ring R
$\mathrm{idem}(R)$	The set of all idempotent elements in a ring R
$\mathrm{reg}(R)$	The set of all regular elements in a ring R
$\mathrm{ureg}(R)$	The set of all unit-regular elements in a ring R
$\mathrm{sr}(R)$	The stable range of a ring R
$\mathrm{isr}(R)$	The idempotent stable range of a ring R
$\mathbb{M}_n(R)$	The ring of $n \times n$ square matrices over a ring R
$\mathbb{CFM}_\Omega(R)$	The ring of $\Omega \times \Omega$ column finite matrices over a ring R
M_R	A unital right R-module over a ring R
Mod- R	The category of right R-modules
$\bigoplus M_i$	Direct sum of right R-modules M_i
$\prod M_i$	Direct product of right R-modules M_i
$M^{(I)}$	$\bigoplus_{(i \in I)} M_i$ with $M_i \cong M$ for every $i \in I$
M^I	$\prod_{(i \in I)} M_i$ with $M_i \cong M$ for every $i \in I$
$A \leq M$	A submodule A of an R-module M
$A \subseteq^{\oplus} M$	A direct summand A of an R-module M
$A \leq_e M$	An essential submodule A of an R-module M
$A \ll M$	A small submodule A of an R-module M
$\mathrm{ann}_R(m)$	$\{r \in R \mid mr = 0\}$, the right annihilator of $m \in M$ in R
$\mathrm{rad}(M)$	The Jacobson radical of M_R
$\mathrm{soc}(M)$	The socle of M_R
$\mathrm{ann}_R(M)$	$\{r \in R \mid Mr = 0\}$, the right annihilator of M in R

$\mathrm{E}(M)$	Injective envelope of M
$\mathcal{L}(M)$	The lattice of all submodules of a module M
$\mathrm{End}_R(M)$	The endomorphism ring of M_R
$\mathrm{Hom}_R(M,N)$	The group of all module homomorphisms from M_R to N_R
$\mathrm{im}(f)$	The image of an R-homomorphism f
$\ker(f)$	The kernel of an R-homomorphism f
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{Z}_n$	The cyclic group $\mathbb{Z}/n\mathbb{Z}$
$\mathbb{Z}_{p^\infty}$	The Prüfer p-group

Acronyms

ACC	Ascending chain condition
DCC	Descending chain condition
UFD	Unique factorization domain
SIP	Summand intersection property
SSP	Summand sum property

1 PRELIMINARIES

In this chapter, we give some basic concepts and facts which will be used frequently. Throughout this dissertation, all rings we consider are associative rings R with identity. Modules are unital right R-modules unless otherwise stated.

1.1 Basic Notions

We begin by recalling some basic definitions and results which can be found in a standard text on ring and module theory such as [5], [54], and [27].

There are many important types of elements in a ring R. But two of them are of great significance; the units and idempotents. An element $u \in R$ is said to be a *unit* in case there is an element $v \in R$, called inverse of u, such that $uv = vu = 1$. The set of all units of R, denoted by $\mathrm{U}(R)$, forms a multiplicative group. An element $e \in R$ is an *idempotent* in case $e^2 = e$. A ring always has at least two idempotents, namely 0 and 1. An idempotent e of R is a *central idempotent* if it is in the center of R. Clearly, if $e \in R$ is an idempotent, then the complementary idempotent to e, namely $1 - e$, is also an idempotent.

Idempotents in the endomorphism ring of a module M determine direct summand submodules of that module. Recall that a submodule N of M is called a *direct summand* of M in case there is a submodule N' of M with $M = N \oplus N'$, that is,

$$M = N + N' \text{ and } N \cap N' = 0.$$

In that case, N' is also a direct summand, and N and N' are *complementary direct summands* or *direct complements* . It is clear that every non-zero module M has at least two direct summands, namely, 0 and M. A non-zero module M is *indecomposable* if 0 and M are its only direct summands.

A submodule N of a module M is said to be *essential* in M, abbreviated $N \leq_e M$, in case for every submodule $L \leq M$, $N \cap L = 0$ implies $L = 0$. Then we say that M is an *essential extension* of N. A non-zero module M is *uniform* in case each of its non-zero submodules is essential in M.

Now we recall some properties of essential submodules.

Proposition 1.1.1 ([5]) *Let M be an R-module with submodules $K \leq N \leq M$ and $H \leq M$. Then the following hold:*

(1) *$K \leq_e M$ if and only if for each $0 \neq x \in M$, there exists an $r \in R$ such that $0 \neq xr \in K$.*

(2) *$K \leq_e M$ if and only if $K \leq_e N$ and $N \leq_e M$.*

(3) *$H \cap K \leq_e M$ if and only if $H \leq_e M$ and $K \leq_e M$.*

(4) *If $K \leq_e M$ and $f\colon L \longrightarrow M$ a homomorphism, then $f^{-1}(K) \leq_e L$.*

(5) *Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$; then $K_1 \oplus K_2 \leq_e M$ if and only if $K_1 \leq_e M_1$ and $K_2 \leq_e M_2$.*

Recall that a module M is a *simple module* if it is non-zero and its only submodules are 0 and M. The *socle* of a module M, denoted by $\mathrm{soc}(M)$, is the sum of all simple submodules of M, or equivalently is the intersection of all essential submodules of M. If M fails to have a simple submodule, then $\mathrm{soc}(M) = 0$. An R-module M is called *semisimple* (or *completely reducible*) if $M = \mathrm{soc}(M)$. Clearly, every simple module is semisimple. As is well known, a module M is semisimple if and only if every submodule of M is a direct summand.

The right socle S_r of a ring R is the sum of all minimal right ideals of R. The left socle S_l of a ring R is defined analogously. Note that S_r and S_l are not equal in general. In case they are equal, we may write $\mathrm{soc}(R)$ for either socle. A ring R is said to be *(right) semisimple* if the right regular module R_R is semisimple. Thanks to the celebrated works of Wedderburn and Artin, there is no distinction between right and left semisimplicity of a ring, as the following theorem shows.

Theorem 1.1.2 (Wedderburn-Artin) *For a ring R, the following are equivalent:*

(1) *The ring R is semisimple as a right (left) module over itself.*

(2) *Every right (left) module over R is semisimple.*

(3) *The ring R is a finite direct product of simple artinian rings.*

(4) *The ring R is a finite direct product of matrix rings over division rings.*

The notion of an essential submodule can be dualized in a natural way. A submodule K of a module M is called *small* in M if, for every submodule $L \leq M$, $K + L = M$ implies $L = M$. It is denoted by $N \ll M$. If every proper submodule of M is small, then M is called a *hollow* module. Dually, we have the following proposition.

Proposition 1.1.3 ([5]) *Let M be an R-module. Then the following statements hold for $K \leq N \leq M$ and $H \leq M$:*

(1) *$N \ll M$ if and only if $K \ll M$ and $N/K \ll M/K$.*

(2) *$H + K \ll M$ if and only if $H \ll M$ and $K \ll M$.*

(3) *If $K \ll M$ and $f\colon M \longrightarrow L$ is an R-homomorphism, then $f(K) \ll L$. In particular, if $K \ll M \leq L$, then $K \ll L$.*

(4) *Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll M$ if and only if $K_1 \ll M_1$ and $K_2 \ll M_2$.*

(5) *Let $K \leq L \leq M$. If $K \ll M$ and $L \subseteq^{\oplus} M$, then $K \ll L$. In particular, if $K \ll M$ and $K \subseteq^{\oplus} M$, then $K = 0$.*

The *(Jacobson) radical* of a module M is defined to be the intersection of all maximal submodules of M, and is denoted by $\mathrm{rad}(M)$. Equivalently, it is the sum of all small submodules of M. If the module M has no maximal submodules, then $\mathrm{rad}(M) = M$. For a ring R, $\mathrm{rad}(R_R) = \mathrm{rad}({}_RR)$ is an ideal of R, and we simply denote it by $\mathrm{J}(R)$.

Recall that if M is a module, then its *(right) annihilator*

$$ann_R(M) = \{r \in R \mid Mr = 0\}$$

and that M is faithful if $ann_R(M) = 0$. For any $m \in M$, the set $ann_R(m) = \{r \in R \mid mr = 0\}$ is the annihilator of m in R, and it is a right ideal of R. In the ring case, for an element a in R, we will denote by $l_R(a)$ and $r_R(a)$, the left and right annihilators of a in R, respectively.

Let M be a module. A set $\mathcal{L}$ of submodules of M satisfies the *ascending chain condition* (ACC) if, for every chain

$$M_1 \leq M_2 \leq \ldots \leq M_n \leq \ldots$$

in $\mathcal{L}$, there is an n with $M_{n+i} = M_n$ $(i = 1, 2, \ldots)$. The *descending chain condition* (DCC) is analogously defined. A module M is *noetherian* in case the lattice $\mathcal{L}(M)$ of all submodules of M satisfies the ACC. A module M is *artinian* in case $\mathcal{L}(M)$ satisfies the DCC. A ring R is called *right noetherian (resp., right artinian)* if R_R is noetherian (resp., artinian). Left-handed versions are defined similarly. R is *noetherian (resp., artinian)* if it is both right and left noetherian (resp., artinian). The artinian and noetherian properties are inherited by submodules and factor modules (see [5]).

Recall that a commutative integral domain with unique factorization of ideals into prime ideals is called a *Dedekind domain*; such a ring is necessarily noetherian, i.e., it satisfies the ascending chain condition for ideals. Any noetherian unique factorization domain, briefly UFD, is a Dedekind domain, but there are UFD's that are not noetherian, and hence not Dedekind, e.g., the polynomial ring in infinitely many indeterminates over a field (see [19]).

A module is *local* if it has a greatest proper submodule. Equivalently, a module is local if and only if it is cyclic, non-zero, and has a unique maximal proper submodule. It is known that every local module is indecomposable (see [27]).

Proposition 1.1.4 ([27]) *The following conditions are equivalent for a ring R:*

(1) $R/\mathrm{J}(R)$ *is a division ring.*

(2) R_R *is a local module (that is, R has a unique maximal proper right ideal).*

(3) *The sum of two non-invertible elements of R is non-invertible.*

(4) $\mathrm{J}(R)$ *is a maximal right ideal.*

(5) $\mathrm{J}(R)$ *is the set of all non-invertible elements of R.*

Since the condition (1) of the above proposition is left-right symmetric, for any ring R, the right module R_R is local if and only if the left module ${}_RR$ is local. A ring R which satisfies the equivalent conditions of Proposition 1.1.4 is said to be a *local ring* (see [27]). As a generalization of a local ring, a ring R is called *semilocal* if $R/\mathrm{J}(R)$ is a semisimple artinian ring.

Definition 1.1.5 ([54]) Two rings R, S are said to be Morita equivalent ($R \approx S$, for short) if there exists a category equivalence $F\colon$ Mod-$R \to$ Mod-S. A ring-theoretic property $\mathcal{P}$ is said to be Morita invariant if, whenever R has the property $\mathcal{P}$, so does every ring S with $S \approx R$.

1.2 Stable Range One and Idempotent Stable Range One Conditions

The concept of stable range was initiated by Bass [9] in the context of Algebraic K-theory, and thereafter, many authors have worked on the simplest case of stable range one (see, e.g., [14, 31, 34, 40, 48, 56, 85]).

Definition 1.2.1 ([56]) A sequence $\{a_1, \ldots, a_n\}$ in a ring R is said to be *left unimodular* if $Ra_1 + \cdots + Ra_n = R$. In case $n \geqslant 2$, such a sequence is said to be *reducible* if there exist $r_1, \ldots, r_{n-1} \in R$ such that $R(a_1 + r_1 a_n) + \cdots + R(a_{n-1} + r_{n-1}a_n) = R$.

This definition directs us to the definition of stable range.

Definition 1.2.2 ([56]) A ring R is said to have *left stable range* $\leqslant n$ if every left unimodular sequence of length $> n$ is reducible. The smallest such n is said to be the *left stable range* of R; we write simply $\mathrm{sr}_l(R) = n$. (If no such n exists, we say $\mathrm{sr}_l(R) = \infty$.) The right stable range is defined similarly, and is denoted by $\mathrm{sr}_r(R)$.

The stable range condition for a ring R is left-right symmetric due to Vaserstein (see [85]). Thus, we can omit the subscripts and call it simply the stable range of a ring R.

Proposition 1.2.3 ([56]) *The following statements hold:*

(1) *If S is a factor ring of R, then* $\mathrm{sr}(S) \leqslant \mathrm{sr}(R)$.

(2) $\mathrm{sr}(R) = \mathrm{sr}(R/\mathrm{J}(R))$.

Examples 1.2.4 ([56], [85])

(1) $\mathrm{sr}(\mathbb{Z}) = 2$.

(2) For any field $k \subseteq \mathbb{R}$, $\mathrm{sr}(k[x_1, \ldots, x_n]) = n + 1$.

(3) For any field k, $\mathrm{sr}(k[[x_1, \ldots, x_n]]) = n + 1$.

Throughout this work, we focus on the simplest case, that is, the case of stable range one. Recall that a ring R is *Dedekind-finite* (or sometimes, *directly finite*) if $uv = 1$ for $u, v \in R$ implies that $v \in \mathrm{U}(R)$.

Theorem 1.2.5 ([56]) *Let R be a ring. Then the following hold:*

(1) *If $\mathrm{sr}(R) = 1$, then R is Dedekind-finite.*

(2) $\mathrm{sr}(R) = 1$ *if and only if, whenever $a, b \in R$ and $Ra + Rb = R$, there exists $x \in R$ such that $a + xb \in \mathrm{U}(R)$.*

The following theorem due to Bass was one of the earliest results obtained on the stable range of rings.

Theorem 1.2.6 ([9]) *If R is a semilocal ring, then* $\mathrm{sr}(R) = 1$.

The notion of stable range one was transferred from a ring to an element of a ring by Khurana and Lam, as in the definition below.

Definition 1.2.7 ([50]) An element $a \in R$ is said to have *stable range one* (written $\mathrm{sr}(a) = 1$) if, for any $b \in R$, $Ra + Rb = R$ implies that $a + xb \in \mathrm{U}(R)$ for some $x \in R$.

Here we use the left version of the definition of a stable range one element. Unfortunately, it is not known whether the notion of stable range one for a given element $a \in R$ is left-right symmetric.

A new kind of stable range one condition was provided by Chen in 1999, as follows.

Definition 1.2.8 ([14]) A ring R is said to have *idempotent stable range one* (written $\mathrm{isr}(R) = 1$) if, whenever $a, b \in R$ and $Ra + Rb = R$, there exists $e \in \mathrm{idem}(R)$ such that $a + eb \in \mathrm{U}(R)$.

As in the case of stable range one, idempotent stable range one condition for rings is left-right symmetric (see [14]).

Theorem 1.2.9 ([14]) *The following are equivalent for a ring R:*

(1) $\mathrm{isr}(R) = 1$.

(2) $\mathrm{isr}(R/\mathrm{J}(R)) = 1$ *and idempotents can be lifted modulo* $\mathrm{J}(R)$.

Corollary 1.2.10 ([14]) *If* R *is a local ring, then* $\mathrm{isr}(R) = 1$.

In general, every ring satisfying idempotent stable range one condition has stable range one, but the converse is not true as the following example shows.

Example 1.2.11 ([14])Consider the semilocal commutative domain

$$R = \{\tfrac{m}{n} \in \mathbb{Q} \mid 2 \not| n, 3 \not| n\}$$

with two maximal ideals $M_1 = 2R$ and $M_2 = 3R$. Then $\mathrm{J}(R) = M_1 \cap M_2$ and $R/\mathrm{J}(R) \cong R/M_1 \times R/M_2$. Then the factor ring $R/\mathrm{J}(R)$ has two non-trivial idempotents which do not lift to idempotents in R, because R has no non-trivial idempotents. Since R is semilocal, $\mathrm{sr}(R) = 1$ by Theorem 1.2.6. However, $\mathrm{isr}(R) \neq 1$ via Theorem 1.2.9.

Recently, Wang et al. introduced an element-wise definition for idempotent stable range one condition, as follows.

Definition 1.2.12 ([86]) An element $a \in R$ is said to have *idempotent stable range one* (written $\mathrm{isr}(a) = 1$) if $Ra + Rb = R$ for any $b \in R$ implies $a + eb \in \mathrm{U}(R)$ for some $e \in \mathrm{idem}(R)$.

Clearly, for any unit u in a ring R, $\mathrm{isr}(u) = 1$. Moreover, any regular element in a ring R with $\mathrm{sr}(R) = 1$ has idempotent stable range one, as we shall see in the next section (see Theorem 1.3.19).

1.3 Cancellation and Regularity

To begin the groundwork of cancellation, we first introduce the substitution notion, due to P. Crawley and L. Fuchs.

Definition 1.3.1 ([18], [56]) A module M is said to have the *substitution property* or be *substitutable* if, given a module A with internal decompositions

$$A = M_1 \oplus N_1 = M_2 \oplus N_2$$

where $M_1 \cong M \cong M_2$, then the summands N_1 and N_2 have a common complement M_0, necessarily isomorphic to M, that is, there is a submodule M_0 of A for which

$$A = M_0 \oplus N_1 = M_0 \oplus N_2$$

Proposition 1.3.2 ([56]) *Let K and L be modules. Then $M = K \oplus L$ has the substitution property if and only if K and L have both the substitution property.*

We now relate the substitution property and stable range one condition.

Theorem 1.3.3 ([56]) *An R module M has the substitution property if and only if $S = \mathrm{End}_R(M_R)$ has stable range 1.*

For R-modules M, X, Y over a ring R, $M \oplus X \cong M \oplus Y$ in general does not imply $X \cong Y$. In fact, given non-isomorphic modules X and Y, if we let

$$M := Y \oplus X \oplus Y \oplus X \ldots,$$

then $X \oplus M \cong Y \oplus M$, and we cannot cancel M. This construction is often referred to as "Eilenberg's trick" (see [56]).

Definition 1.3.4 ([18]) Given a collection $\mathcal{C}$ of right R-modules, a module $M \in \mathcal{C}$ is said to be *cancellable* in $\mathcal{C}$ if, whenever $M \oplus X \cong M \oplus Y$ for $X, Y \in \mathcal{C}$, then $X \cong Y$. In the case where $\mathcal{C}$ is the category of all right R-modules, we simply say that M is *cancellable* or has the *cancellation property.*

Proposition 1.3.5 ([56]) *Let K and L be modules. Then $K \oplus L$ is cancellable if and only if K and L themselves are.*

Proposition 1.3.6 ([18]) *A substitutable module M is cancellable.*

Theorem 1.3.3 together with Proposition 1.3.6 yield the following.

Theorem 1.3.7 (Evans [26]) *If the endomorphism ring S of a module M_R has stable range 1 (e.g., S is semilocal), then M_R is cancellable.*

The following result provides an interesting example of cancellable modules.

Proposition 1.3.8 ([56]) *Let R be a Dedekind domain. Then the module R_R is cancellable.*

Example 1.3.9 ([18]) The $\mathbb{Z}$-module $\mathbb{Z}$ is cancellable but not substitutable. The proposition above clearly implies that $\mathbb{Z}_{\mathbb{Z}}$ is cancellable. However, $\mathrm{sr}(\mathbb{Z}) \neq 1$, so Theorem 1.3.3 gives us the latter.

An element a in R is called *(von Neumann) regular* provided there exists an element $x \in R$ such that $a = axa$. The set of all regular elements in R will be denoted by $\mathrm{reg}(R)$. Following [34], the ring R is called *(von Neumann) regular* if every element in R is regular. It is well-known that for any $a \in R$, a is regular $\Leftrightarrow aR$ is a direct summand of $R_R \Leftrightarrow Ra$ is a direct summand of ${}_RR$.

The following well-known result gives a criterion for the regularity of an element in the endomorphism of a module.

Proposition 1.3.10 ([56]) *Let $S = \mathrm{End}_R(M_R)$ where M_R is a right module over the ring R. Then $f \in S$ is regular if and only if* $\ker(f)$ *and* $\mathrm{im}(f)$ *are both direct summands of M_R.*

Corollary 1.3.11 ([56]) *The endomorphism ring of any semisimple module M_R is regular.*

Following Ehrlich [24], an element a in R is called *unit-regular* provided that there exists a unit element $u \in R$ such that $a = aua$. The set of all unit-regular elements in R will be denoted by $\mathrm{ureg}(R)$. The ring R is called *unit-regular* if every element in R is unit-regular. Moreover, any element $a \in R$ is unit-regular $\Leftrightarrow$ there exist $u \in \mathrm{U}(R)$ and $e \in \mathrm{idem}(R)$ such that $a = ue \Leftrightarrow$ there exist $v \in \mathrm{U}(R)$ and $f \in \mathrm{idem}(R)$ such that $a = fv$.

Fact 1.3.12 ([38, Theorem 2B(14)]) *If $x, y \in \mathrm{ureg}(R)$ and $Rx = Ry$, then $x = uy$ for some $u \in \mathrm{U}(R)$* (see also, [50, Lemma 3.3]).

Theorem 1.3.13 (Ehrlich-Handelman [24], [37]) *Let M_R be a module with a regular endomorphism ring $S = \mathrm{End}_R(M_R)$. Then S is unit-regular if and only if, whenever $M = M_1 \oplus M_2 = N_1 \oplus N_2$ (in the category of right R-modules) with $M_1 \cong N_1$, then $M_2 \cong N_2$.*

The property of the module M of Theorem 1.3.13 motivates a definition as follows.

Definition 1.3.14 ([56]) A right R-module M is said to be *internally cancellable* if, whenever $M = M_1 \oplus M_2 = N_1 \oplus N_2$ with $M_1 \cong N_1$, then $M_2 \cong N_2$. A ring R is said to be *internally cancellable* (IC, for short) in case the right R-module R_R is internally cancellable.

An obvious necessary condition for a module M to satisfy internal cancellation is that it is *Dedekind-finite*, in the sense that $M = M \oplus X \Rightarrow X = 0$. In general, however, this condition is only necessary, but not sufficient (see [56]).

Corollary 1.3.15 ([56])

(1) *A regular ring R is unit-regular iff R is internally cancellable.*

(2) *Any semisimple ring is unit-regular.*

Recall that a ring R is *abelian* if all idempotents in R are central.

Theorem 1.3.16 ([56]) *Abelian regular rings are unit-regular. (In particular, commutative regular rings are unit-regular.)*

Now we establish the relation between unit-regularity and stable range one condition with the help of a result proved by Fuchs [31], Henriksen [40], and Kaplansky [48] independently.

Theorem 1.3.17 ([56]) *If R is a unit-regular ring, then* $\mathrm{sr}(R) = 1$.

Nicholson defined a ring element a of R to be *clean* if it can be written as a sum of a unit and an idempotent. If every $a \in R$ is clean, R is said to be a *clean ring* (see [69]). Nicholson also asked whether unit-regular rings are clean. The first attempt to give a positive answer to this question was by Camillo and Yu. However, their proof had a gap. Later, Camillo and Khurana proved the following result, which, in particular, shows that a unit regular ring is clean.

Theorem 1.3.18 (Camillo-Khurana [12]) *A ring R is unit-regular if and only if every element a of R can be written as $e+u$ such that $aR \cap eR = 0$, where e is an idempotent and u is a unit in R.*

Recently, Wang et al. have proved that any unit-regular ring has idempotent stable range one with the help of the theorem below.

Theorem 1.3.19 ([86]) *Let $a \in \mathrm{reg}(R)$ where R is a ring with $\mathrm{sr}(R) = 1$. Then, whenever $Ra + Rb = R$, there exists an idempotent $e \in R$ such that $a + eb \in \mathrm{U}(R)$ and $aR \oplus eR = R$. In particular, $\mathrm{isr}(a) = 1$, and a is clean.*

Indeed, any element $a \in R$ with $\mathrm{isr}(a) = 1$ is clean. This can easily be seen by considering the equality $Ra + R(-1) = R$. Also, Theorem 1.3.19 gives immediately the following improvement of Theorem 1.3.17.

Corollary 1.3.20 ([86]) *If R is a unit-regular ring, then $\mathrm{isr}(R) = 1$, and so R is clean.*

The Camillo-Khurana theorem can be refined by adding an equivalent statement with the help of Theorem 1.3.19.

Theorem 1.3.21 ([86]) *For any ring R, the following are equivalent:*

(1) *R is unit-regular.*

(2) *For any $a \in R$, there exist $u \in \mathrm{U}(R)$ and an idempotent $e \in R$ such that $a = e+u$ and $aR \cap eR = 0$.*

(3) *Whenever $Ra+Rb = R$, there exists an idempotent $e \in R$ such that $a+eb \in \mathrm{U}(R)$ and $aR \cap eR = 0$.*

1.4 The Exchange Property

Crawley and Jónsson introduced the exchange property in their study on the decompositions of algebraic systems in 1964. Here we restrict our attention to the category of modules instead of considering general algebraic systems.

Definition 1.4.1 ([20]) Let $\aleph$ be a cardinal number. A module M is said to have the $\aleph$-*exchange property* if, for any module G and any internal direct sum decompositions

$$G = M' \oplus N = \oplus_{i \in I} A_i,$$

where $M' \cong M$ and $|I| \leqslant \aleph$, there are submodules B_i of A_i, $i \in I$, such that $G = M' \oplus (\oplus_{i \in I} B_i)$.

A module M has the *exchange property* if it has the $\aleph$-exchange property for every cardinal $\aleph$. A module M has the *finite exchange property* if it has the $\aleph$-exchange property for every finite cardinal $\aleph$ (see [20]).

It is clear from the definition that a finitely generated module with the finite exchange property has the full exchange property. However, it is unknown whether any module with the finite exchange property also has the full exchange property (see [20]).

The following result shows that the class of modules with the $\aleph$-exchange property is closed under direct summands and finite direct sums.

Lemma 1.4.2 ([20]) *Suppose $\aleph$ is a cardinal and $M = M_1 \oplus M_2$. The module M has the $\aleph$-exchange property if and only if both M_1 and M_2 have the $\aleph$-exchange property.*

In general, however, the class of modules with the $\aleph$-exchange property is not closed under arbitrary direct sums. Crawley and Jónsson showed that if B is a $\mathbb{Z}$-module such that $B = \oplus_{i \in I} B_i$ where, for $i = 1, 2, 3, \ldots$, B_i is a cyclic $\mathbb{Z}$-module of order p^i, then B does not have the 2-exchange property even though each of the cyclic $\mathbb{Z}$-modules of order p^i has the exchange property (see [20]).

Clearly, every module has the 1-exchange property. The behavior of the modules with the 2-exchange property is quite surprising, as follows.

Lemma 1.4.3 ([20]) *If a module M has the 2-exchange property, then M has the finite exchange property.*

Now we introduce the definition of exchange rings due to Warfield.

Definition 1.4.4 ([88]) A ring R is called a *exchange ring* if the module ${}_RR$ has the (finite) exchange property.

Moreover, he proved that the above definition is left-right symmetric and provided an important connection between the modules with the finite exchange property and exchange rings (see [88]).

Theorem 1.4.5 ([88]) *A module M has the finite exchange property if and only if its endomorphism ring* $\mathrm{End}(M)$ *is an exchange ring.*

Later Monk [65] gave a ring-theoretic description of these exchange rings. This result of Monk implies that any direct product of exchange rings is again an exchange ring.

Theorem 1.4.6 ([65]) *A ring R is an exchange ring if and only if for any $a \in R$, there exist $b, c \in R$ such that $bab = b$ and $c(1-a)(1-ba) = 1-ba$*

Subsequently, Nicholson provided another characterization of exchange rings. For this characterization, recall that if I is a right (or left) ideal of R, we say that idempotents lift modulo I if, given any $a \in R$ with $a^2 - a \in I$, there exists $e^2 = e \in R$ with $e - a \in I$.

Theorem 1.4.7 ([69]) *A ring R is an exchange ring if and only if idempotents lift modulo I for every right (equivalently, left) ideal I of R.*

It is immediate from the above result that any factor ring of an exchange ring is an exchange ring. Moreover, we have:

Proposition 1.4.8 ([69]) *A ring R is an exchange ring if and only if $R/\mathrm{J}(R)$ is exchange and idempotents lift modulo $\mathrm{J}(R)$.*

This result enables us to show that the class of exchange rings is quite large and, in fact, contains all (von Neumann) regular rings. Call a ring R *semiregular* if $R/\mathrm{J}(R)$ is (von Neumann) regular and idempotents lift modulo $\mathrm{J}(R)$ (see [69]).

Proposition 1.4.9 ([69]) *Every semiregular ring is exchange.*

The next result due to Nicholson provides another class of exchange rings and gives a characterization of exchange rings among rings with central idempotents.

Proposition 1.4.10 ([69]) *The following hold:*

(1) *Every clean ring is exchange.*

(2) *An abelian ring is clean if and only if it is exchange. In particular, commutative clean rings are precisely commutative exchange rings.*

Indecomposable modules with the (finite) exchange property have a special importance. They are exactly the modules with a local endomorphism ring as the following theorem states.

Theorem 1.4.11 ([20, 88]) *The following conditions are equivalent for an indecomposable module M_R:*

(1) *The endomorphism ring of M_R is local.*

(2) *M_R has the finite exchange property.*

(3) *M_R has the exchange property.*

Theorem 1.4.12 ([56]) *Let M be a finite direct sum of indecomposable modules. If M satisfies the 2-exchange property, then M has the substitution property.*

Here we should note that, without the assumption on M in the above theorem (namely, that it is a finite direct sum of indecomposables), the 2-exchange property on M need not imply the substitution property. Nevertheless, it turns out that, in the presence of 2-exchange, we have the following result due to H.-P. Yu.

Theorem 1.4.13 ([96]) *Let M be a module with the 2-exchange (or equivalently, finite exchange) property. Then the following conditions on M are equivalent:*

(1) *M is internally cancellable.*

(2) *M is cancellable.*

(3) *M has the substitution property.*

The following result unites the various criteria introduced in the last two sections for the right R-module R_R under the additional assumption that R has the exchange property.

Theorem 1.4.14 ([86]) *Let R be an exchange ring. Then the following statements are equivalent:*

(1) $\mathrm{sr}(R) = 1$.

(2) *If $a, b \in R$ are such that $aR = bR$, then $b = au$ for some $u \in \mathrm{U}(R)$.*

(3) *R_R is internally cancellable.*

(4) *R_R is cancellable in the category of all right R-modules.*

(5) *R_R is cancellable in the category $\mathcal{P}(R)$ of finitely generated projective right R-modules.*

(6) *Every module in $\mathcal{P}(R)$ is cancellable in $\mathcal{P}(R)$.*

(7) *The left analogues of* (2), (3), (4), (5), *and* (6).

1.5 Perspectivity, Summand Sum and Intersection Properties

Continuous geometry was invented by von Neumann in the fall of 1935 in [66]. He set out to formulate suitable axioms to characterize this new structure. It happened that just previously, K. Menger and G. Birkhoff had characterized L_n (L_n denotes the lattice of all linear subsets of an $n-1$ dimensional projective geometry), by lattice-type axioms; in particular, Birkhoff had shown the structures L_n could be characterized as the complemented modular irreducible lattices which satisfy a chain condition. Von Neumann dropped the chain condition and replaced it by two of its weak consequences: (i) order completeness of the lattice, and (ii) continuity of the lattice operations. Lattices which are complemented, modular, irreducible, satisfy (i) and (ii), but do not satisfy a chain condition, were called by von Neumann: continuous geometries. One of the von Neumann's fundamental results was the construction, for an arbitrary continuous geometry, of a dimension function with values ranging over the interval $[0, 1]$. The construction was based on the definition: x and y are to be called equidimensional if x and y are in perspective relation, that is: for some w the lattice join and meet of x with w are identical with those of y with w (see [66]). This summary is taken from Halperin's foreword of von Neumann's ground-breaking book "Continuous Geometry" (see [68]).

After von Neumann's above definition, not only the perspectivity but also the transitivity of perspectivity has been studied by von Neumann and Halperin in a series of papers, see [66, 36, 67]. Holland [42] studied the perspectivity notion in orthomodular lattices. Later Handelman considered the transitivity of perspectivity in the lattice $\mathcal{L}(R)$ of principal right ideals of a von Neumann regular ring R. Handelman showed that a von Neumann regular ring R is unit-regular if and only if the transitivity holds on a two by two matrix ring over R.

Throughout this work, we place a particular emphasize on the notion of perspectivity in the context of rings and modules. But first we need to recall some lattice theoretical notions.

A relation $\leqslant$ on a set P is a *partial order* on P in case it is reflexive ($a \leqslant a$), transitive ($a \leqslant b$ and $b \leqslant c \Rightarrow a \leqslant c$), and anti-symmetric ($a \leqslant b$ and $b \leqslant a \Rightarrow a = b$). A pair $(P, \leqslant)$ consisting of a set and a partial order on the set is called a *partially ordered set* or a *poset* (see [5]).

Let P be a poset and let $A \subseteq P$. An element $e \in A$ is a *greatest* (*resp., least*) element of A in case $a \leqslant e$ (resp., $e \leqslant a$) for all $a \in A$. Not every subset of a poset has a greatest or a least element, but clearly if one does exist, it is unique. An element $b \in P$ is an *upper bound* (*resp., lower bound*) for A in case $a \leqslant b$ (resp., $b \leqslant a$) for all $a \in A$. So the greatest (resp., least) element, if it exists, is an upper (resp., lower) bound for A. If the set of upper bounds of A has a least element, it is called the *least upper bound*, *join*, or *supremum* of A; if the set of lower bounds has a greatest element, it is called the *greatest lower bound*, *meet*, or *infimum* of A. A *lattice* (*resp., complete lattice*) is a poset P in which every pair (resp., every subset) of P has both a least upper bound and a greatest lower bound in P (see [5]).

Let L be a lattice. Then each pair $a, b \in L$ has both a join and a meet in L; let us denote these by $a \vee b$ and $a \wedge b$, respectively. The lattice L is said to be *modular* in case it satisfies the modularity condition: for all $a, b, c \in L$,

$$b \leqslant a \text{ implies } a \wedge (b \vee c) = b \vee (a \wedge c).$$

The lattice L is called *bounded* in case L has two elements 0 and 1 satisfying the following conditions:

(1) for all a in L, $a \vee 1 = 1$ and $a \wedge 1 = a$;

(2) for all a in L, $a \vee 0 = a$ and $a \wedge 0 = 0$.

The elements 1 and 0 are called *top* and *bottom* of L, respectively. Furthermore, by a *complement* of an element a in a bounded lattice L, we mean an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$; and a bounded lattice L is called *complemented* if all its elements have complements (see [10]).

Definition 1.5.1 ([10]) Let L be a complemented modular lattice. Two elements a and b of L are *perspective,* denoted by $a \sim b$, if they have a common complement.

As we mentioned above, we come back to the ring and module theoretical aspects of perspectivity. Two direct summands A and B of a module M are called *perspective*, denoted by $A \sim B$, if they have a common complement, i.e., there exists a submodule C such that

$$M = A \oplus C = B \oplus C.$$

It is clear that $A \sim B$ implies $A \cong B$. In a recent work by Garg et al. [33], the modules in which any two isomorphic summands have a common complement have been studied. These modules are called *perspective modules*. Indeed, a module M is perspective exactly when, for any two summands A, B of M, $A \cong B$ implies $A \sim B$. Clearly, every perspective module is internally cancellable. Further, we have

M has the substitution property $\Rightarrow$ M is perspective $\Rightarrow$ M is internally cancellable.

However, the converse of the above implications is not true in general. See [33], for more information.

As expected, a ring R is called (right) *perspective* if the right regular module R_R is perspective. Since perspectivity is a left-right symmetric property for rings, it is enough to call such rings simply perspective [33]. Abelian rings and rings with stable range one are examples of perspective rings. As we mentioned earlier, Wang et al. [86] proved that in a ring with stable range one, every regular element is clean (see Theorem 1.3.19). Garg et al. both generalized this result and established the following characterization of perspective rings.

Theorem 1.5.2 ([33]) *For a ring R, the following conditions are equivalent:*

(1) *R is perspective.*

(2) *If $Ra + Rb = R$ for some $a, b \in R$ and if $aR \oplus X = R$ for some right ideal X of R, then $br_R(a)$ and X have a common complement.*

(3) *If $Ra + Rb = R$ for some $a, b \in R$ and if $aR \oplus X = R$ for some right ideal X of R, then there exists $e \in \mathrm{idem}(R)$, such that $eR = X$ and $a + eb$ is a unit.*

(4) *If* $aR \oplus X = R$ *for some* $a \in R$*, then* $r_R(a)$ *and* X *have a common complement.*

In particular, every regular element of a perspective ring has idempotent stable range one and is thus clean.

After the above characterization, the authors ask the following question in [33]:

Question 1.5.3 If every regular element of R has idempotent stable range one, is R perspective?

We will give a partial answer to this question in Chapter 2.

Remark 1.5.4 ([33]) In [49, Example 4.5], it was proved that in the ring $\mathbb{M}_2(\mathbb{Z})$ the element $\left(\begin{smallmatrix} 12 & 5 \\ 0 & 0 \end{smallmatrix}\right)$ is unit-regular but not clean. Hence, $\mathbb{M}_2(\mathbb{Z})$ is not a perspective ring by Theorem 1.5.2. Note further that $\mathbb{M}_2(\mathbb{Z})$ is also an example of an IC ring which is not perspective by [50, (5.9)(1)].

In the last part of this section, we recall two important classes of modules, namely, modules with summand intersection property and modules with summand sum property. Modules with summand intersection property were first studied by Kaplansky [47] and he showed that if F is a free module over a principal ideal domain R, then the intersection of any two summands of F is again a summand. This result motivated Fuchs [30] to consider the problem of characterizing abelian groups in which the intersection of two direct summands is again a summand. This problem was addressed by Wilson [92] for modules over a ring. Later, Garcia [32] studied modules M with the property that the sum of any pair of direct summands of M is a direct summand of M.

Definition 1.5.5 ([92]) A module M is said to have the *summand intersection property* (SIP, for short) if the intersection of two direct summands of M is a direct summand.

Definition 1.5.6 ([32]) A module M is said to have the *summand sum property* (SSP, for short) if the sum of two direct summands of M is again a direct summand.

Proposition 1.5.7 ([32]) *If a direct sum* $L \oplus N$ *of two modules* L *and* N *has the SSP and* $f\colon L \to N$ *a homomorphism, then* $\mathrm{im}(f)$ *is a direct summand of* N*.*

The following results concern the question of whether a quasi-projective module has the SSP (or a quasi-injective module has the SIP) can be settled by looking at the SSP of rings (see [32]). Note that we shall consider injectivity and projectivity concepts in the next two sections.

Theorem 1.5.8 ([32]) *A module M has both SSP and SIP if and only if* $\mathrm{End}(M)$ *has SSP.*

Corollary 1.5.9 ([32]) *Let M be a module with its endomorphism ring $S = \mathrm{End}_R(M)$. Then:*

(1) *If M is quasi-projective, then M has SSP if and only if* $\mathrm{End}_R(M)$ *has SSP.*

(2) *If M is quasi-injective, then M has SIP if and only if* $\mathrm{End}_R(M)$ *has SSP.*

More recently, Alkan and Harmancı [3] studied modules having the SSP and the SIP and their relations with some generalizations of quasi-injective modules and quasi-projective modules.

Theorem 1.5.10 ([3]) *A module M has SSP if and only if for every decomposition $M = A \oplus B$ and every homomorphism $f\colon A \to B$, the image of f is a direct summand of B.*

1.6 Injectivity and Related Concepts

Baer [8] initiated the study of abelian groups which are summands whenever they are subgroups. Modules which are summands of every containing module were studied by a number of authors. Eckmann and Schopf [23] introduced the terminology "injective" (see also [63]). In this section, we not only consider injective modules, but we also investigate some generalizations.

Definition 1.6.1 ([54]) A right R-module M is said to be *injective* if, for any monomorphism $f\colon A \to B$ of right R-modules and any homomorphism $g\colon A \to M$, there exists an R-homomorphism $h\colon B \to M$ such that $g = hf$.

Next, recall that a ring R is called *right self-injective* if the right regular module R_R is injective. Left self-injective rings are defined similarly.

Proposition 1.6.2 ([11]) *If M is an injective module that is also a submodule of an R-module N, then M is a direct summand of N.*

Proposition 1.6.3 ([54]) *A direct product $M = \prod_\alpha M_\alpha$ of right R-modules is injective if and only if each M_α is.*

It is not true that every direct sum of injective modules is injective. Indeed, it is precisely the right noetherian rings over which every direct sum of injective right modules is injective (see [5]).

Proposition 1.6.4 (Baer's Criterion [8]) *A right R-module M is injective if and only if, for any right ideal I of R, any R-homomorphism $f\colon I \to M$ can be extended to $f'\colon R \to M$.*

The notion of an essential extension is closely related to the concept of injectivity. First we give a characterization of injective modules in terms of essential extensions.

Theorem 1.6.5 ([76]) *A module M_R is injective if and only if it has no proper essential extensions.*

Now we present the main result of Eckmann-Schöpf and Baer on the basic theory of injective envelopes of arbitrary modules.

Theorem 1.6.6 ([76]) *Let M be an R-module. Then there exists an R-module E satisfying the following equivalent conditions:*

(1) *E is an essential injective extension of M.*

(2) *E is a maximal essential extension of M.*

(3) *E is a minimal injective extension of M.*

Moreover, if E_1 and E_2 are both essential injective extensions of M, then there is an isomorphism $\theta\colon E_1 \to E_2$ which is the identity on M.

Definition 1.6.7 ([76]) Let M be an R-module. An R-module E satisfying the conditions of Theorem 1.6.6 is called an *injective envelope* (or *injective hull*) of M; we use the symbol $\mathrm{E}(M)$ to denote an injective envelope of M.

Remark 1.6.8 Let M be a module. Sometimes it is possible to denote an injective envelope of the module M as a pair (E, i) where E is an injective R-module and $0 \to M \xrightarrow{i} E$ is an essential monomorphism, that is, $\mathrm{im}(i) \leqslant_e E$.

Proposition 1.6.9 ([5]) *Let M be an injective right R-module with its endomorphism ring $S = \mathrm{End}(M)$. Let $f \in S$. Then*

$$f \in \mathrm{J}(S) \text{ if and only if } \ker(f) \leqslant_e M.$$

Theorem 1.6.10 ([63]) *An injective module M has the cancellation property if and only if M is Dedekind-finite.*

Next we recall the notion of a quasi-injective module which generalizes that of an injective module due to Johnson and Wong [46].

Definition 1.6.11 ([54]) An R-module M is said to be *quasi-injective* if, for any submodule $N \subseteq M$, any homomorphism $g\colon N \to M$ can be extended to an endomorphism of M.

Clearly, any injective module is always quasi-injective. The converse is not true in general; it is not difficult to find a simple module which is always quasi-injective but need not always be injective. Furthermore, a direct summand of a quasi-injective module is always quasi-injective. However, in general, a direct sum of two quasi-injective modules need not be quasi-injective (see [54]).

The following result is an interesting characterization of quasi-injective modules M in terms of its injective envelope $\mathrm{E}(M)$.

Theorem 1.6.12 ([54]) *A module M_R is quasi-injective if and only if M is fully invariant in $\mathrm{E}(M)$ i.e., M is stabilized by every endomorphism of $\mathrm{E}(M)$.*

The following theorem was first proved by Warfield [87] for injective modules, and the proof was generalized to quasi-injective modules by Fuchs [29].

Theorem 1.6.13 ([29]) *Every quasi-injective module M has the exchange property.*

Utumi studied continuity concept for rings in a series of papers (see [82, 83, 84]) and introduced three conditions for a ring. These conditions were extended to modules by Jeremy [45] and Mohamed and Bouhy[61], as follows.

Definitions 1.6.14 ([63]) A module M is called a *Ci-module* if it satisfies the following Ci-conditions.
$C1$: Every submodule of M is essential in a direct summand of M.
$C2$: Whenever A and B are submodules of M such that $A \cong B$ and B is a direct summand of M, then A is a direct summand of M.
$C3$: Whenever A and B are direct summands of M with $A \cap B = 0$, then $A + B$ is a direct summand of M.

Here there is a point that needs mentioning. $C1$-modules are also known as *extending modules* or *CS-modules* (complements are summands), and $C2$-modules are also known as *direct-injective modules* in the literature. Moreover, a ring R is said to be *right CS* if the right regular module R_R is $C1$, and is said to be *right* $C2$ (*right* $C3$, respectively) if R_R is $C2$ ($C3$, respectively).

Definitions 1.6.15 ([63]) Let M be a module. M is called *continuous* if it satisfies both the $C1$- and $C2$-conditions, and is called *quasi-continuous* if it satisfies both the $C1$- and $C3$-conditions.

Now we should note the following hierarchy of the above-mentioned definitions for modules:

$$\text{Injective} \Rightarrow \text{quasi-injective} \Rightarrow \text{continuous} \Rightarrow \text{quasi-continuous} \Rightarrow C1.$$

Further, it is well known that every $C2$-module is a $C3$-module, and each of the Ci-properties of modules is inherited by direct summands, and thus direct summands of (quasi-) continuous modules are (quasi-) continuous. However, in general, a direct sum of (quasi-) continuous modules need not be (quasi-) continuous (see [63] for more information).

Continuous and quasi-continuous modules have a particular importance because of their relationship with the exchange property. To establish this relationship, the following decomposition theorem is crucial. First, recall that a module M is called *(summand-)*

square-free if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$ (and Y_1, $Y_2 \subseteq^{\oplus} M$), then $Y_1 = Y_2 = 0$.

Theorem 1.6.16 ([62]) *If M is a quasi-continuous module, then we can write $M = M_1 \oplus M_2$ where M_1 is quasi-injective and M_2 is square-free.*

Theorem 1.6.17 ([62]) *Every continuous module has the exchange property.*

Unfortunately, quasi-continuous modules do not necessarily enjoy the finite exchange property (e.g., the abelian group $\mathbb{Z}$), but when they do then they also have full exchange property.

Theorem 1.6.18 ([64, 73]) *Every quasi-continuous module with the finite exchange property has the exchange property.*

We conclude this section by a result of Nicholson and Yousif that will be needed later.

Proposition 1.6.19 ([71]) *The following conditions are equivalent for a local ring R:*

(1) *R is a right $C2$-ring.*

(2) $\mathrm{J}(R) = \{a \in R \mid r_R(a) \neq 0\}$.

In particular, any local ring with nil Jacobson radical is a right and left $C2$-ring.

1.7 Projectivity and Related Concepts

The notion of projective module was introduced by Cartan and Eilenberg in their revolutionary book "Homological Algebra" (see [91]). Major contributions to this concept were made by Kaplansky and Bass. In this section, we review various forms of projectivity.

Definition 1.7.1 ([54]) A right R-module M is said to be *projective* if, for any epimorphism $f\colon B \to C$ of right R-modules and any homomorphism $g\colon M \to C$, there exists an R-homomorphism $h\colon M \to B$ such that $g = fh$.

An injective R-module M is a direct summand of each R-module N that extends M. Projective modules enjoy a dual property.

Proposition 1.7.2 ([11]) *If $f\colon N \to M$ is an epimorphism and M is a projective R-module, then M is isomorphic to a direct summand of N.*

Proposition 1.7.3 ([54]) *A direct sum $M = \oplus_\alpha M_\alpha$ of right R-modules is projective if and only if each summand M_α is projective.*

Note that a ring is a projective module over itself. Moreover, a *free module*, that is, a module isomorphic to a (possibly infinite) direct sum of copies of R_R is a projective module. On the other hand, the direct product of projective modules need not be projective in general. For example, the direct product $M = \mathbb{Z}\times\mathbb{Z}\times\cdots$ is not a projective $\mathbb{Z}$-module. This example is attributed to R. Baer [54]. The next result provides a basic characterization of a projective module M in terms of its (first) dual $M^* := \mathrm{Hom}_R(M, R)$.

Lemma 1.7.4 (Dual Basis Lemma [54]) *A right R-module M is projective if and only if there exist a family of elements $\{a_i\colon i \in I\} \subseteq M$ and linear functionals $\{f_i\colon i \in I\} \subseteq M^*$ such that, for any $a \in P$, $f_i(a) = 0$ for almost all i, and $a = \sum_i a_i f_i(a)$.*

It is well known that every R-module M is the homomorphic image of a projective module. Among the projective modules that "cover" M, there may be one that is, in some sense, minimal. Such a cover of M, if it exists, can be viewed as a "best approximation" of M by a projective module (see [11]).

Definition 1.7.5 ([5]) A projective R-module P is a *projective cover* of M if there exists an epimorphism $p\colon P \to M$ with small kernel, i.e., $\ker(p) \ll P$.

Remark 1.7.6 Let M be a module. It is possible to denote a projective cover of the module M as a pair (P, p) where P is a projective R-module and $P \xrightarrow{p} M \to 0$ is a small epimorphism, that is, $\ker(p) \ll P$.

Proposition 1.7.7 ([5]) *Let R be a ring. If M is a projective right R-module, then* $\mathrm{rad}(M) = M\mathrm{J}(R)$.

Proposition 1.7.8 ([5]) *Let M be a projective right R-module with its endomorphism ring $S = \mathrm{End}(M)$. Let $f \in S$. Then*

$$f \in \mathrm{J}(S) \text{ if and only if } \mathrm{im}(f) \ll M.$$

The following lemma will be used frequently in Chapter 2.

Lemma 1.7.9 (Nicholson's Lemma [69, Lemma 2.8]) *Let P be a projective module over any ring R and let A and B submodules of P such that $P = A + B$. If A is a direct summand of P, then there exists a submodule $C \subseteq B$ such that $P = A \oplus C$.*

Definition 1.7.10 ([54]) An R-module M is said to be *quasi-projective* if, for any quotient module Q of M, any homomorphism $g\colon M \to Q$ can be lifted to an endomorphism of M.

Clearly, any projective module is always quasi-projective. The converse is not true in general; it is not difficult to find a simple module which is always quasi-projective but need not always be projective. Furthermore, a direct summand of a quasi-projective module is always quasi-projective. However, in general, a direct sum of two quasi-projective modules need not be quasi-projective.

Recently, the above result of Nicholson has been generalized to π-projective modules. Recall that a module M is said to be *π-projective* if, for every two submodules U, V of M with $U + V = M$, there exists $f \in \mathrm{End}(M)$ with $\mathrm{im}(f) \subseteq U$ and $\mathrm{im}(1 - f) \subseteq V$. It is easy to see that every quasi-projective module is also π-projective. Hollow (and hence local) modules trivially have this property (see [93]).

There are modules that fail to have a projective cover, this brings up the question "Are there rings over which every module has a projective cover?" Such rings do indeed exist. In the process, we first describe rings over which every finitely generated module has a projective cover (see [11]).

Definition 1.7.11 ([55]) A ring R is called *semiperfect* if R is semilocal, and idempotents of $R/\mathrm{J}(R)$ can be lifted to R.

Theorem 1.7.12 ([55]) *A ring R is semiperfect if and only if every finitely generated right R-module has a projective cover.*

Our next goal is to introduce the notion of left and right perfect rings. This depends on a new notion of nilpotency called T-nilpotency, where the letter "T" apparently stands for "transfinite" (see [55]).

Definition 1.7.13 ([55]) A subset A of a ring R is called *left* (resp., *right*) *T-nilpotent* if, for any sequence of elements $\{a_1, a_2, a_3, \dots\} \subseteq A$, there exists an integer $n \geqslant 1$ such that $a_1 a_2 \dots a_n = 0$ (resp., $a_n \dots a_2 a_1 = 0$).

Definition 1.7.14 ([55]) A ring R is called *right* (resp., *left*) *perfect* if $R/\mathrm{J}(R)$ is semisimple and $\mathrm{J}(R)$ is right (resp., left) T-nilpotent. If R is both left and right perfect, we call R a perfect ring.

Theorem 1.7.15 ([55]) *A ring R is right perfect if and only if every right R-module has a projective cover.*

Now we introduce concepts dual to continuity and quasi-continuity. For this, we need the following definitions.

Definitions 1.7.16 ([63]) A module M is called a *Di-module* if it satisfies the following Di-conditions.
$D1$: For every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2$ is small in M_2.
$D2$: Whenever A and B are submodules of M with $M/A \cong B$ and B is a direct summand of M, then A is a direct summand of M.
$D3$: Whenever A and B are direct summands of M with $A + B = M$, then $A \cap B$ is a direct summand of M.

It is worthwhile to note that $D1$-modules are also known as lifting modules, and $D2$-modules are also known as direct-projective modules.

Definitions 1.7.17 ([63]) Let M be a module. M is called *discrete* if it is both a $D1$- and a $D2$-module, and is called *quasi-discrete* if it is both a $D1$- and a $D3$-module.

It is clear that continuity generalizes injectivity. On the other hand, discreteness generalizes projectivity if and only if the ring is perfect. Hence, in contrast to the hierarchy of the injectivity and its related concepts, we have the following:

$$\text{Projective} \;\Rightarrow\; \text{quasi-projective} \;\not\Rightarrow\; \text{discrete} \;\Rightarrow\; \text{quasi-discrete} \;\Rightarrow\; D1.$$

It is widely known that every quasi-projective module is a $D2$-module, and every $D2$-module is a $D3$-module. Each of the Di-properties of modules is inherited by direct summands, and thus direct summands of (quasi-) discrete modules are (quasi-) discrete. However, in general, a direct sum of (quasi-) discrete modules need not be (quasi-) discrete (see [63] for more information).

We end this section by a decomposition theorem for quasi-discrete modules and its application.

Theorem 1.7.18 ([62]) *A quasi-discrete module M has a decomposition, unique up to isomorphism, $M = \oplus_{i\in I} H_i$, where each H_i is hollow; moreover if M is discrete, then each H_i has a local endomorphism ring.*

Corollary 1.7.19 ([62]) *A quasi-discrete module M has the exchange property if and only if every hollow summand of M has a local endomorphism ring.*

2 INTERNALLY CANCELLABLE RINGS

A right R-module M is said to be *internally cancellable* if, whenever $M = M_1 \oplus M_2 = N_1 \oplus N_2$ with $M_1 \cong N_1$, then $M_2 \cong N_2$. This property of modules was first considered by Ehrlich [24] and Handelman [37]; they proved independently that for a module M with a regular endomorphism ring $S = \mathrm{End}_R(M_R)$,

$$M \text{ is internally cancellable if and only if } S \text{ is unit-regular.}$$

In fact, the regularity condition on the endomorphism ring was not necessary. Guralnick and Lanski [35] dropped this condition, and showed that

$$M \text{ is internally cancellable if and only if every regular element in } S \text{ is unit-regular.}$$

There is another result of Guralnick and Lanski [35] which provide a characterization for internally cancellable modules in terms of "pseudo-similarity" in the endomorphism rings of these modules.

All of the characterizations mentioned above address the class of rings that are endomorphism rings of internally cancellable modules. Following [50], a ring R is said to be *internally cancellable* (IC, for short) in case the right R-module R_R is internally cancellable. The class of IC rings is quite large and contains abelian rings (i.e., rings with all idempotents central), unit-regular rings, and right artinian rings by [50]. It follows from the above result due to Guralnick and Lanski that, for any ring R, R_R is IC if and only if every regular element in R is unit-regular, and hence the internal cancellation property of rings is left-right symmetric.

First, we record a well-known characterization of IC rings in terms of isomorphic idempotents.

Theorem 2.0.1 ([18]) *The following are equivalent for a ring R:*

(1) *R is an IC ring.*

(2) *Given idempotents e, $f \in R$, if $eR \cong fR$, then $(1-e)R \cong (1-f)R$.*

(3) *Given idempotents e, $f \in R$, if $eR \cong fR$, then e and f are conjugates, that is, $ueu^{-1} = f$ for some $u \in U(R)$.*

(4) *The left analogues of* (2) *and* (3).

Next, consider the following characterization of IC rings where

$$(\diamondsuit): Ra + Rb = R \text{ implies that } a + xb \in \mathrm{U}(R) \text{ for some } x \in R.$$

Lemma 2.0.2 ([79], [50]) *The following are equivalent for a ring R:*

(1) *R is IC.*

(2) *For each $a \in \mathrm{reg}(R)$ and $b \in R$, $(\diamondsuit)$ holds.*

(3) *For each $a, b \in \mathrm{reg}(R)$, $(\diamondsuit)$ holds.*

(4) *For each $a \in \mathrm{reg}(R)$ and $b \in \mathrm{idem}(R)$, $(\diamondsuit)$ holds.*

(5) *For each $a \in R$ and $b \in \mathrm{idem}(R)$, $(\diamondsuit)$ holds.*

(6) *For each $a \in R$ and $b \in \mathrm{reg}(R)$, $(\diamondsuit)$ holds.*

The first four conditions in Lemma 2.0.2 were given by Song et al. in [79], and they define a ring R to be *regularly stable* if R satisfies the condition (3) in Lemma 2.0.2. Other conditions were given by Khurana and Lam [50], and they define a ring R to have *regular stable range one* (written $\mathrm{rsr}(R) = 1$) if R satisfies the condition (2) in Lemma 2.0.2, since the condition (2) means that every regular element has stable range one.

Now let us summarize the following hierarchy of the rings we have considered so far.

$$R \text{ is unit-regular} \implies isr(R) = 1 \implies sr(R) = 1 \implies rsr(R) = 1 \iff R \text{ is IC}$$

Note that the converse of the above implications are true when the ring is regular by Corollary 1.3.15. On the other hand, if R is both exchange and IC, then $sr(R) = 1$ by Theorem 1.4.14.

This chapter seeks to continue the study of internally cancellable rings and find some new characterizations. In the first section, inspired by Lemma 2.0.2, we consider the following condition:

$(*)$: $Ra + Rb = R$ implies that $a + xb \in \mathrm{U}(R)$ and $aR \cap xR = 0$ for some $x \in R$,

where the elements $a, b \in R$ are to be quantified. There are nine combinations, and we obtain new characterizations of unit-regular rings (Corollary 2.1.4) and IC rings (Theorem 2.1.5) via these combinations. It is observed that $(*)$ and $(\diamondsuit)$ have different behaviors. The cases where the element $a + xb$ is unit-regular in the condition $(\diamondsuit)$ are also considered, and then rings with stable range one (Theorem 2.1.9) and IC rings (Theorem 2.1.10) are characterized.

In Section 2.2, we consider IC rings with the summand sum property. Following [33], a ring R is called *perspective* if any two isomorphic direct summands of R_R have a common complement, i.e., if $eR \cong fR$ for any $e, f \in \mathrm{idem}(R)$, then there exists a direct summand C of R_R such that $R = eR \oplus C = fR \oplus C$. Perspective rings include abelian rings, rings with stable range one, and right or left quasi-duo rings (see [33, Section 2 and Corollary 4.8]). Clearly, any perspective ring is IC. We prove that an IC ring with SSP is a perspective ring (Theorem 2.2.2). This generalizes Handelman's result [37, Theorem 2] saying that unit regular rings (which are both IC and SSP) are always perspective. On the other hand, it gives a partial answer to a question posed in [33] (Corollary 2.2.6).

Section 2.3 includes a direct proof to the result [15, Corollary 2.7] saying that for any abelian ring R and for any $a \in \mathrm{reg}(R)$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \mathrm{idem}(R)$, $u \in \mathrm{U}(R)$ is given (Theorem 2.3.2). As a final result, IC rings are characterized by special clean elements (Proposition 2.3.5).

2.1 Unit-Regular Elements and Internal Cancellation

Consider the following statement:

$$(*):\ Ra + Rb = R \Rightarrow a + xb \in \mathrm{U}(R) \text{ and } aR \cap xR = 0 \text{ for some } x \in R,$$

where the elements $a, b \in R$ are to be quantified. We deal with the nine combinations arising from the quantifiers "for all", "for all regular elements", and "for all idempotents elements" for each of a and b.

First we recall two theorems from Khurana and Lam [50].

Theorem 2.1.1 [50, Theorem 3.2] *If a is a unit-regular element in a ring R, then* $\mathrm{sr}(a) = 1$.

Proof. Consider any $a \in \mathrm{ureg}(R)$, and let b be any element of R such that $Ra+Rb = R$. Since a is unit-regular, Ra is a direct summand of ${}_RR$. Then there exists a left ideal $C \subseteq Rb$ such that $R = Ra \oplus C$ by Nicholson's Lemma. Write (uniquely) $1 = e_1 + f_1$ where $e_1 \in Ra$ and $f_1 \in C$. Then e_1, f_1 are complementary idempotents with $Ra = Re_1$ and $C = Rf_1$. Thus, Fact 1.3.12 implies that $a = u_1e_1$ for some $u_1 \in \mathrm{U}(R)$. Writing $f_1 = yb$ for some $y \in R$, and left-multiplying $1 = e_1 + f_1$ by u_1, we get $a + u_1yb = u_1 \in \mathrm{U}(R)$. This checks that $\mathrm{sr}(a) = 1$. □

Theorem 2.1.2 [50, Theorem 3.5] *Let R be a ring and $a \in \mathrm{reg}(R)$. Then a is unit-regular if and only if* $\mathrm{sr}(a) = 1$.

Proof. The "only if" part is true for any element $a \in R$, by Theorem 2.1.1. For the "if" part, write $a = axa$ for some $x \in R$, and assume that $\mathrm{sr}(a) = 1$. The following familiar argument is from the proof of [19, (4.12)]. In view of $Ra + R(1 - xa) = R$, we get an element $y \in R$ such that $a + y(1 - xa) \in \mathrm{U}(R)$. Letting u be the inverse of this unit, we have

$$a = axa = au\,[a + y(1 - xa)]\,xa = auaxa = aua,$$

so we have $a \in \mathrm{ureg}(R)$. □

On the other hand, any unit regular element in a ring R need not have idempotent stable range 1. Khurana and Lam [49, Example 4.5] showed that in the ring $\mathbb{M}_2(\mathbb{Z})$ the element $A = \left(\begin{smallmatrix} 12 & 5 \\ 0 & 0 \end{smallmatrix}\right)$ is unit-regular but not clean. As we mentioned in the preliminary chapter, every element having idempotent stable range 1 is clean. Hence, $\mathrm{isr}(A) \neq 1$.

Now we characterize unit-regular elements.

Theorem 2.1.3 *For any element a in a ring R, the following are equivalent:*

(1) *a is unit-regular.*

(2) *Whenever $Ra + Rb = R$ with $b \in R$, there exists $x \in R$ such that $a + xb \in \mathrm{U}(R)$ and $aR \cap xR = 0$.*

(3) *Whenever $Ra + Rb = R$ with $b \in \mathrm{reg}(R)$, there exists $x \in R$ such that $a + xb \in \mathrm{U}(R)$ and $aR \cap xR = 0$.*

(4) *Whenever* $Ra + Rb = R$ *with* $b \in \mathrm{idem}(R)$, *there exists* $x \in R$ *such that* $a + xb \in \mathrm{U}(R)$ *and* $aR \cap xR = 0$.

Proof. (1) $\Rightarrow$ (2) Assume a is unit-regular and let $Ra + Rb = R$. Since Ra is a direct summand of ${}_RR$, there exists $B \subseteq Rb$ such that $Ra \oplus B = R$ by Nicholson's Lemma. Then we can write $1 = e + f$ where $Ra = Re$ and $B = Rf$ for $e, f \in \mathrm{idem}(R)$. Since a is unit-regular, there exists a unit u in R such that $a = ue$ by Fact 1.3.12. Write $f = rb$ for some $r \in R$. Then $a + (urbr)b = ue + uf = u$. Finally, we show that $aR \cap urbrR = 0$. For, if $ax = urbry$ for some x, $y \in R$, then $ex = fry = (1 - e)ry = 0$, and hence $ax = urbry = 0$.
(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (1) Since $Ra + R(1) = R$, there exists $x \in R$ such that $a + x = u \in \mathrm{U(R)}$ and $aR \cap xR = 0$ by hypothesis. Now we can follow the proof of [12, Theorem 1]. Multiplying $a - u = -x$ by $u^{-1}a$ from the right gives that $au^{-1}a - a = -xu^{-1}a \in aR \cap xR = 0$. Thus, $au^{-1}a = a$, i.e., a is unit-regular. □

Theorem 2.1.3 immediately gives the following corollary that characterizes rings satisfying $(*)$ "for each $a, b \in R$", or "for each $a \in R$ and $b \in \mathrm{reg}(R)$", or "for each $a \in R$ and $b \in \mathrm{idem}(R)$". As is well-known, the class of rings with idempotent stable range one is properly contained in the class of rings with stable range one, but with the additional condition "$aR \cap xR = 0$", we see that these classes coincide. Note that the equivalence of the conditions (1) and (2) in Corollary 2.1.4 is due to Wang et al. [86, Theorem 3.6].

Corollary 2.1.4 *The following are equivalent for a ring* R*:*

(1) R *is unit-regular.*

(2) *Whenever* $Ra + Rb = R$ *with* $a, b \in R$, *there exists* $e \in \mathrm{idem}(R)$ *such that* $a + eb \in \mathrm{U}(R)$ *and* $aR \cap eR = 0$.

(3) *Whenever* $Ra + Rb = R$ *with* $a, b \in R$, *there exists* $x \in R$ *such that* $a + xb \in \mathrm{U}(R)$ *and* $aR \cap xR = 0$.

(4) *Whenever* $Ra + Rb = R$ *with* $a \in R$ *and* $b \in \mathrm{reg}(R)$, *there exists* $x \in R$ *such that* $a + xb \in \mathrm{U}(R)$ *and* $aR \cap xR = 0$.

(5) *Whenever* $Ra + Rb = R$ *with* $a \in R$ *and* $b \in \mathrm{idem}(R)$*, there exists* $x \in R$ *such that* $a + xb \in \mathrm{U}(R)$ *and* $aR \cap xR = 0$*.*

Next, we will consider the element "a" as regular in $(*)$ whenever $b \in R$, $b \in \mathrm{reg}(R)$, or $b \in \mathrm{idem}(R)$ and characterize IC rings. Compare it with Lemma 2.0.2.

Theorem 2.1.5 *The following are equivalent for a ring* R*:*

(1) R *is IC.*

(2) *Whenever* $Ra + Rb = R$ *with* $a \in \mathrm{reg}(R)$ *and* $b \in R$*, there exists* $x \in R$ *such that* $a + xb \in \mathrm{U}(R)$ *and* $aR \cap xR = 0$*.*

(3) *Whenever* $Ra + Rb = R$ *with* $a \in \mathrm{reg}(R)$ *and* $b \in \mathrm{reg}(R)$*, there exists* $x \in R$ *such that* $a + xb \in \mathrm{U}(R)$ *and* $aR \cap xR = 0$*.*

(4) *Whenever* $Ra + Rb = R$ *with* $a \in \mathrm{reg}(R)$ *and* $b \in \mathrm{idem}(R)$*, there exists* $x \in R$ *such that* $a + xb \in \mathrm{U}(R)$ *and* $aR \cap xR = 0$*.*

Proof. (1) $\Rightarrow$ (2) is true by Theorem 2.1.3 since every regular element is unit-regular. (2) $\Rightarrow$ (1) Every regular element has stable range one by hypothesis. Thus, R is IC by Theorem 2.0.2. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (2) Let $Ra + Rb = R$ where $a \in \mathrm{reg}(R)$ and $b \in R$. Since Ra is a direct summand of R, there exists $e \in \mathrm{idem}(R)$ with $Re \subseteq Rb$ such that $R = Ra \oplus Re$ by Nicholson's Lemma. By assumption, there exists $x \in R$ such that $a + xe \in \mathrm{U}(R)$ and $aR \cap xR = 0$. Let $e = rb$ for some $r \in R$. Thus, $a + (xr)b \in \mathrm{U}(R)$ and $aR \cap xrR = 0$. □

Now we consider the case where the elements "a" and "b" are idempotents in $(*)$ and see that this always holds.

Theorem 2.1.6 *For any idempotents* e, f *in a ring* R*, if* $Re + Rf = R$*, then there exists a unit-regular element* $x \in Rf$ *such that* $e + xf \in \mathrm{U}(R)$ *and* $eR \cap xR = 0$*.*

Proof. Assume that $Re + Rf = R$ where $e, f \in \mathrm{idem}(R)$. By Nicholson's Lemma, there exists $g \in \mathrm{idem}(R)$ such that $Rg \subseteq Rf$ and $Re \oplus Rg = R$. Write $1 = t_1 + t_2$ where $t_1, t_2 \in \mathrm{idem}(R)$, $t_1 \in Re$, and $t_2 \in Rg$. Then $Re = Rt_1$ and $Rg = Rt_2$. Let u and v be units such that $e = ut_1$ and $t_2 = vg$ by Fact 1.3.12. Since $g \in Rf$, $gf = g$. Now $e + uvgf = e + uvg = ut_1 + ut_2 = u$. Clearly, $eR \cap uvgR = 0$. Thus, we are done with $x = uvg$. □

Remark 2.1.7 In other words, Theorem 2.1.6 says that $(*)$ always holds for each $a \in \text{idem}(R)$ and $b \in \text{idem}(R)$. This condition is also equivalent to the following conditions.

(i) $(*)$ holds for each $a \in \text{idem}(R)$ and $b \in R$.
(ii) $(*)$ holds for each $a \in \text{idem}(R)$ and $b \in \text{reg}(R)$.

It is enough to prove that if $(*)$ holds for each $a \in \text{idem}(R)$ and $b \in \text{idem}(R)$, then (i) holds. Let $Ra + Rb = R$ where $a \in \text{idem}(R)$ and $b \in R$. According to Nicholson's Lemma, there exists $B \subseteq Rb$ such that $R = Ra \oplus B$. Take $B = Re$ where $e \in \text{idem}(R)$. Then there exists $x \in R$ such that $a + xe \in \text{U}(R)$ and $aR \cap xR = 0$ by hypothesis. Write $e = rb$ where $r \in R$. Thus $a + (xr)b \in \text{U}(R)$ and $aR \cap xrR = 0$. Hence, (i) holds.

The Case Where "$a + xb$" is Unit-Regular

Any unit element in a ring is clearly unit-regular. Based upon this fact, one can consider the cases where the element $a+xb$ is unit-regular instead of being unit in the statement $(\diamondsuit)$. We see that rings with stable range one and IC rings can also be characterized with the cases including unit-regular elements.

Theorem 2.1.8 *For any element a in a ring R, the following are equivalent:*

(1) *a has stable range one.*

(2) *Whenever $Ra + Rb = R$ with $b \in R$, there exists an element x in R such that $a + xb$ is* ***unit-regular****.*

Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (1) Let $Ra + Rb = R$. By assumption, there exists $x \in R$ such that $a + xb = ug$ where $u \in \text{U}(R)$ and $g \in \text{idem}(R)$. Then $Rg + Rb = R$. Since any idempotent has stable range one by Theorem 2.1.1, there exists $y \in R$ such that $g + yb = v$ where $v \in \text{U}(R)$. It follows that $a + xb = ug = uv - uyb$, and so $a + (x + uy)b = uv$ which is a unit in R. Therefore, a has stable range one. □

The next corollary generalizes [89, Theorem 2.4] by removing the unnecessary "exchange" assumption placed on R.

Corollary 2.1.9 *The following are equivalent for a ring R:*

(1) $sr(R) = 1$.

(2) *Whenever $Ra + Rb = R$, there exists an element x in R such that $a + xb$ is* ***unit-regular****.*

Lemma 2.1.10 *Let $a \in \mathrm{reg}(R)$. Then, a is unit-regular if and only if whenever $Ra + Rb = R$, there exists an element x in R such that $a + xb$ is* ***unit-regular****.*

Proof. If a is unit-regular, then we can take $x = 0$ to prove the necessity. For the sufficiency, we have $\mathrm{sr}(a) = 1$ by Theorem 2.1.8. Since $a \in \mathrm{reg}(R)$, it is unit-regular by Theorem 2.1.2. □

Now Lemma 2.0.2 and Lemma 2.1.10 together give the following result.

Theorem 2.1.11 *The following are equivalent for a ring R:*

(1) *R is IC.*

(2) *For each $a \in \mathrm{reg}(R)$ and $b \in R$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb$ is* ***unit-regular****.*

(3) *For each $a \in \mathrm{reg}(R)$ and $b \in \mathrm{reg}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb$ is* ***unit-regular****.*

(4) *For each $a \in \mathrm{reg}(R)$ and $b \in \mathrm{idem}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb$ is* ***unit-regular****.*

(5) *For each $a \in R$ and $b \in \mathrm{reg}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb$ is* ***unit-regular****.*

(6) *For each $a \in R$ and $b \in \mathrm{idem}(R)$, if $Ra + Rb = R$, then there exists $x \in R$ such that $a + xb$ is* ***unit-regular****.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 2.0.2. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.
(4) $\Rightarrow$ (1) Let $a \in \mathrm{reg}(R)$. Using Lemma 2.1.10, we will show that a is unit-regular. Let $Ra + Rb = R$. Then $Ra \oplus Rf = R$ for some $f \in \mathrm{idem}(R)$ with $Rf \subseteq Rb$. By hypothesis, there exists an element y in R such that $a + yf$ is unit-regular. If $f = rb$,

then $a + yf = a + yrb$ is unit-regular. This proves (1).

(1) $\Rightarrow$ (5) follows from Lemma 2.0.2.

(5) $\Rightarrow$ (6) is trivial.

(6) $\Rightarrow$ (1) Let $Ra + Rf = R$ where $a \in R$ and $f \in \text{idem}(R)$. To complete the proof, according to Lemma 2.0.2, it suffices to show that there exists an element x in R such that $a + xf \in \text{U}(R)$. Since $a + yf$ is unit-regular for some $y \in R$ by hypothesis, write $a + yf = vg$ where $v \in \text{U}(R)$, $g \in \text{idem}(R)$. Then $a \in Rg + Rf$, and so $R = Rg + Rf$. According to Theorem 2.1.6, there exists $z \in R$ such that $g + zf = u \in \text{U}(R)$. Then $a + yf + vzf = vu$, and hence $a + (y + vz)f = vu \in \text{U}(R)$, as desired. □

In Theorem 2.1.11, we can add trivially the condition "$aR \cap xR = 0$" to the items (2-4). However, when we consider Corollary 2.1.9 and the items (5) and (6) of Theorem 2.1.11, the following question arises.

Question 2.1.12 What is the structure of a ring R with the property that whenever $Ra + Rb = R$ with "$a \in R$" and "$b \in R$ or $b \in \text{reg}(R)$ or $b \in \text{idem}(R)$", there exists $x \in R$ such that $a + xb$ is ***unit-regular*** and $aR \cap xR = 0$?

2.2 Internal Cancellation with Summand Sum Property

This section will be devoted to IC rings with SSP. As we mentioned in Section 1.5, any perspective ring is IC. However, the converse does not hold in general (see Remark 1.5.4). Indeed, as Garg et al. states that it is not difficult to see that the converse is true if eR and ueR (equivalently, Re and Reu) have a common complement for every $e \in \text{idem}(R)$ and $u \in \text{U}(R)$ [33].

We first recall the following result of Handelman because it has a close connection with the main result (Theorem 2.2.2) of this section.

Theorem 2.2.1 ([37, Theorem 2]) *For a regular ring R, the following are equivalent:*

(1) *R is unit-regular.*

(2) *For idempotents e, f in R, $eR \cong fR$ implies $(1 - e)R \cong (1 - f)R$.*

(3) *For idempotents e, f in R, $eR \cong fR$ implies there exists a unit x such that $x^{-1}ex = f$*

(4) *If* $I \oplus J, I' \oplus K \in \mathcal{L}(R)$*, and also* $I \oplus J \cong I' \oplus K$ *and* $I \cong I'$*, then* $J \cong K$*.*

(5) *For* $I, J \in \mathcal{L}(R)$*,* $I \cong J$ *implies* I *is perspective to* J*.*

(6) R *satisfies the cancellation law for finitely generated projective modules.*

Now we prove our main result. On one hand, this generalizes Handelman's above result [37, Theorem 2] saying that unit regular rings (which are both IC and SSP) are always perspective. On the other hand, it gives a partial answer to the Question 1.5.3 posed in [33] (See Corollary 2.2.6).

Theorem 2.2.2 *If* R *is an IC ring with SSP, then* R *is perspective.*

Proof. Let e, f be idempotents in R such that $eR \cong fR$. Since R has SSP, $eR + fR$ is a direct summand of R_R, and hence projective. According to Nicholson's Lemma, there exist idempotents g and h in R such that $eR + fR = eR \oplus gR = hR \oplus fR$ where $gR \subseteq fR$ and $hR \subseteq eR$. Write $R = (eR + fR) \oplus T$ for some right ideal T of R. Then $R = eR \oplus gR \oplus T = hR \oplus fR \oplus T$. Since R is IC, $gR \cong hR$. Let φ be the isomorphism from gR to hR. Since $gR \cap hR = 0$, a routine argument shows that $gR \oplus hR = gR \oplus C = C \oplus hR$ where $C = \{\, x + \varphi(x) \,|\, x \in gR \,\}$. Then it follows that $eR+fR = eR \oplus C = C \oplus fR$, and hence $R = (eR+fR) \oplus T = eR \oplus C \oplus T = fR \oplus C \oplus T$. Thus, eR and fR have a common complement, and so R is perspective. □

Example 2.2.3 SSP is not superfluous in Theorem 2.2.2; let $R = \mathbb{M}_2(\mathbb{Z})$. As $R \cong \mathrm{End}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z})$ and $\mathbb{Z} \oplus \mathbb{Z}$ is an IC $\mathbb{Z}$-module, it follows that the ring R is IC. As we mentioned earlier, the element $\left(\begin{smallmatrix} 12 & 5 \\ 0 & 0 \end{smallmatrix}\right)$ is unit-regular but not clean. Hence, R is not perspective by Remark 1.5.4. On the other hand, $\mathbb{Z} \oplus \mathbb{Z}$ does not have SSP as a $\mathbb{Z}$-module by Proposition 1.5.7. Thus, R does not have SSP by Corollary 1.5.9.

Recall that a module M is called *perspective* if any two isomorphic direct summands have a common direct complement [33]. A module-theoretic property $\mathcal{P}$ is called *an endomorphism ring property* ("ER-property", for short) if for any module M_R, M_R has $\mathcal{P}$ if and only if $\mathrm{End}_R(M)$ has $\mathcal{P}$ as a module over itself (see [56, 8.1] for details). According to [56, Proposition 8.5] and [33, Theorem 3.4], the module properties "internal cancellation" and "perspectivity" are ER-properties. Thereby, we get the following module-theoretic version of Theorem 2.2.2.

Corollary 2.2.4 *If M is a quasi-projective*[1] *right R-module with SSP, then the following are equivalent:*

(1) *M satisfies internal cancellation.*

(2) *M is perspective.*

Proof. We should note that if M is quasi-projective, then M has SSP if and only if $\mathrm{End}_R(M)$ has SSP by Corollary 1.5.9. Hence an application of Theorem 2.2.2 finishes the proof. □

A version of the following corollary was proved by Chen in [15, Theorem 2.4]. Here we will be able to prove this result by effectively dropping the unnecessary regularity condition on b.

Corollary 2.2.5 *Let R be a ring with SSP. Then the following are equivalent:*

(1) *R is IC.*

(2) *Whenever $Ra + Rb = R$ with $a \in \mathrm{reg}(R)$ and $b \in R$, there exists $e \in \mathrm{idem}(R)$ such that $a + eb \in \mathrm{U}(R)$ and $aR \cap eR = 0$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in \mathrm{reg}(R)$ and $b \in R$ such that $Ra + Rb = R$. Then $R = aR \oplus T$ for some right ideal T. According to Theorem 2.2.2, R is perspective, and so there exists $e \in \mathrm{idem}(R)$ such that $eR = T$ and $a + eb \in \mathrm{U}(R)$ by Theorem 1.5.2. This completes the proof.
(2) $\Rightarrow$ (1) Let $a \in \mathrm{reg}(R)$. Since $Ra + R(-1) = R$, there exists $e \in \mathrm{idem}(R)$ such that $a - e \in \mathrm{U}(R)$ and $aR \cap eR = 0$. Then a is unit-regular, and hence R is IC. □

In [33], the authors ask the following question:

If every regular element of R has idempotent stable range one, then is R perspective?

According to Theorem 2.2.2, we have a partial answer as follows.

[1] This can be replaced by (D3) condition by [3, Theorem 22]. Recall that M has (D3) if for any direct summands A and B of M with $A + B = M$, then $A \cap B$ is a direct summand in M. Any quasi-projective module has (D3).

Corollary 2.2.6 *Let R be a ring with SSP. Then the following are equivalent:*

(1) *R is perspective.*

(2) *Every regular element of R has idempotent stable range one.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 1.5.2(3).
(2) $\Rightarrow$ (1) Since every regular element has idempotent stable range one, R is IC by Lemma 2.0.2(2). Now the perspectivity of R is an immediate consequence of Theorem 2.2.2. $\square$

It is well known that IC rings are exactly the rings in which every regular element is unit-regular. Recently, Chen et al. have studied the rings in which the product of two regular elements is unit-regular as an extension of unit-regular rings [16]. Actually, this new class of rings coincides with the class of IC rings with SSP, as the following lemma shows.

Lemma 2.2.7 [16, Lemma 2.2] *Let R be a ring. Then the product of two regular elements in R is unit-regular if and only if R is an IC ring with SSP.*

Proof. ($\Rightarrow$) Suppose that the product of two regular elements in R is unit-regular. Then, in particular, every regular element in R is unit-regular. This implies that R is IC. Let $e, f \in R$ be idempotents. Then, $(1-e)f \in R$ is unit-regular. Hence, $(1-e)fR \subseteq^{\oplus} R$. Write $(1-e)fR \oplus A = R$. Then, $(1-e)R = (1-e)fR \oplus ((1-e)R \cap A)$. Hence, since $eR + fR = eR \oplus (1-e)fR$, we get $R = eR \oplus (1-e)fR \oplus ((1-e)R \cap A) = (eR + fR) \oplus ((1-e)R \cap A)$. This shows that R has SSP.
($\Leftarrow$) Let $a, b \in R$ be regular. Since R is IC, a and b are unit-regular. Write $a = ue$, $b = fv$, where $e, f \in R$ are idempotents and $u, v \in \mathrm{U}(R)$. Then, $ab = u(ef)v$. Since R has SSP, we see that $(1-e)R + fR \subseteq^{\oplus} R$, i.e., $(1-e)R \oplus efR \subseteq^{\oplus} R$. This implies that $efR \subseteq^{\oplus} R$, and so, $ef \in R$ is regular. By hypothesis, $ef \in R$ is unit-regular. Therefore, $ab \in R$ is unit-regular and hence the result follows. $\square$

Proposition 2.2.8 [16, Proposition 2.3] *Let R be an IC ring. Then the following are equivalent:*

(1) *R has SSP.*

(2) *Every product of two idempotents is unit-regular.*

(3) *Every finite product of idempotents is unit-regular.*

Proof. (1) $\Rightarrow$ (2) Since every idempotent is regular, this is clear.
(2) $\Rightarrow$ (1) It is enough to apply ($\Rightarrow$) part of the proof of Lemma 2.2.7.
(1) $\Leftrightarrow$ (3) This follows from (1) $\Leftrightarrow$ (2) and induction. □

2.3 Special Clean Elements

Following [1], a ring R is called *special clean* if every element a can be decomposed as the sum of a unit u and an idempotent e with $aR \cap eR = 0$. The Camillo-Khurana Theorem in [12] states that R is unit-regular if and only if R is a special clean ring.

Inspired by this notion, we call an element a in R *special clean* if there exists a decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{idem}(R)$ and $u \in \text{U}(R)$. It is easy to see that any special clean element is unit-regular (see (4) $\Rightarrow$ (1) in Theorem 2.1.3). This gives the following fact for a ring R:

$$\text{Every regular element is special clean} \Longrightarrow \text{IC}$$

This implication is irreversible, because the ring $R = \mathbb{M}_2(\mathbb{Z})$ is IC, but it has a unit-regular element which is not clean (see Example 2.2.3).

Chen proved that the converse of the above implication is true if R has SSP [15, Theorem 2.4]. After that result abelian rings are considered to obtain a corollary, that is, if R is an abelian ring, then for any $a \in \text{reg}(R)$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \text{idem}(R)$ and $u \in \text{U}(R)$ [15, Corollary 2.7]. The proof of Chen involves SSP and IC property of abelian rings and a technical result. We will offer an independent and more elementary proof here. First, we need the following lemma.

Lemma 2.3.1 *Any left non zero-divisor regular element in an abelian ring is a unit.*

Proof. Let R be an abelian ring and x a left non zero-divisor regular element of R. Let $y \in R$ be such that $xyx = x$. Then $x(1 - yx) = 0$ implies that $yx = 1$. On the other hand, since $e = xy$ is an idempotent, $x = x^2y$, and $x(1 - xy) = 0$. Hence $xy = 1$, and so x is a unit. □

Theorem 2.3.2 *Let R be an abelian ring. Then, for every $a \in \mathrm{reg}(R)$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \mathrm{idem}(R)$ and $u \in \mathrm{U}(R)$.*

Proof. Let $a \in \mathrm{reg}(R)$. Then the right annihilator of a, $ann_r(a)$, is equal to eR for some idempotent $e \in R$. Write $a = e + (a - e)$. First, we claim that $a - e$ is a left non zero-divisor. Let $r \in R$ be such that $(a - e)r = 0$. Since $ae = 0$, $a(1 - e) = a$. Then, $0 = (1 - e)(a - e)r = (1 - e)ar = a(1 - e)r = ar$. So $r \in eR$. On the other hand, $0 = e(a - e)r = -er$. This gives $(1 - e)r = r - er = r$. Then $r \in (1 - e)R$, and hence $r = 0$. Next, we show that $a - e$ is regular. Since a is regular, there exists b such that $aba = a$. Let $b' = bab$. Then $ab'a = a$, and hence $(a-e)(b'-e)(a-e) = a-e$. According to Lemma 2.3.1, $a - e$ is a unit. It is easy to see that $aR \cap eR = 0$. The uniqueness follows from the proof of [2, Proposition 5.1], and we include it here for completeness. Let $a = e + u = e' + u'$ where $e, e' \in \mathrm{idem}(R)$, $u, u' \in \mathrm{U}(R)$ with $aR \cap eR = 0$ and $aR \cap e'R = 0$. Then $au^{-1} = eu^{-1} + 1$ and $au^{-1}(1 - e) = 1 - e \in aR$ since R is abelian. Also, $e'(1 - e) = (1 - e)e' \in e'R \cap aR = 0$, and so $e'(1 - e) = (1 - e)e' = 0$. Since R is abelian, $e = e'e = ee' = e'$, and hence $u = a - e = a - e' = u'$. □

Recently, it has been proved that the converse of the above theorem is also true by Chen et al. [16]. We include it here for completeness.

Theorem 2.3.3 [16, Corollary 3.7] *Let R be a ring. If, for every $a \in \mathrm{reg}(R)$, there exists a unique decomposition $a = e + u$ such that $aR \cap eR = 0$ where $e \in \mathrm{idem}(R)$ and $u \in \mathrm{U}(R)$, then R is abelian.*

Proof. Let $e \in R$ be an idempotent, and let $x \in R$. Choose $a = e - 1$. Then, $a \in R$ is regular. Clearly, $e + ex(1 - e) \in R$ is an idempotent. It is easy to verify that $a - e, a - (e + ex(1 - e)) \in \mathrm{U}(R)$ with $aR \cap eR = aR \cap (e + ex(1 - e))R = 0$. The uniqueness forces that $e + ex(1 - e) = e$, hence, $ex = exe$. Likewise, $xe = exe$. Accordingly, $ex = xe$, which completes the proof. □

Theorem 2.3.2 cannot be generalized to perspective rings: let $R = \mathbb{M}_3(\mathbb{Z}_2)$ be the ring of 3×3 matrices over $\mathbb{Z}_2$. It is a perspective ring since the stable range of $\mathbb{M}_3(\mathbb{Z}_2)$ is one, but the idempotent $a = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ can be written in two different ways: $a = e+u = f+v$, $e, f \in \mathrm{idem}(R)$, $u^2 = 1$, $v^2 = 1$, $aR \cap eR = 0$, and $aR \cap fR = 0$ where $e = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $u = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$.

Corollary 2.3.4 [2, Proposition 5.1] *If R is abelian, then R is unit-regular if and only if for every $a \in R$, there exists a unique decomposition $a = e+u$ such that $aR \cap eR = 0$ where $e \in \mathrm{idem}(R)$ and $u \in \mathrm{U}(R)$.*

Chin and Qua proved that R is unit-regular if and only if for any $a \in R$, there exists $e \in \mathrm{idem}(R)$ such that $a - e$ or $a + e$ is unit and $aR \cap eR = 0$ in [17, Theorem 2.2]. It can also be observed that if $a + e$ is unit and $aR \cap eR = 0$, then a is unit-regular. Thus, by the proof of Theorem 2.3.2, we see that if R is an abelian ring, then for any regular element $a \in R$, there exists a unique $e \in \mathrm{idem}(R)$ such that $a - e$ or $a + e$ is unit in R and $aR \cap eR = 0$. Moreover, one can see that the converse of this statement is also true by an argument similar to that used in the proof of Theorem 2.3.3.

We conclude this section with the following characterization of IC rings.

Proposition 2.3.5 *The following are equivalent for a ring R:*

(1) *R is IC.*

(2) *For every $a \in \mathrm{reg}(R)$, there exists $u \in \mathrm{U}(R)$ such that au is special clean.*

Proof. (1) $\Rightarrow$ (2) Let $a \in \mathrm{reg}(R)$. Since $Ra + R(-1) = R$, there exist $x \in R$ and $v \in \mathrm{U}(R)$ such that $a+x(-1) = v$ and $aR \cap xR = 0$ by Theorem 2.1.5. Then $R = aR \oplus xR$. Let $g \in \mathrm{idem}(R)$ be such that $aR = (1 - g)R$ and $xR = gR$. Since x is regular and R is IC, x is unit-regular. Then there exists $u \in \mathrm{U}(R)$ such that $xu = g$ by Fact 1.3.12. By considering $a + x(-1) = v$, we see that $au = g + vu$, $g \in \mathrm{idem}(R)$, $vu \in \mathrm{U}(R)$, and $auR \cap gR = 0$. Hence, au is special clean.
(2) $\Rightarrow$ (1) Let a be a regular element of R. Then there exists $u \in \mathrm{U}(R)$ such that au is special clean. Since special clean elements are unit-regular, au is unit-regular. Hence, a is unit-regular. $\square$

3 LIFTING STABLE RANGE ONE ELEMENTS

Lifting of some special elements modulo an ideal of a ring is a quite substantial subject in ring theory. The structure of many classes of rings, including exchange, semiperfect, and semiregular rings is described in terms of lifting idempotents (for a detailed account of this, see for example [5] and [69]). On the other hand, a particular emphasis has been placed on lifting units by Menal and Moncasi [60] for certain types of self-injective rings; by Perera [75] for exchange rings and certain classes of C*-algebras with real rank zero; and by Šter [80] for clean rings. Recently, Khurana et al. [51], besides lifting of idempotents and units, have considered lifting of different types of elements; such as (von Neumann) regular elements, unit-regular elements, conjugate idempotents, etc.

In this chapter, inspired by the work in [51], we present lifting of elements having (idempotent) stable range one modulo an ideal and investigate several properties and applications of such ideals. As we mentioned earlier, the concept of stable range was introduced by Bass [9] in the context of algebraic K-theory and the simplest case of stable range one has attracted attention (see for example [14, 31, 34, 40, 48, 56, 85]). A new characterization of rings with stable range one is provided in this chapter, too.

Now, it is useful to state all the lifting properties in terms of some special classes together for having a complete interpretation.

Let I be an ideal of a ring R and $\mathcal{C}(R)$ be a class of some elements having a property $\mathcal{C}$ in R. An element a in R is called $\mathcal{C}$ *lifting modulo* I if, whenever $a+I \in \mathcal{C}(R/I)$, then there exists $b \in \mathcal{C}(R)$ such that $a+I = b+I$. The ideal I is called $\mathcal{C}$*-lifting* if every element of R is $\mathcal{C}$ *lifting modulo* I. Throughout this chapter, we will consider the following classes.

$\mathrm{U}(R) = \{x \in R \,|\, x \text{ is a unit}\}$ $\qquad$ $\mathrm{idem}(R) = \{x \in R \,|\, x \text{ is an idempotent}\}$

$\mathrm{reg}(R) = \{x \in R \,|\, x \text{ is regular}\}$ $\qquad$ $\mathrm{ureg}(R) = \{x \in R \,|\, x \text{ is unit-regular}\}$

$\mathcal{SR}_1(R) = \{x \in R \,|\, \mathrm{sr}(x) = 1\}$ $\qquad$ $\mathcal{ISR}_1(R) = \{x \in R \,|\, \mathrm{isr}(x) = 1\}$

For clarity, an ideal I is called an *idempotent lifting* in case I is $\mathrm{idem}(R)$-lifting; *unit lifting* if I is $\mathrm{U}(R)$-lifting; *regular lifting* if I is $\mathrm{reg}(R)$-lifting; *unit-regular lifting* if I is $\mathrm{ureg}(R)$-lifting *stable range one lifting* if I is $\mathcal{SR}_1(R)$-lifting; and *idempotent stable range one lifting* if I is $\mathcal{ISR}_1(R)$-lifting.

The first section of this chapter is devoted to stable range one lifting ideals and Section 2 is concerned with idempotent stable range one lifting ideals. We obtain that these two lifting conditions properly imply lifting of units modulo an ideal. It is well-known that the Jacobson radical $\mathrm{J}(R)$ of a ring R is always unit lifting. Further, we see that it is also stable range one lifting (Corollary 3.1.4). Moreover, if R is a regular ring, then an ideal I is unit lifting if and only if I is stable range one lifting (Proposition 3.1.8). If R is a left or a right duo ring, then every ideal is stable range one lifting if and only if $\mathrm{sr}(R) = 1$ (Theorem 3.1.10). Among other results, it is proved in Section 2 that if I is an idempotent stable range one lifting ideal such that R/I is perspective, then it is regular lifting (Theorem 3.2.11). This result yields some important corollaries. We characterize rings with idempotent stable range one, that is, we prove that $\mathrm{isr}(R) = 1$ iff $\mathrm{isr}(R/I) = 1$ and I is idempotent stable range one lifting for any ideal I contained in the Jacobson radical (Corollary 3.2.12). The Jacobson radical $\mathrm{J}(R)$ is idempotent stable range one lifting in case it is idempotent lifting (Corollary 3.2.3). The converse of this statement is true if R is a left quasi-duo ring (Corollary 3.2.14). Last but not least, we prove that if R is a (right and left) duo ring, then every ideal is idempotent stable range one lifting iff every ideal is regular lifting iff $\mathrm{isr}(R) = 1$ iff R is exchange (Theorem 3.2.15).

3.1 Stable Range One Lifting Ideals

In this section, we introduce stable range one lifting ideals. First, recall that an element $a \in R$ is said to have *stable range one* (written $\mathrm{sr}(a) = 1$) if, whenever $Ra + Rb = R$ for any $b \in R$, then there exists $x \in R$ such that $a + xb \in \mathrm{U}(R)$. Clearly, every unit element and every element in the Jacobson radical $\mathrm{J}(R)$ of a ring R have stable range one.

Definition 3.1.1 Let I be an ideal of a ring R. I is called a *stable range one lifting ideal* if, for every $a \in R$ with $a + I \in \mathcal{SR}_1(R/I)$, there exists $b \in \mathcal{SR}_1(R)$ such that $a + I = b + I$.

Obviously, the trivial ideals of R are stable range one lifting. On the other hand, if $\mathrm{sr}(R) = 1$, then every ideal of R is stable range one lifting. Local rings, unit-regular

rings, and semilocal rings are some examples of rings with stable range one, see [85] for more examples.

Example 3.1.2 Consider the ring $\mathbb{Z}$. We first note that the only elements of $\mathbb{Z}$ with stable range one are $0, 1, -1$, because if a is an integer different from $0, 1, -1$, then one can choose $b \in \mathbb{Z}$ such that $gcd(a, b) = 1$, $b \not| 1 - a$, and $b \not| 1 + a$, so that $a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$, but there do not exist $x \in \mathbb{Z}$ such that $a + xb = 1$ or $a + xb = -1$
This fact gives immediately that the ideals $2\mathbb{Z}$ and $3\mathbb{Z}$ are stable range one lifting. But the ideal $4\mathbb{Z}$ is not stable range one lifting. To see this, consider the element $2 + 4\mathbb{Z}$. Clearly, $\mathrm{sr}(2+4\mathbb{Z}) = 1$, but there do not exist $b \in \mathbb{Z}$ with $2+4\mathbb{Z} = b+4\mathbb{Z}$ and $\mathrm{sr}(b) = 1$. Indeed, b cannot be $0, 1$, or -1.
More generally, if $n \geq 4$, then $n\mathbb{Z}$ is not stable range one lifting. To show the last sentence, one can consider the unit-regular elements different from $0 + n\mathbb{Z}, 1 + n\mathbb{Z}$, and $-1 + n\mathbb{Z}$ in the ring $\mathbb{Z}/n\mathbb{Z}$.

Recall that a two-sided ideal I of a ring R is called a *radical ideal* if $1 + x \in \mathrm{U}(R)$ for every $x \in I$ (see [90]). Obviously, every ring has a largest radical ideal, namely, the Jacobson radical of R.

Proposition 3.1.3 *Let I be a proper ideal of a ring R. For any $a \in R$, if $\mathrm{sr}(a) = 1$ in R, then $\mathrm{sr}(a + I) = 1$ in R/I. The converse is true if I is a radical ideal.*

Proof. Let $a \in R$. Set $\overline{a} := a+I$ and $\overline{R} := R/I$. Assume that $\mathrm{sr}(a) = 1$. If $\overline{R}\overline{a}+\overline{Rb} = \overline{R}$, then $Ra + Rb + I = R$. Then we can find $r, s \in R$ and $y \in I$ such that $1 = ra + sb + y$. This implies that $Ra + R(sb + y) = R$. By assumption, there exists $x \in R$ such that $a + x(sb + y) \in \mathrm{U}(R)$. Hence $\overline{a} + \overline{x}(\overline{sb + y}) = \overline{a} + (\overline{xs})\overline{b} \in \mathrm{U}(\overline{R})$. Thus $\mathrm{sr}(a + I) = 1$.
For the converse, assume that I is a radical ideal and $\mathrm{sr}(a + I) = 1$. If $Ra + Rb = R$, then there exists $x \in R$ such that $\overline{a} + \overline{x}\overline{b} = \overline{u}$ where $u \in \mathrm{U}(R)$. This implies that $a + xb = u + j$ for some $j \in I$. Since I is a radical ideal, $1 + u^{-1}j$ is a unit. It follows that $a + xb = u + j = u(1 + u^{-1}j)$ is a unit, too. Thus, $\mathrm{sr}(a) = 1$. □

It is well known that the Jacobson radical $\mathrm{J}(R)$ of a ring R need not be idempotent lifting, but the following result shows an interesting property of $\mathrm{J}(R)$. Recall that $\mathrm{sr}(R) = 1$ iff $\mathrm{sr}(R/\mathrm{J}(R)) = 1$.

Corollary 3.1.4 *Any radical ideal of a ring R is stable range one lifting. In particular,* $\mathrm{J}(R)$ *is stable range one lifting.*

Proof. Let I be a radical ideal and $a \in R$ with $\overline{a} = a + I \in \mathcal{SR}_1(R/I)$. We claim that $a \in \mathcal{SR}_1(R)$. Suppose that $Ra + Rb = R$ for some $b \in R$. Then there exist $x \in R$ and $\overline{u} \in \mathrm{U}(R/I)$ such that $\overline{a} + \overline{x}\overline{b} = \overline{u}$. Let $\overline{v}$ be the inverse of $\overline{u}$. Multiplying the last equality on the left by $\overline{v}$, we get $\overline{va} + \overline{vx}\overline{b} = \overline{1}$. Since $v(a + xb) - 1 \in I$, $v(a + xb)$ is invertible in R by assumption. This implies that $a + xb$ is left invertible. Similarly, the multiplication on the right by $\overline{v}$ will imply that $a + xb$ is right invertible. Hence $a \in \mathcal{SR}_1(R)$. □

Lemma 3.1.5 *Let $\varphi : R \longrightarrow S$ be a ring isomorphism. If* $\mathrm{sr}(a) = 1$ *in R, then* $\mathrm{sr}(\varphi(a)) = 1$ *in S.*

Example 3.1.6 Let $R = \{(x_1, \ldots, x_n, s, s, \ldots) \,|\, x_1, \ldots, x_n \in \mathbb{Q}, s \in \mathbb{Z}, n \geq 1\}$. Then R is a commutative ring with $\mathrm{J}(R) = 0$ and every regular element of R is unit-regular by [50, Remark 6.6]. Further, every regular element of R has idempotent stable range one by Theorem 1.5.2. Set $I := \{(x_1, \ldots, x_n, 0, 0, \ldots) \,|\, x_1, \ldots, x_n \in \mathbb{Q}, n \geq 1\}$. Then $R/I \cong \mathbb{Z}$ via the map $\varphi : (x_1, \ldots, x_n, s, s, \ldots) + I \mapsto s$. For any $a \in R$, we claim that

$$sr(a) = 1 \Longleftrightarrow a \text{ is unit-regular.}$$

Assume that $sr(a) = 1$. Proposition 3.1.3 implies that $sr(a+I) = 1$, and then $sr(\varphi(a+I)) = 1$ in $\mathbb{Z}$ by Lemma 3.1.5. So $\varphi(a + I) = 0, 1$, or -1. This implies that $a = (x_1, \ldots, x_n, s, s, \ldots)$ where $s = 0, 1$, or -1, and so a is unit-regular. Since any unit-regular element has stable range one by Theorem 2.1.1, the claim follows.

Now we show that the ideal I is stable range one lifting. For this, assume that a is an element in R such that $sr(a+I) = 1$. As in the above discussion, a is unit-regular, and hence $sr(a) = 1$.

Lemma 3.1.7 *Any stable range one lifting ideal is unit lifting.*

Proof. Let I be a stable range one lifting ideal of a ring R. Take an invertible element $\overline{a} \in R/I$ with the inverse $\overline{b}$. Since $\overline{a}$ is unit-regular, $\mathrm{sr}(\overline{a}) = 1$. By hypothesis, there exists an $x \in R$ such that $\overline{a} = \overline{x}$ and $\mathrm{sr}(x) = 1$. Then $\overline{b}\overline{a} = \overline{b}\overline{x} = \overline{1}$, and so $c := 1 - bx \in I$. This

gives us that $Rx + Rc = R$. Since $\mathrm{sr}(x) = 1$, there exists $y \in R$ such that $x + yc = v$ is a unit in R, and hence $\overline{a} = \overline{x} = \overline{x + yc} = \overline{v}$. Thus v is the required element. □

The converse of Lemma 3.1.7 is not true in general, because the ideal $4\mathbb{Z}$ of $\mathbb{Z}$ is unit lifting ($\overline{1}$ and $\overline{3} = \overline{-1}$ in $\mathbb{Z}/4\mathbb{Z}$ lift to units) but not stable range one lifting by Example 3.1.2. In the following result, we see that the converse of Lemma 3.1.7 is true if R is a regular ring.

Proposition 3.1.8 *If R is a regular ring, then any ideal I of a ring R is unit lifting if and only if it is stable range one lifting.*

Proof. Assume that I is unit lifting and let $\overline{a} \in \mathcal{SR}_1(R/I)$. Then the regular element $\overline{a}$ is unit-regular by the fact that a regular element has stable range one iff it is unit-regular (see Theorem 2.1.1 and Theorem 2.1.2). Since any regular ring is exchange, every ideal is idempotent lifting by Theorem 1.4.7. Hence I is unit lifting and idempotent lifting. This is equivalent to saying that I is unit-regular lifting by [51, Theorem 6.2]. Thus there exists a unit-regular element $b \in R$ such that $\overline{a} = \overline{b}$. Since b is unit-regular, $\mathrm{sr}(b) = 1$, and so b is the desired element. □

Bacella showed in [6, Lemma 3.5] that a regular ring R with an ideal I is unit-regular if and only if eRe is unit-regular for every idempotent $e \in I$, R/I is unit-regular, and I is unit lifting. Hence Proposition 3.1.8 implies that "unit lifting" can be interchanged with "stable range one lifting" in Bacella's result.

In [51, Theorem 6.2], the authors proved that an ideal is unit-regular lifting if and only if it is both unit lifting and idempotent lifting. This leads us to suspect that stable range one lifting ideals need not be unit-regular lifting. For this, it suffices to consider a ring R such that $\mathrm{J}(R)$ is not idempotent lifting (see Example 3.2.16), and thus $\mathrm{J}(R)$ is the required ideal by Corollary 3.1.4. The following corollary is immediate by the fact that over a regular (or an exchange) ring any ideal is idempotent lifting by Theorem 1.4.7.

Corollary 3.1.9 *If R is a regular ring, then any ideal I of a ring R is stable range one lifting if and only if it is unit-regular lifting.*

Let R be a commutative ring. Then $\mathrm{sr}(R) = 1$ iff the natural map $\mathrm{U}(R) \to \mathrm{U}(R/I)$ is an epimorphism for every ideal I of R by [25, Lemma 6.1]. The latter condition

actually means that every ideal I is unit lifting. More generally, Siddique proved in [78, Theorem 3] that $\mathrm{sr}(R) = 1$ if and only if every left unit lifts modulo every principal left ideal, i.e., if $ba - 1 \in Rc$ for some $a, b, c \in R$, there exists a left unit $u \in R$ such that $a - u \in Rc$.

Now we will characterize a ring R with $\mathrm{sr}(R) = 1$ whenever R is a left or a right duo ring. Recall that a ring R is called *left duo* if every left ideal is a right ideal; equivalently $aR \subseteq Ra$ for every $a \in R$. Right duo rings can be defined analogously. If R is a left and right duo ring, then we say that R is a *duo ring*. Any left duo ring is directly finite (see, for example [74, Corollary 1.11]).

Theorem 3.1.10 *If R is a left or a right duo ring, then the following are equivalent:*

(1) *Every ideal of R is stable range one lifting.*

(2) *Every ideal of R is unit lifting.*

(3) $\mathrm{sr}(R) = 1$.

Proof. (3) $\Rightarrow$ (1) $\Rightarrow$ (2) are obvious.
(2) $\Rightarrow$ (3) Assume that R is left duo. It is enough to show that every left unit lifts modulo every principal ideal left ideal by [78, Theorem 3]. Let $ba - 1 \in Rc$ for some $a, b, c \in R$. Then Rc is an ideal of R and hence R/Rc is a left duo ring. It follows that R/Rc is directly finite. Hence, $a + Rc$ is a unit. The hypothesis implies that there exists a unit $u \in R$ such that $a - u \in Rc$. Thus, $\mathrm{sr}(R) = 1$. By the left-right symmetry of stable range one condition for rings, the right duo case has a similar proof. □

Proposition 3.1.11 *Let I and K be ideals of a ring R with $I \subseteq K$. If K is stable range one lifting, then K/I is stable range one lifting. The converse is true if, in addition, I is stable range one lifting.*

Proof. Assume that K is stable range one lifting. Let $a \in R$ with $\mathrm{sr}(a + I + K/I) = 1$. The mapping $\varphi : \frac{R/I}{K/I} \to R/K$, defined by $\varphi(r + I + K/I) = r + K$ for every $r \in R$, is a ring isomorphism, so that $\mathrm{sr}(a + K) = 1$ by Lemma 3.1.5. By hypothesis, we can find an element $b \in \mathcal{SR}_1(R)$ such that $a + K = b + K$. Further, $b + I \in \mathcal{SR}_1(R/I)$ by Proposition 3.1.3. Thus $a + I + K/I = b + I + K/I$ and $\mathrm{sr}(b + I) = 1$.

Conversely, assume that I and K/I are stable range one lifting. Let $a \in R$ with $\mathrm{sr}(a+K) = 1$. By the above isomorphism and Lemma 3.1.5, $\mathrm{sr}(a+I+K/I) = 1$. Since K/I is stable range one lifting, there exists $b+I \in \mathcal{SR}_1(R/I)$ such that $a+I+K/I = b+I+K/I$. Then $a-b \in K$. Since I is stable range one lifting, there exists $c \in \mathcal{SR}_1(R)$ such that $b+I = c+I$. Now $b-c \in I \subseteq K$ implies that $a-c \in K$. Hence $a+K = c+K$ and $c \in \mathcal{SR}_1(R)$. □

If $I \subseteq K$ and K is stable range one lifting, then I need not be stable range one lifting. For example, take $I = 4\mathbb{Z}$ and $K = 2\mathbb{Z}$ in $\mathbb{Z}$.

Following [97], we denote by $\delta_r := \delta(R_R)$ the ideal which is the intersection of all essential maximal right ideals of a ring R. Clearly, $\mathrm{J}(R) \subseteq \delta_r$ and $S_r \subseteq \delta_r$ (see also [97, Lemma 1.9]). In view of [97, Corollary 1.7], $\mathrm{J}(R/S_r) = \delta_r/S_r$; in particular, R is semisimple if and only if $\delta_r = R$. Now as a consequence of Proposition 3.1.11 and Corollary 3.1.4, we can get the following result.

Corollary 3.1.12 (1) *δ_r is stable range one lifting if and only if $\delta_r/\mathrm{J}(R)$ is stable range one lifting in $R/\mathrm{J}(R)$.*
(2) *If S_r is stable range one lifting, then δ_r is stable range one lifting.*

Proof. Take $I = \mathrm{J}(R)$ and $K = \delta_r$ for (1) and $I = S_r$ and $K = \delta_r$ for (2) in Proposition 3.1.11. □

Note that, for any ring R, S_r is always idempotent lifting by [7], but it need not be stable range one lifting as the following example shows.

Example 3.1.13 [59, Example 1] Let F be a field, V_F a countably infinite-dimensional vector space, and $Q = \mathrm{End}_F(V)$. Then there exists a regular directly infinite subring R of Q such that $\mathrm{soc}(Q) = \mathrm{soc}(R) \subseteq R$ and $R/\mathrm{soc}(Q)$ is a field. On the other hand, Baccella proved in [6, Lemma 3.4] that for a subring T of Q with $\mathrm{soc}(Q) \subseteq T$, T is directly finite if and only if $T/\mathrm{soc}(T)$ is directly finite and $\mathrm{soc}(T)$ is unit lifting. From this result we deduce that $\mathrm{soc}(Q)$ is not unit lifting. Hence $\mathrm{soc}(Q)$ is not stable range one lifting by Proposition 3.1.7. We further note that $\mathrm{soc}(Q) = \mathrm{soc}(R) = \delta({}_RR) = \delta(R_R)$.

At the end of this section, we discuss about the symmetry of stable range one lifting ideals.

Remark 3.1.14 In this chapter, we define stable range one elements by considering principal left ideals. There is a symmetric right version: an element $a \in R$ is said to have *right stable range one* if, for any $b \in R$, $aR+bR = R$ implies that $a+bx \in \mathrm{U}(R)$ for some $x \in R$. To avoid ambiguity about left and right versions of this definition, we will subscript the notation by l or r. It is not known yet whether the left version of element-wise stable range one condition is equivalent to that of right one. The best result in this direction is that if a is a regular element, then $\mathrm{sr}_l(a) = 1$ iff a is unit-regular iff $\mathrm{sr}_r(a) = 1$ in [50].

There is a more natural situation in which the left-right symmetry of stable range one element and stable range one lifting ideal can occur.

Lemma 3.1.15 *Let R be a duo ring. Then $sr_l(a) = 1$ iff $sr_r(a) = 1$ for any $a \in R$.*

Proof. Assume that $\mathrm{sr}_l(a) = 1$. Write $aR + bR = R$. Since R is left duo, $Ra + Rb = R$. Then there exist an element $x \in R$ such that $a + xb = u \in \mathrm{U}(R)$. Since $Rb \subseteq bR$, $xb = by$ for some $y \in R$. Hence $a + by = u$, i.e., $\mathrm{sr}_r(a) = 1$. □

Corollary 3.1.16 *Let R be a duo ring and $I \lhd R$. Then the following are equivalent:*

(1) *I is right stable range one lifting.*

(2) *I is left stable range one lifting.*

Proof. It is enough to note that if R is a duo ring, then R/I is a duo ring by [74, Proposition 1.4]. □

3.2 Idempotent Stable Range One Lifting Ideals

As we mentioned earlier, an element $a \in R$ is said to have *idempotent stable range one* if, whenever $Ra + Rb = R$ for any $b \in R$, then there exists $e \in \mathrm{idem}(R)$ such that $a + eb \in \mathrm{U}(R)$. Obviously, if u is a unit in R, then $\mathrm{isr}(u) = 1$. Further, if R is a Dedekind-finite ring, then $\mathrm{isr}(0) = 1$.

Recently, Wang et al. discovered that every regular element has idempotent stable range one over a ring R with $\mathrm{sr}(R) = 1$ (see Theorem 1.3.19). As a consequence of this, they showed that any unit-regular ring has idempotent stable range one (see

Corollary 1.3.20). However, by [49], this result does not hold on the element-wise level i.e., unit-regular elements need not have idempotent stable range one. This was shown by finding a unit-regular element which is not clean, i.e., not a sum of an idempotent and a unit. Indeed, any element $a \in R$ with $\text{isr}(a) = 1$ is clean. This can easily be seen by considering the equality $Ra + R(-1) = R$.

Definition 3.2.1 Let I be an ideal of a ring R. If for any element $a \in R$ with $a + I \in \mathcal{ISR}_1(R/I)$, there exists $b \in \mathcal{ISR}_1(R)$ such that $a + I = b + I$, then I is called an *idempotent stable range one lifting ideal*.

In case $\text{isr}(R) = 1$, then every ideal of R is idempotent stable range one lifting. Obviously, if $\text{isr}(R) = 1$, then $\text{sr}(R) = 1$. In this vein, one can ask the following question:

Is any idempotent stable range one lifting ideal stable range one lifting?

We do not have an answer to this question yet, but the converse will be answered in the negative in Example 3.2.16.

Proposition 3.2.2 *Let I be a radical ideal of a ring R. If* $\text{isr}(a) = 1$*, then* $\text{isr}(a+I) = 1$ *for any $a \in R$. The converse is true if, in addition, I is idempotent lifting.*

Proof. Let $a \in R$ with $\text{isr}(a) = 1$. Set $\overline{a} := a + I$ and $\overline{R} := R/I$. Assume that $\overline{R}\overline{a} + \overline{R}\overline{b} = \overline{R}$. Then $Ra + Rb + I = R$. Since I is a radical ideal, $I \subseteq \text{J}(R)$. Now I is a small left ideal of R, and hence $Ra + Rb = R$. By assumption, there exists $e^2 = e \in R$ such that $a + eb$ is a unit in R. Thus $\overline{a} + \overline{e}\overline{b}$ is a unit in R/I.
For the converse, assume in addition that I is idempotent lifting. Let $a \in R$ with $\text{isr}(\overline{a}) = 1$, and let $Ra + Rb = R$. Since I is idempotent lifting, there exists $e \in \text{idem}(R)$ and $\overline{u} \in \text{U}(R/I)$ such that $\overline{a} + \overline{e}\overline{b} = \overline{u}$. Let $\overline{v}$ be the inverse of $\overline{u}$. Multiplying the last equality by $\overline{v}$ on the left, we obtain $\overline{va} + \overline{ve}\overline{b} = \overline{1}$. Since $1 - v(a + eb) \in I$, $v(a + eb)$ is invertible in R. This implies that $a + eb$ is left invertible. Similarly, the multiplication by v on the right will give that $a + eb$ is right invertible. Hence $\text{isr}(a) = 1$. □

Corollary 3.2.3 *Any idempotent lifting radical ideal I of a ring R is idempotent stable range one lifting.*

Remark 3.2.4 Idempotent lifting condition in Corollary 3.2.3 is not superfluous: There exists a ring R such that $\mathrm{J}(R)$ is neither idempotent lifting nor idempotent stable range one lifting by Example 3.2.16.

Remark 3.2.5 The radial ideal condition on I in Corollary 3.2.3 is not superfluous: For example, the ideal $I = 4\mathbb{Z}$ in the ring $R = \mathbb{Z}$ is idempotent lifting but it is not idempotent stable range one lifting. To see this, consider $2 + 4\mathbb{Z}$ in $\mathbb{Z}/4\mathbb{Z}$. Clearly, $\mathrm{isr}(2 + 4\mathbb{Z}) = 1$, but $2 + 4\mathbb{Z}$ does not lift to an idempotent stable range one element in $\mathbb{Z}$ by Example 3.1.2. Observe that I is not a radical ideal.

Example 3.2.6 Let $R = \{(x_1, \ldots, x_n, s, s, \ldots) \,|\, x_1, \ldots, x_n \in \mathbb{Q}, s \in \mathbb{Z}, n \geq 1\}$. By Example 3.1.6, being a unit-regular element is equivalent to being an element with stable range one in R. Since every regular element of R has idempotent stable range one by Theorem 1.5.2, we get that, for any $a \in R$,

$$\mathrm{sr}(a) = 1 \iff a \text{ is unit-regular} \iff \mathrm{isr}(a) = 1.$$

Further, the ideal I in Example 3.1.6 is idempotent stable range one lifting because $R/I \cong \mathbb{Z}$ and the only elements with idempotent stable range one of the ring $\mathbb{Z}$ are $0, 1$ and -1.

Lemma 3.2.7 *Any idempotent stable range one lifting ideal is unit lifting.*

Proof. We proceed with the same argument as in the proof of Lemma 3.1.7. Let I be an idempotent stable range one lifting ideal of a ring R. Take an invertible element $\overline{a} \in R/I$ with the inverse $\overline{b}$. Since $\overline{a}$ is a unit, $\mathrm{isr}(\overline{a}) = 1$. By hypothesis, we can find an element $x \in R$ such that $\overline{a} = \overline{x}$ and $\mathrm{isr}(x) = 1$. Then $\overline{b}\overline{a} = \overline{b}\overline{x} = \overline{1}$, and so $c := 1 - bx \in I$. This implies that $Rx + Rc = R$. Since $\mathrm{isr}(x) = 1$, there exists an idempotent $e \in R$ such that $x + ec = v$ is a unit in R. Hence $\overline{a} = \overline{x} = \overline{x + ec} = \overline{v}$. Thus v is the required element. □

The converse of Lemma 3.2.7 need not be true. For example, the ideal $I = 4\mathbb{Z}$ in the ring $R = \mathbb{Z}$ is unit lifting, but as we have pointed out before, it is not idempotent stable range one lifting.

Following Nicholson [69], an element x in a ring R is called *suitable* if there exists an idempotent $e \in R$ such that $e - x \in R(x - x^2)$. He showed that a ring R is an exchange

ring if and only if every element of R is suitable. He further proved that any clean element is suitable.

In the literature, there are some natural equivalence relations on idempotents: First, two idempotents e and f in a ring R are said to be *isomorphic* if $eR \cong fR$ as right R-modules, and second, they are called *conjugate* if $f = u^{-1}eu$ for some unit $u \in \mathrm{U}(R)$. Close attention to the lifting of isomorphic idempotents and conjugate idempotents has been paid recently in [51]. Now we preface Theorem 3.2.11 with three lemmas from [51] needed for its proof.

Lemma 3.2.8 [51, Theorem 5.2] *Let R be a ring, I be an ideal of R, and let $x \in R$ be an idempotent modulo I. Then x lifts to an idempotent modulo I iff x lifts to a suitable element modulo I.*

Lemma 3.2.9 [51, Proposition 3.11] *Let R be a ring and I be an ideal of R. If R/I is perspective, and I is idempotent lifting, then I is isomorphic idempotent lifting and conjugate idempotent lifting.*

Lemma 3.2.10 [51, Proposition 5.20] *If units and isomorphic idempotents lift modulo an ideal of R, then regular elements lift.*

Theorem 3.2.11 *Let R be a ring and I be an ideal of R such that R/I is perspective. If I is idempotent stable range one lifting, then it is regular lifting (hence it is idempotent lifting).*

Proof. First we claim that I is idempotent lifting. Let $\overline{a}$ be an idempotent in R/I. Since R/I is perspective, every regular element has idempotent stable range one by Theorem 1.5.2. Hence $\mathrm{isr}(\overline{a}) = 1$. Since I is idempotent stable range one lifting, there exists $b \in \mathcal{ISR}_1(R)$ such that $a - b \in I$. Since $\mathrm{isr}(b) = 1$, the equality $Rb + R(-1) = R$ implies that b is clean. Then b is suitable. This means that $\overline{a}$ lifts to a suitable element of R. By Lemma 3.2.8, this is equivalent to saying that $\overline{a}$ lifts to an idempotent of R. Hence I is idempotent lifting. Now by Lemma 3.2.9, I is isomorphic idempotent lifting. Finally, Lemmas 3.1.7 and 3.2.10 yield that I is regular lifting, as desired. □

The converse of Theorem 3.2.11 is not true in general. For example, let $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$. Then I is regular lifting, because all regular elements of R/I are $\overline{0}, \overline{1}$ and $\overline{-1}$

and they are lifted to regular elements of $\mathbb{Z}$. But we have pointed out before that I is not idempotent stable range one lifting.

Note that regular lifting ideals are idempotent lifting in general and they are equivalent for ideals contained in the Jacobson radical by [51, Theorem 5.24].

Before stating the next corollary, we recall that a ring R has idempotent stable range one if each of its elements has idempotent stable range one. A characterization of this class of rings was obtained by Hiremath and Hedge in [41, Proposition 2.18] as follows: If I is an ideal contained in the Jacobson radical of the ring R, then

$$\text{isr}(R) = 1 \text{ if and only if } \text{isr}(R/I) = 1 \text{ and } I \text{ is idempotent lifting.}$$

Note that this result was first proved for $I = \text{J}(R)$ in [14, Theorem 9].

Corollary 3.2.12 *Let R be a ring and I be an ideal of R such that $I \subseteq \text{J}(R)$. Then* $\text{isr}(R) = 1$ *if and only if* $\text{isr}(R/I) = 1$ *and I is idempotent stable range one lifting.*

Proof. ($\Rightarrow$) If $\text{isr}(R) = 1$, then $\text{isr}(R/I) = 1$ by [41, Proposition 2.18] and clearly I is idempotent stable range one lifting.
($\Leftarrow$) Assume that $\text{isr}(R/I) = 1$ and I is idempotent stable range one lifting. Since any ring with stable range one is perspective, R/I is perspective. It follows that I is idempotent lifting by Theorem 3.2.11. Hence $\text{isr}(R) = 1$ by [41, Proposition 2.18].
There is also an alternative (direct) way to get that I is idempotent lifting. Let $\overline{a}$ be an idempotent in R/I. Since $\text{isr}(\overline{1-a}) = 1$, there exists a $c \in R$ such that $\overline{1-a} = \overline{c}$ and $\text{isr}(c) = 1$. Since c is clean, $c = e + u$ for some idempotent e and a unit u in R. Then $\overline{a} = \overline{1-c} = \overline{1-e} - \overline{u} = (\overline{1-e} - \overline{u})^2$ gives that $\overline{eu - u + ue + u^2} = \overline{0}$, and multiplying by $\overline{u^{-1}}$ from the left we have $\overline{a} = \overline{u^{-1}eu}$ where $u^{-1}eu$ is an idempotent in R. □

Corollary 3.2.13 *Let R be a ring and I an ideal of R such that $I \subseteq \text{J}(R)$. Then R is perspective and I is idempotent lifting if and only if R/I is perspective and I is idempotent stable range one lifting.*

Proof. It follows from [33, Proposition 5.7], Corollary 3.2.3 and Theorem 3.2.11. □

Last but not least, we have the following corollary of Theorem 3.2.11, but first recall that a ring R is *left quasi-duo* if every maximal left ideal is a two-sided ideal [95].

Corollary 3.2.14 *If R is a left quasi-duo ring and I is an idempotent stable range one lifting ideal, then I is regular lifting. In particular,* $\mathrm{J}(R)$ *is idempotent stable range one lifting iff* $\mathrm{J}(R)$ *is idempotent lifting.*

Proof. Since R/I is left quasi-duo, it is perspective by [33, Corollary 4.8]. Hence I is regular lifting. The last assertion follows from Corollary 3.2.3. □

There is a close relationship between the class of exchange rings and the lifting property of regular elements modulo left ideals. A ring R is an exchange ring iff every left ideal is regular lifting, i.e., if L is a left ideal and $a - aba \in L$, then there exists a regular element $c \in R$ such that $a - c \in L$ [28, Corollary 5].

Rings with idempotent stable range one were characterized in [14, Theorem 12] over abelian rings. Here we provide a characterization over duo rings. Note that a ring is abelian iff every direct summand left ideal is fully invariant (see [74, p. 536]). Hence any left duo ring is abelian.

Theorem 3.2.15 *If R is a duo ring, then the following are equivalent:*

(1) *Every ideal is idempotent stable range one lifting.*

(2) *Every ideal is regular lifting.*

(3) *R is exchange.*

(4) $\mathrm{isr}(R) = 1$.

(5) *R is clean.*

Proof. First note that (3) − (5) are equivalent for any abelian ring by [14, Theorem 12]. The equivalence of (2) and (3) is true for any ring R by [28, Corollary 5].
(1) ⇒ (2) Since R is a duo ring, R/I is a duo ring for any ideal I of R by [74, Proposition 1.4], and so it is perspective by [33]. Now Theorem 3.2.11 implies that I is regular lifting.
(4) ⇒ (1) It is obvious. □

Hence Theorem 3.1.10 and Theorem 3.2.15 together imply that, over a commutative ring, if every ideal is idempotent stable range one lifting, then every ideal is stable range one lifting.

Now we can present an example of a stable range one lifting ideal which is not idempotent stable range one lifting.

Example 3.2.16 There exists an ideal I of a ring R such that I is stable range one lifting but it is neither idempotent stable range one lifting nor idempotent lifting: Consider a semilocal commutative domain with two maximal ideals M_1 and M_2 (for example, take $R = \{\frac{m}{n} \in \mathbb{Q} \mid 2 \not| n, 3 \not| n\}$). Then $\mathrm{J}(R) = M_1 \cap M_2$ and $R/\mathrm{J}(R) \cong R/M_1 \times R/M_2$. The factor ring $R/\mathrm{J}(R)$ has two non-trivial idempotents which do not lift to idempotents in R, because R has no non-trivial idempotents. Hence $\mathrm{J}(R)$ is not idempotent lifting. However, it is stable range one lifting by Corollary 3.1.4. Moreover, since any commutative ring is perspective, $R/\mathrm{J}(R)$ is perspective. Thus, $\mathrm{J}(R)$ is not idempotent stable range one lifting by Theorem 3.2.11.

Finally, we investigate some extensions of idempotent stable range one lifting ideals.

Lemma 3.2.17 *Let $\varphi : R \longrightarrow S$ be a ring isomorphism with $\varphi(1_R) = 1_S$. If $\mathrm{isr}(a) = 1$ in R, then $\mathrm{isr}(\varphi(a)) = 1$ in S.*

Proposition 3.2.18 *Let I and K be ideals of a ring R with $I \subseteq \mathrm{J}(R) \cap K$. If K is idempotent stable range one lifting, then K/I is idempotent stable range one lifting. The converse is true if, in addition, I is idempotent stable range one lifting.*

Proof. The proof is similar to that of Proposition 3.1.11. Assume that K is idempotent stable range one lifting. Let $a \in R$ with $\mathrm{isr}(a + I + K/I) = 1$. The mapping $\varphi : \frac{R/I}{K/I} \to R/K$, defined by $\varphi(r + I + K/I) = r + K$ for every $r \in R$, is a ring isomorphism, so that $\mathrm{isr}(a + K) = 1$ by Lemma 3.2.17. By hypothesis, there exists $b \in R$ such that $a + K = b + K$ and $\mathrm{isr}(b) = 1$. On the other hand, $\mathrm{isr}(b + I) = 1$ by Proposition 3.2.2. Thus $a + I + K/I = b + I + K/I$ and $\mathrm{isr}(b + I) = 1$.
Conversely, assume that I and K/I are idempotent stable range one lifting. Let $a \in R$ with $\mathrm{isr}(a+K) = 1$. The above mentioned isomorphism and Lemma 3.2.17 implies that $\mathrm{isr}(a + I + K/I) = 1$. Since K/I is idempotent stable range one lifting, there exists $b + I \in R/I$ such that $a + I + K/I = b + I + K/I$ and $\mathrm{isr}(b + I) = 1$. Then $a - b \in K$. Since I is idempotent stable range one lifting, there exists $c \in R$ such that $b + I = c + I$ and $\mathrm{isr}(c) = 1$. Now $b - c \in I \subseteq K$ gives that $a - c \in K$. Hence $a + K = c + K$ and $\mathrm{isr}(c) = 1$. □

In particular, taking $K = \mathrm{J}(R)$ or $I = \mathrm{J}(R)$ in Proposition 3.2.18 respectively yields the following corollary.

Corollary 3.2.19 *The following hold for a ring R:*

(1) *Let I be an ideal of R with $I \subseteq \mathrm{J}(R)$. If $\mathrm{J}(R)$ is idempotent stable range one lifting, then $\mathrm{J}(R)/I$ is idempotent stable range one lifting. The converse is true if I is idempotent stable range one lifting.*

(2) *Let K be an ideal of R with $\mathrm{J}(R) \subseteq K$. If K is idempotent stable range one lifting, then $K/\mathrm{J}(R)$ is idempotent stable range one lifting. The converse is true if $\mathrm{J}(R)$ is idempotent stable range one lifting.*

As an example, consider any ring R. If δ_r is idempotent stable range one lifting, then $\delta_r/\mathrm{J}(R)$ is idempotent stable range one lifting. The converse is true if $\mathrm{J}(R)$ is idempotent stable range one lifting.

We end this section with the following questions that we were unable to answer.

Question 3.2.20 Is any idempotent stable range one lifting ideal stable range one lifting? Since the Jacobson radical is always stable range one lifting, it is necessary to consider an ideal different from $\mathrm{J}(R)$.

Question 3.2.21 Is the converse of Corollary 3.2.3 true? This is equivalent to asking that whether there is a (non-quasi-duo) ring R with $\mathrm{J}(R)$ idempotent stable range one lifting but not idempotent lifting?

Question 3.2.22 Is the element-wise definition of (idempotent) stable range one is left-right symmetric?

4 $C4$- and $D4$-MODULES VIA PERSPECTIVITY

Utumi introduced continuity concept for rings in a series of papers (see [82, 83, 84]) and established three conditions for a ring that are satisfied if the ring is self-injective. The driving force behind all of these conditions is the von Neumann's continuous geometries which are the analogues of projective geometries, except that they have no points. Subsequently, Utumi's conditions were extended to modules by Jeremy [45] and Mohamed and Bouhy [61], as follows.

A module M is called a *Ci-module* if it satisfies the following Ci-conditions.
$C1$: Every submodule of M is essential in a direct summand of M.
$C2$: Whenever A and B are submodules of M such that $A \cong B$ and B is a direct summand of M, then A is a direct summand of M.
$C3$: Whenever A and B are direct summands of M with $A \cap B = 0$, then $A + B$ is a direct summand of M.

Moreover, M is called *continuous* if it is both a $C1$- and $C2$-module, and is called *quasi-continuous* if it is both a $C1$- and $C3$-module. It is well known that every $C2$-module is a $C3$-module, and every quasi-injective module is continuous. For a full account on these conditions, see [63].

Recently, the class of $C3$-modules have been thoroughly investigated by Amin et al. in [4] and Ibrahim et al. in [43]. In these articles, the authors extended many well-known results on rings and modules in terms of (quasi-)continuous to $C3$-modules. In [4, Proposition 2.3], it was proved that if M is a $C3$-module, then for every decomposition $M = A \oplus B$ and every homomorphism $f : A \to B$ with $\ker(f)$ a direct summand of A, then $\mathrm{im}(f)$ is a direct summand of B. This result was recently considered in [21], and a module M is called a $C4$-module if it satisfies the latter property. An example was provided in [21, Example 2.10] to show that the class of $C4$-modules is a non-trivial generalization of the class of $C3$-modules.

Dually, a module M is called a *Di-module* if it satisfies the following Di-conditions.
$D1$: For every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2$ is small in M_2.

$D2$: Whenever A and B are submodules of M with $M/A \cong B$ and B is a direct summand of M, then A is a direct summand of M.
$D3$: Whenever A and B are direct summands of M with $A + B = M$, then $A \cap B$ is a direct summand of M.

A module M is called *discrete* if it is both a $D1$- and a $D2$-module, and is called *quasi-discrete* if it is both a $D1$- and a $D3$-module. Every $D2$-module is a $D3$-module and every quasi-projective module is a $D2$-module. Again we refer the interested reader to [63].

The class of $D3$-modules has also been investigated by Amin et al. in [94]. It was shown in [94, Proposition 4] that if M is a $D3$-module, then for every decomposition $M = A \oplus B$ and every homomorphism $f : A \to B$ with $\mathrm{im}(f)$ a direct summand of B, then $\ker(f)$ is a direct summand of A. A module M that satisfies the latter property is called a $D4$-*module* in [22].

In this chapter, we continue the study of $C4$- and $D4$-modules, providing several new characterizations and results on the subject. Recall that two direct summands A and B of a module M are called *perspective* exactly when they have a common (direct sum) complement C, i.e., $M = A \oplus C = B \oplus C$. We will call two idempotents e and f of a ring R *perspective* if, eR and fR have a common complement.

In Section 4.1, we use the notion of perspective submodules to prove in Theorem 4.1.4 that, a module M is a $C4$-module if and only if whenever A and B are perspective (direct) summands of M with $A \cap B = 0$, then $A \oplus B \subseteq^{\oplus} M$. Moreover, arbitrary direct sum of $C4$-modules are also investigated. Furthermore, we introduce the notion of restricted ACC on summands, and show in Proposition 4.1.15 that a $C4$-module M with the restricted ACC on summands can be decomposed as $M = A \oplus B \oplus K$ where $A \cong B$ is a $C2$-module and K is a summand-square-free module.

In Section 4.2, $C4$-modules are characterized by their endomorphism rings. It is proved that a right R-module M is a $C4$-module if and only if for any idempotents $e, f \in \mathrm{End}_R(M)$, if $\ker(e) = \ker(f) = \ker(e - f)$, then $(1 - e)fM$ is a direct summand of M. We provide an example of a $C4$-module whose endomorphism ring is not $C4$, and provide several conditions under which the endomorphism ring of a $C4$ -module is a

right $C4$-ring. Section 4.3 is devoted to right $C4$-rings. For example, corner rings and trivial extensions are investigated in terms of $C4$ property.

In Section 4.4, we consider the $D4$-modules, dualizing many of our results on $C4$-modules and providing several new characterizations of $D4$-modules. We prove in Proposition 4.4.14 that a module M is both a $D4$-module and a summand-square-free module iff M is a $C4$-module and a summand-dual-square-free module. In Proposition 4.4.17 we show that, if M is a $D4$-module that satisfies the restricted DCC on (direct) summands, then $M = A \oplus B \oplus K$ where $A \cong B$, A and B are $D2$-modules, and K is a summand-dual-square-free module. As a result, we prove in Proposition 4.4.20 that a quasi-discrete module M with DCC on summands can be decomposed as $M = A \oplus B \oplus K$ where $A \cong B$ are quasi-projective modules and K is both a summand-square-free and a summand-dual-square-free module.

4.1 $C4$-Modules

In this section, we provide some basic properties of $C4$-modules. We start with the following lemma that has been established in [21].

Lemma 4.1.1 [21, Theorem 2.2] *The following are equivalent for a module M:*

(1) *If $M = A \oplus B$ and $f : A \to B$ is a monomorphism, then* $\mathrm{im}(f) \subseteq^{\oplus} B$.

(2) *If $M = A \oplus B$ and $f : A \to B$ is a homomorphism with* $\ker(f) \subseteq^{\oplus} A$, *then* $\mathrm{im}(f) \subseteq^{\oplus} B$.

(3) *If $B \cong A \subseteq^{\oplus} M$, $B \subseteq M$, and $A \cap B = 0$, then $A \oplus B \subseteq^{\oplus} M$.*

(4) *If $B \cong A \subseteq^{\oplus} M$, $B \subseteq M$, and $A \cap B = 0$, then $B \subseteq^{\oplus} M$.*

(5) *If A, $B \subseteq^{\oplus} M$, $A \cong B$, and $A \cap B = 0$, then $A \oplus B \subseteq^{\oplus} M$.*

(6) *If $M = A \oplus A' = B \oplus B'$ and $A \cap B = A \cap B' = 0$, then $A \oplus B \subseteq^{\oplus} M$.*

Definition 4.1.2 [21] A module M is called a *$C4$-module* if it satifies any of the equivalent conditions in Lemma 4.1.1. A ring R is called a *right $C4$-ring* if R is a $C4$-module as a right R-module.

Examples 4.1.3 (1) Clearly, any $C3$-module is a $C4$-module. Thus all quasi-continuous modules, uniform modules, indecomposable modules, semisimple modules, regular modules, and modules with the summand sum property are $C4$, all being examples of $C3$-modules.

(2) Recall that a module M is called *(summand-) square-free* if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$ (and Y_1, $Y_2 \subseteq^{\oplus} M$), then $Y_1 = Y_2 = 0$. Using the above notions, it is easy to see that any summand-square-free module is $C4$. But the converse is not true in general. Let F be a field and $R = \mathbb{M}_2(F)$. Now R_R is continuous by [72, Theorem 1.35], and so it is a $C4$-module. Consider the idempotents $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then eR and fR are non-zero non-intersecting isomorphic direct summands of R_R. Thus, R_R is not a summand-square-free module. See also [21, Example 2.8] for another example.

Note that any direct summand of a $C4$-module is again a $C4$-module by [21, Proposition 2.15]. The following characterization of $C4$-modules in terms of perspective direct summands will be used frequently throughout this chapter.

Theorem 4.1.4 *The following are equivalent for a module M:*

(1) *M is a $C4$-module.*

(2) *If A and B are perspective direct summands of M with $A \cap B = 0$, then $A \oplus B \subseteq^{\oplus} M$.*

(3) *If A and B are perspective direct summands of M with $A \cap B \subseteq^{\oplus} M$, then $A + B \subseteq^{\oplus} M$.*

Proof. (1) $\Rightarrow$ (2) Let A and B be perspective direct summands with common direct sum complement C and $A \cap B = 0$. Let $\pi : M \to C$ be the projection with $\ker(\pi) = B$. Consider the restriction map $\pi|_A : A \to C$. Since $\ker(\pi|_A) = A \cap B = 0 \subseteq^{\oplus} A$, we have $\mathrm{im}(\pi|_A) \subseteq^{\oplus} C$ by the $C4$ property of M. Then we could write $C = \pi(A) \oplus X$ for some submodule $X \subseteq C$. It can easily be seen that $A \oplus B = B \oplus \pi(A)$. Hence $M = B \oplus C = B \oplus \pi(A) \oplus X = A \oplus B \oplus X$, and so $A \oplus B \subseteq^{\oplus} M$.

(2) $\Rightarrow$ (3) Let A and B be perspective direct summands of M with $A \cap B \subseteq^{\oplus} M$. Then there exist C, $D \subseteq^{\oplus} M$ such that $M = A \oplus C = B \oplus C = (A \cap B) \oplus D$. By modularity law, we have $A = (A \cap B) \oplus (A \cap D)$ and $B = (A \cap B) \oplus (B \cap D)$. Now $A \cap D$ and $B \cap D$

are perspective direct summands of M and $(A \cap D) \cap (B \cap D) = (A \cap B) \cap D = 0$. Now $(A \cap D) \oplus (B \cap D) \subseteq^{\oplus} M$ by assumption, and so $(A \cap D) \oplus (B \cap D) \subseteq^{\oplus} D$. Write $D = (A \cap D) \oplus (B \cap D) \oplus D'$ for a submodule $D' \subseteq D$. Then $M = (A \cap B) \oplus D = (A \cap B) \oplus (A \cap D) \oplus (B \cap D) \oplus D'$. Since $A+B = [(A \cap B) \oplus (A \cap D)] \oplus [(A \cap B) \oplus (B \cap D)] = (A \cap B) \oplus [(A \cap D) \oplus (B \cap D)]$, we obtain $A + B \subseteq^{\oplus} M$.

(3) $\Rightarrow$ (1) Let $M = A \oplus B$ and $f : A \to B$ be a monomorphism. Consider the graph submodule $T = \{a + f(a) : a \in A\}$ of M. It can easily be checked that $M = T \oplus B$. So A and T are perspective direct summands of M. We claim that $A \cap T = 0$. For, if $x \in A \cap T$, then there exists an $a \in A$ such that $x = a + f(a)$. Since $x - a = f(a) \in A \cap B = 0$ and f is a monomorphism, we have $a = 0$, and so $x = 0$. By assumption, we have $A \oplus T \subseteq^{\oplus} M$. Now we show that $A \oplus T = A \oplus \mathrm{im}(f)$. To see this, take $x \in \mathrm{im}(f)$. Then $x = f(a)$ for some $a \in A$ and we can write $x = -a + a + f(a) \in A + T$. Since $A \oplus T \subseteq^{\oplus} M$, $\mathrm{im}(f) \subseteq^{\oplus} M$, and hence $\mathrm{im}(f) \subseteq^{\oplus} B$. □

Corollary 4.1.5 *The following conditions on a module M are equivalent:*

(1) *M is a $C4$-module.*

(2) *If A and B are perspective direct summands of M with $A \cap B = 0$, then there exists $B' \subseteq M$ with $B \subseteq B'$ such that $M = A \oplus B'$.*

Proof. (1) $\Rightarrow$ (2) Let A and B are perspective direct summands of M with $A \cap B = 0$. Then $M = (A \oplus B) \oplus C$ for some $C \subseteq M$ by Theorem 4.1.4. Taking $B' := B \oplus C$ gives the result.

(2) $\Rightarrow$ (1) Let A and B are perspective direct summands of M with $A \cap B = 0$. Write $M = A \oplus C = B \oplus C$ for some $C \subseteq M$. By assumption, there exists $B' \subseteq M$ with $B \subseteq B'$ such that $M = A \oplus B'$. Now $B' = B \oplus (C \cap B')$ by the modular law, and so $M = A \oplus B \oplus (C \cap B')$. Hence M is a $C4$-module by Theorem 4.1.4. □

In the next result we will replace the condition $A \cap B = 0$ in Lemma 4.1.1 by the weaker one $A \cap B \subseteq^{\oplus} M$.

Theorem 4.1.6 *The following are equivalent for a module M:*

(1) *M is a $C4$-module.*

(2) *If $B \cong A \subseteq^{\oplus} M$, $B \subseteq M$, and $A \cap B \subseteq^{\oplus} M$, then $A + B \subseteq^{\oplus} M$.*

(3) *If* $B \cong A \subseteq^{\oplus} M$, $B \subseteq M$, *and* $A \cap B \subseteq^{\oplus} M$, *then* $B \subseteq^{\oplus} M$.

(4) *If* A, $B \subseteq^{\oplus} M$, $A \cong B$, *and* $A \cap B \subseteq^{\oplus} M$, *then* $A + B \subseteq^{\oplus} M$.

Proof. (1) $\Rightarrow$ (2) Let A and B be two submodules of M with $A \cap B \subseteq^{\oplus} M$ and $B \overset{\sigma}{\cong} A \subseteq^{\oplus} M$. Write $M = A \oplus T$ for a submodule $T \subseteq M$, and let $\pi : A \oplus T \to T$ be the natural projection. Clearly, $A + B = A + \pi(B)$. Now consider the restriction $\pi|_B : B \to T$. Since $M = A \oplus T$, $\pi|_B \circ \sigma^{-1} : A \to T$ is a homomorphism with $\ker(\pi|_B \circ \sigma^{-1}) = \sigma(A \cap B)$ and $\sigma(A \cap B) \subseteq^{\oplus} A$, the $C4$ property of M implies that $\mathrm{im}(\pi|_B \circ \sigma^{-1}) = \pi(B) \subseteq^{\oplus} T$. If $T = \pi(B) \oplus K$ for a submodule K of T, then $M = A \oplus T = A \oplus (\pi(B) \oplus K) = (A + \pi(B)) \oplus K = (A + B) \oplus K$, as desired.
(2) $\Rightarrow$ (3) Let A and B be two submodules of M with $A \cap B \subseteq^{\oplus} M$ and $B \cong A \subseteq^{\oplus} M$. By hypothesis, $A + B \subseteq^{\oplus} M$. Write $M = (A \cap B) \oplus Y$ and $M = (A + B) \oplus X$ for some submodules X, $Y \subseteq M$. Then $A = (A \cap B) \oplus (A \cap Y)$ by the modular law. Now $A + B = (A \cap B) + (A \cap Y) + B = (A \cap Y) \oplus B$. Thus, $M = (A \cap Y) \oplus B \oplus X$ and so $B \subseteq^{\oplus} M$.
(3) $\Rightarrow$ (1) is clear by Lemma 4.1.1.
(2) $\Rightarrow$ (4) is clear.
(4) $\Rightarrow$ (1) Let A and B be perspective direct summands of M with $A \cap B \subseteq^{\oplus} M$. Then $A \cong B$. By hypothesis, $A + B \subseteq^{\oplus} M$. Hence M is $C4$ by Theorem 4.1.4. □

It was mentioned in [21] that the direct sum of two $C4$-modules need not be $C4$ (see Remarks 2.31 and 3.2 in [21]). In the next theorem we consider a specific case where the direct sum of $C4$-modules is a again a $C4$-module. Recall first that a submodule N of a module M is called *fully invariant* in M if $f(N) \subseteq N$ for every endomorphism f of M.

Theorem 4.1.7 *Let* $M = \oplus_{i \in I} M_i$ *be a module, where* M_i *is fully invariant in* M *for every* $i \in I$. *Then* M *is a* $C4$*-module if and only if each* M_i *is a* $C4$*-module.*

Proof. Suppose that M_i is a $C4$-module for every $i \in I$. Let $M = A \oplus C = B \oplus C$ such that $A \cap B = 0$. Since each M_i is fully invariant, $M_i = (A \cap M_i) \oplus (C \cap M_i) = (B \cap M_i) \oplus (C \cap M_i)$ for every $i \in I$. It follows that $M = \oplus_{i \in I} M_i = \oplus_{i \in I} [(A \cap M_i) \oplus (C \cap M_i)] = [\oplus_{i \in I}(A \cap M_i)] \oplus [\oplus_{i \in I}(C \cap M_i)]$, $A = \oplus_{i \in I}(A \cap M_i)$, and $B = \oplus_{i \in I}(B \cap M_i)$. Thus $A \oplus B = [\oplus_{i \in I}(A \cap M_i)] \oplus [\oplus_{i \in I}(B \cap M_i)] = \oplus_{i \in I} [(A \cap M_i) \oplus (B \cap M_i)]$. Since $A \cap M_i$

and $B \cap M_i$ are perspective direct summands of M_i with zero intersection, $(A \cap M_i) \oplus (B \cap M_i) \subseteq^{\oplus} M_i$ for every $i \in I$. Hence $A \oplus B$ is a direct summand of M, and so M is a $C4$-module, as required. The converse is obvious, since a direct summand of a $C4$-module is again $C4$ by [21, Proposition 2.15]. □

As we mentioned earlier, a module M is said to have the *summand intersection property* (SIP, for short) if the intersection of any two direct summands of M is a direct summand. Dually, M is said to have the *summand sum property* (SSP, for short) when the sum of any two direct summands of M is a direct summand. In the next proposition, we characterize modules with the SIP (SSP) in terms of perspective direct summands.

Proposition 4.1.8 *Let M be a module. Then we have the following:*

(1) *M has the SIP if and only if the intersection of any two perspective direct summands of M is a direct summand.*

(2) *M has SSP if and only if the sum of any two perspective direct summands of M is a direct summand.*

Proof. 1) ($\Rightarrow$) Obvious. ($\Leftarrow$) Let $M = A \oplus B$ and $f : A \to B$ be a homomorphism. It is enough to show that $\ker(f) \subseteq^{\oplus} A$ by [39, Proposition 1.4]. Consider the graph submodule $T = \{a + f(a) \mid a \in A\}$ of M. Then $M = A \oplus B = T \oplus B$. Hence A and T are perspective direct summands. By the hypothesis, $A \cap T \subseteq^{\oplus} M$. Now we claim that $A \cap T = \ker(f)$. To see this, let $a = a' + f(a') \in A \cap T$. Then $a - a' = f(a') \in A \cap B = 0$, and so $a = a' \in \ker(f)$. Hence $\ker(f) \subseteq^{\oplus} A$.

2) ($\Rightarrow$) Obvious. ($\Leftarrow$) Similarly, let $M = A \oplus B$ and $f : A \to B$ be a homomorphism. It is enough to show that $\mathrm{im}(f) \subseteq^{\oplus} B$ by [3, Theorem 8]. Let T be the graph submodule as above. By the hypothesis, $M = (A+T) \oplus D$ for some D. Since $A+T = \{a+f(a') \mid a, a' \in A\}$, we have that $B = \mathrm{im}(f) \oplus D'$ where $D' = \{b \in B \mid \exists\, a \in A \text{ such that } a + b \in D\}$. Hence M has SSP. □

Every module with SSP is a $C3$-module. The converse is true if the module has SIP by [3, Corollary 20]. The next corollary was established in [21, Example 2.9], and here we provide a slightly different proof.

Corollary 4.1.9 *If M is a $C4$-module with SIP, then M has SSP.*

Proof. Let A and B be perspective direct summands of M. Then $A \cap B \subseteq^{\oplus} M$ by Proposition 4.1.8(1). Hence $A + B \subseteq^{\oplus} M$ by Theorem 4.1.4. Thus M has SSP again by Proposition 4.1.8(2). □

In [32, Theorem 2.3] it was proved that a module M has both SSP and SIP if and only if $\mathrm{End}(M)$ has SSP. Therefore, Corollary 4.1.9 implies the following.

Corollary 4.1.10 *Let M be a module. Then M is a $C4$-module with SIP if and only if* $\mathrm{End}(M)$ *has SSP.*

Example 4.1.11 There exists a module with SIP which is not C_4. Let $R = \begin{pmatrix} \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{pmatrix}$ and e_{ij} be the 3×3 matrix with (i, j)-entry 1 and all other entries zero. Then all idempotents of R are 0, 1, e_{11}, e_{22}, e_{33}, $e_{11} + e_{22}$, $e_{11} + e_{33}$, $e_{22} + e_{33}$, $e_{11} + e_{13}$, $e_{13} + e_{33}$, $e_{13} + e_{22} + e_{33}$, $e_{11} + e_{22} + e_{13}$. So it can easily be checked that the intersection of any two direct summands of R_R is a direct summand, i.e., R_R has SIP. Now consider the idempotents $e = e_{33}$, $f = e_{13} + e_{33}$, and $g = e_{11} + e_{22}$. An easy computation shows that $R = eR \oplus gR = fR \oplus gR$ and $eR \cap fR = 0$, but $eR + fR$ is not a direct summand of R_R. Thus R is not right C_4.

One can also observe that for the idempotents $e = e_{11}$, $f = e_{11} + e_{13}$, and $g = e_{22} + e_{33}$, $R = Re \oplus Rg = Rf \oplus Rg$ and $Re \cap Rf = 0$, but $Re + Rf$ is not a direct summand of ${}_RR$. Thus R is not left C_4 either.

Camillo et al. restricted the $C3$ ($C2$) property to the class of simple modules in [13], where the following result was established.

Proposition 4.1.12 [13, Proposition 2.1] *The following are equivalent for a module M:*

(1) *For any simple submodules A, B of M with $A \cong B \subseteq^{\oplus} M$, $A \subseteq^{\oplus} M$.*

(2) *For any simple direct summands A, B of M with $A \cap B = 0$, $A \oplus B \subseteq^{\oplus} M$.*

(3) *If $M = A_1 \oplus A_2$ with A_1 simple and $f\colon A_1 \to A_2$ an R-homomorphism, then* $\mathrm{im}(f) \subseteq^{\oplus} A_2$.

Subsequently, a module M is called *simple-direct-injective* in [13] if it satisfies any of the equivalent conditions of Proposition 4.1.12. When we restrict the $C4$ property to the class of simple modules we see that it coincides with the class of simple-direct-injective modules.

Proposition 4.1.13 *M is simple-direct-injective if and only if for any simple perspective direct summands A, B of M with $A \cap B = 0$, $A \oplus B \subseteq^{\oplus} M$.*

Proof. Necessity is obvious. Assume that for any simple perspective direct summands A, B of M with $A \cap B = 0$, $A \oplus B \subseteq^{\oplus} M$. Let $M = A_1 \oplus A_2$ with A_1 simple and $f : A_1 \to A_2$ an R-homomorphism. Without loss of generality we may assume that $f \neq 0$. Then f is an R-monomorphism. Let $T = \{a + f(a) \,|\, a \in A_1\}$ be the graph submodule. We have that $M = T \oplus A_2$ and $A_1 \cap T = 0$. Since $T \cong M/A_2 \cong A_1$, T and A_1 are simple perspective direct summands of M with $A_1 \cap T = 0$. Thus $A_1 \oplus T = A_1 \oplus \mathrm{im}(f) \subseteq^{\oplus} M$ by assumption. This means that $\mathrm{im}(f) \subseteq^{\oplus} M$, and so $\mathrm{im}(f) \subseteq^{\oplus} A_2$. By Proposition 4.1.12, M is simple-direct-injective. □

Definition 4.1.14 A module M is said to satisfy *the restricted ascending chain condition (ACC) on summands* if, M has no strictly ascending chains of non-zero summands

$$\begin{aligned} A_1 &\subsetneq A_2 \subsetneq \cdots \\ B_1 &\subsetneq B_2 \subsetneq \cdots \end{aligned}$$

with $A_i \cong B_i$ and $A_i \cap B_i = 0$ for all $i \geqslant 1$.

Clearly, every summand-square-free module and every module with the ACC on summands (equivalently, DCC on summands) satisfies the restricted ACC on summands. In particular, modules with finite Goldie (or dual Goldie) dimension are examples of modules with the restricted ACC on summands. Thus semilocal rings and rings with no infinite sets of orthogonal idempotents satisfy both the left and right restricted ACC on summands.

Proposition 4.1.15 *If M is a $C4$-module that satisfies the restricted ACC on summands, then $M = A \oplus B \oplus K$ where $A \cong B$ is a $C2$-module and K is a summand-square-free module.*

Proof. There is nothing to prove if M is a summand-square-free module. Assume that M is not a summand-square-free module, and let A_1, B_1 be non-zero summands of M with $A_1 \cong B_1$ and $A_1 \cap B_1 = 0$. Since M is a $C4$-module, by Lemma 4.1.1, $A_1 \oplus B_1 \subseteq^{\oplus} M$. Write $M = A_1 \oplus B_1 \oplus T_1$ for a submodule $T_1 \subsetneqq M$. By [21, Proposition 2.15] , both A_1 and B_1 are $C2$-modules. We are done if T_1 is a summand-square-free module. Otherwise, by repeating the argument we can find two non-zero summands A_2, B_2 of T_1 with $A_2 \cap B_2 = 0$ and $A_2 \cong B_2$. In this case, since T_1 is a $C4$-module, we can write $M = A_1 \oplus B_1 \oplus A_2 \oplus B_2 \oplus T_2$ for a submodule $T_2 \subsetneqq M$. Clearly $A_1 \oplus A_2 \cong B_1 \oplus B_2$ and so, by [21, Proposition 2.15], both $A_1 \oplus A_2$ and $B_1 \oplus B_2$ are $C2$-modules with $(A_1 \oplus A_2) \cap (B_1 \oplus B_2) = 0$ and $A_1 \subsetneqq A_1 \oplus A_2$ and $B_1 \subsetneqq B_1 \oplus B_2$. By repeating the process, we obtain proper ascending chains of non-zero summands

$$\begin{aligned} A_1 &\subsetneqq A_1 \oplus A_2 \subsetneqq A_1 \oplus A_2 \oplus A_3 \subsetneqq \cdots \\ B_1 &\subsetneqq B_1 \oplus B_2 \subsetneqq B_1 \oplus B_2 \oplus B_3 \subsetneqq \cdots \end{aligned}$$

of M with $A_1 \oplus A_2 \oplus \cdots \oplus A_k \cong B_1 \oplus B_2 \oplus \cdots \oplus B_k$ and $(A_1 \oplus A_2 \oplus \cdots \oplus A_k) \cap (B_1 \oplus B_2 \oplus \cdots \oplus B_k) = 0$ for all k. Since M satisfies *the restricted ACC condition on summands,* the two chains must terminate. This means $M = A_1 \oplus B_1 \oplus A_2 \oplus B_2 \oplus \cdots \oplus A_n \oplus B_n \oplus T_n$, with $A_1 \oplus A_2 \oplus \cdots \oplus A_n \cong B_1 \oplus B_2 \oplus \cdots \oplus B_n$ are $C2$-modules and T_n is a summand-square-free module. Now we are done by setting $A := A_1 \oplus A_2 \oplus \cdots \oplus A_n$, $B := B_1 \oplus B_2 \oplus \cdots \oplus B_n$, and $K := T_n$. □

4.2 Endomorphism Rings of $C4$-Modules

Recently, the endomorphism rings of $C2$, $C3$, and $C4$-modules were investigated by Nicholson and Yousif [72], Mazurek et al. [58], and Ding et al. [21], respectively. Let M be a right R-module and $S = \mathrm{End}_R(M)$. If S_S is a right $C2$-module, then M_R is $C2$; the converse is true if $\ker(\alpha)$ is generated by M, i.e., $\ker(\alpha) = \sum\{\, \theta(M) \,|\, \theta \in S,\, \theta(M) \subseteq \ker(\alpha)\}$, whenever α is such that $r_S(\alpha)$ is a direct summand of S_S [72, Theorem 7.14]. If S_S is a right $C3$-module, then M_R is $C3$ [4, Proposition 2.8] (see also [58, Proposition 4.6]); the converse is true if for every pair of idempotents $e, f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ by [58, Proposition 4.6]. Similarly, if S_S is a right $C4$-module, then M_R is $C4$; the converse is true if for every pair of idempotents $e, f \in S$ with $eS \cap fS = 0$ we have $eM \cap fM = 0$ [21, Proposition 2.13].

We will now give some interesting results concerning the endomorphism rings of $C4$-modules. Just as with the $C3$-condition, the $C4$-condition is not an endomorphism ring invariant (see [21, Example 2.14]). The next result shows that this is almost the case.

Lemma 4.2.1 *Let M be a right R-module with $S = \mathrm{End}_R(M)$. Then the following are equivalent:*

(1) *M is a $C4$-module.*

(2) *For every pair of perspective idempotents $e, f \in S$ with $eM \cap fM = 0$, there exist orthogonal idempotents $g, h \in S$ such that $eM = gM$ and $fM = hM$.*

(3) *For every pair of perspective idempotents $e, f \in S$ with $eM \cap fM = 0$, there exists an idempotent g of S such that $eM = gM$ and $fM \subseteq (1-g)M$.*

Proof. (1) $\Rightarrow$ (2) This proof is similar to the proof of Lemma 4.5 in [58]. Let $e, f \in S$ be perspective idempotents in S with $eM \cap fM = 0$. Write $S = eS \oplus X = fS \oplus X$ for some right ideal X in S. Then there exist idempotents $p, q \in S$ such that $eS = pS$, $fS = qS$, and $X = (1-p)S = (1-q)S$. It follows that $M = eM \oplus (1-p)M = fM \oplus (1-q)M$ and $(1-p)M = (1-q)M$. Hence eM and fM are perspective direct summands of M. Since M is $C4$, $M = eM \oplus fM \oplus N$ for some N. If g is the projection to eM with kernel $fM \oplus N$, and h is the projection to fM with kernel $eM \oplus N$, then g and h are orthogonal idempotents such that $gM = eM$ and $hM = fM$.

(2) $\Rightarrow$ (3) Let $e, f \in S$ be perspective idempotents with $eM \cap fM = 0$. By hypothesis, there exist orthogonal idempotents $g, h \in S$ such that $gM = eM$ and $hM = fM$. Since $fM = hM \subseteq \ker(g) = (1-g)M$, g is the desired idempotent.

(3) $\Rightarrow$ (1) Let e, f be perspective idempotents in S with $eM \cap fM = 0$. By hypothesis, there exists an idempotent $g \in S$ such that $eM = gM$ and $fM \subseteq (1-g)M$. Since $M = fM \oplus (1-f)M$, $(1-g)M = fM \oplus [(1-f)M \cap (1-g)(M)]$. Now $M = gM \oplus [fM \oplus ((1-f)M \cap (1-g)M)] = eM \oplus [fM \oplus [(1-f)M \cap (1-g)M]]$. Hence $eM \oplus fM$ is a direct summand of M. $\square$

The following lemma will enable us to prove another characterization of $C4$-modules in terms of the endomorphism ring.

Lemma 4.2.2 *Let M be a right R-module with $S = \mathrm{End}_R(M)$. For any idempotents $e, f \in S$, we have the following:*

(1) *$eM + fM \subseteq^{\oplus} M$ if and only if $(1-e)fM \subseteq^{\oplus} M$.*

(2) *$(1-e)M = (1-f)M$ and $eM \cap fM = 0$ if and only if $\ker(e) = \ker(f) = \ker(e-f)$.*

Proof. (1) is by the fact that $eM + fM = eM \oplus (1-e)fM$. (2) is obvious. □

Proposition 4.2.3 *A right R-module M is $C4$ if and only if for any idempotents $e, f \in \mathrm{End}_R(M)$, if $\ker(e) = \ker(f) = \ker(e-f)$, then $(1-e)fM \subseteq^{\oplus} M$.*

Proof. Let A and B be perspective direct summands of M with $A \cap B = 0$. Then we can find idempotents $e, f \in \mathrm{End}_R(M)$ such that $A = eM$, $B = fM$, $M = eM \oplus (1-e)M = fM \oplus (1-f)M$, and $(1-e)M = (1-f)M$. By Lemma 4.2.2 and the hypothesis, $eM + fM \subseteq^{\oplus} M$. Hence M is $C4$. The converse is obvious by Lemma 4.2.2. □

Following [57], a right R-module M is called *k-local-retractable* if $r_M(\varphi) = r_S(\varphi)M$ for every $\varphi \in S = \mathrm{End}_R(M)$ (It was called "*P-flat* over S" in [70]). For example, free modules, regular modules, and modules whose all non-zero endomorphisms are monomorphisms are k-local retractable (see [57]).

Theorem 4.2.4 *Let M be a right R-module with $S = \mathrm{End}_R(M)$. Then S is a right $C4$-ring, if M is a $C4$-module and one of the following is satisfied:*

(1) *M is k-local-retractable.*

(2) *For any $\alpha \in S$, $\ker(\alpha)$ is generated by M.*

(3) *For every pair of perspective idempotents $e, f \in S$ with $eS \cap fS = 0$, we have $eM \cap fM = 0$.*

Proof. (1) Suppose that M is $C4$ and k-local-retractable. Let eS and fS be perspective direct summands of S with zero intersection. Then we could write $S = eS \oplus (1-e)S = fS \oplus (1-f)S$ and $(1-e)S = (1-f)S$. Since $(1-e)S = (1-f)S$ and $eS \cap fS = 0$, we have $r_S(e) = r_S(f) = r_S(e-f)$. Now the k-local-retractable property of M gives that $r_M(e-f) = r_S(e-f)M = r_S(f)M = (1-f)M = (1-e)M$. We claim that $eM \cap fM = 0$. Let $x = em = fm' \in eM \cap fM$. Then $m - em = (1-e)m = m - fm' = (1-f)m''$ for

some $m'' \in M$, and so $fm = fm'$. This implies that $m \in r_M(e-f) = (1-e)M$, and thus $x = em = 0$. By Lemma 4.2.1, there exist orthogonal idempotents $g, h \in S$ such that $gS = eS$ and $hS = fS$. So $eS \oplus fS = gS \oplus hS = (g+h)S \subseteq^{\oplus} S$. It follows that S is right $C4$.

(2) Assume that M is $C4$, and $\ker(\alpha)$ is generated by M for any $\alpha \in S$, i.e., $\ker(\alpha) = \sum\{\, \theta(M) \,|\, \theta \in S, \theta(M) \subseteq \ker(\alpha)\}$. Let e, f be idempotents in S such that $r_S(e) = r_S(f) = r_S(e-f)$. We claim that $(1-e)fS \subseteq^{\oplus} S$. By hypothesis, $\ker(e-f) = \sum\{\, \theta(M) \,|\, \theta \in S, \theta(M) \subseteq \ker(e-f)\}$. If $\theta(M) \subseteq \ker(e-f)$, then $\theta \in r_S(e-f) = r_S(e)$, and so $\theta(M) \subseteq \ker(e)$. This implies that $\ker(e-f) \subseteq \ker(e)$. Similarly, $\ker(e) \subseteq \ker(e-f)$. Hence $\ker(e-f) = \ker(e) = \ker(f)$. By Proposition 4.2.3, $(1-e)fM \subseteq^{\oplus} M$. It follows that $(1-e)fS \subseteq^{\oplus} S$. By Proposition 4.2.3, S_S is $C4$.

(3) It can easily be seen by a proof similar to [58, Proposition 4.6]. □

Consider the free right R-module $F = R^{(\Omega)}$ on Ω generators. Then $\mathrm{End}_R(F)$ can be identified with $\mathbb{CFM}_\Omega(R)$, the ring of $\Omega \times \Omega$ matrices where each column has only finitely many non-zero entries. The ring $\mathbb{CFM}_\Omega(R)$ is called the ring of column finite matrices. The $C2$ and $C3$ properties of column finite matrices were investigated in [77].

Corollary 4.2.5 *If M is a free right R-module, then M is a $C4$-module if and only if $\mathrm{End}_R(M)$ is a right $C4$-ring. In particular, the following assertions hold for $n \in \mathbb{Z}^+$ and any infinite set Λ:*

(1) *R^n is a right $C4$-module if and only if $\mathbb{M}_n(R)$ is a right $C4$-ring.*

(2) *$R^{(\Lambda)}$ is a right $C4$-module if and only if $\mathbb{CFM}_\Lambda(R)$ is a right $C4$-ring.*

Proof. It follows from Theorem 4.2.4(i) since every free module is k-local-retractable (or a generator for right R-modules). □

Corollary 4.2.6 *The following conditions are equivalent for a ring R:*

(1) *$\mathbb{CFM}_\mathbb{N}(R)$ is a right $C2$-ring.*

(2) *$\mathbb{CFM}_\mathbb{N}(R)$ is a right $C3$-ring.*

(3) *$\mathbb{CFM}_\mathbb{N}(R)$ is a right $C4$-ring.*

(4) *$\mathbb{CFM}_\Lambda(R)$ is a right $C4$-ring for any infinite set Λ.*

Proof. The equivalence of the first two conditions was proved by [77, Theorem 3]. Next, (2) $\Rightarrow$ (3) is obvious. Now suppose that (3) holds. Then $R_R^{(\mathbb{N})}$ is $C4$. For any infinite set Λ, $R_R^{(\Lambda)} \cong R_R^{(\Lambda)} \oplus R_R^{(\Lambda)}$. This fact and [21, Proposition 2.15] give that $R_R^{(\mathbb{N})}$ is $C2$. Since $R_R^{(\mathbb{N})}$ is free, it is a generator for right R-modules. Hence (1) follows by [72, Theorem 7.14]. (2) $\Rightarrow$ (4) is obvious by [77, Theorem 3]. (4) implies (3) because if Λ is an infinite set, then $R_R^{(\mathbb{N})}$ can be viewed as a direct summand of $R_R^{(\Lambda)}$, and any direct summand of a $C4$-module is $C4$. □

A ring R is called *right strongly $C2$-ring* [71] if R_R^n is a $C2$-module for every $n \geq 1$, equivalently if $\mathbb{M}_n(R)$ is a right $C2$-ring for every $n \geq 1$. Similarly, right strongly $C3$-rings and right strongly $C4$-rings can be defined. But by [21, Proposition 2.15] and [4, Proposition 2.10], we see that they are all equivalent, i.e., R is a right strongly $C2$-ring if and only if R is a right strongly $C3$-ring if and only if R is a right strongly $C4$-ring.

4.3 Right $C4$-Rings

An example of a right $C4$-ring which is not left $C4$ was given by [21, Example 2.12]. Here we provide another example of a left $C4$-ring which is not right $C4$ which shows that the $C4$ property of rings is not left-right symmetric.

Example 4.3.1 There exists a left $C4$-ring which is not right $C4$. Let R be the ring of matrices, over a division ring D, of the form

$$\gamma = \begin{pmatrix} a & 0 & b & c \\ 0 & a & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{pmatrix}$$

R is artinian, right Kasch by [54, Examples 8.29(6)], and so R is left C_2-ring by [72, Proposition 1.46]. Now we claim that R is not right $C4$. The set of all idempotents of R is $\{0, 1, E_{c,d}, F_{c,d}\}$ where

$$E_{c,d} = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } F_{c,d} = \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

for any $c, d \in D$. Consider the idempotents $E_{1,0}$ and $E_{0,1}$. Then

$$R = E_{1,0}R \oplus F_{c,d}R = E_{0,1}R \oplus F_{c,d}R \quad \text{and} \quad E_{1,0}R \cap E_{0,1}R = 0$$

for any $c, d \in D$. Hence $E_{1,0}R$ and $E_{0,1}R$ are perspective direct summands of R with zero intersection. But their sum is not generated by an idempotent. For, any non-zero idempotent in $E_{1,0}R + E_{0,1}R$ is of the form $E_{x,1-x}$ where $x \in D$, so if $E_{1,0}R + E_{0,1}R = E_{x,1-x}R$ for some x, then we would have the following relations:

$$E_{x,1-x}E_{1,0} = E_{1,0} \quad \text{and} \quad E_{x,1-x}E_{0,1} = E_{0,1}$$

But in this case, $1 = x = 0$, a contradiction.

Proposition 4.3.2 *Let R_i $(i \in I)$ be any collection of rings, and let R be the direct product $\prod_{i\in I} R_i$. Then R is a right $C4$-ring if and only if every R_i is a right $C4$-ring.*

Proof. ($\Rightarrow$) Suppose that R is a right $C4$-ring. Let e_iR_i and f_iR_i be perspective direct summands of R_i with common direct complement X_i and zero intersection. Then there exist idempotents g_i, $h_i \in R_i$ such that $e_iR_i = g_iR_i$, $f_iR_i = h_iR_i$, and $X_i = (1 - g_i)R_i = (1 - h_i)R_i$. Now write e (resp. f) for the element in R with ith component e_i (resp. f_i) and all other components 0, and $1 - g$ (resp. $1 - h$) for the element in R with ith component $1 - g_i$ (resp. $1 - h_i$) and all other components 1. Then $eR \oplus (1 - g)R = R = fR \oplus (1 - h)R$. It is easy to see that eR and fR are perspective direct summands of R with zero intersection. Hence, $eR \oplus fR \subseteq^{\oplus} R$ by assumption. It follows that $e_iR_i \oplus f_iR_i \subseteq^{\oplus} R_i$, and so R_i is a right $C4$-ring.

($\Leftarrow$) Suppose that each R_i is a right $C4$-ring. Let $\{e_i\}_iR$ and $\{f_i\}_iR$ be perspective direct summands of R with zero intersection where $e_i^2 = e_i$ and $f_i^2 = f_i$ for each $i \in I$. Then there exist idempotents $\{g_i\}_i$, $\{h_i\}_i \in R$ such that $\{e_i\}_iR = \{g_i\}_iR$, $\{f_i\}_iR = \{h_i\}_iR$, and $\{1 - g_i\}_iR = \{1 - h_i\}_iR$. It can easily be seen that e_iR_i and f_iR_i are perspective direct summands of R_i with zero intersection for each $i \in I$. Since each R_i is a right $C4$-ring, $e_iR_i \oplus f_iR_i = k_iR_i \subseteq^{\oplus} R_i$ for some idempotent $k_i \in R_i$. Thus, $\{e_i\}_iR \oplus \{f_i\}_iR = \{k_i\}_iR \subseteq^{\oplus} R$, and so R is a right $C4$-ring. □

Proposition 4.3.3 *If R is a right $C4$-ring, then so is eRe for any idempotent $e \in R$ such that $ReR = R$.*

Proof. If R is right $C4$, then eR_R is $C4$ as a direct summand of R_R. Note that $eRe \cong \mathrm{End}_R(eR)$. So it is enough to show that $fS \cap gS = 0$ implies $f(eR) \cap g(eR) = 0$ for any pair of (perspective) idempotents $f, g \in S = eRe$ by Theorem 4.2.4(iii). Assume that $fS \cap gS = 0$ for some idempotents $f, g \in S$. Let $fr = gt \in f(eR) \cap g(eR)$. Then for every $x \in R$, $ferxe = getxe \in fS \cap gS = 0$. Since $ReR = R$, we have that $fer = 0$. Hence $fr = 0$. □

Example 4.3.4 The condition $ReR = R$ is not superfluous in Proposition 4.3.3: Let R be the algebra of matrices, over a field F, of the form

$$\begin{pmatrix} a & x & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & y & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & z \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}$$

Let $e = e_{11} + e_{22} + e_{33} + e_{44} + e_{55}$, where e_{ij} are the matrices with (i, j)-entry 1 and all other entries zero. Then e is an idempotent of R such that $ReR \neq R$. Since R is a quasi-Frobenius ring by [52, Example 9], R is a right $C4$-ring. However, $eRe \cong \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right) = S$ is not a right $C4$-ring. To prove that, consider the idempotents $e = e_{12} + e_{22}$ and $f = e_{22}$ of S. Then it can easily be seen that $S = eS \oplus e_{11}S = fS \oplus e_{11}S$ and $eS \cap fS = 0$. But $eS + fS$ is not a direct summand of S_S because it is the second column of S.

By Proposition 4.3.3, the right $C4$-property for rings is Morita invariant if and only if for every $n \geq 1$, $\mathbb{M}_n(R)$ is a right $C4$-ring whenever R is a right $C4$-ring. But this does not necessarily hold. Because there exists a right $C4$-ring which is not right strongly $C2$ (equivalently, strongly $C4$): Let A be a commutative local UFD that is not principal ideal domain (for instance, A can be the ring of formal power series in two variables over a field). Let M be the direct sum of all A/pA where p ranging over the primes of A. Let $R = A \propto M$, the trivial extension of A by M (see below for the definition). Then R is a local ring and every (right) non zero-divisor element of R is invertible [53]. Hence, by Proposition 1.6.19, R is a $C2$-ring, and so is $C4$. But, it is not a strongly $C2$-ring, as was shown in [53, p. 285].

Let R be a ring and M an R-R-bimodule. Then the trivial extension $R \propto M$ is a ring whose underlying group is $R \times M$ with the multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms)$$

where $r, s \in R$ and $m, n \in M$. In fact, $R \propto M$ is isomorphic to the subring $\{\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right) \mid r \in R, m \in M\}$ of the formal 2×2 triangular matrix ring $\left(\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix}\right)$ and $R \propto R \cong R[x]/(x^2)$. For convenience, let $(I, N) = \{(r, n) : r \in I, n \in N\}$ where I is a subset of R and N is a subset of M.

Proposition 4.3.5 *Let R be a ring and M an R-R-bimodule. Then*

(1) *If $R \propto M$ is a right $C4$-ring, and for any idempotents $e, f \in R$, $eR \cap fR = 0$ implies $eM \cap fM = 0$, then R is a right $C4$-ring.*

(2) *If R is a right $C4$-ring, and $eM(1 - e) = 0$ for any idempotent $e \in R$, then $R \propto M$ is a right $C4$-ring.*

Proof. Set $T = R \propto M$.

(1) Let e, f, g be idempotents in R such that $R = eR \oplus gR = fR \oplus gR$ and $eR \cap fR = 0$. Then $E := (e, 0)T$, $F := (f, 0)T$ and $G := (g, 0)T$ are direct summands of T. Now let $(ea, em) = (fb, fn) \in E \cap F$ where $a, b \in R$ and $m, n \in M$. Then $ea = fb \in eR \cap fR = 0$, and $em = fn \in eM \cap fM = 0$, by hypothesis. This implies that $E \cap F = 0$. Similarly, $E \cap G = 0$ and $F \cap G = 0$. Also, it can easily be seen that $T = E \oplus G = F \oplus G$. Then E and F are perspective direct summands. Since T is right $C4$, $E + F \subseteq^{\oplus} T$. Then there exists an idempotent $(h, m) \in T$ such that $E + F = (h, m)T$. It follows that $h^2 = h$. Now we claim that $eR + fR = hR$. Since $(h, m) \in E + F$, $hR \subseteq eR + fR$. If $ea + fb \in eR + fR$ for some $a, b \in R$, $(e, 0)(a, 0) + (f, 0)(b, 0) = (ea + fb, 0) \in E + F$. Hence $ea + fb \in hR$. This proves the claim. Therefore, R_R is $C4$.

(2) If $eM(1-e) = 0$ for any idempotent $e \in R$, then any direct summand of T is of the form (eR, eM) ($e^2 = e \in R$) by the proof of [32, Proposition 4.5]. Now let (eR, eM) and (fR, fM) be perspective direct summands of T with a common direct sum complement (gR, gM) and zero intersection. Then it is easy to see that $R = eR \oplus gR = fR \oplus gR$ and $eR \cap fR = 0$. Since R is right $C4$, $eR + fR \subseteq^{\oplus} R$. Let $h^2 = h \in R$ be such that $eR + fR = hR$. Now $(eR, eM) + (fR, fM) = (eR + fR, eM + fM) = (eR + fR, (eR + fR)M) = (hR, hM) \subseteq^{\oplus} T$. □

Corollary 4.3.6 *Let R and S be rings and M an R-S-bimodule. Consider the formal triangular matrix ring $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$. Then the following hold.*

(1) *If T is right $C4$, and for any idempotents $e, f \in R$, $eR \cap fR = 0$ implies $eM \cap fM = 0$, then R and S are right $C4$.*

(2) *If R and S are right $C4$ and $M = 0$, then T is right $C4$.*

Proof. (1) Note that $T = \left(\begin{smallmatrix} R & M \\ 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix} R & M \\ 0 & 0 \end{smallmatrix}\right)$ and S are right $C4$ by Proposition 4.3.2. Since $\left(\begin{smallmatrix} R & M \\ 0 & 0 \end{smallmatrix}\right) \cong (R \times 0) \propto M$, R is right $C4$ by Proposition 4.3.5.
(2) It is obvious. □

Note that Garcia in [32, Proposition 4.5] proved that if the ring $R \propto M$ has SSP, then $eM(1-e) = 0$ for any idempotent $e \in R$. But this property need not hold for $C4$-modules.

Example 4.3.7 There exist an idempotent e of a ring R and an R-R-bimodule M such that $R \propto M$ is right C_4 and $eM(1-e) \neq 0$: Let $T = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z}_{p^\infty} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$. Then all idempotents of T are 0, 1, $E_m = \left(\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right)$, and $F_m = \left(\begin{smallmatrix} 0 & m \\ 0 & 1 \end{smallmatrix}\right)$ where $m \in \mathbb{Z}_{p^\infty}$. By direct calculations, $E_m T \cap F_{m'} T = 0$ and $T = E_m T \oplus F_{m'} T$ for any $m, m' \in \mathbb{Z}_{p^\infty}$. For any $m \neq m'$ in $\mathbb{Z}_{p^\infty}$, $E_m T \cap E_{m'} T \neq 0$ and $F_m T \cap F_{m'} T \neq 0$. Hence after checking all direct summands regarding to C_4, we see that T_T is right C_4. Now we shall note the fact that $T \cong (\mathbb{Z} \times \mathbb{Z}) \propto \mathbb{Z}_{p^\infty}$. So take $R = \mathbb{Z} \times \mathbb{Z}$, $M = \mathbb{Z}_{p^\infty}$ and $e = (1, 0)$, as desired.

Recently, in [43], rings whose cyclic right modules are $C3$ are investigated in great detail. Commutative rings, abelian exchange rings, and local rings are examples of such rings. Here we will first notice that this kind of property of rings is equivalent to the property of rings whose cyclic right modules are right $C4$.

Proposition 4.3.8 *For a ring R, every cyclic right R-module is $C4$ if and only if every cyclic right R-module is $C3$.*

Proof. It can be obtained by the result that every factor module of M is $C4$ if and only if every factor module of M is SSP (equivalently $C3$, by [43]) [21, Proposition 2.28(1)]. □

Hence a structure theorem, Theorem 3.18 in [43], can be restated for $C4$-modules: Over a semiperfect ring R, every cyclic right R-module is $C4$ if and only if R is a direct product of a semisimple artinian ring and a ring which is a finite direct product of local rings. More generally, we have the following result inspired by [43, Lemma 2.4].

Corollary 4.3.9 *Let $n \geq 2$. Every n-generated module is $C4$ if and only if every n-generated module is $C3$.*

Proof. ($\Leftarrow$) is obvious. ($\Rightarrow$) Let $P_R = R^n$ and $S = \mathrm{End}_R(P)$. Then Mod-R and Mod-S are Morita equivalent categories with functors $\mathrm{Hom}_R({}_SP_R, \_)$ and $\_\otimes_S P$. It is known that for any n-generated module N, $\mathrm{Hom}_R(P, N)$ is a cyclic S-module, and for any cyclic S-module M, $M \otimes_S P$ is an n-generated R-module. Hence every cyclic S-module is a $C3$-module if and only if every n-generated R-module is a $C3$-module. Further, by [4, Remark 2.11], the $C4$ property of modules are preserved under Morita equivalences. Hence every cyclic S-module is a $C4$-module if and only if every n-generated R-module is a $C4$-module. By Proposition 4.3.8, the proof is completed. □

4.4 $D4$-Modules

The following lemma was established by Ding et al. in [22] and will be used frequently throughout this section.

Lemma 4.4.1 [22, Theorem 2.2] *The following conditions are equivalent for a module M:*

(1) *If $M = A \oplus B$ with $A, B \subseteq M$ and $f : A \to B$ is an epimorphism, then* $\ker(f) \subseteq^{\oplus} A$.

(2) *If $M = A \oplus B$ with $A, B \subseteq M$ and $f : A \to B$ is a homomorphism with* $\mathrm{im}(f) \subseteq^{\oplus} B$, *then* $\ker(f) \subseteq^{\oplus} A$.

(3) *If A and B are submodules of M with $A \subseteq B$ and $M/B \cong A \subseteq^{\oplus} M$, then $B \subseteq^{\oplus} M$.*

(4) *If A and B are submodules of M with $M = A + B$, $A \subseteq^{\oplus} M$ and $M/A \cong M/B$, then $A \cap B \subseteq^{\oplus} M$.*

(5) *If A and B are direct summands of M with $M = A + B$ and $M/A \cong M/B$, then $A \cap B \subseteq^{\oplus} M$.*

(6) *If A and B are submodules of M with $M = A + B$, $A \subseteq^{\oplus} M$ and $M/A \cong M/B$, then $B \subseteq^{\oplus} M$.*

(7) *If $M = A \oplus A' = B \oplus B' = A + B = A + B'$, where A, A', B and B' are submodules of M, then $A \cap B \subseteq^{\oplus} M$.*

(8) *If A and B are direct summands of M with $M = A + B$ and $A \cong B$, then $A \cap B \subseteq^{\oplus} M$.*

Definition 4.4.2 A module M is called a $D4$-*module* if it satisfies any of the equivalent conditions in Lemma 4.4.1.

In the next theorem we provide new characterizations of $D4$-modules in terms of perspective direct summands.

Theorem 4.4.3 *The following conditions on a module M are equivalent:*

(1) *M is a $D4$-module.*

(2) *If A and B are perspective direct summands of M with $A+B = M$, then $A \cap B \subseteq^{\oplus} M$.*

(3) *If A and B are perspective direct summands of M with $A + B \subseteq^{\oplus} M$, then $A \cap B \subseteq^{\oplus} M$.*

Proof. (1) $\Rightarrow$ (2) Let A and B be perspective direct summands with common direct sum complement C, and with $A + B = M$. Let $\pi : M \to C$ be the projection with $\ker(\pi) = B$. Consider the restriction map $\pi|_A : A \to C$. Now $A + B = M$ implies that $\pi(A) = C \subseteq^{\oplus} C$, and hence $\ker(\pi|_A) \subseteq^{\oplus} A$. Since $\ker(\pi|_A) = A \cap B \subseteq^{\oplus} A \subseteq^{\oplus} M$, $A \cap B \subseteq^{\oplus} M$.
(2) $\Rightarrow$ (3) Let A and B be perspective direct summands of M with $A+B \subseteq^{\oplus} M$. Then there exist C, $D \subseteq^{\oplus} M$ such that $M = A \oplus C = B \oplus C = (A+B) \oplus D$. By modularity, we have $A + B = A \oplus (C \cap (A + B))$ and $A + B = B \oplus (C \cap (A + B))$. Now $A \oplus D$ and $B \oplus D$ are perspective direct summands of M and $(A \oplus D) + (B \oplus D) = M$. Then the hypothesis implies that $(A \oplus D) \cap (B \oplus D) = (A \cap B) \oplus D \subseteq^{\oplus} M$. Thus $A \cap B \subseteq^{\oplus} M$.

(3) $\Rightarrow$ (1) Let $M = A \oplus B$ and $f : A \to B$ be an epimorphism. Consider the graph submodule $T = \{a + f(a) : a \in A\}$ of M. Obviously, $M = T + B$ and $T \cap B = 0$. Then $M = T \oplus B = A \oplus B$, i.e., A and T are perspective direct summands of M. Since f is an epimorphism, we have $A + T = M$. According to the hypothesis, $A \cap T \subseteq^{\oplus} M$. It can easily be shown that $A \cap T = \ker(f)$, and so $\ker(f) \subseteq^{\oplus} M$. Hence $\ker(f) \subseteq^{\oplus} A$. □

Corollary 4.4.4 *The following conditions on a module M are equivalent:*

(1) *M is a D4-module.*

(2) *If $M = A + B$ for any perspective direct summands A and B of M, then there exists $B' \subseteq B$ such that $M = A \oplus B'$.*

Proof. (1) $\Rightarrow$ (2) Let $M = A + B$ where A and B are perspective direct summands of M. Then $M = (A \cap B) \oplus C$ for some C by Theorem 4.4.3. By the modular law, $B = (A \cap B) \oplus (C \cap B)$. It follows that $M = A \oplus (C \cap B)$. So $B' := C \cap B$.
(2) $\Rightarrow$ (1) Let $M = A + B$ where A and B are perspective direct summands of M. Then there exists $B' \subseteq B$ such that $M = A \oplus B'$. This implies that $B = (A \cap B) \oplus B'$. Hence $A \cap B$ is a direct summand of B, and so of M. □

In the following theorem, we see that it is enough to assume that $A + B \subseteq^{\oplus} M$ in the conditions (4)-(6) and (8) in Lemma 4.4.1.

Theorem 4.4.5 *The following conditions on a module M are equivalent:*

(1) *M is D4-module.*

(2) *If A and B are submodules of M with $A+B \subseteq^{\oplus} M$, $A \subseteq^{\oplus} M$ and $M/A \cong M/B$, then $A \cap B \subseteq^{\oplus} M$.*

(3) *If A and B are direct summands of M with $A + B \subseteq^{\oplus} M$ and $M/A \cong M/B$, then $A \cap B \subseteq^{\oplus} M$.*

(4) *If A and B are submodules of M with $A+B \subseteq^{\oplus} M$, $A \subseteq^{\oplus} M$ and $M/A \cong M/B$, then $B \subseteq^{\oplus} M$.*

(5) *If A and B are direct summands of M with $A + B \subseteq^{\oplus} M$ and $A \cong B$, then $A \cap B \subseteq^{\oplus} M$.*

Proof. (1) $\Rightarrow$ (2) Let A and B be submodules of M with $A+B \subseteq^{\oplus} M$, $A \subseteq^{\oplus} M$ and $M/B \overset{\sigma}{\cong} M/A$. Write $M = A \oplus Y$ and $M = (A+B) \oplus X$ for some submodules X, $Y \subseteq M$. Now consider the morphisms: $\pi : M \to M/(A \cap B)$ the natural epimorphism with the $\ker(\pi) = A \cap B$, $f : M/(A \cap B) \to M/B$ defined by $f(m + (A \cap B)) = m + B$ (for every $m \in M$), and $\phi : M/A \to Y$ an isomorphism. Then we define $g = \phi\sigma f\pi|_A$. Now since $f(A/(A \cap B)) = (A+B)/B \subseteq^{\oplus} M/B$, $\mathrm{im}(g) \subseteq^{\oplus} Y$. Clearly, $\ker(g) = A \cap B$, and so $A \cap B \subseteq^{\oplus} A$. Thus, $A \cap B \subseteq^{\oplus} M$.

(2) $\Rightarrow$ (4) Let A and B be submodules of M with $A + B \subseteq^{\oplus} M$, $A \subseteq^{\oplus} M$ and $M/A \cong M/B$. By hypothesis, $A \cap B \subseteq^{\oplus} M$. Write $M = (A \cap B) \oplus X = (A + B) \oplus Y$ for some submodules X, $Y \subseteq M$. Then $A = (A \cap B) \oplus (A \cap X)$ by the modular law. Now $A + B = (A \cap B) + (A \cap X) + B = B \oplus (A \cap X)$ and so $B \subseteq^{\oplus} M$.

(4) $\Rightarrow$ (1) is clear by Lemma 4.4.1.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Let A and B be perspective direct summands of M with $A + B \subseteq^{\oplus} M$. Since A and B are perspective direct summands of M, $M/A \cong M/B$. By hypothesis, $A \cap B \subseteq^{\oplus} M$. Hence M is $D4$ by Theorem 4.4.3.

(1) $\Rightarrow$ (5) Let A and B be direct summands of M with $A+B \subseteq^{\oplus} M$ and $A \overset{\phi}{\cong} B$. Write $M = A \oplus A'$ for some submodule $A' \subseteq M$. Consider $\pi : M \to M/A$ as the natural epimorphism. Let ψ denote the isomorphism $M/A \cong A'$ and set $f = \psi \circ \pi|_B \circ \phi$. Since $A+B \subseteq^{\oplus} M$ and $\mathrm{im}(f) = \psi((A+B)/A)$, $\mathrm{im}(f) \subseteq^{\oplus} A'$. Note that $\ker(f) = \phi^{-1}(A \cap B)$ implies that $\phi^{-1}(A \cap B) \subseteq^{\oplus} A$ by hypothesis and hence $A \cap B \subseteq^{\oplus} B$. Thus $A \cap B \subseteq^{\oplus} M$.

(5) $\Rightarrow$ (1) is clear by Theorem 4.4.1. $\square$

In general the direct sum of two $D4$-modules need not be a $D4$-module, see [22, Example 2.12]. Indeed it was shown in [22, Theorem 2.13] that the direct sum of any two $D4$-modules is a $D4$-module if and only if R is a semisimple ring. In the next theorem we provide a specific case where the direct sum of a set of $D4$-modules is again a $D4$-module. Note the fact that if N is a fully invariant submodule of M, then $N = \oplus_{i \in I}(N \cap M_i)$ for any decomposition $M = \oplus_{i \in I} M_i$ (see [74, Lemma 2.1]).

Theorem 4.4.6 *Let $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i. If $N = \oplus_{i \in I}(N \cap M_i)$ for every submodule N of M, then M is a $D4$-module if and only if each M_i is a $D4$-module, $i \in I$.*

Proof. Suppose that M_i is a $D4$-module for every $i \in I$. Let $M = A \oplus C = B \oplus C$ such that $A + B = M$. By hypothesis, we have $A = \oplus_{i\in I}(A \cap M_i)$, $B = \oplus_{i\in I}(B \cap M_i)$, and $C = \oplus_{i\in I}(C \cap M_i)$. Since $M = A \oplus C = B \oplus C$, $M = \oplus_{i\in I}\left[(A \cap M_i) \oplus (C \cap M_i)\right] = [\oplus_{i\in I}(B \cap M_i)] \oplus [\oplus_{i\in I}(C \cap M_i)]$. Therefore, $M_i = (A \cap M_i) \oplus (C \cap M_i) = (B \cap M_i) \oplus (C \cap M_i)$ for every $i \in I$. Also, $M = A+B$ implies that $M = \oplus_{i\in I}\left[(A \cap M_i) + (B \cap M_i)\right]$, and so $M_i = (A \cap M_i) + (B \cap M_i)$. Since $A \cap M_i$ and $B \cap M_i$ are perspective direct summands of M_i with $(A \cap M_i) + (B \cap M_i) = M_i$, $(A \cap M_i) \cap (B \cap M_i) \subseteq^{\oplus} M_i$ for every $i \in I$. Now $A \cap B = [\oplus_{i\in I}(A \cap M_i)] \cap [\oplus_{i\in I}(B \cap M_i)] = \oplus_{i\in I}\left[(A \cap M_i) \cap (B \cap M_i)\right] \subseteq^{\oplus} M$, and hence M is a $D4$-module. The converse is obvious, since a direct summand of a $D4$-module is again a $D4$-module, see [22, Proposition 2.11]. □

It is known that if $N = \oplus_{i\in I}(N \cap M_i)$ for every $N \leqslant M$, then $\mathrm{Hom}(M_i, M_j) = 0$ for every $i \neq j$ in I (see [74, Lemma 2.4]), so it is natural to ask the following question.

Question 4.4.7 Can the above hypothesis in Theorem 4.4.6 be omitted if we assume instead that $\mathrm{Hom}(M_i, M_j) = 0$ for every $i \neq j$ in I?

The following proposition is a dual to Corollary 4.1.9 and it can be proved by a similar argument.

Proposition 4.4.8 *If M is a $D4$-module with SSP, then M has SIP.*

Proposition 4.4.9 *Let M be a right R-module with $S = \mathrm{End}_R(M)$. Then the following are equivalent:*

(1) *M is a $D4$-module.*

(2) *For every pair of perspective idempotents $e, f \in S$ with $eM + fM = M$, there exists an idempotent g of S such that $gM = eM$ and $(1 - g)M \subseteq fM$.*

Proof. Follows from [81, Lemma 5.3]. □

Lemma 4.4.10 *Let M be a right R-module and $S = \mathrm{End}_R(M)$. For any idempotents $e, f \in S$, $eM \cap fM \subseteq^{\oplus} M$ if and only if $\ker((1 - e)f) \subseteq^{\oplus} M$.*

Proof. ($\Rightarrow$) Suppose $eM \cap fM \subseteq^{\oplus} M$ for idempotents $e, f \in S$. Then there exist $C \subseteq^{\oplus} M$ such that $M = (eM \cap fM) \oplus C$. By the modular law, $fM = (eM \cap fM) \oplus$

$(C \cap fM)$. Hence $M = (eM \cap fM) \oplus (C \cap fM) \oplus (1-f)M$. Since $\ker((1-e)f) = (eM \cap fM) \oplus (1-f)M$, it follows that $\ker((1-e)f) \subseteq^{\oplus} M$.

($\Leftarrow$) is obvious by the fact that $\ker((1-e)f) = (eM \cap fM) \oplus (1-f)M$. □

Proposition 4.4.11 *A right R-module M is a $D4$-module if and only if for any pair of idempotents $e, f \in \mathrm{End}_R(M)$, if $M = eM + fM$ and $\ker(e) = \ker(f)$, then $\ker(1-e)f \subseteq^{\oplus} M$.*

Proof. ($\Rightarrow$) is obvious by Lemma 4.4.10.
($\Leftarrow$) Let A and B be perspective direct summands of M with $M = A+B$. Then we can find idempotents $e, f \in \mathrm{End}_R(M)$ such that $A = eM$, $B = fM$, $M = eM \oplus (1-e)M = fM \oplus (1-f)M$, and $(1-e)M = (1-f)M$. By the hypothesis and Lemma 4.4.10, $eM \cap fM \subseteq^{\oplus} M$. □

In [43], Ibrahim et al. restricted the submodules in the $D2$-condition to the the class of simple modules, and call a module M *simple-direct-projective* if it satisfies any of the equivalent conditions in the following lemma:

Lemma 4.4.12 ([44, Proposition 2.1], [43]) *The following are equivalent for a module M:*

(1) *If A and B are submodules of M with B simple and $M/A \cong B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$.*

(2) *$M = A \oplus B$ with B simple and $f : A \to B$ is an R-homomorphism, then $\ker(f) \subseteq^{\oplus} M$.*

(3) *If A and B are direct summand of M with B maximal, then $A \cap B \subseteq^{\oplus} M$.*

(4) *If A and B are maximal direct summands of M, then $A \cap B \subseteq^{\oplus} M$.*

In the next proposition we characterize simple-direct-projective modules in terms of perspective direct summands.

Proposition 4.4.13 *M is simple-direct-projective if and only if for any perspective direct summands A, B of M with B maximal, $A \cap B \subseteq^{\oplus} M$.*

Proof. Necessity is obvious. Assume that for any perspective direct summands A, B of M with B maximal, $A \cap B \subseteq^{\oplus} M$. Let $M = A_1 \oplus A_2$ with A_2 simple and $f : A_1 \to A_2$ an R-homomorphism. We claim that $\ker(f)$ is a direct summand of A_1. Without loss of generality we may assume that $f \neq 0$. Then f is an R-epimorphism. Let $T = \{a + f(a) \,|\, a \in A_1\}$ be the graph submodule. We have that $M = T \oplus A_2$. Since $A_2 \cong M/T \cong M/A_1$, T and A_1 are perspective direct summands of M with A_1 maximal. Thus $T \cap A_1 \subseteq^{\oplus} M$ by assumption. Clearly, $\ker(f) \subseteq T \cap A_1$. Since $\ker(f)$ is maximal in A_1 and $M = T + A_1$, we have $T \cap A_1 = \ker(f)$. Hence $\ker(f)$ is a direct summand in M, and so in A_1. By Lemma 4.4.12, M is simple-direct-projective. □

A module M is called *summand-dual-square-free* [22] if M has no proper direct summands A and B with $M = A + B$ and $M/A \cong M/B$. Any summand-dual-square-free module is a $D4$-module by [22, Proposition 5.4].

Proposition 4.4.14 *The following conditions on a module M are equivalent:*

(1) *M is a $D4$- and summand-square-free module.*

(2) *M is a $C4$- and summand-dual-square-free module.*

Proof. (1) ⇒ (2) Clearly, M is a $C4$-module. Now, we show that M is summand-dual-summand-square-free. Assume that M is not summand-dual-square-free, then there exists two non-zero proper summands A_1, B_1 of M with $A_1 + B_1 = M$ and $M/A_1 \cong M/B_1$. Since M is a $D4$-module, $A_1 \cap B_1 \subseteq^{\oplus} M$. Write $M = (A_1 \cap B_1) \oplus T_1$ and so $A_1 = (A_1 \cap B_1) \oplus (A_1 \cap T_1)$ and $B_1 = (A_1 \cap B_1) \oplus (B_1 \cap T_1)$. Therefore, we have $A_1 \cap T_1 \cong A_1/(A_1 \cap B_1) \cong M/B_1 \cong M/A_1 \cong B_1/(A_1 \cap B_1) \cong B_1 \cap T_1$ with $(A_1 \cap T_1) \cap (B_1 \cap T_1) = (A_1 \cap B_1) \cap T_1 = 0$ and both $A_1 \cap T_1$ and $B_1 \cap T_1$ summands of M. Since M is summand square-free, $A_1 \cap T_1 = B_1 \cap T_1 = 0$. Thus $A_1 = (A_1 \cap B_1) = B_1$ and so $M = A_1 + B_1 = A_1 = B_1$ a contradiction. Hence M is summand-dual-square-free.

(2) ⇒ (1) Clearly, M is a $D4$-module. Now, we show that M is summand-square-free. Assume that M is not a summand-square-free module, and let A_1, B_1 be non-zero summands of M with $A_1 \cong B_1$ and $A_1 \cap B_1 = 0$. Since M is a $C4$-module, $A_1 \oplus B_1 \subseteq^{\oplus} M$. Write $M = A_1 \oplus B_1 \oplus T_1$ for a submodule $T_1 \subsetneq M$. Now, $M/(A_1 \oplus T_1) \cong B_1 \cong A_1 \cong M/(B_1 \oplus T_1)$ with $M = A_1 \oplus B_1 \oplus T_1 = (A_1 \oplus T_1) + (B_1 \oplus T_1)$ and both $A_1 \oplus T_1$ and $B_1 \oplus T_1$ summands of M. Since M is summand-dual-square-free,

$M = A_1 \oplus T_1 = B_1 \oplus T_1$ and so $A_1 = B_1 = 0$, a contradiction. Hence M is summand-square-free. □

Corollary 4.4.15 *A ring R is summand-square-free as a right R-module if and only if R is a right $C4$-module and summand-dual-square-free as a right R-module.*

Definition 4.4.16 A module M is said to satisfy *the restricted descending chain condition on direct summands* if, M has no strictly descending chains of non-zero direct summands

$$A_1 \supsetneqq A_2 \supsetneqq \cdots$$
$$B_1 \supsetneqq B_2 \supsetneqq \cdots$$

with $M/A_i \cong M/B_i$ and $A_i + B_i \subseteq^{\oplus} M$ for all $i \geqslant 1$.

Proposition 4.4.17 *If M is a $D4$-module that satisfies the restricted DCC on summands, then $M = A \oplus B \oplus K$ where $A \cong B$, A and B are $D2$-modules, and K is a summand-dual-square-free module.*

Proof. If M is summand-dual-square-free, then the proof is done by setting each $A = B = 0$ and $K = M$. Suppose that M is not summand-dual-square-free, then there exist two non-zero proper summands A_1, B_1 of M with $A_1 + B_1 = M$ and $M/A_1 \cong M/B_1$. Since M is a $D4$-module, $A_1 \cap B_1 \subseteq^{\oplus} M$. Write $M = (A_1 \cap B_1) \oplus T_1$ and $M = A_1 \oplus A_1'$. Now, since $T_1 \cong M/(A_1 \cap B_1) \cong A_1/(A_1 \cap B_1) \oplus A_1' \cong M/B_1 \oplus A_1' \cong M/A_1 \oplus A_1' \cong A_1' \oplus A_1'$ and T_1 is a $D4$-module, $A_1' \oplus A_1'$ is a $D4$-module, and then by [22, Proposition 2.11], A_1' is a $D2$-module. Therefore, $T_1 = K_1 \oplus K_1'$ where $K_1 \cong A_1' \cong K_1'$ are $D2$-modules. Clearly $Y_1 := A_1 \cap B_1 \neq M$. If $Y_1 = 0$, then $M = T_1 = K_1 \oplus K_1'$ and the proof is done.

Now, suppose that $Y_1 \neq 0$; and so we have $0 \subsetneqq Y_1 \subsetneqq M$. If Y_1 is summand-dual-square-free, the proof is done. Suppose that Y_1 is not summand-dual-square-free, then there exist two non-zero proper summands A_2, B_2 of Y_1 with $A_2 + B_2 = Y_1$ and $Y_1/A_2 \cong Y_1/B_2$. Since Y_1 is a $D4$-module, $Y_2 := A_2 \cap B_2 \subseteq^{\oplus} Y_1$. Write $Y_1 = (A_2 \cap B_2) \oplus T_2$, $Y_1 = A_2 \oplus A_2' = B_2 \oplus B_2'$, $M = A_2 \oplus A_2' \oplus T_1$, and $M = (A_2 \cap B_2) \oplus T_2 \oplus T_1$. Then $Y_1/A_2 \cong Y_1/B_2$ implies that $M/A_2 \cong M/B_2$. Now, we have $T_2 \cong Y_1/(A_2 \cap B_2) = (A_2 + A_2')/(A_2 \cap B_2) \cong A_2/(A_2 \cap B_2) \oplus A_2' \cong Y_1/B_2 \oplus A_2' \cong Y_1/A_2 \oplus A_2' \cong A_2' \oplus A_2'$.

Therefore, $T_1 \oplus T_2 \cong (A_1' \oplus A_1') \oplus (A_2' \oplus A_2') \cong (A_1' \oplus A_2') \oplus (A_1' \oplus A_2')$ is a $D4$-module and so by [22, Proposition 2.11], $A_1' \oplus A_2'$ is a $D2$-module. Therefore, $T_1 \oplus T_2 = K_2 \oplus K_2'$ with $K_2 \cong A_1' \oplus A_2' \cong K_2'$ are $D2$-modules. Clearly $Y_2 := A_2 \cap B_2 \neq Y_1$. If $Y_2 = 0$, then $M = T_2 \oplus T_1 = K_2 \oplus K_2'$ and the proof is done.

Now, suppose that $Y_2 \neq 0$. If Y_2 is summand-dual-square-free, the proof is done. Suppose that Y_2 is not summand-dual-square-free, then by continuing the process and if each Y_i is not summand-dual-square-free, we get proper descending chains,

$$\begin{aligned} A_1 &\supsetneq A_2 \supsetneq \cdots \\ B_1 &\supsetneq B_2 \supsetneq \cdots \end{aligned}$$

with $M/A_i \cong M/B_i$ and $A_i + B_i \subseteq^{\oplus} M$ for all $i \geqslant 1$, contradicting the hypothesis that M has the restricted DCC on summands. Therefore, there exists a summand-dual-square-free module Y_n such that $M = T_1 \oplus T_2 \oplus \cdots \oplus T_n \oplus Y_n$ with $T_1 \oplus T_2 \oplus \cdots \oplus T_n = K_n \oplus K_n'$ with $K_n \cong K_n'$ a $D2$-module and Y_n summand-dual-square-free. Hence the proof is done by setting $A := K_n$, $B := K_n'$ and $K := Y_n$. □

Recall that a ring R is called *I- finite* if it contains no infinite orthogonal family of idempotents (see [72, Lemma B.6.]).

Corollary 4.4.18 *If R is I-finite, then $R_R = A \oplus B \oplus K$ with $A \cong B$ and K a summand-dual-square-free module. Moreover, if R is also a right $C4$-ring, then $R_R = A \oplus B \oplus K$ where $A \cong B$ are $C2$-modules and K is both a summand-dual-square-free as well as a summand-square-free module.*

Lemma 4.4.19 [21, Theorem 2.27] *If M is a module whose local summands are summands, then $M = A \oplus B \oplus K$ where $A \cong B$ and K is a summand-square-free module.*

Proposition 4.4.20 *If M is a quasi-discrete module with DCC on summands, then $M = A \oplus B \oplus K$ where $A \cong B$ are quasi-projective modules and K is both a summand-square-free and a summand-dual-square-free module.*

Proof. By Proposition 4.4.17, $M = X \oplus Y \oplus T$ where $X \cong Y$ and T is a summand-dual-square-free module. Now, since T is quasi-discrete, every local summand of T is a summand, see [63, Corollary 4.13]. By Lemma 4.4.19, $T = C \oplus D \oplus K$ where $C \cong D$ and K is both a summand-square-free module as well as a summand-dual-square-free

module. Now, if we set $A := X \oplus C$ and $B := Y \oplus D$, then $M = A \oplus B \oplus K$ with $A \cong B$. By [22, Proposition 4.12], since M is quasi-discrete, both A and B are quasi-projective as required. □

Corollary 4.4.21 *If R is a semiperfect ring, then $R_R = A \oplus B \oplus K$ with $A \cong B$ and K is both a summand-dual-square-free as well as a summand-square-free module.*

We end this chapter with a number of related questions that we were unable to answer.

Question 4.4.22 If M is a $C4$-module, does the finite exchange property imply the full exchange property?

Question 4.4.23 Is there a $C1$- and $C4$-module that is not $C3$?

Question 4.4.24 If M is a $D4$-module, does the finite exchange property imply the full exchange property?

Question 4.4.25 Is there a $D4$-module that is not $D3$?

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