MULTIVARIATE STOCHASTIC PRIORITIZATION OF DEPENDENT ACTUARIAL RISKS UNDER UNCERTAINTY

BAĞIMLI AKTÜERYAL RİSKLERİN BELİRSİZLİK ALTINDA ÇOKDEĞİŞKENLİ STOKASTİK ÖNCELİKLENDİRİLMESİ

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27 / 07 / 2018

Ezgi Nevruz

"If there were no difference between essence and appearance,

there would be no need for science."

Karl Marx

ETHICS

In this thesis study, prepared in accordance with the spelling rules of Institute of Graduate Studies in Science of Hacettepe University,

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- I did not do any distortion in the data set
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27/07/2018

EZGİ NEVRUZ

ABSTRACT

MULTIVARIATE STOCHASTIC PRIORITIZATION OF DEPENDENT ACTUARIAL RISKS UNDER UNCERTAINTY

Ezgi NEVRUZ

Doctor of Philosophy, Department of Actuarial Sciences

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The main prompting factor behind decision making is comparing or ordering risks. Risk management strategies should be based on the dynamics of stochastic ordering relations and influences of decision makers' tendencies on risk prioritization. The objective of this thesis is to construct a concept for stochastic risk prioritization of multivariate aggregate claims.

The definition of risk from perspectives of individuals, companies or governments may vary according to their risk perceptions, as risk is indicated not only by objective measures but also by subjective characteristics. In order to describe the risk accurately, the theoretical background of multivariate stochastic prioritization of dependent actuarial risks should be understood. For this aim, we familiarize ourselves with order theory that allows comparing and ordering objects

characterized by multiple indicators.

Being an important issue of human behaviour, this area falls within the boundaries of several fields, one of which - public health - is our specific interest. We intend to apply the order theory to a chosen risk area such as foodborne or agricultural risks, since they are rather vulnerable aspects of public health. Analytic tools may not always be sufficient for prioritization especially when we work on environmental risks. Hence, geographic information system is a useful tool for risk prioritization in such cases.

In this thesis, we aim to prioritize aggregate claim vectors of different risk clusters in agricultural insurance under the assumption that the individual claims exposed to similar environmental risks are dependent. For this purpose, first we obtain risk clusters for a crop-hail insurance portfolio considering spatial and temporal features of hazard regions. We propose an extended approach for differential evolution optimization which determines the optimal sample set used in inverse distance weighting with reduction technique. Second, we prioritize the aggregate claims taken as actuarial risks by using various stochastic ordering relations that are studied within the framework of partial order theory. These relations are stochastic dominance, stochastic majorization and stop-loss dominance. Having discussed the concept of risk itself, we also investigate the risk measures which could be sufficient and accurate criteria for determining the riskiness of a portfolio.

The classical first-order stochastic dominance is useful to design the risk prioritization context. We also suggest stochastic majorization relation according to multivariate representation of actuarial risks. This relation is very beneficial for our study since it enables us to order aggregate claim vectors partially using Schur-convex risk measures.

On the other hand, we consider the impacts of risk perception on prioritization of risks. Working within this context and attempting to contribute to it, we seek for a behavioral approach which could enhance and facilitate the description of the choices individuals make in risky situations. An example of such approaches could be cumulative prospect theory (CPT), as a more accurate alternative to expected utility theory. In the stop-loss dominance context, we adapt the zero-utility premium principle in order to obtain solutions for stop-loss premiums and propose stop-loss dominance relation under CPT.

Keywords: Cumulative prospect theory, differential evolution algorithm, geographic information system, risk clustering, Schur-convexity, spatiotemporal interpolation, stochastic majorization, stop-loss dominance.

ÖZET

BAĞIMLI AKTÜERYAL RİSKLERİN BELİRSİZLİK ALTINDA ÇOKDEĞİŞKENLİ STOKASTİK ÖNCELİKLENDİRİLMESİ

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Riskleri karşılaştırmak; riskleri ortaya çıkaran faktörlerin karakteristiklerini dikkate alarak, karar verme aşamalarında adil ve doğru standartlar altında uygulanan en temel süreçtir. Risk yönetimi açısından ele alındığında ise riskleri sıralandırmak, özellikle süre ve maliyet optimizasyonu açısından önemli bir bileşendir. Çevresel riskleri önceliklendirmek ise riskleri coğrafik olarak karşılaştıran ve sıralandıran spesifik bir alandır. Analitik araçlar, önceliklendirme için her zaman yeterli olmayabilir. Bazı durumlarda, özellikle çevresel risklerin değerlendirilmesi ile ilgilenildiğinde, verideki coğrafik bilgi dikkate alınmalıdır. Bu çalışmada, benzer çevresel risklere maruz olan tarım sigortası hasarların bağımlılığı dikkate alındığında; tarımsal sigortalarda aktüeryal risklerin meteorolojik karakteristiklerinin tahmin edilmesi, kümelendirilmesi ve stokastik sıralandırma bağıntıları kullanılarak önceliklendirilmesi amaçlanmaktadır. Çalışmada ele alınan

sıralandırma bağıntıları birinci dereceden stokastik baskınlık, toplam hasar fazlası baskınlığı ve stokastik baskılama bağıntılarıdır.

Tezde, TARSİM tarım sigortaları havuzunun sağladığı veri setinde 2014 yılında gerçekleşen hasarlara ilişkin bilgiler kullanılmıştır. 100'den fazla ürünü, 5 farklı tehlikeye karşı teminat altına alan sigorta ürünlerine ilişkin prim, sigorta bedeli, hasar tarihi, hasar nedeni, ödenen/muallak tazminat tutarı, hasar oranı vb. bilgileri kapsayan veride yapılan düzenlemeler ile veri seti kendine özgü bir hale dönüştürülmüştür.

Aktüeryal risklerin öncelendirilmesi için bitkisel dolu sigortası portföyünde ortaya çıkan bireysel hasarlar ve her bir risk kümesine ait toplam hasarlar ele alınmıştır. Tehlike bölgeleri ve bitkisel ürün sınıfları dikkate alınarak kurulan modelde, toplam hasar vektörleri elde edilmiştir. Bu amaçla öncelikle; meteorolojik değişkenlerin girdi olarak ele alındığı mekansal-zamansal interpolasyon teknikleri kullanılarak bireysel poliçelere ait meteorolojik değişkenler tahmin edilip aktüeryal riskler kümelendirilmiştir. Ardından, vektör sıralamasında kolaylık sağlayan kısmî sıralama teorisi çerçevesinde önerilen stokastik sıralandırma bağıntıları kullanılarak aktüeryal riskler önceliklendirilmiştir.

Aktüeryal risklerin yönetimi ve değerlendirilmesi sürecinde; çevresel riskler, sigorta türünün dinamikleri açısından sıklıkla kullanılan hayat ve hayat-dışı sigorta ürünlerinden farklılık arz etmektedir. Klasik sigorta ürünlerinin risklerinin aktüeryal değerlemesinde analitik araçlar etkili performans göstermektedir; ancak tarımsal sigortalarda bu araçların etkinliği azalabilir. Bu çalışmada, farklı meteorolojik değişkenlerin ve coğrafi bilginin tarımsal sigorta risklerine etkisi birlikte incelenmiştir. Klasik risk kuramında bireysel hasarların bağımsız olduğu varsayılmasına rağmen, literatürde bu varsayımın gerçekçi olmadığını ileri süren ve doğrulayan birçok çalışma bulunmaktadır [1–3]. Aynı tehlikelere veya fiziksel ve finansal çevre gibi benzer olumsuz etkilere maruz oldukları için bireysel riskler bağımlı olabilirler. Örneğin, bir tarımsal sigorta portföyündeki hasarlar, aynı meteorolojik olayın meydana gelme olasılığına ve sonuçlarına maruz olabilir. Aynı coğrafik alanlarda üretilen bitkisel ürünler aynı fiziksel çevredeki benzer tehlikelerle karşı karşıyadır. Bu çalışmada, ayrık risk kümelerine ait bağımlı toplam hasar raslantı değişkenlerinin önceliklendirilmesi tartışılmaktadır. Bunun yanı sıra, aynı risk kümesindeki hasarların bağımlı oldukları varsayılmaktadır. Model kurulumu, bu bağımlılığı yansıtacak şekilde vektörel tanım ile yapılmıştır.

Model kurulumuna ek olarak, portföy hakkında yeterli bilgiyi içeren ve bunu doğru yansıtan bir risk ölçütü tanımlamak riskin değerlendirilmesi için en önemli işlevlerden birisidir. Risk ölçütü belirlenirken, ortalamanın yanı sıra ortalama etrafında yayılma ve korelasyonun da dikkate alınması gerekir. Bağımlılık durumunda toplam hasarın ortalaması bağımsızlık durumundan farksız olarak elde edildiğinden; bu çalışmada hem değişkenliği, hem de bağımlılığı hesaba katan "değişim katsayısı" ve "(standartlaştırılmış) genelleştirilmiş varyans" gibi risk ölçütleri ele alınmıştır. Aktüeryal risk olarak toplam hasar raslantı değişkeni kullanılacağından bu ölçütler çokdeğişkenli analiz çerçevesinde tanımlanmıştır.

Risk değerlendirmesi için riskin nasıl tanımlandığını göz önünde bulundurmak önem taşımaktadır. Literatürde yer alan birçok risk tanımı incelendiğinde riskin miktarının belirlenebileceği, ölçülebileceği ve mevcut veri yardımıyla matematiksel bir ilişki olarak ifade edilebileceği sonucuna ulaşılmıştır. Bunun yanı sıra, riski değerlendirirken sadece etkilenen kişi sayısının veya sadece sonuç senaryo üzerindeki finansal etkinin dikkate alınmasının yeterli olmadığı ve bu faktörlerin beraber ele alınması gerektiği düşünülmektedir. Riskin teknik açıdan değerlendirilmesine ilişkin bu tarştışmalara ek olarak, belirsizlik altında karar verme ve tercihlerin modellenmesi kapsamında riskin subjektif yanının da göz ardı edilmemesi gerekmektedir.

Karar vericilerin rasyonel olduğu varsayımına dayalı olan geleneksel beklenen fayda kuramının aksine, bireylerin veya kurumların karar verirken yansız olmadıklarını gösteren birçok çalışma bulunmaktadır [4–9]. Bu çalışmada, bireylerin risk algısını ve risk önceliklendirmede yanlı karar verme süreçlerini dikkate alan bir modelleme geliştirmek amacıyla *kümülatif beklenti teorisi* ele alınmıştır. Sıfır fayda prim ilkesi çerçevesinde, bazı özel tanımlı değer fonksiyonları için toplam hasar fazlası reasürans primleri elde edilmiştir. Elde edilen bu çözümler yardımıyla, kısmî sıralama teorisi çerçevesinde *toplam hasar fazlası sıralandırma bağıntısı* önerilmiştir.

Çalışmanın temelini oluşturan bir diğer stokastik sıralandırma ise *stokastik baskılamadır*. Stokastik baskılama geleneksel vektör sıralamasına kıyasla daha kullanışlı bir bağıntıdır. Schur-konveks fonksiyonlar bu bağıntının en temel araçlarıdır. Riskleri sıralandırmak için kullanılan birçok risk ölçütü arasında yer alan ve Schur-konveks fonksiyonlar olan varyans ve değişim katsayısı önemli değişkenlik ölçütleri olarak bu çalışmada kullanılmıştır. Ayrıca, toplam hasar raslantı değişkeninin sürekli olması nedeniyle çalışmada önerilen model kurulumuna bağlı olarak baskılama aksiyomları adapte edilmiştir. Uygulamada, veride yer alan mevcut tehlike bölgesi gruplandırmasının bu bağıntının aksiyomlarını sağlamadığı ve risk kümelendirilmesinin yeniden yapılması gerektiği anlaşılmıştır.

Çalışmanın ilk aşaması olan mekansal-zamansal interpolasyon için literatürde en sık kullanılan kriging, ters mesafe ağırlıklandırma, rasgele orman ve bu yöntemlerin çeşitli genişletilmiş versiyonları arasından *azaltma yaklaşımıyla ters mesafe ağırlıklandırma yöntemi* kullanılmıştır. Bir mekansal-zamansal interpolasyon yönteminin performansı, örneklemin dinamikleri ve dizaynına, veri setine ve örneklem üzerinde etkisi olan faktörler arasındaki ilişkiye göre değişkenlik göstermektedir. Sözü geçen nedenlerden ötürü bu çalışmada, tarımsal hasarların gerçekleştiği konum ve zamana ilişkin bilinmeyen meteorolojik verilerin tahmin edilmesi için; daha etkin ve daha iyi performans gösterdiği birçok çalışma ile gösterilen ters mesafe ağırlıklandırma yöntemi tercih edilmiştir. Bu yöntem uzaklık esaslı bir yöntem olduğundan; örneklem noktalarından hem mekansal, hem de zamansal olarak uzaklıkları dikkate alan mekansal-zamansal interpolasyon tekniği olarak kullanılabilir.

Mekansal-zamansal interpolasyon tekniği için gerekli olan optimal örneklem kümesinin, her bir hasar için ayrı ayrı belirlenmesi gerekmektedir. Bunun için, literatürde yer alan paralel stokastik optimizasyon teknikleri arasından dolaysız arama yöntemi olan *diferansiyel gelişim algoritması* kullanılmıştır. Model kurulumuna dayalı olarak tek değişkenli değişkenler için önerilen algoritma, bu çalışmada çokdeğişkenli duruma uyarlanarak geliştirilmiştir. Ayrıca algoritmanın karar verici kriterlerinden birisi olan popülasyon büyüklüğünün seçimi için de bir çözüm önerisi sunulmuştur. Hasar verisine ilişkin enlem, boylam ve yükseklik bilgileri ele alınarak optimal girdi olarak kullanılacak örneklem noktaları, tahmin hatasını ve noktalar arasındaki mesafenin değişim katsayısını içeren bir maliyet fonksiyonunu minimize edecek şekilde belirlenmiştir.

Tahmin edilen meteorolojik değişkenlerin tarımsal risklerin kümelendirmesine etkisini dikkate alan *model-bazlı kümeleme çalışması* ile toplam hasar vektörleri olarak ifade edilen riskler gruplandırılmıştır. Risk kümelerine ait toplam hasar rasgele vektörleri kısmî sıralama teorisi çerçevesinde önerilen stokastik bağıntılar ile önceliklendirilmiştir.

Anahtar Kelimeler: Kümülatif beklenti teorisi, diferansiyel gelişim algoritması, coğrafik bilgi sistemi, risk kümelendirme, Schur-konvekslilik, mekansal-zamansal interpolasyon, stokastik baskılama, toplam hasar fazlası baskınlığı.

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CONTENTS

Page

LIST OF TABLES

Page

LIST OF FIGURES

Page

LIST OF SYMBOLS AND ABBREVIATIONS

Symbols

$\boldsymbol{S}^{(i)}$	Aggregate claim random vector of the i-th risk cluster
$S_j^{(i)}$	Aggregate claim random variable of the i-th risk cluster and j-th crop class
$\precsim_{\text{sd}}$	First-order stochastic dominance relation
$\precsim_{\text{sl}}$	Stop-loss dominance relation
$\precsim_{\text{maj}}$	Stochastic majorization relation

Abbreviations

CPT	Cumulative Prospect Theory
POT	Partial Order Theory
rv	random variable
DM	Decision maker
r.vector	random vector
EUT	Expected Utility Theory
YDT	Yaari's Dual Theory
GIS	Geographic Information System
STI	Spatiotemporal Interpolation
DE	Differential Evolution
wrt	with respect to
iff	if and only if
s.t.	such that
df	distribution function
iid	independent and identically distributed

1. INTRODUCTION

In one of Plato's dialogues, the Theaetetus, "knowledge" is defined in three different ways: "knowledge as nothing but perception", "knowledge as true judgement" and "knowledge as a true judgement with an account". This situation may also arise in daily life. A question of how to measure a person's knowledge can be thought as an example because many answers may arise to this question. For instance, an individual's knowledge can be evaluated by her educational status, her age, the number of books she has read etc. Various measures can be taken for the evaluation of an indicator. Should we choose the measures that are more efficient or should we combine them? Are these measurements comparable or not? Finally, if we want to rank people according to their knowledge, what should we do? Likewise, there is also an interesting question: "Is the abundance of alien species risky or an enrichment for a given ecosystem?". We should answer this depending on what the investigator's attitude is [10]. It stands to reason that risk could not only be assessed according to feelings, however paying no attention to the social interpretation of risk and performing a technical analysis alone may cause a misunderstanding about decision makers' (DMs') judgments of risk.

Comparing or ranking things is one of the main motivations of the choices in human nature. Rational people always consider the riskiness of things when they need to compare them. They tend to rank risks in their minds before they act. We can thus adapt the questions about 'knowledge' to 'risk' topic because the same problematique exists for risk prioritization. The starting point here should be to ask the question of what the risk is. The *risk perception* phenomenon has a huge amount of influence on the definition of risk. Cumulative prospect theory (CPT) could be a commonly used approach taking the subjective part of risk into account. In our study, we will adapt CPT in order to take consideration of the impact of risk perception.

Risk analysts need to understand a problem related to risk management, determine potential solutions and deliberate on the criteria according to decisions to be evaluated before starting the analysis. For this aim, some difficulties could be handled: choosing a simple problem among alternatives, sorting actions into classes or categories, placing actions in a form of preference ordering, evaluating these actions by describing their consequences in a formalized and systematic manner, searching for, identifying or creating new decision alternatives, and choosing a subset of alternatives considering characteristics, interactions and synergies between them [11, 12]. In this study, we obtain risk clusters and we propose a multivariate model setting which reflects the dependencies among these clusters considering that similar environmental features cause similar hazards.

Problem structuring, which is the process of understanding a problem, prevents one from addressing the wrong problem since it is oriented to problem finding rather than discovering the answer to problem. If problems are unstructured, no clear formulation could exist to solve them. Formulation of a problem and solving it helps to reduce the uncertainty. Uncertainty might arise not only because of the lack of knowledge, but also due to different interpretations. Since different stakeholders have different perceptions about the problem, interaction between them helps to have a negotiated knowledge [13]. From this point of view, thinking about specific actions or alternatives, adapting alternative perspectives, considering both positive and negative reviews and handling barriers and constraints are the key points for switching from problem structuring to model building [12].

1.1. Problem Definition and Objectives of the Study

Decision making and risk prioritization cannot be dissociated from each other since both of them are the essential parts of risk assessment, which is the procedure of the risk value hierarchy. As risk management is becoming more complex, describing the factors of risk characterization efficiently is becoming obligatory. Risk management, which is completely related to risk assessment, is also conducted to help ranking a number of alternatives based on the risk criteria. These debates on risk notion mostly depend on discussions about defining and accordingly managing the risk.

In terms of risk analysis, there have been many conflicts between social scientists working on risk management and technical risk assessors. Social scientists claim that risk should be handled as a human value and cannot be calculated objectively, whereas the other camp manages risk only in consideration of technical calculation. At this point of the discussion, we think that risk can be evaluated using a mathematical model with including the social characteristics of it. If we ignore either approach, we will tackle an imperfect modeling problem. Since both views are reasonable, combining these two aspects may be the best way to cope with both social and technical features of risk. For this aim, we propose to use actuarial approaches taking the technical part of the insurance risks and adapt the preference perspective to these models.

In this study, we consider impacts of risk perception on prioritization of risks. For this reason, we handle various risk definitions as an evidence of the influence of how risk is perceived. On the other hand, we present a discussion about risk definition emphasizing the importance of including of uncertainty in this study. To provide a better understanding, we investigate numerous areas of studies about the approaches for ranking of risks.

To understand the theoretical background of multivariate ranking and prioritization, we focus on the order theory that allows comparing and ordering objects characterized by multiple indicators. Unlike total ordering, we do not need to rank all sets of risks when we use *partial order theory (POT)* [1]. POT is more appropriate for our study because this approach does not need all pairs of elements to be comparable [14].

It is assumed in the classical risk theory that individual claims are independent. However, Dhaene et al. [2] give some examples attesting that independency assumption is unrealistic. Individual claims may be dependent since they are exposed to similar hazards or affected by similar adverse effects such as physical or financial environment. For instance, the claims of an agricultural insurance policy are contingent on the occurrence probabilities and the consequences of the same meteorological event. Crops are subjected to the same physical environment by being produced in the same geographic area. Hereby, we discuss the prioritization of the dependent aggregate claim random variables (rvs) of the disjoint risk clusters. We assume that claims within each cluster are correlated. Ambagaspitiya [3] proposes a general method for derivation of formulas for aggregate claims' distributions under dependency assumption. Likewise, we assume that the claims arised in similar environment are dependent. Since the claims of a risk cluster are determined according to the environmental factors, which are correlated, we use the general vectorial definition proposed by Ambagaspitiya [3].

In our study, first we cluster aggregate claims of crop-hail insurance wrt meteorological features under dependency of individual claims. Having obtained the risk clusters, we prioritize the aggregate claim vectors related to each cluster using the stochastic ordering relations proposed within the framework of POT.

The common tendency of the existing studies for comparing portfolios of correlated risks shows that distributional properties and moments of the aggregate claim rv are useful tools for the aim of prioritizing actuarial risks. Defining a sufficient measure, which includes enough information about the portfolio and reflects it accurately, is one of the most important tasks to analyze the risk. While comparing risks, the mean and the variance of aggregate claims and the correlation between risks are usually considered as risk measures. Although some studies take only the expected value as a measure, it can be disadvantageous. A risk measure should reflect both mean and dispersion. The calculation of the expected value of total claim rv is same under both dependency and independency cases. Thus, in order to take both the dispersion and the dependency into account, one should consider a different measure including additional information to overcome this drawback. In order to represent both variation and dependency as well as the mean, we suggest to use "coefficient of variation" and "(standardized) generalized variance" as risk measures in this study. According to our definition of risk as an aggregate claim rv, we use these measures in the context of *multivariate analysis*. Hence, we propose to use the multivariate definition of the coefficient of variation.

The major aim of this study is to suggest a multivariate stochastic ordering of the actuarial risks under dependency assumption considering the impacts of DMs' risk perceptions. This thesis consists of four main parts. First of all, it is very essential to examine the dependency structure of the aggregate claims. Secondly, risks should be conveniently clustered for the chosen ordering relation and the risk measure. The third important part is to propose an appropriate stochastic ordering for prioritizing them. Lastly, the risk perception under uncertainty must be reflected within the prioritization procedure. These parts are summarized in Figure 1.1.

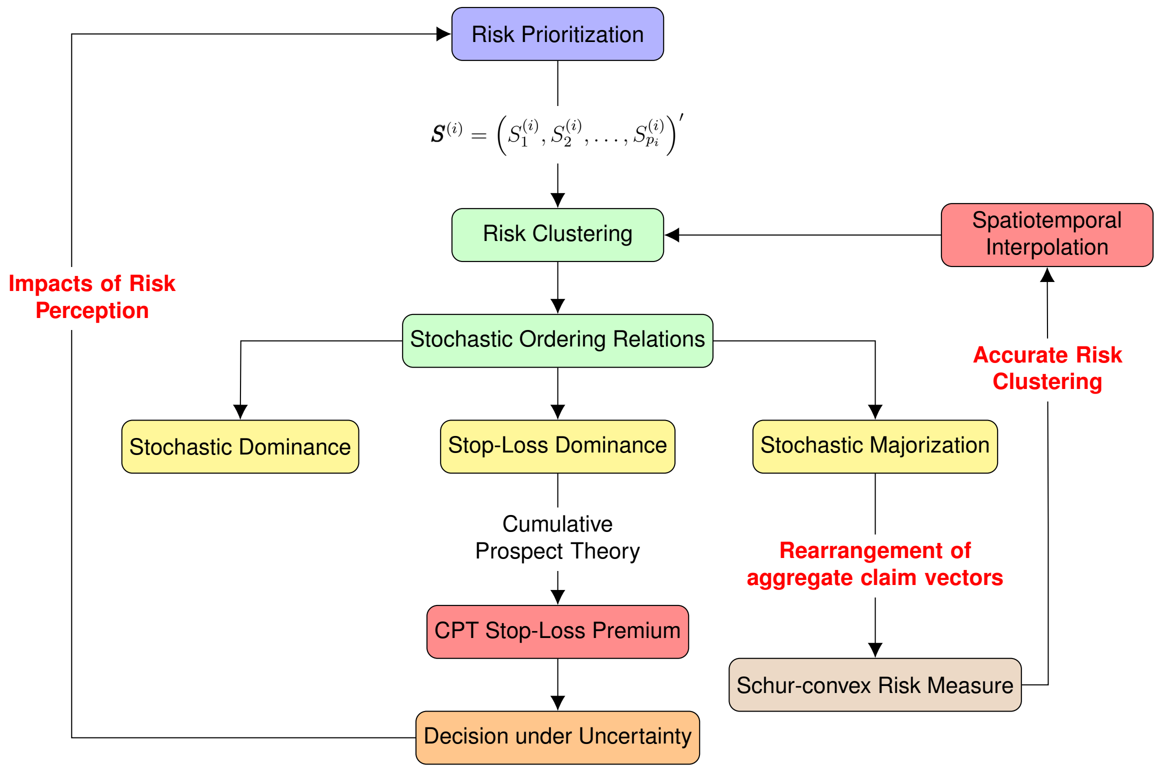


Figure 1.1: The summary diagram for the fundamental parts of this study

Firstly, by setting a multivariate model as $\boldsymbol{S}^{(i)} = \left(S_1^{(i)}, S_2^{(i)}, \ldots, S_{p_i}^{(i)}\right)'$, we take the dependency into account. Here, the elements of the aggregate claim random vector (r.vector) $\boldsymbol{S}^{(i)}$ are assumed to be dependent. The data is organized according to risk clusters and crop classes. Our model setting is discussed in Section 1.3 in detail. We first use the existing hazard clustering in order to prioritize the aggregate r.vectors after setting the model.

In the literature on prioritization of actuarial risks, aggregate claims are ordered according to their distributions, which is called *stochastic ordering*. The prioritization of the aggregate claims is complex because they are calculated as the sum of dependent individual claim amount rvs. Dealing with multivariate representation of aggregate claims helps us to prioritize dependent actuarial risks accurately.

By the help of stochastic dominance, which is one of the main types of stochastic ordering, the aggregate claim r.vectors could be ordered. In addition to that, the premiums could also be ordered since stochastic ordering helps to order moments of the claim rv and premiums are represented as functions of these moments when they are calculated according to well-known premium principles. The main advantage of stochastic ordering is the adaptability of the relations for convolution and mixing.

We choose three stochastic orderings for the third part of our study. These orderings are stochastic dominance, stop-loss dominance and stochastic majorization that are studied under POT. Here, the stochastic dominance ordering is the traditional relation used for ranking purposes. We use this relation in order to build a bridge between the relations we propose in this study and the existing relations in the literature. Having provided this interconnection, it is possible to take advantage of the properties of the stochastic dominance.

The stochastic majorization is a useful and powerful relation to derive inequalities. It has some advantages in comparison with traditional vector orderings. Schur-convexity is the most useful tool of this relation. We use the variance and the coefficient of variation for ordering risks. However, the given classification in our data set seems unsuitable for majorization purposes. It means that the data is not organized and grouped with regard to actuarial risks to be majorized. Therefore, we need to cluster the risks and to check if these clusters are appropriate for majorization.

The last relation called stop-loss ordering is used in our study to obtain an ordering relation considering the risk perception, which is the tendency of DMs who are exposed to risk. People respond to a risk or a hazard in accordance with their perceptions of that risk. That is the reason why the term “prioritization” is used instead of “ordering” under preference context. Ordering risks technically means that DMs are assumed to order risks objectively. However, their perceptions play a crucial role for risk prioritization [10]. Therefore, it is obvious that preference modeling is the basic notion arising within the decision theory. Utility theory is the main concept reflecting the preferences. Stop-loss premium calculated under the utility theorem is a useful tool to compute the aggregate risk of an insurer. Hence, the stochastic stop-loss dominance taking into account DMs’ choices is an efficient relation under these discussions.

The most common method for modeling preferences of DMs is the expected utility theory (EUT), which is also very practical in actuarial literature for many insurance applications. One of the most frequent usage of EUT is to obtain results with regard to suitable premium levels for an insurance contract, from both the perspective of an insured and an insurer's point of view. Thus, the highest amount of premium that the insured accepts to pay to be covered and the lowest amount of premium that the insurer tolerates for providing assurance are determined using this approach. EUT is extended by some modifications of its axioms and some changes of its properties. Yaari's dual theory (YDT), the approach of subjective probabilities suggested by Schmeidler [4] and the prospect theory are famous methods known as the extensions of EUT. In this study, these approaches are discussed in detail in terms of their appropriateness to model risk perception under decision theory.

We propose CPT stop-loss dominance which reflects the impacts of the risk perceptions of DMs. In comparison with CPT, traditional preference models are insufficient to handle bias of DMs' choices. Among the most common approaches in preference modeling, we suggest to use CPT approach for the decision making part. In our study, we obtain CPT stop-loss premiums by solving different utility equilibrium equations under the zero-utility premium principle.

The thesis is organized as follows:

In introduction, we review the literature after defining our problem and summarizing the thesis. We also introduce the dependency structure of the actuarial risks through a representation of aggregate claims in the frame of multivariate analysis.

In Chapter 2, we explain the crucial role of risk perception and give various risk definitions in detail in terms of assessment and ranking of risks. We discuss the controversial issues and different approaches about defining risk to figure out the concept of risk. In this chapter, we also discuss approaches of preference modeling under risk and under uncertainty in terms of their advantages and drawbacks.

In Chapter 3, we discuss stochastic ordering relations as various approaches adapting order theory to actuarial models for comparing and ordering risks. At first, we present axioms of POT and representations of partially ordered sets as useful tools for ordering. We also introduce geographic information system (GIS), which provides convenience for environmental studies. Moreover, we handle various actuarial applications of ordering relations as well as ordering under risk.

Chapter 4 provides a relationship between stochastic ordering notion and CPT. In this chapter, we adapt risk perception to the technical assessment of risk. We obtain solutions for stop-loss premiums and we propose stop-loss dominance relations by the help of zero-utility premium principle under CPT.

In Chapter 5, we introduce the theory of majorization and Schur-convexity first. We discuss stochastic majorization and its applications as one of the stochastic ordering approaches. We propose a rearrangement of the aggregate claim vectors for the sake of fulfillment of the majorization axioms. In addition, we present a case study in this chapter in order to show that an accurate risk clustering is needed for majorization purposes.

In Chapter 6, we cluster agricultural claims considering their spatial and temporal characteristics estimated by a spatiotemporal interpolation (STI) technique. This study also differs from the previous ones due to finding optimal sample set for STI method using an extended differential evolution (DE) optimization algorithm. We also suggest a way to choose the values of control variables of DE algorithm. We present the application of this study by introducing our data set of crop-hail insurance and explaining how we organize this data for the prioritization aims. We give the results of STI through optimization, risk clustering and stochastic majorization ordering in this chapter.

Finally, we make our concluding remarks and present our ideas for further research in Chapter 7.

1.2. Literature Review

There are many studies about technical assessment of risk. Jactel et al. [15] adapt multicriteria decision analysis for the comparison of forest management alternatives. In their approach, several risks are considered simultaneously and different evaluation criteria are used to compare these risks. Ball and Golob [16] summarize overall risk management procedure including risk prioritization and explain that there exist no commonly accepted definition of risk ranking. In this context, Klinke and Renn [10] work on two-fold assessment of risk considering both objective and subjective parts of it. Our interest on risk perception impact is arised within these discussions.

Marhavilas et al. [17] claim that risk could be quantified, measured and defined as a mathematical relationship through data. Methodologies on risk assessment between 2000 and 2009 are investigated. This study provides a classification and comparison of the methods. According to Alencar and Almeida [18], there is a tendency on assessing risk considering that either the number of people affected or the financial impact on the resulting scenario is taken into account. However, handling only one of these factors is insufficient when we pay attention to the complexity of the analysis.

In addition, Marsaro et al. [19] argue that multicriteria decision model incorporates multiattribute utility theory, which considers DMs' preferences and some aspects of the decision theory. The risk of transporting natural gas by pipeline is assessed in this study. Since the length of pipelines varies from a few meters to hundreds of kilometers, they subject to several types of risk that can cause a gas leak such as corrosion, third party damages, workman faults. Since a pipeline normally extends over different areas, accidents have different consequences from an area to another. For instance, an accident in an uninhabited area does not influence people in a similar way compared to an accident in a residential area. Thus, using multiple criteria according to the risk characteristics is very important. The risk management process, which is updated by the risk assessment, is organized to support the decision with regard to the resource constraints. By means of this risk management mechanism, resources could be allocated ideally and risk is mitigated and managed more appropriately according to the usage of resources by DM. Leung et al. [20] investigate and examine similarities or dissimilarities among the existing risk approaches. In their study, it is aimed to combine more than 300 individual risk assessment models in the literature and to integrate the main concepts into an appropriate model in both verbal and mathematical ways. Detailed information about different areas especially the ones considering environmental risk prioritization is given in Section 2.3.2.

As a discussion of risk concept, Aven [21] presents various perspectives of risk consisting of events, probabilities and consequences. Škulj [22] discusses subjective vs objective probability concepts suggesting to use non-additive probabilities. Schmeidler [4] handles this topic under traditional EUT proposed by von Neumann and Morgenstem [23].

In addition to EUT, which is the most commonly-used preference model, various preference approaches such as distorted expectation theory, expected utility with non-additive probabilities and (cumulative) prospect theory are studied by Savage [5], Ellsberg [6], Yaari [7], Tversky and Kahneman [8], Kahneman and Tversky [9], Denuit et al. [1].

Heilmann and Schröter [24] and Denuit et al. [1] suggest several applications of risk ordering under independence and dependence assumptions. Dhaene et al. [2] provide a different view to risk ordering by working on comonotonicity property. As for POT and its applications, Patil and Taillie [25] and Brüggemann and Patil [14, 26] provide a significant contribution to literature.

Kahneman and Tversky propose prospect theory in 1979 and modify it in 1992 by discussing violations of EUT [9, 27]. Kaluszka ve Krzeszowiec [28] provide an understanding of zero-utility premium principle under CPT, which helps us to contribute to it by suggesting stop-loss dominance within the frame of CPT. The axioms given by Choquet [29] and the concepts of CPT discussed by Eckles and Wise [30] are very useful for our study to obtain CPT stop-loss premium solutions.

Marshall et al. [31] define majorization inequalities and provide their properties. In addition to CPT stop-loss dominance, majorization relation is another valuable ordering relation in this thesis. In order to apply this relation, we need to cluster aggregate claims in our data set through STI. The setting given by Susanto et al. [32] enables us to form the optimization problem for the STI part. DE algorithm is proposed by Storn and Price [33] as an efficient optimization approach. We extend this algorithm to find the optimal sample set for the estimation of environmental characteristics of our portfolio.

1.3. Multivariate Representation of Actuarial Risks

In order to introduce our model setting, let us consider a crop-hail insurance portfolio. We suppose that there are m risk clusters and p_i crop classes for i-th risk cluster with $i = 1, 2, \ldots, m$. We define the aggregate claims of the i-th risk cluster as a p_i-variate r.vector as follows:

$$\boldsymbol{S}^{(i)} = \left(S_1^{(i)}, S_2^{(i)}, \ldots, S_{p_i}^{(i)}\right)', \tag{1.1}$$

where p_i is the number of crop classes. Here, the aggregate claim of the i-th risk cluster and j-th crop class can be represented by the rv $S_j^{(i)}$ and it is obtained as

$$S_j^{(i)} = \sum_{k=1}^{N_j} X_{jk}^{(i)},$$

where N_j is the claim number of the j-th crop class and $X_{jk}^{(i)}$ is the claim amount of the k-th individual in the j-th crop class and i-th risk cluster with $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, p_i$ and $k = 1, 2, \ldots, N_j$.

Figure 1.2 helps us to understand the prioritization of the aggregate claim r.vectors better.

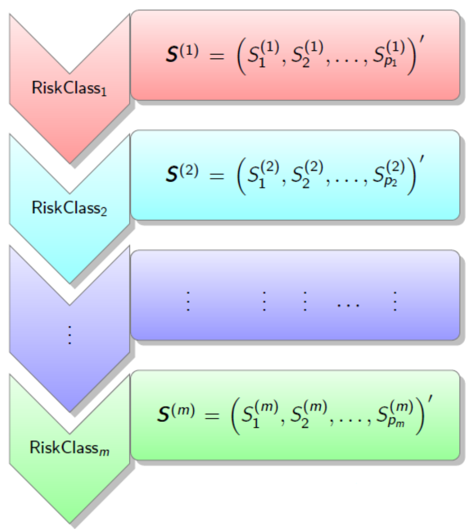


Figure 1.2: Priozitization of aggregate claim r.vectors of risk clusters

Here, $\boldsymbol{S}^{(1)}, \boldsymbol{S}^{(2)}, \ldots, \boldsymbol{S}^{(m)}$ for $i = 1, 2, \ldots, m$ denote aggregate claim r.vectors for the first risk cluster, second risk cluster, ..., m-th risk cluster, respectively. There are p_i aggregate claim rvs $S_j^{(i)}, j = 1, 2, \ldots, p_i$ within each vector.

If we order the risk clusters from the least risky one to the most risky one, where the least risky one is on the top, then a general ordering inequality could be denoted as

$$\boldsymbol{S}^{(1)} \precsim \boldsymbol{S}^{(2)} \precsim \cdots \precsim \boldsymbol{S}^{(m)},$$

which means that the first risk cluster is less riskier than the second risk cluster, and so on. Here, "$\precsim$" is a general notation for a binary relation over a partially ordered set. In the following chapters, we use this relation as $\precsim_{\text{sd}}$, $\precsim_{\text{sl}}$ and $\precsim_{\text{maj}}$ for first-order stochastic dominance, stop-loss dominance and stochastic majorization, respectively.

Ambagaspitiya [3] defines the aggregate claim rv for each group according to the collective risk model considering the distributions of both the claim number rv and the claim size rv separately. In order to model the r.vector $\boldsymbol{S}^{(i)}$, a multivariate definition is used. For the evaluation of aggregate claim r.vectors, we consider various risk measures such as generalized variance (GV), standardized generalized variance (SGV) and coefficient of variation (CV). A general dependency structure of a single risk cluster i is represented in Figure 1.3.

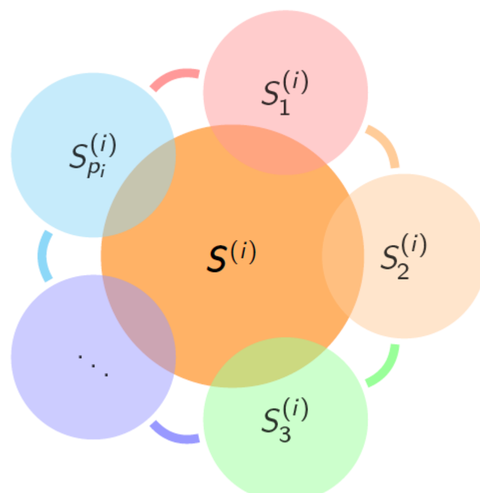


Figure 1.3: Dependency of aggregate claim rvs for a single risk cluster

This figure reflects the dependency of aggregate claims for multivariate case where each risk cluster i consists of p_i crop classes. The aggregate claim rvs for the i-th risk cluster are included in the aggregate claim r.vector $\boldsymbol{S}^{(i)}$. Here, the claims within each risk cluster are correlated whereas the risk clusters are disjoint. According to our data set, the risk clusters are determined with respect to (wrt) the environmental similarities. Therefore, we assume that the claims exposed to similar environmental risks, i.e. the claims arising in the same hazard region, are dependent.

We can compare the risk clusters using risk measures in two ways: (i) Ordering the aggregate claim r.vectors using the risk measure $\gamma\left(\boldsymbol{S}^{(i)}\right)$, and (ii) Ordering the overall aggregate claim rvs given in Equation (1.5) using the risk measure $\gamma\left(S^{(i)}\right)$. According to the common tendency in the literature of assessing the riskiness of a rv, these measures should reflect both the expected value and the dispersion. One of the common examples for measuring both statistical mean and variation together is using CV. In addition to CV, aggregate claim r.vectors could be ordered by GV or SGV which reflects variation. The following equations show the calculations of these risk measures:

(i) For ordering aggregate claim size r.vectors $\boldsymbol{S}^{(i)}$ of each risk cluster, we use the measure $\mathbb{GV}\left(\boldsymbol{S}^{(i)}\right)$ or $\mathbb{SGV}\left(\boldsymbol{S}^{(i)}\right)$ proposed by SenGupta [34]. They represent overall variabilities. GV is calculated as

$$\mathbb{GV}\left(\boldsymbol{S}^{(i)}\right) = det(\boldsymbol{\Sigma}^{(i)}) = \left|\boldsymbol{\Sigma}^{(i)}\right|, \tag{1.2}$$

and SGV is calculated as

$$\mathbb{SGV}\left(\boldsymbol{S}^{(i)}\right) = \left|\boldsymbol{\Sigma}^{(i)}\right|^{\frac{1}{p_i}}. \tag{1.3}$$

In addition to GV and SGV, we propose to use a definition for the multivariate CV as follows:

$$\mathbb{CV}\left(\boldsymbol{S}^{(i)}\right) = \left(\boldsymbol{\mu}^{(i)\prime}{\boldsymbol{\Sigma}^{(i)}}^{-1}\boldsymbol{\mu}^{(i)}\right)^{-\frac{1}{2}}. \tag{1.4}$$

This equation is a generalization of the univariate CV given in Equation (1.6). Here, the mean vector of the r.vector $\boldsymbol{S}^{(i)}$ is represented as

$$\boldsymbol{\mu}^{(i)} = \left(\mu_1^{(i)}, \mu_2^{(i)}, \ldots, \mu_n^{(i)}\right)',$$

whereas the covariance matrix of the r.vector $\boldsymbol{S}^{(i)}$ is obtained as

$$\boldsymbol{\Sigma}^{(i)} = \begin{pmatrix} \left(\sigma_1^{(i)}\right)^2 & \sigma_{12}^{(i)} & \ldots & \sigma_{1n}^{(i)} \\ \sigma_{12}^{(i)} & \left(\sigma_2^{(i)}\right)^2 & \ldots & \sigma_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n}^{(i)} & \sigma_{2n}^{(i)} & \ldots & \left(\sigma_n^{(i)}\right)^2 \end{pmatrix}.$$

(ii) For ordering overall aggregate claim size rvs $S^{(i)}$ of each risk cluster, the overall aggregate claim size rv is calculated by the summation of the aggregate claims of the crop classes for each risk cluster as

$$S^{(i)} = \sum_{j=1}^{p_i} S_j^{(i)} = \mathbf{1}'\boldsymbol{S}^{(i)}, \tag{1.5}$$

where $\mathbf{1}$ is the k-dimensional vector of ones. We use the measure $\mathbb{CV}\left(S^{(i)}\right)$ to order $S^{(i)}$ scalar quantities. Let $\mathbb{CV}\left(S^{(i)}\right)$ be the univariate CV which is calculated as

$$\mathbb{CV}\left(S^{(i)}\right) = \frac{\sigma^{(i)}}{\mu^{(i)}}, \tag{1.6}$$

where $\mu^{(i)} = \mathbf{1}'\boldsymbol{\mu}^{(i)}$ is the mean and $\sigma^{(i)} = \mathbf{1}'\boldsymbol{\Sigma}^{(i)}\mathbf{1}$ is the standart deviation of the rv $S^{(i)}$.

2. DECISION MAKING UNDER UNCERTAINTY

2.1. Introduction

In this chapter, we concentrate on decision making associated with risk prioritization. Since our main aim in this thesis is to make a connection between risk perception and ranking of risks, we consider tendencies of individuals in their choices. In this context, it is very important to assess the structure of DMs' preferences, i.e. risk tendencies and judgments [19]. Multicriteria decision making is an attitude which helps DM to come up with a prior idea when there exist several contradictory views [35]. In other words, this approach deals with multiple angles related to characteristics of DMs. Also, it could be useful to identify outcomes in monetary values and to overcome conflicts among different criteria [36].

We can find many definitions of risk in the literature. According to common tendency in especially engineering area, risk is defined in conjunction with expected loss. However, such an understanding may be insufficient when potential consequences are large with small probabilities. Although the expected value seems to be ordinary in such cases, Haimes [37] claims that expected value is deceptive for rare and extreme events in contrast to common losses. Thus, we need to consider an expression reflecting a quality more than expected value when we evaluate risk.

Having reviewed the existing risk definitions in the literature, we could see at first glance that Campbell [38] defines risk as "expected disutility". According to classical decision theorists, the expected (dis)utility yields rational choices. From this point of view, Paté-Cornell [39] claims that preferences, i.e. tendencies, should not be parts of risk assessment. Description of risk should be possible even when DMs could not define their utility functions.

Aven [21] investigates various definitions of risk in scientific literature. Some definitions of risk are as follows:.

(i) "a measure of probability and severity of adverse effects" [40].

(ii) "a combination of probability of an event and its consequences" [41].

(iii) "(s_i, p_i, c_i), where s_i is the i-th scenario, p_i is the probability of that scenario, and c_i is the consequence of that scenario" [42–44].

(iv) "the uncertainty of an outcome, of an action and of an event" [45].

(v) "the situation or the event where something of human value (including humans themselves) is at stake and where the outcome is uncertain" [46, 47].

(vi) "an uncertain consequence of an event or an activity wrt something that humans value" [48].

(vii) "a two-dimensional combination of events/consequences and associated uncertainties" [49, 50].

(viii) "uncertainty about and severity of the consequences (or outcomes) of an activity wrt something that humans value" [51]

In this chapter, we give various explanations of risk perception and the axiomatic differences between risk and uncertainty. We also provide basic axioms of preference modeling and different approaches handling DMs' choices under risk and uncertainty.

Section 2.2 explains the importance of risk perception and various definitions of risk for risk assessment. In Section 2.3, we present the place of risk prioritization in risk management process through some examples. Section 2.4 provides fundamental conditions of preference theory axiomatically. In Section 2.5, we introduce traditional EUT and its properties related to the given axioms. As an alternative to EUT, we give YDT in Section 2.6. Section 2.7 presents an extension of EUT using non-additive probability approach. In Section 2.8, we provide a general introduction of prospect theory, which reflects bias in DMs' decisions. Finally, Section 2.9 concludes this chapter.

2.2. The Concept of Risk

Aven [21] claims that perspectives of risk depending on probabilities are insufficient. One should handle urcertainties in order to define risk accurately. The concept of risk usually includes events, probabilities and consequences (outcomes). This is formalized as Risk $= (A, C, P)$ where A represents event, C denotes consequence of A and P is the associated probability.

If we adapt the definition of risk to our context, we must note that the severity is just a tool for describing the consequences. In addition to this, uncertainties are expressed through probabilities, however they are connected with events and consequences. Aven [21] concludes that for interpreting probability, there exist two alternatives basically in the literature for practical use of the risk context which are objective relative frequency interpretation (P_f) and subjective probability (P_s) as the measure of uncertainty. Uncertainty is often invisible behind the probabilities, therefore limiting attention to them could camouflage factors that may cause surprising outcomes. An example is given through Ellsberg paradox in Section 2.7 that may help to understand the significance of uncertainty. Therefore, dealing with uncertainty instead of probability is a significant aspect since probability is commonly used as a tool to reflect uncertainty. For the first interpretation of probability, we can define events associated with consequences (losses) and probabilities that reflect the frequency. The estimation of P_f usually depends on analysts' knowledge. On the other hand, for the case that probability is a subjective measure, it can be said that uncertainty is based on DM's knowledge. Hence, risk is not only an analytic concept but also a normative notion [10].

2.3. Risk Management Strategies

The risk management process can be generally characterized in accordance with the steps shown in Figure 2.1 [52].

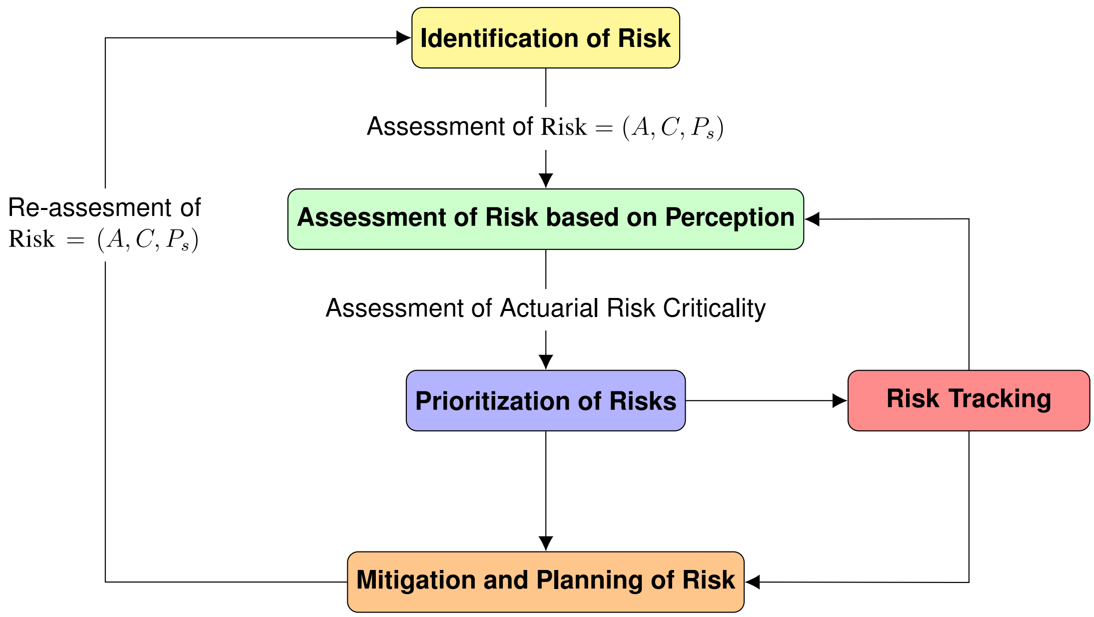


Figure 2.1: Risk management: Fundamental steps

As it is seen from the process that the prioritization of risks is updated by risk assessment through the risk identification, and supports the risk mitigation and risk tracking by providing information and tools. Therefore, we investigate the risk criteria deeply in addition to the concept of risk itself.

According to Ball and Golob [16], ordering risks indicates prioritization or ranking of objectives with regard to a significance level. There does not exist a commonly accepted definition of risk ranking and apparently has different meanings for different individuals, and even for the same individual at different times. One of the most common techniques for prioritizing hazards is to obtain scores for hazards by plotting them according to probability and consequence. Hazards having high probability and severe consequence usually have the first priority. As explained in their study, risk ranking may be performed variously by different actors, who have different roles in risk management, according to their own perceptions of essence and need about their occupations. Among several approaches of risk prioritization, we summarize some examples provided by Ball and Golob [16] in this section.

In engineering, the most common techniques such as checklists, fault trees, event trees and loss control procedures might be counted as precursors to the process of risk ranking. A well-known example widely used in chemical engineering for the identification of safety interventions and process efficiency improvements is called as the Hazard and Operability Study (HAZOP). This technique is used for the identification of hazards. On the other hand, it might not provide enough knowledge about the hazards since the information on tolerable risks or those requiring action is not known in this method.

2.3.1. Debate Topics Related to the Risk Assessment

Klinke and Renn [10] offer an approach to determine criteria for risk assesment, and to classify types of risk and management strategies guaranteeing scientific reliability, demonstration of social variety and political practicability. It is suggested that being aware of the debate problems, risk assessment process can be performed beneficially. In this study, legitimate function of risk management is discussed within the frame of some crucial themes. Since we are interested in "risk perception" notion in our study, we discuss three topics in detail.

Firstly, twofold assessment of risk, which are realist and constructivist approaches, should be considered. The dual nature of risk is one of the major controversial issue arised from examining the possibility [53, 54]. The question here is whether methodical estimations of risk appear objectively or not. According to Hilgartner [55], Luhmann [56] and Adams [57] in which a *constructivist approach* is handled, the procedure of valuing risk depends on mental constructions according to a logical framework. Besides, the *realism perspective* suggested by Catton [58], Dunlap [59], Dickens [60] and Rosa [46] claims that technical risk estimates are accurate indicators of hazards that will probably influence people. In this approach, estimates are obtained by calculated results without paying attention to attitudes or judgments of investigators.

According to Klinke and Renn [10], the term "risk" undoubtedly denotes the negative experience that people fear, and this fear obviously comes from an occurrence that has not yet happened but might happen in the future. Since there exist various aspects of risk identified as negative effect or harm, it is suggested that risks are difficult to examine and to judge from an entirely objectivist viewpoint. As a result of these discussions, we try to include this idea into our definition of risk prioritization by means of the prospect theory.

Secondly, involving public in the risk assessment process is another controversial issue which is related to the first topic in terms of the argument about the subjective vs objective risk analysis. This topic is important for determining acceptable levels of risk to consider the individuals who are likely influenced by hazard. Although it is an incomplete debate and does not seem to be completed more recently, we should consider the variables having impacts on public's judgment of risk. These factors can be summarized as expected perceived loss frequency, catastrophic perspective, qualitative characteristics of risk, stimulated emotions by risk, trust in regulators and institutions managing risk, and beliefs of society related to the reason of risk or of risk-handling actors [61–65].

It is considered that public concerns, which reflects perception, need to be involved into risk evaluation. WGBU [66] suggests a list of risk criteria that are scope of damage, probability of occurance, incertitude (entire uncertainty), omnipresence (geographic dispersion of risks), persistency (temporal characteristics of risk), reversibility (capable of being restored), delay effect (elapsed time from the event and the consequence), violation of equity (difference between risk-averse and risk-seeking DMs), and potential of stimulation. These criteria seem to be appeared as the most effective ones for indicating various risks [10]. By means of these criteria, we can suggest an adequate indicator for a measurement of risk perception. In our study, handling the temporal and spatial characteristics of risk clusters in addition to paying attention to the bias of DMs through prospect theory enables us to take these criteria into account.

Lastly, dealing with uncertainty has become the most significant debate as the literature about this issue is improving. The term "uncertainty" indicates a collection of various aspects that can be ignored in risk assessment. As mentioned before, it is undoubtedly inferred that probability itself is an estimation of an uncertain event. Having worked on CPT, we take into consideration of uncertainty in this concept.

2.3.2. Different Application Areas for Prioritization of Risks

Being an important issue of human behaviour, POT falls within the boundaries of several fields; one of which, public health, is our specific interest. We intend to apply the order theory to a chosen risk area such as foodborne disease, plant health or agricultural risks, because they are rather vulnerable aspects of public health. Since we could obtain agricultural insurance claim data set, the application part of our study consists of the analysis of an agricultural data set. On the other hand, the existing methodology on other areas of environmental risks is enlightening to conduct our study.

The prioritization and management of foodborne risks have accelerated in recent years to reduce the rate and effects of foodborne diseases. There exist various methods for prioritization considering health-related, economic and social factors [48, 67–70]. For instance, Ruzante et al. [71] rank foodborne risks according to PROMETHEE method [72]. In their study, four factors are considered to assess the food safety risks and six pathogen-food combinations are ranked according to these factors. This study provides a transparent multicriteria ranking process in which assessment of various scenarios depending on the researchers' aims can be done systematically.

As for plant health related risks, the ecological balance is distorted by the biological species being under the threat of rapid extinction. One of the most substantial agents which adversely affect the plant health is invasive alien species (IAS). Even though there exist innocuous invaders, some of them may seriously damage especially environment and economy.

IAS is any animal or plant that has a tendency to extend over an area and leads to ecological and social problems causing harm to economy and human welfare. Since ecological balance is like a chain, some links in this chain are broken due to IAS. Invaders threaten not only plants and animals, but also humans as the link of this chain. For instance, the plants that are nutritional source of humans will be imperiled when an invasive plant arises in that ecology [73].

Although plant health risk is an international phenomenon causing ecological, economic, social and public health problems; there is a restricted awareness about the risk perception of environmental managers or experts [74–77]. Perception of risk is very significant for thinking about and beginning to deal with the issue of IAS [78–80]. Accordingly, Andreu et al. [81] survey environmental managers' thoughts about noxious alien plants for the purpose of identifying and analyzing the impacts of their perceptions of IAS.

2.4. Fundamental Axioms and Theorems in Preference Modeling

In decision making, we pay attention to the process that DMs' preferences are modeled. Explaining behaviors while making a decision is a complex problem due to the difference among decisions which is directly related to ascertaining the perceptions of individuals. Preferences differ significantly among individuals, thus values are identified as prospects in decision making. This divergence leads us to seek a method modeling DMs' tendencies accurately.

"Subjective (personal) probability" concept is proposed to measure the impacts of DMs' tendencies on preferences. Škulj [22] suggests to use non-additive probabilities because these probabilities have been emerging as easily modified measures to evaluate subjective probabilities. According to this study, subjective probability is the probability that DMs assign to events when real probability is not known. It is also claimed that non-additive probabilities can provide satisfactory results even if accurate probabilities are known. In addition, Schmeidler [4] proposes an approach justifying the usage of non-additive probabilities for the definition of risk and acknowledges adaptations of these probabilities to EUT.

The subjective probability notion also initiates a discussion about risk and uncertainty terms. Škulj [22] compares definitions of risk and uncertainty, and explains their dissimilarity in this point of view. Risk is identified as a term connected with DMs' choices through known probabilities of events whereas uncertainty is about decisions made with unknown probabilities. Therefore, dealing with uncertainties in place of probabilities is necessary since probabilities represent uncertainties. Schmeidler [4] also discusses assigning probabilities to uncertain events. It is proposed in this study that enabling non-additive probabilities provides flexibility for DMs to assess probabilities.

In our study, several different approaches such as *EUT*, *distorted expectation theory (YDT)*, *expected utility with non-additive probabilities*, and *prospect theory* are discussed and compared in terms of reflecting the impacts of risk perception. Among all approaches discussed, we suggest to use non-additive probabilities to define the prospects because they provide a flexibility to model DMs' preferences even if they are biased.

Schmeidler [4] compiles several properties used for the preference relation theories. These axioms are *weak order, comonotonic independence, independence, continuity, (strict) monotonicity, and nondegeneracy*. They are very useful to understand the difference among preference models. For detailed information, see [4].

The property of completeness may be the most restrictive assumption since DMs have to make their choices according to total ordering. Total ordering requires "totality" which means that all pairs of elements in a set are comparable under a defined relation. On the other hand, partial ordering is more useful than total ordering for our context, we can hence rank all risks even if there exist incomparable ones.

von Neumann-Morgernstern Theorem representing traditional utility theorem and Anscombe-Aumann Theorem are two fundamental theorems that provide an insight into the preference modeling. We discuss approaches for relations between risky choices in the following sections. Schmeidler [4] provides an implication about using von Neumann-Morgernstern theorem to adapt subjective probabilities.

As an extension of these fundamental theorems, Schmeidler [4] proposes a new theorem called as "Schmeidler's Theorem" in our study hereafter. It is proven that making Anscombe-Aumann Theorem less restrictive by changing the property *strict monotonicity* to *monotonicity* yields no difference. In addition to this, more extensions to Anscombe-Aumann Theorem are provided. Schmeidler [4] also uses *comonotonic independence* instead of *independence*. Finally, it is suggested to use a non-additive probability measure υ in the place of the finitely additive probability $\mathbb{P}$. For detailed information, see [4].

2.5. Traditional Expected Utility Theory

EUT has widely used in the actuarial literature as a representation of rational decision making under risk and uncertainty. Within the frame of EUT, a value is assigned to monetary amount through a utility function reflecting preferences of DMs. Wang and Young [82] stated that this theory has been used to make a decision in three important insurance applications: (i) optimal (re)insurance policies, (ii) optimal insurance when moral hazard or adverse selection exists, and

(iii) choice between insurance buying and precautionary saving.

EUT has been connected to "stochastic dominance", which is one of the most commonly used stochastic ordering relations, because stochastic dominance can be indicated as a partial ordering in terms of utility functions [82]. Stochastic dominance is often used in decision analysis where one risky option is compared with another superior/inferior one. EUT is discussed in the frame of stochastic dominance in Section 3.3.2.

In this section, the axioms proposed by von Neumann and Morgenstem [23] are provided. They are used for ordering risks according to decision making under EUT. Let $X,\ Y,\ Z \geq 0$ be random claims, or *risks*, and let

$$S_X(t) = \mathbb{P}(X > t), t \geq 0$$

represent the survival function of X. The inverse survival function is defined as [82]

$$S_X^{-1}(q) = \inf\{t \geq 0 :\ S_X(t) \leq q\},\ 0 \leq q \leq 1. \tag{2.1}$$

Yaari [7] presents the axioms of EUT as follows.

Proposition 2.5.1. ***(Axioms of von Neumann and Morgenstem's EUT)*** *Consider that $X \precsim Y$ means "X is less riskier than Y". Then,*

1. *Neutrality: If $S_X = S_Y$, then $X \precsim Y$ and $Y \precsim X$, i.e. X and Y represent equal risks.*
2. *Complete weak order: "$\precsim$" is reflexive, transitive and connected.*
3. *Continuity: "$\precsim$" is continuous in the topology of weak convergence.*
4. *Monotonocity: If $S_X \leq S_Y$, then $X \precsim Y$.*
5. *Independence: If $X \precsim Y$, and Z is any risk, then*

$$\{(\alpha, X), (1-\alpha, Z)\} \precsim \{(\alpha, Y), (1-\alpha, Z)\}, \forall \alpha \in [0, 1],$$

where $\{(\alpha, X), (1-\alpha, Z)\}$ *represents probabilistic mixture defined as*

$$S_{\{(\alpha,X),(1-\alpha,Z)\}}(t) = \alpha S_X(t) + (1-\alpha) S_Z(t), \forall t \geq 0.$$

Let a utility function be u s.t. [23]

$$X \precsim Y \Leftrightarrow \mathbb{E}[u(-X)] \geq \mathbb{E}[u(-Y)].$$

Here, the utility function and expected utility are defined in the following definition [82]:

Definition 2.5.2. *Consider that* u *is a (normalized) utility function. For* $X \geq 0$*, the expected utilities are given by*

$$\begin{aligned} &\mathbb{E}[u(X)] = \textstyle\int_0^\infty S_X(t)\,\mathrm{d}u(t) = \int_0^1 u[S_X^{-1}(q)]\,\mathrm{d}q, \text{ and} \\ &\mathbb{E}[u(-X)] = -\mathbb{E}[\tilde{u}(X)], \end{aligned} \tag{2.2}$$

where $\tilde{u}$ *is the utility defined as* $\tilde{u}(w) = -u(-w)$*. Here,* u *is defined on* $\mathbb{R}$ *with* $u(0) = 0$*.*

DMs do not prefer one risky choice over another for the reason that it has higher expected utility. Rather, a risky choice yields a higher expected utility because of the fact that it is preferred to the other. DMs do not have utilities, they have preferences. Utilities are only representations of these preferences. DMs behave as if they maximize their utilities when they make choices based on their preferences [83]. Although EUT has some problems with reflecting non-objective preferences, this theory has contributed to the literature by providing an understanding on management of risk and assessment of uncertainty [1].

For ordering risks, it is commonly accepted that tendencies of individuals should be considered as well as the technical calculation of risk. In this context, EUT has widely adapted to the actuarial literature with the motivation of understanding managerial and decision economics of risk and uncertainty.

In this study, we take the notation suggested by Kahneman and Tversky [9] to define prospects and preference relations. Considering that x_i represents outcome with probability p_i in the prospect $(x_1, p_1; \ldots; x_n, p_n)$, the sum of the probabilities of outcomes must be equal to 1, i.e. $p_1 + \ldots + p_n = 1$ according to EUT. Three axioms of EUT applied to preferences are given as follows.

Proposition 2.5.3. *Let $\{x_1, \ldots, x_n\}$ be the set of outcomes with the probabilities $\{p_1, \ldots, p_n\}$. The properties of the utility function u are as follows:*

1. *Expectation: $U(x_1, p_1; \ldots; x_n, p_n) = p_1 u(x_1) + \ldots + p_n u(x_n)$ is the representation of the utility of a prospect where U denotes the expected utility of its outcomes.*
2. *Asset integration: $(x_1, p_1; \ldots; x_n, p_n)$ is acceptable iff $U(w + x_1, p_1; \ldots; w + x_n, p_n) > u(w)$ where $u(w)$ is the utility function of the asset w.*
3. *Risk aversion: If u is concave, i.e. $u'' < 0$, it means that the DM is risk-averse.*

We deal with the "risk aversion" axiom specifically in this thesis because we are interested in DMs' risk tendencies in their preferences. This axiom reflects their attitudes in buying insurance to avoid the uncertain and unpredictable events. If they prefer a certain expected value of a prospect instead of prospect X itself, they are called "risk-averse DMs". Thus, if a DM is risk-averse, then

$$\mathbb{E}\left[u(X)\right] \leq u\left[\mathbb{E}(X)\right]. \tag{2.3}$$

This relation is a result of Jensen's inequality [1].

The meaning of preference relation $\precsim$ may cause an ambiguous situation. Rvs represent positive monetary amounts in economics whereas they represent losses in actuarial literature. Thus, if $X \precsim Y$, rational DMs prefer Y to X according to interpretation of economists because Y has a larger monetary income. On the other hand, actuaries would prefer X to Y according to the same ordering relation since X is a smaller claim than Y. To overcome this doubtful interpretation, we suggest to use this relation as it is considered in actuarial literature by taking rvs as claims.

2.6. Distorted Expectation Theory: Yaari's Dual Theory

Yaari [7] proposes an alternative theory by adjusting the independence axiom of classical EUT. In this study, it is suggested to entail independence property wrt direct mixtures of "payments of risky prospects" instead of entailing independence wrt mixtures of "only risky prospects". Thus, evaluation of risky prospects seems similar with EUT apart from that functions of payments and probabilities exchange [1]. In YDT, risk attitudes are characterized by *distortion* applied to distributions. A duality between concavity and convexity of a utility function is proposed in [7]. The utility function is concave in risk aversion context and it is convex in uncertainty aversion context. Therefore, YDT provides us a different perspective to understand the nature of preferences in terms of subjective preferences.

To provide theory of distorted expectation, we firstly give some definitions. Let X rv denote DM's prospect taking values from $(-\infty, +\infty)$. Then,

$$\mathbb{E}(X) = -\int_{-\infty}^{0} [1 - S_X(x)] \,\mathrm{d}x + \int_{0}^{+\infty} S_X(x) \,\mathrm{d}x. \tag{2.4}$$

The derivation of this equation is given in Appendix A.1.1. Here $S_X(x) = \mathbb{P}(X > x)$, which is defined as survival function in the previous section, could be called as decumulative distribution function (df) or tail function as in [1].

On the other hand, assuming that there exists a non-decreasing function g called "distortion function" for each DM, Yaari [7] takes the "distorted expectation" of X as the same derivation in Equation (2.4) to value the future prospect. Considering $g : [0, 1] \to [0, 1]$ where $g(0) = 0$ and $g(1) = 1$, the distorted expectation of X is defined as

$$\mathbb{E}_g(X) = -\int_{-\infty}^{0} [1 - g\left(S_X(x)\right)] \,\mathrm{d}x + \int_{0}^{+\infty} g\left(S_X(x)\right) \mathrm{d}x. \tag{2.5}$$

If X is a non-negative rv, then Equation (2.5) is replaced by

$$\mathbb{E}_g(X) = \int_{0}^{+\infty} g\left(S_X(x)\right) \mathrm{d}x. \tag{2.6}$$

Since $S_X(x)$ is a non-increasing function of x, $g\left(S_X(x)\right)$ is also a non-increasing function of x, as well. Thus, Denuit et al. [1] introduce $g\left(S_X(x)\right)$ as "risk-adjusted tail function". As a result, the aim of DM is to maximize $\mathbb{E}_g(X)$ when X is taken as wealth. Let X and Y be two prospects, then DM prefers wealth Y instead of X, i.e. $X \precsim Y \Leftrightarrow \mathbb{E}_g(X) \leq \mathbb{E}_g(Y)$. On the contrary, if we need to make an inference based on actuarial literature; DM aims to minimize $\mathbb{E}_g(X)$ when X is taken as a claim rv. Consider that X and Y are two prospects of loss, DM prefers X instead of Y, i.e. $X \precsim Y \Leftrightarrow \mathbb{E}_g(X) \leq \mathbb{E}_g(Y)$.

According to Yaari's hypothesis of distorted expectations, DMs are risk-averse if g is convex, i.e. $g(p) \leq p,\ \forall p$ or equivalently $g\left(S_x(x)\right) \leq S_X(x),\ x \in \mathbb{R}$. It shows that risk-averse DMs underestimate the probability of their potential fortune's being more than a certain value, i.e. $\mathbb{P}(X > x)$, when they represent their preferences by distortion function.

As maintained by EUT, for a risk-averse DM, the preference tendency of DM chooses a certain expected value of a prospect rather than an uncertain prospect itself. This statement can be represented as follows.

$$\mathbb{E}_g(X) \leq \mathbb{E}(X) = \mathbb{E}_g\left[\mathbb{E}(X)\right]. \tag{2.7}$$

To clarify the difference between EUT and YDT under preference modeling, the comparison of these two approaches are handled in terms of attractiveness of risk [1]. For a non-negative rv X, it is measured as

$$\mathbb{E}[u(X)] = \int_0^1 u\left(\text{VaR}_p(X)\right) \mathrm{d}p, \tag{2.8}$$

according to EUT whereas it is evaluated as

$$\mathbb{E}_g(X) = \int_0^1 \text{VaR}_p(X)\, \mathrm{d}g(1-p), \tag{2.9}$$

in YDT. Let us assume that $\text{VaR}_p(X)$ is the potential fortune amount, which is also called as value-at-risk. Equations (2.8) and (2.9) show that $\text{VaR}_p(X)$ itself is adjusted by u and tail probability is not changed under EUT whereas tail probability p is modified and the fortune is not adjusted under YDT. For derivations of Equations (2.8) and (2.9), see Appendix A.1.1.

2.7. Expected Utility with Non-additive Probabilities

The most useful characteristic of non-additive subjective probabilities is that they can be adapted to expected utility models. Practicality of non-additive probabilities is discussed through "Ellsberg paradox" in this section. Savage [5] improves von Neumann's EUT by revising it with additive subjective probabilities. However, this extension may still remain insufficient. DMs may have different behavior tendencies when they estimate probabilities. A more flexible model is needed because it is proven that they overestimate (underestimate) small (large) probabilities. As a result of these discussions, Schmeidler [4] proposed a new extension of EUT which includes non-additive probabilities instead of additive ones. In addition to handling additivity of probabilities, this model also explains "uncertainty aversion" which proves the inconsistency of Savage's additive EUT. Škulj [22] defines uncertainty aversion as the situation that DM prefers choices having more information instead of choices where less information is available.

In order to understand uncertainty aversion, we introduce the Ellsberg paradox example which is given by [4]. Consider that urn A and urn B consist of red or black balls. It is known that the urn A contains 50 red and 50 black balls, but there is no information about the colors of the 100 balls in urn B. One ball is randomly picked from each urn and a person is asked to make a bet on the color of the drawn balls. Four events occur which can be indicated as AR, AB, BR, BB, where the event "the ball drawn from urn A is red" is denoted by AR etc.. Betting on each event, the person takes \$100 if that event happens and \$0 otherwise. When this experiment is performed on a significant number of DMs, it is observed that DMs prefer betting on urn A to betting on urn B. They are indifferent between bets on AR and AB, and similarly bets on BR and BB. Thus, Ellsberg paradox empirically shows that DMs choose the events where more information is available, i.e. $AR \simeq AB \succ BR \simeq BB$. If we examine this example in detail, we see that outcomes are $x_i = 0,\ 100\ ;\ i = 1,\ 2$ and the events are denoted by $AR = (100, p_1; 0, q_1)$, $AB = (100, p_2; 0, q_2)$, $BR = (100, p_3; 0, q_3)$, and $BB = (100, p_4; 0, q_4)$ where p_i is the probability that i-th event occurs. Consider that a utility function is chosen arbitrarily such as $u(x) = x$. Since there are only two outcomes, and the preference alternatives are not very complicated, we could choose a simple utility function. Also, we do not discuss how to choose the most suitable utility function here, we deal with assigning probabilities to events.

From Proposition 2.5.3, the expected utility of i-th event is $U(x_1, p_i; x_2, q_i) = p_i u(x_1) + q_i u(x_2) = p_i(100) + q_i(0) = 100p_i$. Since $AR \simeq AB$, $100p_1 = 100p_2 \Rightarrow p_1 = p_2$. In addition, $q_1 = p_2$ and $q_2 = p_1$ because the ball chosen from urn A is either red or black. Hence, it is obtained that $p_i = q_i = 1/2$ for $i = 1, 2$ since $p_1 + q_1 = 1$ according to the additivity property of the probabilities in EUT. Similarly, $p_i = q_i = 1/2$ for $i = 3, 4$ because $BR \simeq BB$. As a result, for all events $U(x_1, p_i; x_2, q_i) = 50$; $i = 1, 2, 3, 4$ but this situation causes a contradiction with the relation $AR \succ BR$. As a result, it is deduced that Ellsberg paradox cannot be explained through additive probabilities. This problem can be solved by using non-additive probabilities. For example, if we choose $p_i = q_i = 2/5$ for $i = 3, 4$; then we have $U(x_1, p_i; x_2, q_i) = 50$ for $i = 1, 2$ and $U(x_1, p_i; x_2, q_i) = 40$ for $i = 3, 4$ which would not be contradictory with the relation $AR \simeq AB \succ BR \simeq BB$. Here, the difference $1 - (2/5 + 2/5) = 1/5$ can be assigned as a penalty because of the lack of information for urn B [22].

It is proven by Ellsberg [6] that uncertainty aversion can only be expressed through non-additive probabilities. The Ellsberg paradox can be used as a demonstration of this discussion [22]. According to this paradox, uncertainty aversion is important for decision making context because it provides weighting of preferences. It also prevents some limitations on DMs. Schmeidler [4] proposes a mathematical characterization of uncertainty aversion.

Definition 2.7.1. ***(Uncertainty Aversion)*** *A binary relation $\precsim$ on L is called "uncertainty aversion" s.t.*

(i) If $f \precsim h$ and $g \precsim h$, then $\alpha f + (1-\alpha)g \precsim h$; $\forall$ f, g and h in L and any $\alpha \in [0, 1]$.

(ii) Equivalent to (i), if $f \precsim g$, then $\alpha f + (1-\alpha)g \precsim g$; $\forall$ f and g in L and any $\alpha \in [0, 1]$.

2.8. Decision under Uncertainty: Prospect Theory

Although EUT is uncomplicated reflection of reality, the mathematical accuracy of this theory and the fact that it is widely used in the literature do not ensure that it is an appropriate model for all DMs' behaviors. As an approach of modeling individuals' making decisions rationally, EUT is showed to be insufficient to reflect bias in their decisions. Tendencies of people can vary according to how they identify risk behind their decision. We have mentioned that decisions of individuals seem to depend on their perceptions. Tversky and Kahneman [8] examine this social side of risk in a comprehensive manner. The term "heuristic", which lowers the complexity of likelihood assessment and simplifies the management tasks, is used as a reference of "belief" in their study. On the other hand, it is claimed that heuristics may cause serious and systematic errors in spite of their usefulness. In their study, an example on estimating the distance of an object is very interesting in this context. In order to attest the influence of heuristics; a situation is provided. It shows that the visible distance of an object is partially ascertained by its sharpness. As result of observations, it is realized that people often overestimate (underestimate) distances when visibility is poor (good). It can be infered from these results that intuitive decisions are not always reliable.

Considering all of these discussions, Kahneman and Tversky [9] develop the prospect theory as a criticism of EUT's assumptions to model decision making under uncertainty. EUT has been used as the most powerful approach for over 250 years in different areas. However, it is shown that there exists a significant number of violations with regard to the assumption of individuals' being rational. As an explanation of this situation, we consider two people having same wealth. According to EUT, these two individuals should be happy identically. Contrary to this, if one has this wealth after losing a huge amount of money whereas the other one has the same wealth with gaining a significant percent (say 1000%) of his prior money, they will not be equally happy. Therefore, the basic assumption of this theory is violated by perceptions of individuals. Starting from this point of view, a lot of empirical studies try to create new perspectives. Among these studies, Kahneman and Tversky come up with prospect theory which represents inconsistency of individuals' preferences among similar choices [9].

A prospect could be edited differently according to the context. Here, different editing operations are described [9]:

1. *Coding:* Outcomes are perceived as gains and losses, instead of final positions of wealth or welfare;
   (This operation is discussed in detail in Chapter 4.)

2. *Combination:* Possibilities relating to same outcomes can be merged;
   (For instance, $(100, 0.15; 100, 0.15)$ can be reduced to $(100, 0.30)$)

3. *Segregation:* A riskless part of a prospect can be separated from a risky component;
   (For instance, $(100, 0.15; 50, 0.85)$ can be edited as a certain gain of 50 and a prospect $(50, 0.15)$, or $(-250, 0.30; -150, 0.70)$ can be edited as a certain loss of 150 and a prospect $(-250, 0.30)$)

4. *Cancellation:* Common parts of prospects, which are outcome-probability pairs, can be omitted;
   (For instance, consider $(100, 0.40; 80, 0.50; -30, 0.10)$ and $(120, 0.40; 80, 0.50; -50, 0.10)$. The choice between the prospects can be changed to the selection between $(100, 0.40; -30, 0.10)$ and $(120, 0.40; -50, 0.10)$)

5. *Simplification:* Rounding probabilities or outcomes;
   (For instance $(201, 0.49)$ or $(199, 0.51)$ is probably edited as the same level of chance to win 200, i.e. $(200, 0.50)$), and lastly

6. *The detection of dominance:* Searching offered prospects to discover hidden influential prospects being rejected without a sufficient investigation.

The final value of an "edited" prospect, V, is represented based on two factors, π and υ. Here, π is connected with each probability p, $\pi(p)$, to demonstrate the impact of p on the overall value of the prospect. The other scale, υ, gives each outcome x a number by $\upsilon(x)$. This scale indicates the subjective value of that outcome by measuring the deviation from the reference point.

If we consider the prospect $(x, p; y, q)$ having at most two non-zero outcomes, x is yielded with probability p, y with probability q, and no outcome with probability $1 - p - q$, where $p + q \leq 1$. A prospect is

i. *strictly positive (negative)* if all outcomes are positive (negative), $x, y > 0$ ($x, y < 0$), and $p + q = 1$;

ii. *regular* if either $p + q < 1$, or $x \geq 0 \geq y$ or $x \leq 0 \leq y$.

If a prospect is regular, then

$$V(x, p; y, q) = \pi(p)\upsilon(x) + \pi(q)\upsilon(y), \tag{2.10}$$

where $\upsilon(0) = 0$, $\pi(0) = 0$ and $\pi(1) = 1$. Here, V is defined on prospects whereas υ is defined on outcomes, i.e. $V(x, 1.0) = V(x) = \upsilon(x)$.

If the prospect is either strictly positive or strictly negative, Equation (2.10) is modified as

$$V(x, p; y, q) = \upsilon(y) + \pi(p)\left[\upsilon(x) - \upsilon(y)\right]. \tag{2.11}$$

As a result of provided introduction of prospect theory, we can develop a notion by means of the idea given in this chapter. There is no doubt that not only tendencies in the decision making, but also inconsistency of individuals' preferences should be managed in risk assessment. Therefore, assuming that risk identification is the fundamental step of risk prioritization, we will try to adapt this approach to incorporate perceptions of individuals into the model of preferences.

2.9. Interim Conclusion: Risk Perception in Decision Making

Our study is mainly based on the effects of risk perception on risk prioritization. In this chapter, we give some examples about different approaches for definition of risk and uncertainty to explain our motivation in this thesis. We introduce some fundamental properties of preference models which helps providing a theoretical explanation for our main model setting of prioritization.

The impacts of preferences of DMs lead us to "utility" and "prospect" terms. In this sense, we give some notation and definitions to explain approaches for modeling preferences such as EUT, distorted expectation theory (YDT), expected utility with non-additive probabilities, and prospect theory.

Moreover, we introduce some application areas to provide an understanding about the ways of applying the theoretical information to real world. We also focus on social features of risk.

For this reason, we investigate the theorems which handle tendencies of individuals in decision making and considering bias of DMs' choices. It is understood from these approaches that controversial issues associated with risk assessment should be taken into account. In addition to this, although EUT is commonly used in decision making, it is inferred that this approach may not sufficiently capture inconsistency of DMs' choices. For this aim, prospect theory is developed as an alternative to EUT. In order to model the risk perception in the decision making, we work on prospect theory in detail in this thesis.

3. STOCHASTIC ORDERING RELATIONS FOR RISK PRIORITIZATION

3.1. Introduction

For an actuary, it is crucial to deal with preferences of DMs when they make choices between potential gains and losses. Stochastic ordering, which is partially ordering among distributions of risks, is a very useful and convenient tool for this aim. In our study, we handle actuarial risks as claim rvs of an insurance product. In this sense, a risk could be more desirable to another one due to its structural and distributional properties such as being "smaller" or "thinner-tailed" (more predictable and thus less risky). Distributions which have thinner tails are explained as "less spread" risks.

Kaas et al. [84] and Denuit et al. [1] study on ordering of actuarial risks, i.e. claims. They apply an ordering method for several actuarial applications in order to obtain premiums according to zero-utility principle, probability of ruin, stop-loss reinsurance, value-at-risk etc.. Deciding whether a rv is riskier than another becomes "mean-variance ordering" without stochastic ordering relations. Mean-variance ordering method is a general use in which the rv with smaller mean is preferred and the variance of that rv arises as a determinant. However, this approach might be ambiguous since it could cause debates among DMs. Combining risk measures, which are referred as indicators in the frame of POT, could be complex due to the reasons given in that context. Therefore, many studies on stochastic ordering of risks have been arisen in actuarial literature.

Various applications, especially in non-life insurance markets, are initially based on modeling total claims. There are two substantial and restricting assumptions which are non-negativity and independence of summands of the aggregate claim. Kaas et al. [84] provide invariance axioms depending on the non-negativity assumption. An example for restrictive invariance properties is about stop-loss ordering. In this axiom, stop-loss ordering remains unchanged regardless of the change in distribution of claim severity or of claim frequency. We must be aware of consequences of this constraint especially when we take both gains and losses into account. Additionally, for individual and collective risk models, individual claims are assumed independent. This assumption is widely used in insurance applications because of two main reasons. Firstly, the fundamental laws in statistics, “Law of Large Numbers” and “Central Limit Theorem” both of which are based on independence assumption, allow insurers to predict their future potential position. The second reason is that obtaining information about risks is easier both statistically and mathematically in terms of computational purposes because statistics on marginal distributions are easy to estimate compared to joint distributions.

On the other hand, even if it provides an understanding for ordering actuarial risks, independency assumption is not realistic. As in some examples, we can consider dependent environmental risks that are influenced by the same environment such as flood and earthquake risks, or the dependent mortality risk called “broken heart syndrome”. From this point of view, Dhaene et al. [2] suggest using comonotonicity property to order aggregate claims without assuming independence. Moreover, Denuit et al. [1] propose a procedure consisting of dependence ordering and integral stochastic ordering.

In Section 3.2, we introduce POT as a basis of all stochastic ordering relations studied in this thesis. Section 3.3 presents most commonly used ordering relations and use of these relations in actuarial literature. In this section, we also examine these stochastic ordering relations in terms of choice under risk. Finally, we summarize this chapter in Section 3.4.

3.2. Partial Order Theory

Ordering of risks taken as rvs is possible by using stochastic ordering relations, which are studied under POT. Therefore, we provide an understanding for risk prioritization and suggest stochastic ordering relations by taking the advantage of properties of the POT.

3.2.1. Combination of Multiple Indicators into a Single Index

Brüggemann and Patil [26] state that complexity of data-based studies leads researchers to improve prioritization based on partial orders. Since indicators are often aggregated by a weighted sum, they firstly discuss the concept of “compound indicators”. Although considerable effort is made to find suitable weights for determining the compound indicator, there are some disadvantages given below:

- Information about indicators could be lost as a consequence of the aggregation (the information collected in a single compound indicator gets interlaced).
- Determining weights could be difficult, and
- If some indicators have same features, compounding them could cause a disadvantage which makes DMs give them more importance than required.

If we examine the construction of compound indicators expressed above, obtaining these indicators has two main difficulties: (i) finding basic indicators and (ii) finding the weights, i.e. deriving the compound indicator. According to Brüggemann and Patil [14], it is very useful that compound indicators provide not only rankings but also an efficient metric system of measurement.

If we return examples related to definition of "knowledge" discussed in introduction chapter, there are different ways for quantification, which are called "indicators". In this perspective, Patil and Taillie [25] study on ranking a finite set of objects when each object has an indicator family. Since dissimilar indicators can cause several comparative assessments, the difficulty of combining them into a single index is investigated in their study.

A nonempty set of objects, X, is considered where each object has $(I_1, I_2, \ldots, I_n)$ of real-valued indicators. Here, small value of an indicator denotes "poor" conditions and large value represents for "good" conditions.

Let $x, y, z, \ldots$ indicate the elements in X. The comparative statement is that x' is "better" or "bigger" than x (we write $x' \geq x$ or $x \leq x'$) if $I'_j \geq I_j$ for all j. Here, the objects x and x' are based on their indicator values $(I_1, I_2, \ldots, I_n)$ and $(I'_1, I'_2, \ldots, I'_n)$, respectively.

On the other hand, objects are not compared precisely since different researchers might rank x and x' differently. This situation could cause unanimous ordering. If we consider that there are

$n = 2$ indicators, all possible decisions are represented by Figure 3.1. In this figure, object x separates indicator space into four quadrants.

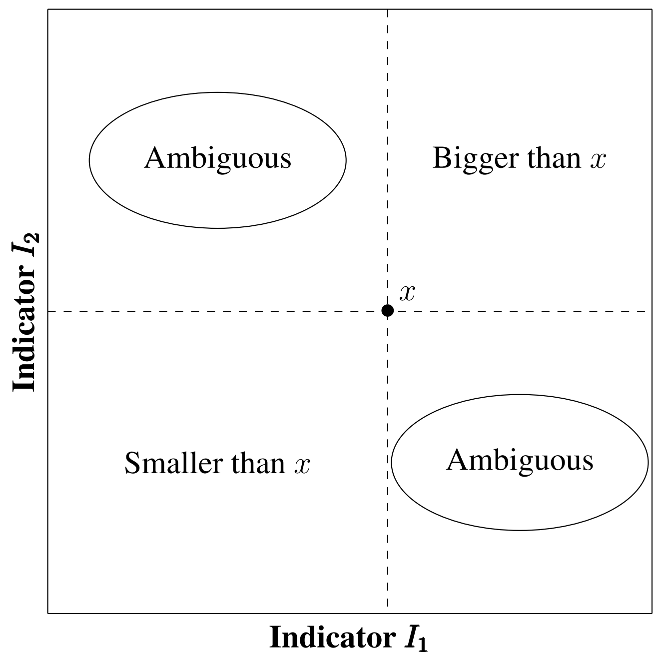


Figure 3.1: Comparison of objects according to indicators I_1 and I_2 when $n = 2$

According to Figure 3.1, it can be obviously seen that object x' which falls in the first quadrant (including its boundary) is bigger than x. x' that falls in the third quadrant is smaller than x. The other two quadrants (excluding their boundaries) are the quadrants of "ambiguity", i.e. objects that fall into these parts are not comparable with x.

In order to solve this ambiguity, Patil and Taillie [25] suggest to combine indicators into an index defined as $\gamma(I_1, I_2, \ldots, I_n) = \gamma(x)$. Here, the index γ could be taken as a risk measure introduced in Section 1.3. For instance, a linear compounding $\gamma = w_1 I_1 + w_2 I_2 + \cdots + w_n I_n$ could be the simplest combination.

According to this setting, the index γ defines a linear ordering on the set of objects X as follows:

$$x \leq_\gamma x' \text{ iff } \gamma(x) \leq \gamma(x') \tag{3.1}$$

The rule represented in Equation (3.1) can be displayed by Figure 3.2 in terms of the shape of γ that passes through object x. In both subfigures, the contour of index γ separates indicator space into two regions. The γ index given on the left is for linear combination whereas the one on the right is for non-linear combination.

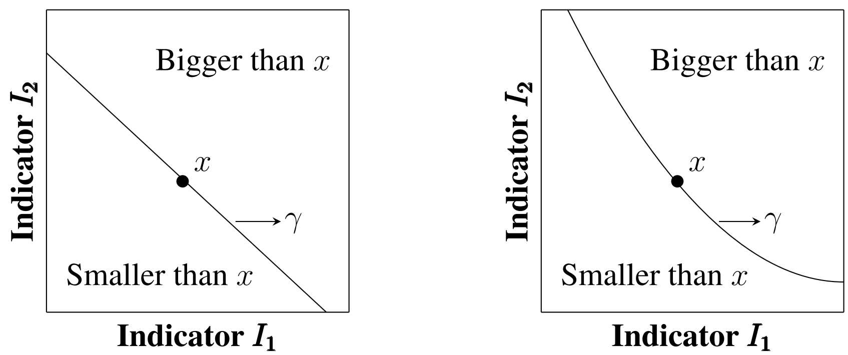


Figure 3.2: Shape of linear (left) and non-linear (right) index γ passing through object x

According to Figure 3.2, it can be seen that object x' which falls into the upper right region is bigger than x and the one that falls into the lower left region is smaller than x.

The index $\gamma(I_1, I_2, \ldots, I_n)$ which yields the relation given in Equation (3.1) is valid if the following conditions are fulfilled:

i. The index $\gamma(I_1, I_2, \ldots, I_n)$ must be monotone increasing for each variable individually.
ii. If the derivatives of the index $\gamma(I_1, I_2, \ldots, I_n)$ exist, then $\dfrac{\partial \gamma}{\partial I_j} \geq 0, \forall j$ must be true.
iii. If the index $\gamma(I_1, I_2, \ldots, I_n)$ is linear, i.e. $\gamma = w_1 I_1 + w_2 I_2 + \cdots + w_n I_n$, then $w_j \geq 0, \forall j$ must be true.

3.2.2. Some basic notations of POT

POT allows one to compare and order objects, characterized by multiple indicators, when there is an acceptable binary relation between two objects. It is a discipline related to discrete mathematics and graph theory, which is a subdiscipline of discrete mathematics. Many applications of POT and graph theory are common in terms of modeling pairwise relations between objects. Let

$X = \{x, y, z, \ldots\}$ be the set of objects. The binary relation "$\leq$" is used to order these objects [14].

Proposition 3.2.1. ***(Axioms of POT)*** *The relation "$\leq$" on the set X is a partial order when it fulfils the following conditions:*

(i) Reflexivity: $x \leq x$ for all $x \in X$.

(ii) Transitivity: $x \leq y$ and $y \leq z$ implies $x \leq z$.

(iii) Antisymmetry: $x \leq y$ and $y \leq x$ implies $x = y$.

(i) and (ii) in Proposition 3.2.1 are the axioms of the relation of "pre-order". Thus, pre-order is partial order if antisymmetry property holds. A set X which fulfils the above conditions is a partially ordered set, "poset".

There is another assumption in this concept when the set X is finite. Patil and Taillie [25] suggest one further relation "$\prec$" in a poset represented in the following proposition.

Proposition 3.2.2. *$x \prec y$ (y covers x) if*

i. $x < y$

ii. A relation $x < a < y$ does not exist for any object a.

3.2.3. Representations of Posets

There are three ways of representing posets which are Hasse diagrams, zeta matrices and cover matrices. Hasse diagram is a very useful planar graph for visualizing a poset when the set of objects X is not very large. Zeta matrix and cover matrix are more preferable for analytical purposes [25].

(a) Hasse diagrams: These graphs are important visual representations of posets especially when there exist incomparable objects in X. Incomparability appears when the order of objects wrt an indicator is different from the order of them wrt another indicator [14].

We use an example from the monograph of Brüggemann and Patil [14] to explain Hasse diagrams. It is supposed that the object set is $X = \{x, y, z, v, w\}$ and the indicator set is $\{I_1, I_2, I_3\}$. If we want to compare a pair of objects, we should check each indicator value related to these objects. An illustration for the partial order through Hasse diagram is represented in Table 3.1 and Figure 3.3.

Table 3.1: The values of objects wrt each indicator

Object	I_1	I_2	I_3
x	3.0	8.3	2.0
y	4.1	9.4	2.5
z	5.2	9.1	3.3
v	2.7	3.6	1.1
w	6.8	13.3	4.7

According to Table 3.1, we can write the following relations:

- $x < y$, $x < z$, $x < w$;
- $y < w$;
- $z < w$;
- $v < x$, $v < y$, $v < z$, $v < w$;
- $y \mid\mid z$

where $y \mid\mid z$ denotes that y and z are incomparable since $y < z$ according to I_1 and I_3, and $y > z$ according to I_2. Considering these relations, the diagrammatic representation of this poset is displayed in the following figure.

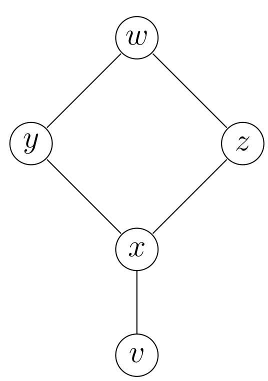


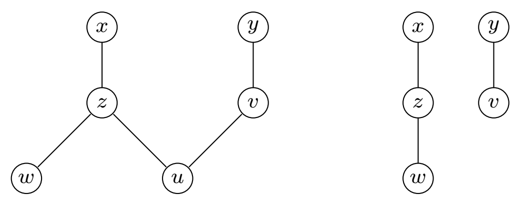
Figure 3.3: Hasse diagram of the data given in Table 3.1

In Figure 3.3, it can be seen that v is the smallest object whereas w is the biggest one. There are 4 levels, i.e. Level 1 (top level): $\{w\}$, Level 2: $\{y, z\}$, Level 3: $\{x\}$ and Level 4 (bottom level): $\{v\}$. Here, Level 2 denotes the incomparability between objects y and z. For more complicated Hasse diagram examples, see [25].

(b) Zeta matrices: These matrices are square matrices which have rows and columns that are filled by binary numbers determined according to the relation between the elements of posets. Inputs of the zeta matrix $\zeta_{x,y}$ are obtained as follows:

$$\zeta_{x,y} = \begin{cases} 1 & , x \leq y \\ 0 & , \text{otherwise} \end{cases} \tag{3.2}$$

In order to illustrate zeta matrices, we consider 2 posets (Poset I, II) [25] having Hasse diagrams given in Figure 3.4.



(a) **Poset I** (b) **Poset II**

Figure 3.4: Hasse diagrams for Posets I, II

In this figure, Poset I has three levels and Poset II has a disconnected Hasse diagram with two connected components which are $\{x, z, w\}$ and $\{y, v\}$.

According to Figure 3.4, the zeta matrix of Poset I, ζ_I is obtained as

$\zeta_I =$

	x	y	z	v	w	u
x	1	0	0	0	0	0
y	0	1	0	0	0	0
z	1	0	1	0	0	0
v	0	1	0	1	0	0
w	1	0	1	0	1	0
u	1	1	1	1	0	1

Here, it is easier to fill the entries row by row. For instance, the entries for the object u in ζ_I is obtained by checking if u is less than or equal to other objects. As it is seen from the Hasse diagram of (a) Poset I in Figure 3.4, u is less than the objects of the set $\{x, y, z, v\}$ and it is equal to itself. Hence, all entries excluding the one related to the object w are 1. u and w are not comparable in Poset I.

The zeta matrix is a very useful tool for comparison purposes because it prevents "maximal" and "minimal" elements of posets. An element is maximal (minimal) iff it is greater (less) than all objects apart from itself. It can be controlled through zeta matrix by checking if all entries in the row (column) of the related object are 0 apart from the entry 1 in the diagonal. For instance, it can be seen from Poset I that x and y are the maximal elements whereas w and u are the minimal elements.

If posets are disconnected, grouping the elements of each connected component together could be practical before filling the entries of the zeta matrix. Considering (b) Poset II, zeta matrix is obtained after grouping the connected components $\{x, z, w\}$ and $\{y, v\}$ as follows:

$\zeta_{II} =$

	x	z	w	y	v
x	1	0	0	0	0
z	1	1	0	0	0
w	1	1	1	0	0
y	0	0	0	1	0
v	0	0	0	1	1

Having obtained the zeta matrix ζ of a poset, the axioms of POT are revised wrt zeta matrix. Determining if zeta matrix fulfils the partial order conditions is complicated, especially for the transitivity axiom. In order to control three axioms of POT given in Proposition 3.2.1 in terms of zeta matrix ζ, Patil and Taillie [25] discuss them as follows:

- *Reflexivity*: The condition $(x \leq x,\ \forall x \in X)$ is controlled by checking the entries in the diagonal of ζ. If all elements of zeta matrix's diagonal are 1, then the object set X is reflexive.

- *Transitivity*: The condition ($x \leq y$ and $y \leq z \Rightarrow x \leq z$) is proven with finding an object a fulfilling the conditions $\zeta_{x,a} = 1$ and $\zeta_{a,y} = 1$ together when $\zeta_{x,y} = 1$. In other words, if the condition ($\sum_a \zeta_{x,a}\zeta_{a,y} \neq 0 \Rightarrow \zeta_{x,y} \neq 0$) is satisfied, then the object set X is transitive.
- *Antisymmetry*: The condition ($x \leq y$ and $y \leq x \Rightarrow x = y$) is checked with controlling entries of the diagonal of $\zeta_{x,y}$ and $\zeta_{y,x}$ simultaneously. Both of them are not 1 when $x \neq y$ if X is antisymmetric.

If the dimension of ζ matrix is larger, testing the partial order axioms is more complex. Therefore, we need more practical control mechanism. For this aim, we use the notation and conditions that Patil and Taillie [25] suggest. We change the axiom (iii) suggested by [25] and we give the revised condition in Proposition 3.2.3 and Remark 3.2.4. We explain the reason of this correction in Appendix A.2.1.

Before giving the axioms of POT in terms of zeta matrix, the following notation used in [25] is needed:

- Consider a non-negative matrix $\boldsymbol{X}$. The logical form of $\boldsymbol{X}$ is obtained as

$$\mathscr{L}(x) = \begin{cases} 1 & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

  for each object x in the object set X.

- Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be non-negative conformable matrices. Then,
  (a) $\boldsymbol{X} \circ \boldsymbol{X} = \mathscr{L}(\boldsymbol{XY})$ where $\boldsymbol{XY}$ is matrix multiplication of $\boldsymbol{X}$ and $\boldsymbol{Y}$.
  (b) $\boldsymbol{X} * \boldsymbol{Y}$is the component-wise multiplication of $\boldsymbol{X}$ and $\boldsymbol{Y}$ when they have same dimensions.

We follow the axioms given in Proposition 3.2.1 to show that zeta matrix ζ defines partial order.

Proposition 3.2.3. *Zeta matrix ζ, in which elements are either 0 or 1, is a representation of a poset iff the following conditions are fulfilled.*

(i) Reflexivity: All elements of ζ's diagonal are 1,

(ii) Transitivity: $\zeta \circ \zeta \leq \zeta$,

(iii) Antisymmetry: $\zeta * \zeta^T = \boldsymbol{I}_{n \times n}$.

Here, the third axiom involves the first axiom. Working with another matrix η obtained as $\zeta = \boldsymbol{I}_{n \times n} + \eta$ might sometimes be more sufficient. The axioms for ζ is revised for η in the following remark.

Remark 3.2.4. *Zeta matrix ζ is a representation of a poset iff the following conditions are fulfilled.*

(i) Reflexivity: All elements of η's diagonal are 0,

(ii) Transitivity: $\eta \circ \eta \leq \eta$,

(iii) Antisymmetry: $\eta * \eta^T = \mathbf{0}_{n \times n}$.

As in Proposition 3.2.3, the antisymmetry axiom given in (iii) involves the reflexivity axiom in (i).

(c) Cover matrices: These matrices are square matrices which have elements of binary numbers obtained as follows:

$$\xi_{x,y} = \begin{cases} 1 & , x \prec y \\ 0 & , \text{otherwise} \end{cases} \tag{3.3}$$

In Equation 3.3, the relation $\prec$ is defined before in Proposition 3.2.2. This equation means that $\eta_{x,y} = 1$ but the summation $\sum_a \eta_{x,a}\eta_{a,y} = 0$. Another formula for cover matrix ξ is obtained from zeta matrix as

$$\xi = \eta - \eta \circ \eta \tag{3.4}$$

Having presented a general framework of POT, we provide a perspective of partial ordering for the sake of clarity of stochastic ordering in risk prioritization context. There exist specific concepts of POT such as ambiguity graph, canonical orders, rank ambiguity etc. which are not studied in this thesis. For detailed information about these topics, see [14, 26, 85].

3.2.4. Using GIS as a tool for prioritization of risks

As it is explained earlier, analytic tools given in the previous section might not always be enough for risk prioritization purposes. Especially for environmental studies where geography-related risks are managed, GIS appears as an efficient tool. Patil and Taillie [25] represent an example about using landscape metrics to prioritize watersheds. In their study, they choose 9 indicators to assess and to order environmental effect across 114 watersheds. In order to show the importance of GIS, Figure 3.5 and Figure 3.6 are borrowed from [25].

The Hasse diagram for the primary part of watersheds is represented in Figure 3.5.

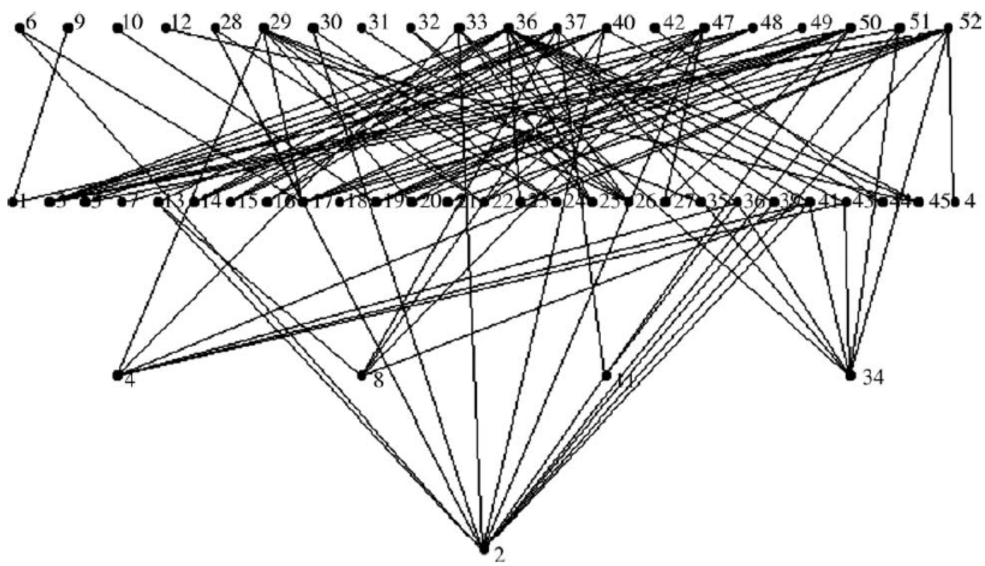


Figure 3.5: The Hasse diagram for the 52 watersheds in the primary part

In this figure, the numeric labels stand for the i-th watershed in the data set. It can be inferred from Figure 3.5 that the diagram is very disconnected. Among 114 watersheds, there are 60 connected parts 58 of which are isolated. There exist 4 watersheds in the secondary part whereas 52 watersheds are included in the primary part.

As it is easily understood from the Hasse diagram, comparing the environmental effect of watersheds seems to be very problematic since more than half of the watersheds are isolated. In Figure 3.5, there exist 4 levels two of which includes only 5 watersheds. This figure indicates the complexity of the comparison since there is no reasonable connection. However, the map of the watersheds in Figure 3.6 provides an information about watersheds. In this map, the shaded areas represent the primary Hasse part including 52 watersheds. Hence, it can be said that the primary part is geographically connected.

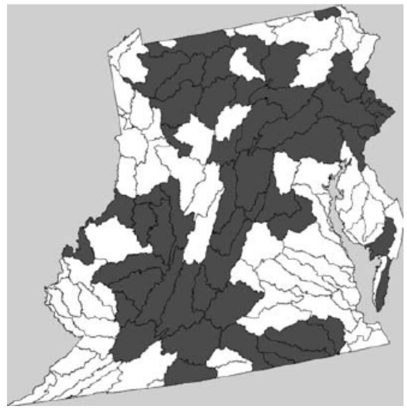


Figure 3.6: Mid-Atlantic region map

As it can be evaluated geographically from the figure that there are three connected components two of which are small and located near the outer edge of the biggest component of the region.

Since we prioritize risks environmentally due to the dynamics of the agricultural insurance data that we use for the application of this study, we deal with the geographic information. Therefore, we suggest to use GIS as a mechanism for risk prioritization in some cases such as environmental risk assessment.

3.3. Specific Relations for Stochastic Ordering: Partial ordering of DFs

We give some definitions and properties of dfs and inverse dfs suggested by Dhaene et al. [2] before introducing stochastic ordering of actuarial risks.

Let X rv has df $F_X(x) = \mathbb{P}(X \leq x)$. Inverse of df, which is also called as *quantile function*, is defined as

$$F_X^{-1}(p) = \inf \{x \in \mathbb{R} : \ F_X(x) \geq p\}, \tag{3.5}$$

for $p \in [0, 1]$.

General properties of dfs and inverse dfs help us to examine the partial order among dfs, and so rvs. Stochastic ordering is taken as partial ordering when dfs are prioritized. Denuit et al. [1] suggest an axiomatic methodology for partially ordered dfs. In this section, we present useful properties for specific stochastic ordering relations.

Definition 3.3.1. *Consider that $F_X(s), F_Y(s)$ and $F_Z(s)$ with $s \geq 0$ are dfs of rvs X, Y and Z, respectively. The binary relation $\precsim$ is a partial order on a set $P = \{F_X, F_Y, F_Z, \ldots\}$ if the axioms of POT are fulfilled as follows:*

(i) Reflexivity: $F_X(s) \precsim F_X(s)$ *for all* $F_X \in P$ *with* $\forall s \geq 0$.

(ii) Transitivity: $F_X(s) \precsim F_Y(s)$ *and* $F_Y(s) \precsim F_Z(s)$ *implies* $F_X(s) \precsim F_Z(s)$ *for all* $s \geq 0$.

(iii) Antisymmetry: $F_X(s) \precsim F_Y(s)$ *and* $F_Y(s) \precsim F_X(s)$ *implies* $F_X \equiv F_Y$ *where "*$\equiv$*" indicates* $F_X \equiv F_Y \Leftrightarrow F_X(s) = F_Y(s),\ \forall s \geq 0$.

In stochastic ordering, we deal with marginal distributions rather than joint distributions. Thus, if $X \precsim Y$, then $X \precsim Y'$ is also true for any rv Y' which is identically distributed with Y. In order to claim that a stochastic order relation is effective, it is expected that the relation fulfil the properties given below [1]. Let $\precsim$ be any binary relation indicating stochastic ordering.

i. *Shift invariance:* $X \precsim Y \Rightarrow X + c \precsim Y + c;\ \forall c$ where c is a constant.

ii. *Scale invariance:* $X \precsim Y \Rightarrow cX \precsim cY;\ \forall c$ where c is a positive constant.

iii. *Closure under convolution:* $X \precsim Y \Rightarrow X + Z \precsim Y + Z;\ \forall Z$ where Z is independent of both X and Y.

iv. *Closure wrt weak convergence:* $X_n \precsim Y_n;\ \forall n = 1, 2, \ldots$ and $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{d} Y \Rightarrow X \precsim Y$ where $\xrightarrow{d}$ indicates the convergence in distribution.

v. *Closure under mixing:* $(X|Z = z) \precsim (Y|Z = z) \Rightarrow X \precsim Y;\ \forall z$ where z is the support of Z.

The stability properties of the binary relation $\precsim$, which is used as a stochastic ordering relation, are given in the following proposition [1].

Proposition 3.3.2. *The binary relation $\precsim$ is said to be stable under some special cases such as mixture, convolution, compounding, and limit if it fulfils certain conditions.*

(i) Mixture: If $X \precsim Y$ is true in terms of conditional probabilities given $\Lambda = \lambda$ for each λ, then $X \precsim Y$ is true in terms of unconditional probabilities. If this is the case, $\precsim$ is said to be stable under mixture.

Consider two sets of independent rvs $\{X_1, X_2, X_3, \ldots\}$ and $\{Y_1, Y_2, Y_3, \ldots\}$ s.t. $X_i \precsim Y_i$ is true for all i.

(ii) Convolution: $\sum_{i=1}^{n} X_i \precsim \sum_{i=1}^{n} Y_i$ is true for all $n \in \mathbb{N}$. If this is the case, $\precsim$ is said to be stable under convolution.

Let $N \in \mathbb{Z}$ be independent of X_i and Y_i. If $\precsim$ satisfies the conditions (i) and (ii), it is clearly obtained that $\sum_{i=1}^{N} X_i \precsim \sum_{i=1}^{N} Y_i$.

(iii) Compounding: $\sum_{i=1}^{N} X_i \precsim \sum_{i=1}^{M} Y_i$ is also true for $N, M \in \mathbb{Z}$ s.t. $N \precsim M$. If this is the case, $\precsim$ is said to be stable under compounding.

(iv) Limit: Consider that X_i converges to X in distribution and Y_i converges to Y in distribution, i.e. $X_i \xrightarrow{d} X$ and $Y_i \xrightarrow{d} Y$. Then $X \precsim Y$ is true. If this is the case, $\precsim$ is said to be stable under limit.

3.3.1. From Ordering Dfs to Ordering Rvs

Heilmann and Schröter [24] introduce "ordering risks" within the concept of actuarial theory. They consider quantities or functions which produce order relations on a set of rvs such as claim sizes (e.g. individual claims, total claims of a single contract in a single period, or aggregate claims of a portfolio in a single period). They suggest five different ordering relations and their interpretations with regard to actuarial applications. These orderings are traditional first-order stochastic dominance and four relations connected with stochastic dominance.

Consider that rvs $X, Y, \ldots$ are risks with dfs $F_X, F_Y, \ldots$ s.t. $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{E}(X) > 0$. Five ordering relations are defined as follows.

1. ($\precsim_{\textbf{sd}}$) *First-order stochastic dominance:*

$$X \precsim_{\textbf{sd}} Y \Leftrightarrow 1 - F_X(s) \leq 1 - F_Y(s),\ \forall s \geq 0. \tag{3.6}$$

Applications for this relation is given below [1, 24].

(i) Ordering of stop-loss premiums defined in Equation (3.11):

$$X \precsim_{\text{sd}} Y \Rightarrow \int_s^\infty (1 - F_X(t))\,\mathrm{d}t \leq \int_s^\infty (1 - F_Y(t))\,\mathrm{d}t,\ \forall x \geq 0.$$

(ii) For all non-decreasing t, $X \precsim_{\text{sd}} Y \Rightarrow t(X) \precsim_{\text{sd}} t(Y)$.

(iii) Consider independent r.vectors $\boldsymbol{X} = \{X_1, X_2, \ldots, X_n\}$ and $\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$.

$$X_i \precsim_{\text{sd}} Y_i,\ \forall i = 1, 2, \ldots, n \Rightarrow \psi(\boldsymbol{X}) \precsim_{\text{sd}} \psi(\boldsymbol{Y})$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a non-decreasing function.

(iv) Ordering of net premiums: If a premium charged for the risk X is denoted by $\pi_X = \mathbb{E}(X)$ according to the "net premium principle", then

$$X \precsim_{\text{sd}} Y \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y) \Rightarrow \pi_X \leq \pi_Y.$$

(v) If $X \precsim_{\text{sd}} Y$, then $X \stackrel{d}{=} Y$ when $\mathbb{E}(X) = \mathbb{E}(Y)$. Here, $\stackrel{d}{=}$ implies $F_X \equiv F_Y$ defined in Definition 3.3.1.

(vi) Order statistics: Let independent rvs $X_1, X_2, \ldots, X_n$ be independent and identically distributed (iid) rvs having the same df F_X and suppose that $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the related order statistics. Two properties are as follows:

- Stochastic dominance ordering of order statistics:

$$X_{(1)} \precsim_{\text{sd}} X_{(2)} \precsim_{\text{sd}} \dots \precsim_{\text{sd}} X_{(n)}. \tag{3.7}$$

- Suppose another sequence of iid rvs $Y_1, Y_2, \dots, Y_n$. Then

$$X_1 \precsim_{\text{sd}} Y_1 \Rightarrow X_{(i)} \precsim_{\text{sd}} Y_{(i)},\ \forall i = 1, 2, \dots, n. \tag{3.8}$$

(vii) Transition from the "net premium principle" to the "expected value principle": Consider $X \precsim_{\text{sd}} Y$. Then the net premium π_Y for risk Y can be indicated as a transition to a premium calculated according to expected value principle as

$$\begin{aligned} \pi_Y &= \mathbb{E}(Y) = \int_0^\infty (1 - F_Y(s))\,\mathrm{d}s = \left[1 + \frac{\int_0^\infty (F_X(s) - F_Y(s))\,\mathrm{d}s}{\int_0^\infty (1 - F_X(s))\,\mathrm{d}s}\right] \mathbb{E}(X) \\ &= [1 + \theta]\,\mathbb{E}(X) \end{aligned} \tag{3.9}$$

Here, $\theta = \dfrac{\int_0^\infty (F_X(s) - F_Y(s))\,\mathrm{d}s}{\int_0^\infty (1 - F_X(s))\,\mathrm{d}s}$ is the premium loading factor applied to the net premium for X. The premium for the rv X is calculated as $\pi_X = (1 + \theta)\mathbb{E}(X)$ according to the expected value principle where θ is the premium loading factor.

For the derivation of Equation (3.9), see Appendix A.2.2, Equation (A.2).

2. ($\precsim_{\text{sl}}$) *Stop-loss dominance:*

$$X \precsim_{\text{sl}} Y \Leftrightarrow \mathbb{E}\left[(X - d)_+\right] \leq \mathbb{E}\left[(Y - d)_+\right],\ \forall d \geq 0. \tag{3.10}$$

Here, the net stop-loss premium $\mathbb{E}\left[(X - d)_+\right]$ with the retention limit d of the stop-loss treaty is defined as

$$\mathbb{E}\left[(X - d)_+\right] = \int_d^\infty (1 - F_X(x))\,\mathrm{d}x = \int_d^\infty (x - d)\,F_X(\mathrm{d}x). \tag{3.11}$$

Applications for this relation is given below [1, 24].

(i) Ordering of expected values: $X \precsim_{\text{sl}} Y \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y)$

For the proof, see Appendix A.2.2, Equation (A.3).

(ii) Ordering of variances: $\{X \precsim_{\text{sl}} Y\} \wedge \{\mathbb{E}(X) = \mathbb{E}(Y)\} \Rightarrow \mathbb{V}(X) \leq \mathbb{V}(Y)$

For the proof, see Appendix A.2.2, Equation (A.5).

(iii) Ordering premiums calculated according to "variance principle": If a premium charged for the risk X is denoted by $\pi_X = \mathbb{E}(X) + \alpha\mathbb{V}(X)$; $\alpha > 0$ according to the variance principle, then $\{X \precsim_{\text{sl}} Y\} \wedge \{\mathbb{E}(X) = \mathbb{E}(Y)\} \Rightarrow \pi_X \leq \pi_Y$ by (ii).

3. ($\precsim_\ell$) *Survival distribution of $\widetilde{X}$:*

$$X \precsim_\ell Y \Leftrightarrow \widetilde{X} \precsim_{\textbf{sd}} \widetilde{Y} \Leftrightarrow \ell_X(s) \leq \ell_Y(s), \; \forall s \geq 0. \tag{3.12}$$

Given that the surplus falls below the initial surplus for the first time, $\widetilde{X}$ is the amount of that fall. The density function of $\widetilde{X}$ is defined as

$$f_{\widetilde{X}} : x \to \frac{1}{\mathbb{E}(X)}[1 - F_X(x)]1_{(0,\infty)}(x).$$

The surplus process of an insurer, which helps us to understand the relation for $\widetilde{X}$ rv, is given in Appendix A.2.3.

Therefore, the survival function of $\widetilde{X}$ is defined as

$$\ell_X(x) = S_{\widetilde{X}}(x) = 1 - F_{\widetilde{X}}(x) = \int_x^\infty f_{\widetilde{X}}(t)\,\mathrm{d}t = \frac{1}{\mathbb{E}(X)}\int_x^\infty [1 - F_X(t)]\,\mathrm{d}t = \frac{\mathbb{E}\left[(X - x)_+\right]}{\mathbb{E}(X)}. \tag{3.13}$$

where $F_{\widetilde{X}}$ is the df of $\widetilde{X}$.

Applications for this relation is given below [1, 24].

(i) Ordering of variances: $\{X \precsim_\ell Y\} \wedge \{\mathbb{E}(X) = \mathbb{E}(Y)\} \Rightarrow \mathbb{V}(X) \leq \mathbb{V}(Y)$

For the proof, see Appendix A.2.2, Equation (A.6).

(ii) Ordering premiums calculated according to "variance principle": If a premium charged for the risk X is denoted by $\pi_X = \mathbb{E}(X) + \alpha\mathbb{V}(X)\ ;\ \alpha > 0$ according to the variance principle, then $\{X \precsim_\ell Y\} \wedge \{\mathbb{E}(X) = \mathbb{E}(Y)\} \Rightarrow \pi_X \leq \pi_Y$ by (i).

(iii) Relationship between mean residual life of X and hazard function of $\widetilde{X}$: In actuarial sciences, hazard function is also called as hazard rate or failure rate, and also as force of mortality in survival analysis literature. Thus, the hazard rate function of $\widetilde{X}$ is given by

$$h_{\widetilde{X}}(x) = \frac{f_{\widetilde{X}}(x)}{S_{\widetilde{X}}(x)} = \frac{1 - F_X(x)}{\int_x^\infty [1 - F_X(t)]\,\mathrm{d}t}.$$

For the proof, see Appendix A.2.2, Equation (A.7).

On the other hand, mean residual lifetime, which is also called expected future lifetime, of X is defined by

$$\mathbb{E}(X_P) = \frac{\int_x^\infty [1 - F_X(t)]\,\mathrm{d}t}{1 - F_X(x)}. \tag{3.14}$$

where the excess-loss rv X_P is defined only when the insurer makes a payment ($X > x$).

For the proof, see Appendix A.2.2, Equation (A.8).

Since X_P is denoted as a conditional rv by $X_P = (X - x|X > x)$ [86], the mean residual life time function at x can be given as the above equation. Therefore, the mean residual life of X is decreasing (increasing) as the hazard of $\widetilde{X}$ is increasing (decreasing) since $\mathbb{E}(X_P) = \dfrac{1}{h_{\widetilde{X}}(x)}$.

(iv) Selection of the premium calculation principle: It is considered that the premium charged for the risk X is denoted by

$$\pi_X = \begin{cases} \mathbb{E}(X) + \alpha\dfrac{\mathbb{V}(X)}{\mathbb{E}(X)} & , \mathbb{E}(X) > 0 \\ 0 & , \mathbb{E}(X) = 0 \end{cases}$$

according to the modified variance principle [87].

If we take the security loading coefficient for the expected value principle as $\theta = 1$ to calculate the premium for $\widetilde{X}$, $\pi_{\widetilde{X}}$, and that for modified variance principle as $\alpha = 1$ to calculate the premium for X, π_X; then

$$\pi_X = \mathbb{E}(X) + \frac{\mathbb{V}(X)}{\mathbb{E}(X)} = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)} \xrightarrow{\text{(**) in Equation A.6}} = 2\mathbb{E}(\widetilde{X}) = \pi_{\widetilde{X}}$$

As a result, the insurer may prefer to calculate premiums for the risk $\widetilde{X}$ that has a decreasing hazard rate under expected value principle rather than for the risk X that has an increasing mean residual life under modified variance principle [24].

4. ($\precsim_k$) *Survival distribution of $\breve{X}$:*

$$X \precsim_k Y \Leftrightarrow \; \breve{X} \precsim_{\textbf{sd}} \breve{Y} \Leftrightarrow k_X(s) \leq k_Y(s), \; \forall s \geq 0. \tag{3.15}$$

Here, $\breve{X}$ rv has the density

$$f_{\breve{X}} : x \to [1 - F_{\hat{X}}(x)] 1_{(0,\infty)}(x)$$

where $F_{\hat{X}}$ is the df of $\hat{X} = \dfrac{X}{\mathbb{E}(X)}$. Therefore, considering that $F_{\breve{X}}$ is the df of $\breve{X}$, the survival function of $\breve{X}$, $k_X(x)$, is defined as

$$k_X(x) = S_{\breve{X}}(x) = 1 - F_{\breve{X}}(x) = 1 - \int_0^x f_{\breve{X}}(t)\,\mathrm{d}t = \frac{1}{\mathbb{E}(X)} \mathbb{E}\left[(X - x\mathbb{E}(X))_+ \right]. \tag{3.16}$$

For the derivation, see Appendix A.2.2, Equation (A.10).

Applications for this relation is given below [1, 24].

(i) Ordering of coefficient of variations: $X \precsim_k Y \Rightarrow \mathbb{CV}(X) \leq \mathbb{CV}(Y)$

For the proof, see Appendix A.2.2, Equation (A.12).

(ii) Ordering of variances: $\{X \precsim_k Y\} \wedge \{\mathbb{E}(X) = \mathbb{E}(Y)\} \Rightarrow \mathbb{V}(X) \leq \mathbb{V}(Y)$

For the proof, see Appendix A.2.2, Equation (A.13).

(iii) Ordering premiums calculated according to “variance principle”: If π_X is the premium calculated according to the variance principle, then $\{X \precsim_k Y\} \wedge \{\mathbb{E}(X) = \mathbb{E}(Y)\} \Rightarrow \pi_X \leq \pi_Y$ by (ii).

5. **($\precsim_r$)** ***Conditional mean residual life-time function:***

$$X \precsim_r Y \Leftrightarrow r_X(x) \leq r_Y(x),\ \forall x \geq 0. \tag{3.17}$$

Here, if $F_X(x) < 1$, then

$$r_X(x) = \mathbb{E}(X_P) = \mathbb{E}[X - x | X > x] = \frac{\int_x^\infty [1 - F_X(t)]\,\mathrm{d}t}{1 - F_X(x)},\ \text{and} \tag{3.18}$$

$r_X(x) = 0$ when $F_X(x) = 1$. This equation is given as an alternative to the definition of $\mathbb{E}(X_P)$ in Equation (3.14). This ordering is important for actuarial applications because the rv X_P is defined only if a payment exists (when $X > x$). If there is no claim, there will be no payment. So insurer cannot capture the loss information. As a result, conditional mean residual life-time is useful to order risks for the policies with deductibles.

In the context of these orderings ($\precsim_{\text{sd}},\ \precsim_{\text{sl}},\ \precsim_{\ell},\ \precsim_{k},\ \precsim_{r}$), Heilmann and Schröter [24] also investigate moment inequalities and accordingly ordering of premiums under some premium calculation principles such as the expected value principle, $\pi_X = (1+\theta)\mathbb{E}(X)$; variance principle, $\pi_X = \mathbb{E}(X) + \alpha\mathbb{V}(X)$; standard deviation principle, $\pi_X = \mathbb{E}(X) + \alpha\sqrt{\mathbb{V}(X)}$; and exponential principle, $\pi_X = \dfrac{1}{\beta}\ln[M_X(\beta)]$ with moment generating function of X, $M_X(\beta)$. According to that paper, ordering between rvs X and Y remains same as that between premiums π_X and π_Y. Thus, considering $\precsim$ denotes any ordering,

$$X \precsim Y \Rightarrow \pi_X \leq \pi_Y \tag{3.19}$$

holds. The following properties could be written according to the properties given in this subsection:

(i) Expected value principle: The relation in Equation (3.19) is fulfilled for ($\precsim_{\text{sd}}$), ($\precsim_{\text{sl}}$) in any case, and for ($\precsim_{\ell}$), ($\precsim_{r}$) if Y is strictly positive almost surely (a.s.).

(ii) Variance and standard deviation principles: The relation in Equation (3.19) is fulfilled for ($\precsim_{\text{sd}}$), ($\precsim_{\text{sl}}$), ($\precsim_{\ell}$), ($\precsim_{k}$), ($\precsim_{r}$) if $\mathbb{E}(X) = \mathbb{E}(Y)$.

(iii) Exponential principle: The relation in Equation (3.19) is fulfilled for ($\precsim_{\text{sd}}$), ($\precsim_{\text{sl}}$) in any case; and for ($\precsim_{\ell}$), ($\precsim_{r}$) if Y is strictly positive a.s..

3.3.2. Choice Under Risk with Stochastic Dominance

Denuit et al. [1] handle the concept of decision making under risk by unifying stochastic dominance with the preference theory. They choose a simple utiliy function to give the general idea of stochastic dominance orderings for making decisions between risky choices. In this point of view, they study classical utility theorem EUT and YDT.

Let us choose a utiliy function defined as

$$u(x) = \begin{cases} 1 & , x \leq s \\ 0 & \text{, otherwise} \end{cases} \tag{3.20}$$

According to Equation (3.20), the aim of DM is to prefer the risky choice when it is smaller than or equal to a constant s because DM's utility is 100%. However, if there are two equivalent choices, both of which are smaller than or equal to s, the preference relation of the DM is shown in the following relation. DMs prefer X instead of Y according to the expected utility which is shown as

$$\mathbb{E}\left[u(X)\right] \leq \mathbb{E}\left[u(Y)\right] \Rightarrow S_X(s) \leq S_Y(s) \tag{3.21}$$

where $\mathbb{E}\left[u(X)\right] = 1\mathbb{P}(X \leq s) + 0\mathbb{P}(X > s) = \mathbb{P}(X \leq s) = 1 - S_X(s)$.

Proposition 3.3.3. *Suppose that all DMs having the same utility function defined in Equation (3.20) prefer X over Y where X and Y are random prospects. $X \precsim_{sd} Y$ holds for these DMs.*

Proof. Let DMs prefer X over Y. From the previous inference,

$$\begin{aligned} &\Rightarrow \quad S_X(s) \leq S_Y(s);\ \forall s \in \mathbb{R} \\ &\xrightarrow{S_X(s)=1-F_X(s)} \quad F_X(s) \geq F_Y(s);\ \forall s \in \mathbb{R} \\ &\Rightarrow \quad X \precsim_{\text{sd}} Y. \end{aligned}$$

□

An important and useful deduction of this proposition is that the larger the probability that a rv being smaller than a specified value, the more likely that rv is preferred. This inference can be shown as

$$X \precsim_{\text{sd}} Y \Leftrightarrow F_X(s) \geq F_Y(s) \Leftrightarrow S_X(s) \leq S_Y(s) \tag{3.22}$$

Remark 3.3.4. *As a generalization of Proposition 3.3.3 for any rvs X and Y, the following results can be obtained.*

$$\begin{aligned} X \precsim_{sd} Y \Leftrightarrow \quad & \mathbb{E}\left[u(X)\right] \leq \mathbb{E}\left[u(Y)\right];\ \textit{for all } u \textit{ s.t. } u \textit{ is non-decreasing function and expectations exist,} \\ \Leftrightarrow \quad & \mathbb{E}\left[u(X)\right] \leq \mathbb{E}\left[u(Y)\right];\ \textit{for all } u \textit{ s.t. } u'(x) \geq 0 \textit{ and expectations exist}. \end{aligned}$$

Focusing on YDT, a similar inference is presented in the following proposition [1].

Proposition 3.3.5. *Suppose that two random prospects X and Y are ordered. Then,*

$$X \precsim_{sd} Y \Leftrightarrow \gamma_g(X) \leq \gamma_g(Y),$$

for all non-decreasing g where g is the distortion function.

Here, γ_g is the Wang risk measure defined as

$$\gamma_g(X) = \int_0^{+\infty} g\left(S_X(x)\right) \mathrm{d}x$$

with a non-decreasing function g. For the proof of Proposition 3.3.5 and more information about the Wang risk measure, see [1].

As a stochastic dominance ordering, stop-loss dominance ordering also provides helpful relations for the procedure of choice under risk. First of all, as in the Remark 3.3.4, expectation of functions can be ordered according to stop-loss dominance relation to provide a framework for preference relations under risk such as EUT and YDT.

Proposition 3.3.6. *For any rvs X and Y, the following results are obtained:*

$$X \precsim_{sl} Y \Leftrightarrow \quad \mathbb{E}\left[\upsilon(X)\right] \leq \mathbb{E}\left[\upsilon(Y)\right]; \quad \textit{for all } \upsilon \textit{ s.t. } \upsilon \textit{ is convex and expectations exist,}$$

$$\Leftrightarrow \quad \mathbb{E}\left[\upsilon(X)\right] \leq \mathbb{E}\left[\upsilon(Y)\right]; \quad \textit{for all } \upsilon \textit{ s.t. } \upsilon''(x) \geq 0 \textit{ and expectations exist.}$$

By using this proposition, we can obtain a relation for utility functions under EUT considering a risk-averse DM. From the insurer's perspective, we can infer Equation (3.21) as provided in the following proposition.

Proposition 3.3.7. *A risk-averse DM has a concave utility function u. Thus, the function $-u(w-x)$ is convex. According to Propositon 3.3.6,*

$$X \precsim_{sl} Y \Leftrightarrow \mathbb{E}\left[-u(w-X)\right] \leq \mathbb{E}\left[-u(w-Y)\right] \Leftrightarrow \mathbb{E}\left[u(w-X)\right] \geq \mathbb{E}\left[u(w-Y)\right].$$

To see the inferences of the adaptation of stop-loss dominance ordering to YDT, the following two propositions are presented.

Proposition 3.3.8. *Suppose two random prospects X and Y. $X \precsim_{sl} Y$ iff*

(i) $\mathbb{E}_g(w-X) \geq \mathbb{E}_g(w-Y)$; *for all non-decreasing and convex g,*

(ii) $\gamma_g(X) \leq \gamma_g(Y)$; *for all non-decreasing and concave g.*

where g is the distortion function.

For the proof, see [1].

3.4. Interim Conclusion: Risk Prioritization through Stochastic Ordering Relations

In this chapter, we firstly give some basic theoretical information about POT which helps us to understand prioritization concept. In addition to POT as an analytical tool, the idea about using GIS as a tool to assess risk offers a diversity to our study.

Moreover, we present various stochastic ordering relations and their properties. Many relations such as first-order stochastic dominance and stop-loss dominance are used to set risk priorities according to the existence of stochastic dominance among the risk distributions. These relations are also investigated under preference modeling compare the orderings according to choice under risk. In order to model bias in decision making, which is one of the main purposes of our study, we investigate CPT in details in Chapter 4. We propose solutions for stop-loss premiums under CPT for the sake of combining stochastic dominance notion with risk perception. Furthermore, as a very suitable relation to multivariate representation of r.vectors, we also provide the stochastic majorization relation in Chapter 5.

4. RISK PRIORITIZATION THROUGH STOP-LOSS DOMINANCE UNDER CUMULATIVE PROSPECT THEORY

4.1. Introduction

In preference theory, the most important approaches of decision making in a risky environment are discussed in Chapter 2. It is generally accepted that EUT does not reflect biased choices and risk perceptions of DMs, so that it needs to be improved. Kahneman and Tversky [9] discuss prospect theory dealing with violations of EUT and suggest two main notions, which are the value function and the probability weighting function. The value function is formed according to the DMs tendencies of making choices for gains or losses. Moreover, the probability weighting function is modified with a non-linear transformation by overweighting or underweighting probabilities.

In addition to the prospect theory, many studies such as Schmeidler [4] and Yaari [7] suggest to transform cumulative probabilities instead of individual ones as an expansion. In the frame of this modification, Tversky and Kahneman [27] develop a new approach, CPT, incorporating above concepts. Before introducing the method, it is useful to discuss fundamental phenomena of the decision making.

i. *Framing effects:* Although it is assumed that identical formulations cause equivalent preference ordering, it is proven that different preferences arise because of dissimilar framing options such as framing with regard to gains or to losses.

ii. *Non-linear preferences:* In EUT, it is assumed that the overall utility of a prospect defined as $U(x_1, p_1; \ldots; x_n, p_n) = p_1u(x_1) + \ldots + p_nu(x_n)$ is linear in outcome. However, non-linear preference formulations are observed in many studies such as Allais [88]. In this study, it is shown that 1% difference between probabilities close to 1.00 has more influence on preferences than those around 0.10.

iii. *Source dependence:* Individuals' preferences about bets on an uncertain event is determined by both the level of uncertainty and its source. Ellsberg paradox given in Section 2.7

shows that the source where more information is available is preferred.

iv. *Risk seeking:* Even if DMs are assumed to be risk-averse in traditional preference theories, researchers come across risk-seeking choices especially when DMs make choices between a certain loss and a high probability of a larger loss or a certain gain and a low probability of a larger gain.

v. *Loss aversion:* Losses seem to be larger and frightening than gains. This asymmetrical situation between losses and gains should not be explained only by income effects or by decreasing risk aversion.

CPT explains non-linear preferences, risk seeking and loss aversion by the value and weighting functions. It deals with the other two phenomena (framing effects and source dependence), as well. The distinguishing feature of the prospect theory is framing and valuation processes. DMs describe acts, possibilities and outcomes at framing step. Valuation step is the part that DM evaluates the value of all prospects and makes decision according to the attained value. Tversky and Kahneman [27] modify the EUT using gains and losses instead of final assets; and multiplying the value of each outcome by a decision weight instead of an additive probability. There exist two problems about the weighting when they first propose the prospect theory: not fulfilling the conditions of stochastic dominance and not being extended to prospects with a large number of outcomes. These problems are solved by CPT.

We introduce CPT and its properties in Section 4.2. We discuss fundamental phenomena wrt our aim. In Section 4.3, we introduce the zero-utility premium principle under CPT proposed by Kaluszka ve Krzeszowiec [28]. This principle is used to compute premiums for individual claims of a single insurance contract. In this section, we also present our contribution based on a stop-loss premium calculation under CPT through some modifications considering aggregate claims of different risk clusters. In Section 4.4, we suggest a stop-loss dominance relation under CPT and our solutions for stop-loss premiums are presented for three different value functions. Finally, we summarize the chapter in Section 4.5.

4.2. Cumulative Prospect Theory and Risk Prioritization

We consider that the set of outcomes, X, consists of outcomes that are neutral (0), positive (gains) and negative (losses). The function $f : P \to X$ is an uncertain prospect assigning a consequence to a state $s \in P$ through $f(s) = x;\ x \in X$.

To obtain the cumulative functional, the outcomes of each prospect are ranked in increasing order. Then, f is represented as a sequence of couples $\{x_i, A_i\}$. Here, if A_i occurs, the prospect yields x_i. Thus, $x_i > x_j \Leftrightarrow i > j$ where A_i is a partition of P. The positive part of f is given as

$$f^+(s) = \begin{cases} f(s) & , f(s) > 0 \\ 0 & \text{, otherwise} \end{cases}$$

whereas the negative part of f is obtained as

$$f^-(s) = \begin{cases} f(s) & , f(s) < 0 \\ 0 & \text{, otherwise} \end{cases}$$

As in EUT, a number $V(f)$ is assigned to each prospect. The relation $V(f) \precsim V(g)$ means that g is preferred to f or DM is indifferent between f and g. We use the term "capacity" suggested by Choquet [29] which is a non-additive set function generalizing the standard probability concept. Let W_i be a capacity assigning a number to each $A \subset P$, i.e. $W(A)$ s.t. $W(\emptyset) = 0$, $W(S) = 1$, and $A \subset B \Rightarrow W(A) \leq W(B)$. According to CPT, $\upsilon : X \to \mathbb{R}$ is a strictly increasing value function s.t. $\upsilon(x_0) = \upsilon(0) = 0$, and there exist capacities W^+ and W^- and prospects $f(x_i, A_i);\ -m \leq i \leq n$. Then,

$$\begin{aligned} V(f) = \quad & V(f^+) + V(f^-), \text{ with} \\ & V(f^+) = \textstyle\sum_{i=0}^{n} \pi_i^+ \upsilon(x_i), \text{ and} \\ & V(f^-) = \textstyle\sum_{i=-m}^{0} \pi_i^- \upsilon(x_i) \end{aligned} \tag{4.1}$$

where $\pi^+(f^+) = \left(\pi_0^+, \pi_1^+, \ldots, \pi_n^+\right)$ and $\pi^-(f^-) = \left(\pi_{-m}^-, \pi_{-m+1}^-, \ldots, \pi_0^-\right)$ are decision weights defined as

$$
\begin{aligned}
\pi_n^+ &= W^+(A_n), \\
\pi_{-m}^- &= W^-(A_{-m}), \\
\pi_i^+ &= W^+(A_i \cup \ldots \cup A_n) - W^+(A_{i+1} \cup \ldots \cup A_n);\ 0 \leq i \leq n-1, \text{ and} \\
\pi_i^- &= W^-(A_{-m} \cup \ldots \cup A_i) - W^-(A_{-m} \cup \ldots \cup A_{i-1});\ 1-m \leq i \leq 0.
\end{aligned}
\tag{4.2}
$$

By taking $i \geq 0 \Rightarrow \pi_i = \pi_i^+$ and $i < 0 \Rightarrow \pi_i = \pi_i^-$, Equation (4.1) reduces to

$$V(f) = \sum_{i=-m}^{n} \pi_i \upsilon(x_i) \tag{4.3}$$

Tversky and Kahneman [27] interpret these decision weights as follows:

(i) π_i^+ *related to positive outcomes*: The difference between the capacities $W^+(A_i \cup \ldots \cup A_n)$, i.e. the events where "the outcome is better than or equal to x_i" and $W^+(A_{i+1} \cup \ldots \cup A_n)$, i.e. the events where "the outcome is strictly better than x_i".

(ii) π_i^- *related to negative outcomes*: The difference between the capacities $W^-(A_{-m} \cup \ldots \cup A_i)$, i.e. the events where "the outcome is worse than or equal to x_i" and $W^-(A_{-m} \cup \ldots \cup A_{i-1})$, i.e. the events where "the outcome is strictly worse than x_i".

Therefore, each decision weight related to an outcome can be explained as the marginal contribution of the relevant event. If each capacity is additive, then W_i becomes the probability of A_i, i.e. $p_i = \mathbb{P}(A_i)$. However, the sum can be smaller or greater than 1 for mixed prospects since the decision weights π_i^+ and π_i^- are defined by different W_i for gains and losses separately. Thus, the decision weights are defined in a different way and Equation (4.2) reduces to the following equation:

$$
\begin{aligned}
\pi_n^+ &= w^+(p_n), \\
\pi_{-m}^- &= w^-(p_{-m}), \\
\pi_i^+ &= w^+(p_i + \ldots + p_n) - w^+(p_{i+1} + \ldots + p_n);\ 0 \leq i \leq n-1, \text{ and} \\
\pi_i^- &= w^-(p_{-m} + \ldots + p_i) - w^-(p_{-m} + \ldots + p_{i-1});\ 1-m \leq i \leq 0,
\end{aligned}
\tag{4.4}
$$

where the functions w^+ and w^- are strictly increasing defined from the unit interval into itself with $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$.

To make the model more clear, a game of chance, given as an example by [27], is provided as follows.

Example 4.2.1. *A dice is rolled once. \$$x$ is received if x is even and \$$x$ is paid if x is odd. Therefore,*

$$f^+ = \left(0, \frac{1}{2}; 2, \frac{1}{6}; 4, \frac{1}{6}; 6, \frac{1}{6}\right), \textit{ and}$$

$$f^- = \left(-5, \frac{1}{6}; -3, \frac{1}{6}; -1, \frac{1}{6}; 0, \frac{1}{2}\right).$$

By Equations (4.1) and (4.4), the value function is obtained as follows:

$$\begin{aligned} V(f) = \;& V(f^+) + V(f^-) \\ = \;& \textstyle\sum_{i=0}^{n} \pi_i^+ \upsilon(x_i) + \sum_{i=-m}^{0} \pi_i^- \upsilon(x_i) \\ = \;& v(2)[w^+(1/2) - w^+(1/3)] + \\ & v(4)[w^+(1/3) - w^+(1/6)] + \\ & v(6)[w^+(1/6) - w^+(0)] + \\ & v(-5)[w^-(1/6) - w^-(0)] + \\ & v(-3)[w^-(1/3) - w^-(1/6)] + \\ & v(-1)[w^-(1/2) - w^-(1/3)]. \end{aligned}$$

After providing a general presentation of CPT, modifications suggested by this theory can be discussed. Firstly, CPT deals with the source dependence problematique by applying to both uncertain and probabilistic prospects. Secondly, CPT allows separate decision weights for gains and losses. As a rank-dependent model, CPT supposes that $w^-(p) = 1 - w^+(1-p)$ or equivalently $W^-(A) = 1 - W^+(S - A)$ whereas some studies such as first version of prospect theory assume that $w^-(p) = w^+(p)$. Thus, the cumulative (rank-dependent) prospect theory fulfils the conditions of stochastic dominance. Thirdly, CPT can be extended to continuous distributions because it can be applied to any finite prospect.

The decision weights for gains and losses shown as $w^+(p)$ and $w^-(p)$, respectively, are henceforth indicated as the probability distortion functions $g(p)$ and $h(p)$ for the sake of consistency with the literature on CPT use in actuarial sciences. Besides, it is not always assumed that $w^-(p) = 1 - w^+(1-p)$, i.e. $h(p) = 1 - g(1-p)$ as in this section. Instead, we indicate this expression as $\bar{g}(p) = 1 - g(1-p)$. We consider the probability distortion functions suggested by Tversky and Kahneman [27]. This function is defined as

$$g(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{1/\gamma}}. \tag{4.5}$$

The probability distortion function used for losses, i.e. $h(p)$, is usually same as $g(p)$ with different values of γ. For instance, Tversky and Kahneman [27] take $\gamma = 0.61$ for gains and $\gamma = 0.69$ for losses.

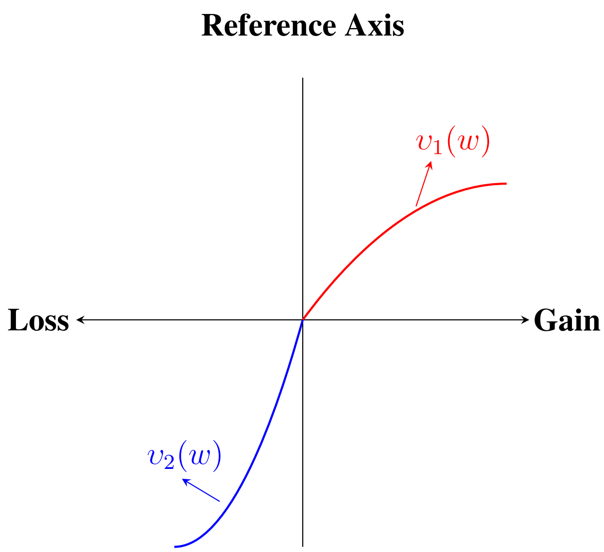
When DMs' preferences are handled in the context of prospect theory, one needs to consider the phenomena given in the following definition and Figure 4.1 reflecting these phenomena.

Definition 4.2.1. ***(Fundamental phenomena of CPT)***

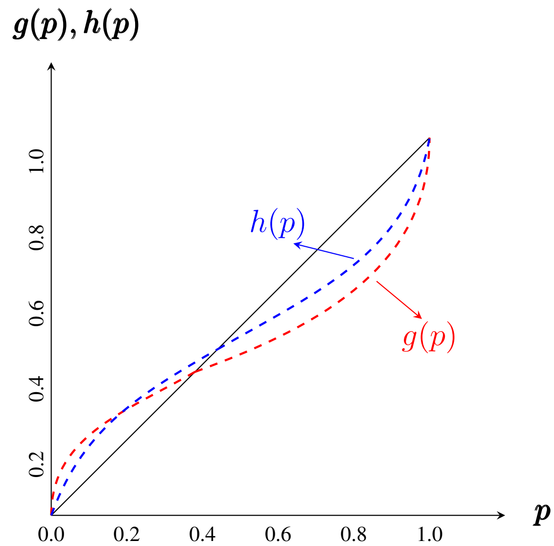
(i) ***Reference point****: It determines values as losses and gains,*

(ii) ***Value function****: It is an S-shaped function that reflects the concavity (convexity) for gains (losses),*

(iii) ***Probability weighting function****: It is probability distortion function that represents overweighting (underweighting) of low (high) probabilities.*



(a) Value functions



(b) Probability distortion functions

Figure 4.1: Value functions (a) and probability distortion functions (b) proposed in prospect theory

In Figure 4.1, the graphs of the typical value functions and probability distortion functions suggested by [9, 27] and studied within the frame of the prospect theory are given . As it is seen from the figure that the value function for losses, $v_2(w)$, is convex , and the value function $v_1(w)$ is concave and more gradual for gains. On the other hand, the graph of the probability distortion functions for the gains and losses show that the low (high) probabilities are overestimated (underestimated).

4.2.1. Adapting CPT to Stochastic Ordering

In order to make a connection between CPT and stochastic ordering relations, we need to repsernt the axiomatic definition of CPT at first. Consider that $F = \{f : P \to X\}$ is the set of prospects where $F^+(F^-)$ indicates the positive (negative) prospects. It is assumed that the binary relation $\precsim$ satisfies the properties of completeness, transitivity, and strictly monotonicity which means that the condition $(\{g \neq h\} \wedge \{g(s) \leq h(s)\}, \forall s \Rightarrow g < h)$ holds.

Definition 4.2.2. ***(Sure thing principle)*** *Define $f = gAh$ as*

$$f(s) = \begin{cases} g(s) & , s \in A \\ h(s) & , s \in P \setminus A \end{cases}$$

for all $g, h \in F$ and $A \subset P$. Then, $\precsim$ on F is independent if

$$gAh \precsim gAh' \Leftrightarrow g'Ah \precsim g'Ah';$$

for all $g, h, g', h' \in F$ and $A \subset P$.

This principle is one of the main properties of EUT [5]. The independence property is not generally held by CPT. In this theorem, they provide a concept relating to *comonotonicity*. Thus, a constant prospect is comonotonic with all prospects if it provides the same outcome in every state. In this study, it is claimed in consideration of many studies that comonotonic independence can be replaced instead of independence axiom. Therefore, if the prospects $gAh, gAh', g'Ah$, and $g'Ah'$ are pairwise comonotonic, it is assumed that it implies independence [27].

In addition, CPT assumes a property called "double matching" defined as below.

Definition 4.2.3. ***(Double matching)*** *For the prospects $g, h \in F$;*

$$\left\{g^+ \approx h^+\right\} \wedge \left\{g^- \approx h^-\right\} \Rightarrow g \approx h.$$

Having provided Definition 4.2.2 and Definition 4.2.3, the following theorem is the representation of CPT within the frame of stochastic ordering.

Theorem 4.2.4. ***($\precsim$ under CPT)*** *Consider that both $(F^+, \precsim)$ and $(F^-, \precsim)$ can be denoted by a cumulative functional. $(F, \precsim)$ fulfils the conditions of CPT iff double matching and comonotonic independence properties hold.*

In this context, we discuss adaptedness of CPT to the stochastic ordering relations. We propose stop-loss dominance ordering under CPT obtaining stop-loss premium solutions for different value functions in forthcoming sections.

4.3. CPT Premium Principle

In order to introduce CPT premium principle, we first give the definition of the Choquet integral. Let $\mathscr{G}$ be the probability distortion function class. The Choquet integral is defined as

$$\mathbb{E}_g(X) := \int_{-\infty}^{0} \left[g\left(\mathbb{P}(X > t)\right) - 1\right] \mathrm{d}t + \int_{0}^{+\infty} g\left(\mathbb{P}(X > t)\right) \mathrm{d}t, \tag{4.6}$$

for the rv X with a fixed $g \in \mathscr{G}$. Here, both integrals are finite. Also, all rvs are assumed to be defined on the probability space $(\Omega, \mathscr{A}, \mathbb{P})$.

The definition and properties of the generalized Choquet integral provided by Kaluszka and Krzeszowiec [28] is presented as follows.

Definition 4.3.1. ***(Generalized Choquet Integral)*** *The generalized Choquet integral is defined as*

$$\mathbb{E}_{gh}\left(X\right) = \mathbb{E}_g\left[(X)_+\right] - \mathbb{E}_h\left[(-X)_+\right] \tag{4.7}$$

for $g, h \in \mathscr{G}$ and rv X.

In this definition $(X)_+$ is obtained as

$$(X)_+ = \max\{X, 0\} = \begin{cases} X & , X \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

Proposition 4.3.2. *The properties of generalized Choquet integral are given as follows:*

(C_1) $\mathbb{E}_{gh}\left(\mathbf{1}_A\right) = g(\mathbb{P}(A))$ *where $\mathbf{1}_A$ is the indicator function of the subset A of the set X,*

(C_2) $\mathbb{E}_{gh}\left(cX\right) = c\mathbb{E}_{gh}\left(X\right);\ \forall c \geq 0$,

(C_3) $\mathbb{E}_{gh}\left(-X\right) = -\mathbb{E}_{hg}\left(X\right)$,

(C_4) If $X \leq Y$, then $\mathbb{E}_{gh}\left(X\right) \leq \mathbb{E}_{gh}\left(Y\right)$,

(C_5) If $g(p) \geq p$ and $h(p) \leq p$ for $p \in [0,1]$, then $\mathbb{E}_{gh}(X) \geq \mathbb{E}(X)$,

(C_5') If $g(p) \leq p$ and $h(p) \geq p$ for $p \in [0,1]$, then $\mathbb{E}_{gh}(X) \leq \mathbb{E}(X)$,

(C_6) If $g(p) = h(p) = p$, then $\mathbb{E}_{gh}(X) = \mathbb{E}(X)$,

(C_7) $\mathbb{E}_{gh}(c) = c \; \forall c \in \mathbb{R}$,

(C_8) $\forall c \in \mathbb{R}$

$$\mathbb{E}_{gh}(X+c) = \mathbb{E}_{gh}(X) + c + \int_0^c \left[h\left(\mathbb{P}(-X > s)\right) - \bar{g}\left(\mathbb{P}(-X > s)\right)\right] \mathrm{d}s \tag{4.8}$$

and

$$\mathbb{E}_{gh}(X+c) = \mathbb{E}_{gh}(X) + c + \int_0^{-c} \left[\bar{h}\left(\mathbb{P}(X \geq s)\right) - g\left(\mathbb{P}(X \geq s)\right)\right] \mathrm{d}s, \tag{4.9}$$

*(C_9) **Jensen's inequality:** If a non-decreasing and concave $\upsilon : \mathbb{R} \to \mathbb{R}$ exists with $\upsilon(0) = 0$, then*

$$\mathbb{E}_{gh}[\upsilon(X)] \leq \upsilon[\mathbb{E}_{gh}(X)] + I_{gh}(X), \tag{4.10}$$

for $g, h \in \mathscr{G}$ and an arbitrary rv X s.t. $\mathbb{E}_{gh}(X)$ exists. Here, $I_{gh}(X)$ is defined as

$$I_{gh}(X) = \int_0^{\upsilon'\left[\mathbb{E}_{gh}(X)\right]\mathbb{E}_{gh}(X) - \upsilon\left[\mathbb{E}_{gh}(X)\right]} \left[\bar{h}\left(\mathbb{P}(\upsilon'\left[\mathbb{E}_{gh}(X)\right] X \geq s)\right) - g\left(\mathbb{P}(\upsilon'\left[\mathbb{E}_{gh}(X)\right] X \geq s)\right)\right] \mathrm{d}s$$

where υ' is the right-sided derivative of υ.

(C_9') Furthermore, if $g(p) \leq \bar{h}(p)$ or $X \geq 0$, then $\mathbb{E}_{gh}[\upsilon(X)] \leq \upsilon[\mathbb{E}_{gh}(X)]$.

Kaluszka and Krzeszowiec [28] consider that an insurance company having a reference point $w \geq 0$ wants to decide whether covering a risk $X \geq 0$ or not. Thus, losses are denoted by $(X - w)_+$ and gains are denoted by $(w - X)_+$. It is assumed that $\upsilon_1 : \mathbb{R} \to \mathbb{R}$ and $\upsilon_2 : \mathbb{R} \to \mathbb{R}$ are nondecreasing value functions for gains and losses, respectively. By the help of zero-utility premium principle, the premium to cover X denoted by π_X is obtained when the following equation is solved.

$$v_1(w) = \mathbb{E}_g\left[\left(v_1\left(w + \pi_X - X\right)_+\right)\right] - \mathbb{E}_h\left[\left(v_2\left(X - w - \pi_X\right)_+\right)\right] \tag{4.11}$$

If the value function $v(w) = v_1((w)_+) - v_2\left((-w)_+\right)$ is an increasing and continuous function s.t. $v(0) = 0$, then the premium can be calculated uniquely. Thus Equation (4.11) is rewritten in terms of generalized Choquet integral as

$$v(w) = \mathbb{E}_{gh}\left[v\left(w + \pi_X - X\right)\right] \tag{4.12}$$

The premium principle defined by Equation (4.12) is also examined in terms of its properties such as non-excessive loading, no unjustified risk loading, translation and scale invariance, additivity for comonotonic and independent risks, subadditivity, risk loading. For details, see [28].

The phenomena given in Definition 4.2.1 are taken into account for the premium principle suggested by Kaluszka and Krzeszowiec [28]. They determine π_X by using Equation (4.12) for the three different value functions which are $v(w) = cw;\ c > 0$, $v(w) = \dfrac{1 - e^{-bw}}{a};\ a, b > 0$, and $v(w) = w$. Instead of the last value function, which is a special form of the first value function, we use the typical value function suggested by Tversky and Kahneman [27] in addition to first and second value function. Therefore, by using a value function defined for losses and gains separately, we calculate the premium without additional restriction as in Equation (4.12). This value function is represented as

$$v(w) = \begin{cases} w^{\alpha} & , w \geq 0 \\ -\lambda(-w)^{\beta} & , w < 0 \end{cases} \tag{4.13}$$

Having provided the zero-utility premium principle suggested by [28], we extend this premium equilibrium for aggregate claims of a stop-loss reinsurance contract. We also use the basic value function proposed in CPT in Equation (4.13) to handle DMs tendencies according to gains and losses.

4.3.1. Determining stop-loss premium under CPT

The stop-loss premium calculation is very important for actuarial applications especially for examining the total risk of the portfolio. To do this, we firstly need to calculate the stop-loss premium by taking into account of the most accurate reference point. Eckles and Wise [30] uses Kahneman-Tversky (KT) framework, which is widely known as CPT, to examine the demand for insurance. In this study, the influence of prospect theory is investigated using two main reference points which are “initial wealth minus premium” and “initial wealth”. According to our experience and the results of this study, taking “initial wealth” as a reference point is more accurate since the amount of the premium is also important for DMs to make decisions about buying an insurance contract with paying a specified premium, so that it should not be included in reference point which determines losses and gains. Therefore, we decide gains and losses in this sense to obtain a function for stop-loss premium by using the zero-utility premium principle.

Considering the existing studies in the literature on stochastic ordering, the contribution of our study is including the impacts of risk perception. For this aim, we take the premium principle suggested in Kaluszka ve Krzeszowiec [28] as a reference. In their study, the distribution of the individual claim rv X is used to calculate the premium π_X. The premium is obtained from the insurer's perspective without considering the dependency between the individual claims of a risk class in the portfolio. However, we suggest that we should take the aggregate claim r.vector of a risk cluster in insurance portfolio as the actuarial risk to evaluate the overall risk $S^{(i)}$ of that cluster to determine the premium. Therefore, we propose to set a model for the total aggregate claim rv $S^{(i)}$ of the r.vector $\boldsymbol{S}^{(i)}$ under the dependency assumption and find the stop-loss premium $\pi_{S^{(i)}}(d)$ with a specified retention limit d. Furthermore, we aim to suggest a stop-loss dominance among risk clusters in Section 4.4. Thus, we adapt the CPT modified zero-utility premium principle from the reinsurer's perspective.

We extend the premium equilibrium for the individual claims given in Equation (4.12) to the zero-utility premium principle for aggregate claims under stop-loss reiunsurance. As in [28], we examine the phenomena presented in Definition 4.2.1, which are remarkable facts of CPT. We also consider the procedure suggested by Eckles and Wise [30] in order to obtain stop-loss premium equilibrium under zero-utility approach. In their study, optimal deductibles are suggested using KT framework. KR (Koszegi-Rabin) framework used by Barseghyan et al. [89] and Sydnor [90] for obtaining optimal deductible is criticized due to their disadvantages given as follows.

(i) Loss rv is assumed to be greater than deductible level.

(ii) It is assumed that insurance policy is bought, thus the decision of whether covering the risk or not is not examined. It can be understood from determining the reference point as "initial wealth minus premium".

(iii) It is also assumed that insurer feels loss when a claim is made, and feels gain when insured person does not make a claim, since the reference point is handled as "wealth minus premium". Here, it is assumed that individuals do not perceive premium as a loss. Instead, the reference point should rely on both wealth and the decision of buying insurance, so that the premium's amount should be influential on the determination of losses and gains, and thus the premium should not be included in reference point.

The determination of the optimal deductible is handled by Eckles and Wise [30] for a single policy. We adapt their proposed procedure to the stop-loss reinsurance by the help of the suggestion proposed by Raviv [91]. Having adapted this, an optimal policy for the individual case imposes a deductible for the aggregate claim case. To calculate the stop-loss premiums, we provide the similar framework considering reinsurer's perspective in order to be consistent with the procedure of premium calculation proposed by [28].

Suppose that the df of an aggregate claim rv S is $F_S(s)$. The stop-loss premium $\pi_S(d)$ with the retention limit d is obtained from both insurer's and reinsurer's point of views by solving Equations (4.11) or (4.12) replacing the insured with the insurer and the insurer with the reinsurer.

In order to decide losses and gains with regard to the reference point "initial wealth (w)", we firstly investigate all cases related to the size of the aggregate claim.

i. *The case that no loss occurs*:

Final wealth is $w - \pi_S(d)$ for insurer and $w + \pi_S(d)$ for reinsurer.

ii. *The case that loss occurs and is smaller than or equal to retention limit ($S \leq d$)*:

Final wealth is $w - \pi_S(d) - S$ for insurer and $w + \pi_S(d)$ for reinsurer.

iii. *The case that loss occurs and is greater than retention limit ($S > d$)*:

Final wealth is $w - \pi_S(d) - d$ for insurer and $w + \pi_S(d) - (S - d)$ for reinsurer.

(a) If $S - d \leq \pi_S(d)$, then insurer feels loss equals to $\pi_S(d) + d - S$ and reinsurer feels gain equals to $-\pi_S(d) - d + S$.

(b) If $S - d > \pi_S(d)$, then insurer feels gain equals to $-\pi_S(d) - d + S$ and reinsurer feels loss equals to $\pi_S(d) + d - S$.

As a result, considering all cases above and adapting the Equations (4.11) or (4.12), we obtain the stop-loss premium $\pi_S(d)$ as a function of w and d. The stop-loss premium equilibrium is provided in Proposition 4.3.3.

Proposition 4.3.3. *Consider that $\pi_S(d)$ is the stop-loss premium of the stop-loss reinsurance contract with the retention limit d. According to the stop-loss contract, the reinsurer will pay the amount of $S - d$ if the loss exceeds the retention limit d, and will not make any payment otherwise, i.e. $(S - d)_+$. The zero-utility equivalence for the reinsurer perspective which shows the minimum premium $\pi_S(d)$ that the reinsurer accepts to cover the risk S under the stop-loss reinsurance is given as:*

$$\upsilon(w) = \mathbb{E}_{gh}\left[\upsilon\left(w + \pi_S(d) - (S - d)_+\right)\right], \tag{4.14}$$

where υ is the value function and $g, h \in \mathscr{G}$ are the probability distortion functions.

4.4. Stop-Loss Dominance under CPT

In addition to first-order stochastic dominance, stop-loss dominance is another important ordering within the frame of POT. In this chapter, we examine impacts of risk perception using the KT framework in order to obtain stop-loss dominance under CPT. We propose a general representation of the stop-loss dominance under CPT as given in the following definition.

Definition 4.4.1. ***(Stop-loss dominance under CPT)*** *Let $S^{(i)}$ be the overall aggregate claim rv and $\pi_{S^{(i)}}(d_i)$ be the stop-loss premium of a stop-loss reinsurance contract with the optimal deductible d_i for the i-th risk cluster. Then,*

$$\pi^{CPT}_{S^{(1)}}(d_1) \leq \pi^{CPT}_{S^{(2)}}(d_2) \Rightarrow S^{(1)} \precsim^{CPT}_{sl} S^{(2)}.$$

Here, the stop-loss premiums $\pi_{S^{(i)}}(d_i)$ are obtained by Proposition 4.3.3.

In this section, we use the following value functions to find stop-loss premium solutions under CPT.

i. **Value function 1:** $v(w) = cw;\ c > 0$,

ii. **Value function 2:** $v(w) = \dfrac{1 - e^{-bw}}{a};\ a, b > 0$, and

iii. **Value function 3:** $v(w) = \begin{cases} w^{\alpha} & , w \geq 0 \\ -\lambda(-w)^{\beta} & , w < 0 \end{cases}$

By the use of Proposition 4.3.3, the minimum premium which the reinsurer accepts to cover the risk $(S - d)_+$ is determined in Equations (4.15), (4.17) and (4.19) for three different value functions. In order to obtain the solutions, we use the properties of the generalized Choquet integral, the definition and properties of df, and the characteristics of probability distortion functions and df. Since the statistical distribution for aggregate claim S varies according to data set, we prefer to present the general solutions in terms of df or survival function of S.

4.4.1. CPT stop-loss premiums

The solutions for the stop-loss premiums considering the risk perception are obtained under CPT using Proposition 4.3.3.

(i) ***CPT stop-loss premiums for Value Function 1***:

Proposition 4.4.2. *Consider that $\pi_S(d)$ is the stop-loss premium of the stop-loss reinsurance contract with the retention limit d. Let the value function*

$$v(w) = cw; \; c > 0.$$

Then, $\pi_S(d)$ is obtained as

$$\pi_S(d) = \varphi^{-1}\left(w + \int_0^\infty h\left[\mathbb{P}\left(S > d + t\right)\right] \mathrm{d}t\right) - w, \tag{4.15}$$

which is the analytical solution of Equation (4.14) for the minimum premium which the reinsurer accepts to cover the risk $(S - d)_+$. Here, φ^{-1} is the inversion of the following function:

$$\varphi(x) = \int_0^x \left\{h\left[\mathbb{P}\left(S > d + t\right)\right] + g\left[\mathbb{P}\left(S \leq d + t\right)\right]\right\} \mathrm{d}t. \tag{4.16}$$

For the derivation of the solution of Equation (4.15), see Appendix A.3.1. Also, for the proof showing that $\varphi(x)$ is invertible, see Appendix A.3.2.

(ii) ***CPT stop-loss premiums for Value Function 2***:

Proposition 4.4.3. *Consider that $\pi_S(d)$ is the stop-loss premium of the stop-loss reinsurance contract with the retention limit d. Let the value function*

$$v(w) = \frac{1 - e^{-bw}}{a}; \; a, b > 0.$$

Then, $\pi_S(d)$ is obtained as

$$\pi_S(d) = \frac{\ln\left[\varphi^{-1}(1)\right]}{b} - w, \tag{4.17}$$

which is the analytical solution of Equation (4.14) for the minimum premium which the reinsurer accepts to cover the risk $(S-d)_+$. Here, φ^{-1} is the inversion of the following function:

$$\varphi(x) = x\left(e^{-bw} - 1\right) + \int_0^x g\left[\mathbb{P}\left(S \leq \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t - \int_x^\infty h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t. \tag{4.18}$$

For the derivation of the solution of Equation (4.17), see Appendix A.3.3. Also, for the proof showing that $\varphi(x)$ is invertible, see Appendix A.3.4.

(iii) ***CPT stop-loss premiums for Value Function 3***:

Proposition 4.4.4. *Consider that $\pi_S(d)$ is the stop-loss premium of the stop-loss reinsurance contract with the retention limit d. Let the value function*

$$\upsilon(w) = \begin{cases} w^\alpha & , w \geq 0 \\ -\lambda(-w)^\beta & , w < 0 \end{cases}$$

Then, $\pi_S(d)$ is obtained as

$$\pi_S(d) = \varphi^{-1}\left(w^\alpha\right) - w - d, \tag{4.19}$$

which is the analytical solution of Equation (4.14) for the minimum premium which the reinsurer accepts to cover the risk $(S-d)_+$. Here, φ^{-1} is the inversion of the following function:

$$\varphi(x) = (x-d)^\alpha + \int_{-\infty}^{x^\alpha} g\left(\mathbb{P}\left[S < s^{1/\alpha}\right]\right) \mathrm{d}s - \int_{-\infty}^{-\lambda x^\beta} \left\{h\left(\mathbb{P}\left[S < \left(-\frac{s}{\lambda}\right)^{1/\beta}\right]\right) - 1\right\} \mathrm{d}s. \tag{4.20}$$

For the derivation of the solution of Equation (4.19), see Appendix A.3.5. Also, for the proof showing that $\varphi(x)$ is invertible, see Appendix A.3.6.

4.5. Interim Conclusion: Risk Perception in Stop-Loss Dominance

EUT, which is the most common traditional approach for modeling preferences, assumes that all decision-makers are rational and their preferences are unbiased under risk. However, this assumption is not realistic. Hence, we seek for a behavioral approach which could reflect inconsistency of decisions when individuals make in situations involving risk. In actuarial studies, decisions under risk are formulated by standard technical calculations, but it is demonstrated that DMs behave differently when outcomes are either gains or losses. Kahneman and Tversky discussed violations of EUT and proposed prospect theory in 1979. Then in 1992, they extended this theory to CPT taking into account "subjective part of risk" [9, 27]. They develop prospect theory in order to provide a cumulative representation of uncertainty. This approach provides a chance of applying both uncertain and risky prospects with any number of outcomes, and it also enables different weighting functions for gains and for losses.

In our thesis, we consider this discussion and reflect DMs' biased choices to risk prioritization. For this aim, we propose stop-loss dominance relation under CPT by obtaining stop-loss premium solutions for specified value functions. Having taken the aggregate claims of each risk cluster, we compute the stop-loss premiums by proposing an extension of the CPT modified zero-utility premium principle for aggregate claims.

5. RISK PRIORITIZATION THROUGH STOCHASTIC MAJORIZATION

5.1. Introduction

Majorization, which is an ordering relation of real-valued vectors, turns out to be a useful tool since we are interested in prioritization of aggregate claim r.vectors in this study. Because it appears within the framework of partial ordering, vectors do not need to be totally ordered, which is very convenient for our study. We will explain this situation later in Section 5.3 by presenting the differences between conditions of stochastic majorization given in Proposition 5.3.2 and conditions of first-order stochastic dominance given in Remark 5.3.1.

Order-preserving functions are very beneficial in this context since we use risk measures defined as functions to evaluate risks. A real-valued function which preserves the ordering of majorization is said to be "Schur-convex" function. Therefore, we choose a risk measure that fulfils the properties of Schur-convexity and we use it to order aggregate claims under majorization.

In order to set priorities through stochastic majorization relation, we propose a rearrangement of aggregate claim vectors. We modify the conditions of majorization ordering according to the structure of aggregate claims. By doing that, we check the convenience of our multivariate model setting introduced in Section 1.3, as well.

After introducing our data set and explaining how we organize the data, we could order some of the predetermined hazard classes using a Schur-convex risk measure under a case study. Although we could not order hazard classes entirely in this case study, it shows us that the existing hazard classification is not suitable for majorization ordering purposes. Therefore, we recluster risks through an extension of STI. Having done this case study, we also see that our model setting for risk prioritization is accurate.

This chapter aims to present a brief introduction and an application of stochastic majorization relation. For this aim, we firstly introduce the theory of majorization and Schur-convexity in Section 5.2. We give the conditions for stochastic majorization and important properties for this relation in terms of parameters in Section 5.3. In Section 5.4, we present our contribution as a modification of majorization conditions and the risk measures that we use. We introduce the data and we also give results of stochastic majorization application through a case study in Section 5.5. Finally, we conclude the chapter in Section 5.6.

5.2. Ordering Risks: Inequalities

Having introduced our model setting in Section 1.3, the following definition clearly demonstrates the convenience of the majorization relation for prioritizing aggregate claim vectors.

Definition 5.2.1. ***(Majorization)*** *"For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, the ordering $\boldsymbol{x} \precsim_{maj} \boldsymbol{y}$ denotes that $\boldsymbol{y}$ majorizes $\boldsymbol{x}$, and it is defined by Hardy et al. [92, 93] as follows:*

$$\boldsymbol{x} \precsim_{maj} \boldsymbol{y} \text{ if } \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, \dots, n-1 \\ \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases} \tag{5.1}$$

Here, $x_{[i]}$ denotes i-th component of $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ in the decreasing order, i.e. the i-th element of the vector $\boldsymbol{x} \downarrow = \left(x_{[1]}, x_{[2]}, \dots, x_{[n]}\right)$ where $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$".

The condition (5.1), which can also be expressed as "$\boldsymbol{x}$ is majorized by $\boldsymbol{y}$", is equavalent to the condition below:

$$\boldsymbol{x} \precsim_{\text{maj}} \boldsymbol{y} \text{ if } \begin{cases} \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, \quad k = 1, \dots, n-1 \\ \\ \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}. \end{cases} \tag{5.2}$$

Here, $x_{(i)}$ denotes i-th component of $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ in the increasing order, i.e. the i-th element of the vector $\boldsymbol{x} \uparrow = \left(x_{(1)}, x_{(2)}, \dots, x_{(n)}\right)$ where $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ [31].

In addition, if we use the strict inequality "$<$" instead of "$\leq$" in Equation (5.1), or "$>$" instead of "$\geq$" in Equation (5.2) for $k = 1, \dots, n-1$, the ordering is called "*strict majorization*".

5.2.1. Schur-convexity

Schur-convex functions are the most important tools of majorization. In 1923, Schur introduced the Schur-convex function that is also referred as "Schur-increasing function". Marshall et al. [31] define these functions that preserve the majorization ordering as follows.

Definition 5.2.2. *"A real function $\phi : \mathscr{A} \to \mathbb{R}$ for some set $\mathscr{A} \subset \mathbb{R}^n$ is Schur-convex on $\mathscr{A}$ if*

$$\boldsymbol{x} \precsim_{maj} \boldsymbol{y} \text{ on } \mathscr{A} \Leftrightarrow \phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y}). \tag{5.3}$$

This relation means that majorization ordering remains for the Schur-convex function values. ϕ is strictly Schur-convex on $\mathscr{A}$ if $\leq$ is replaced by $<$ in Equation (5.3) when $\boldsymbol{x}$ is not a permutation of $\boldsymbol{y}$".

Likewise, "ϕ is said to be Schur-concave on $\mathscr{A}$ if

$$\boldsymbol{x} \precsim_{maj} \boldsymbol{y} \text{ on } \mathscr{A} \Leftrightarrow \phi(\boldsymbol{x}) \geq \phi(\boldsymbol{y}).$$

ϕ is strictly Schur-convex on $\mathscr{A}$ if $\geq$ is replaced by $>$ in the above equation when $\boldsymbol{x}$ is not a permutation of $\boldsymbol{y}$".

Remark 5.2.3. *"$\phi(\boldsymbol{x})$ is Schur-convex on $\mathscr{A}$ iff $-\phi(\boldsymbol{x})$ is Schur-concave on $\mathscr{A}$".*

In order to show that a function $\phi : \mathscr{A} \to \mathbb{R}$ with $\mathscr{A} \subset \mathbb{R}^n$ is Schur-convex (Schur-concave), the following theorem is provided.

Theorem 5.2.4. ***(Schur's Condition)*** *"Consider that $\phi : \mathcal{I}^n \to \mathbb{R}$ is continuously differentiable where $\mathcal{I} \subset \mathbb{R}$ is an open interval. ϕ is Schur-convex on $\mathcal{I}^n$ if*

i. *ϕ is symmetric on $\mathcal{I}^n$, and*

ii. $(x_i - x_j)\left(\dfrac{\partial \phi}{\partial x_i} - \dfrac{\partial \phi}{\partial x_j}\right) \geq 0$ *for all $1 \leq i, j \leq n$".*

Therefore, if we are interested in $\precsim_{\text{maj}}$, then "ϕ is increasing" means "ϕ is Schur-convex".

5.3. Stochastic Majorization

Stochastic majorization is used to order rvs through the majorization relation between two r.vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$. Consider that the ϕ function on a class of well-behaved functions $\mathscr{C}$ defined on $\mathbb{R}^n$ has the property

$$\text{“}\boldsymbol{x} \precsim_{\text{maj}} \boldsymbol{y} \Leftrightarrow \phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y}) \text{ for all } \phi \in \mathscr{C}\text{”}. \tag{5.4}$$

Before introducing conditions for stochastic majorization, in order to provide a relationship between the relations $\precsim_{\text{sd}}$ and $\precsim_{\text{maj}}$, we recall properties of first-order stochastic dominance in terms of the notation to be given hereafter. The properties of the relations $\precsim_{\text{sd}}$ and $\precsim_{\text{maj}}$ are given in Remark 5.3.1 and Proposition 5.3.2, respectively.

Remark 5.3.1. *The following conditions are identical:*

(a) In univariate case, “for X and Y rvs;

- *(i)* $X \precsim_{sd} Y$,
- *(ii)* $\mathbb{P}(X > s) \leq \mathbb{P}(Y > s)$ *for all* $s \in \mathbb{R}$,
- *(iii)* $\mathbb{P}(X \leq s) \geq \mathbb{P}(Y \leq s)$ *for all* $s \in \mathbb{R}$,
- *(iv)* $\mathbb{E}\left[\psi(X)\right] \leq \mathbb{E}\left[\psi(Y)\right]$ *for all non-decreasing functions s.t. the expectations exist,*
- *(v)* $\psi(X) \precsim_{sd} \psi(Y)$ *for all non-decreasing functions* ψ,
- *(vi)* $\mathbb{P}\left\{X \in \mathcal{A}\right\} \leq \mathbb{P}\left\{Y \in \mathcal{A}\right\}$ *for all sets with non-decreasing indicator functions”.*

(b) In multivariate case, “for $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ *r.vectors;*

- *(i)* $\boldsymbol{X} \precsim_{sd} \boldsymbol{Y}$,
- *(ii)* $\mathbb{P}\left(X_1 > s_1, X_2 > s_2, \ldots, X_n > s_n\right) \leq \mathbb{P}\left(Y_1 > s_1, Y_2 > s_2, \ldots, Y_n > s_n\right)$ *for all* $\boldsymbol{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$,
- *(iii)* $\mathbb{P}\left(X_1 \leq s_1, X_2 \leq s_2, \ldots, X_n \leq s_n\right) \geq \mathbb{P}\left(Y_1 \leq s_1, Y_2 \leq s_2, \ldots, Y_n \leq s_n\right)$ *for all* $\boldsymbol{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$,

(iv) $\mathbb{E}\left[\psi(X_1, X_2, \ldots, X_n)\right] \leq \mathbb{E}\left[\psi(Y_1, Y_2, \ldots, Y_n)\right]$ *for all non-decreasing functions* $\psi : \mathbb{R}^n \to \mathbb{R}$ *s.t. the expectations exist,*

(v) $\psi(X_1, X_2, \ldots, X_n) \precsim_{sd} \psi(Y_1, Y_2, \ldots, Y_n)$ *for all non-decreasing functions* $\psi : \mathbb{R}^n \to \mathbb{R}$*,*

(vi) $\mathbb{P}\{\boldsymbol{X} \in \mathcal{A}\} \leq \mathbb{P}\{\boldsymbol{Y} \in \mathcal{A}\}$ *for all measurable sets* $\mathcal{A} \subset \mathbb{R}$ *with non-decreasing indicator functions".*

Proposition 5.3.2. ***(Stochastic Majorization Conditions)*** *For any class* $\mathscr{C}$ *defined on* $\mathbb{R}^n$ *and fulfilling (5.4), the following stochastic majorization conditions are suggested by Marshall et al. [31]. These conditions are defined as follows.*

(i) Condition $\mathbb{P}_{\mathscr{C}}$*:*

$$\text{“}\phi(\boldsymbol{X}) \precsim_{sd} \phi(\boldsymbol{Y}) \text{ for all } \phi \in \mathscr{C}\text{”} \tag{5.5}$$

where $\precsim_{sd}$ *is the first-order stochastic dominance.*

(ii) Condition $\mathbb{E}_{\mathscr{C}}$*:*

$$\text{“}\mathbb{E}\left[\phi(\boldsymbol{X})\right] \leq \mathbb{E}\left[\phi(\boldsymbol{Y})\right] \text{ for all } \phi \in \mathscr{C}\text{”} \tag{5.6}$$

s.t. all expectations exist.

We summarize the stochastic majorization between the r.vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ under the stochastic majorization conditions as:

(i) $\boldsymbol{X} \precsim_{\text{maj}}^{\mathbb{P}_{\mathscr{C}}} \boldsymbol{Y}$ means that $\boldsymbol{X}$ and $\boldsymbol{Y}$ fulfil (5.5), and

(ii) $\boldsymbol{X} \precsim_{\text{maj}}^{\mathbb{E}_{\mathscr{C}}} \boldsymbol{Y}$ means that $\boldsymbol{X}$ and $\boldsymbol{Y}$ fulfil (5.6).

Having provided Remark 5.3.1 and Proposition 5.3.2, if we consider the conditions (iv) and (v) for multivariate case in Remark 5.3.1, "ψ is non-decreasing" means that ψ is non-decreasing in each arguments separately when other arguments are fixed. Therefore, ψ is order-preserving for the ordering $\boldsymbol{x} \leq \boldsymbol{y}$, i.e. $x_i \leq y_i,$ for all i according to first-order stochastic dominance relation.

However, the conditions given in Equations (5.5) and (5.6) in Proposition 5.3.2, ϕ function does not need to be non-decreasing in each arguments separately. "ϕ is non-decreasing" means that "ϕ is Schur-convex" according to stochastic majorization relation. For this reason, stochastic majorization is more useful for our study.

The classes of ϕ are examined by Marshall et al. [31] and could be written as follows:

(i) "$\mathscr{C}_1 = \{\phi : \phi$ is a real-valued Borel measurable Schur-convex function defined on $\mathbb{R}^n\}$", and

(ii) "$\mathscr{C}_2 = \{\phi : \phi$ is a real-valued continuous, symmetric, and convex function defined on $\mathbb{R}^n\}$"

By means of these classes, the stochastic majorization conditions change into the following conditions:

(i) $\mathbb{P}_{\mathscr{C}_1} \equiv \mathbb{P}_1$: The condition where $\mathscr{C} = \mathscr{C}_1$ in (5.5), and

(ii) $\mathbb{E}_{\mathscr{C}_1} \equiv \mathbb{E}_1$: The condition where $\mathscr{C} = \mathscr{C}_1$ in (5.6).

5.3.1. Stochastic Majorization Conditions in terms of Parameters

The r.vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are ordered with stochastic majorization. These vectors often have distributions that belong to the same parametric family where the space of parameter is a subset of $\mathbb{R}^n$. We order $\boldsymbol{X}$ and $\boldsymbol{Y}$ having dfs $F_{\boldsymbol{\theta}_{\boldsymbol{X}}}$ and $F_{\boldsymbol{\theta}_{\boldsymbol{Y}}}$, respectively by stochastic majorization iff the parameter vectors $\boldsymbol{\theta}_{\boldsymbol{X}}$ and $\boldsymbol{\theta}_{\boldsymbol{Y}}$ are ordered by ordinary majorization [31].

In order to define the stochastic majorization conditions used in terms of parameters, we need to provide the following notation.

Definition 5.3.3. *"Let $\mathcal{A} \subset \mathbb{R}^n$ and let $\{F_{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \mathcal{A}\}$ be a family of n-dimensional dfs defined by a vector-valued parameter $\boldsymbol{\theta}$. The probability that $\phi(\boldsymbol{X})$ exceeds t when the df of $\boldsymbol{X}$ is $F_{\boldsymbol{\theta}}$ is denoted as*

$$\mathbb{P}_{\boldsymbol{\theta}}\left(\phi(\boldsymbol{X}) > t\right) = \int_{\{\phi(\boldsymbol{x})>t\}} dF_{\boldsymbol{\theta}}(\boldsymbol{x}),$$

and the expectation of $\phi(\boldsymbol{X})$ is denoted as

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\phi(\boldsymbol{X})\right] = \int_{\mathbb{R}^n} \phi(\boldsymbol{x}) dF_{\boldsymbol{\theta}}(\boldsymbol{x}) \text{''}.$$

The stochastic conditions for the parametric ordering can be seen in the following proposition.

Proposition 5.3.4. ***(Stochastic Majorization Conditions for Parametric Ordering)*** *Let $\mathscr{C}$ be defined on $\mathbb{R}^n$ and fulfil (5.4). Two types of stochastic majorization conditions are suggested by Marshall et al. [31]. These conditions are defined as follows.*

i. *"Condition $\mathbb{P}^*_{\mathscr{C}}$: $\mathbb{P}_{\boldsymbol{\theta}}\left(\phi(\boldsymbol{X}) > t\right)$ is Schur-convex in $\boldsymbol{\theta}$ for all $\phi \in \mathscr{C}$ and for all t".*

ii. *"Condition $\mathbb{E}^*_{\mathscr{C}}$: $\mathbb{E}_{\boldsymbol{\theta}}\left[\phi(\boldsymbol{X})\right]$ is Schur-convex in $\boldsymbol{\theta}$ for all $\phi \in \mathscr{C}$ s.t. expectations exist".*

For the sake of simplicity, we write $\mathbb{P}^*_i$ and $\mathbb{E}^*_i$ in place of $\mathbb{P}^*_{\mathscr{C}_i}$ and $\mathbb{E}^*_{\mathscr{C}_i}$, respectively where $\mathscr{C}_i, i = 1, 2$ is defined as above. Therefore, the meaning of the stochasic majorization conditions in terms of parameters given in Proposition 5.3.4 is presented in the following remark.

Remark 5.3.5. *Consider that df of $\boldsymbol{X}$ is $F_{\boldsymbol{\theta_X}}$ and df of $\boldsymbol{Y}$ is $F_{\boldsymbol{\theta_Y}}$. Then,*

i. *Condition $\mathbb{P}^*_{\mathscr{C}}$ means*

$$\boldsymbol{X} \precsim^{\mathbb{P}_{\mathscr{C}}}_{maj} \boldsymbol{Y} \text{ when } \boldsymbol{\theta_X} \precsim_{maj} \boldsymbol{\theta_Y} \tag{5.7}$$

$$\begin{aligned} &\Rightarrow \boldsymbol{X} \text{ and } \boldsymbol{Y} \text{ satisfy } \mathbb{P}_{\mathscr{C}} \text{ in (5.5) when } \boldsymbol{\theta_X} \precsim_{maj} \boldsymbol{\theta_Y}, \\ &\Rightarrow \phi(\boldsymbol{X}) \precsim_{sd} \phi(\boldsymbol{Y}) \text{ when } \boldsymbol{\theta_X} \precsim_{maj} \boldsymbol{\theta_Y}, \\ &\Rightarrow \mathbb{P}\left(\phi(\boldsymbol{X}) > t\right) \leq \mathbb{P}\left(\phi(\boldsymbol{Y}) > t\right) \text{ when } \boldsymbol{\theta_X} \precsim_{maj} \boldsymbol{\theta_Y} \text{ for all } t. \end{aligned}$$

ii. *Condition $\mathbb{E}^*_{\mathscr{C}}$ means*

$$\boldsymbol{X} \precsim^{\mathbb{E}_{\mathscr{C}}}_{maj} \boldsymbol{Y} \text{ when } \boldsymbol{\theta_X} \precsim_{maj} \boldsymbol{\theta_Y} \tag{5.8}$$

$$\begin{aligned} &\Rightarrow \boldsymbol{X} \text{ and } \boldsymbol{Y} \text{ satisfy } \mathbb{E}_{\mathscr{C}} \text{ in (5.6) when } \boldsymbol{\theta_X} \precsim_{maj} \boldsymbol{\theta_Y}, \\ &\Rightarrow \mathbb{E}\left[\phi(\boldsymbol{X})\right] \leq \mathbb{E}\left[\phi(\boldsymbol{Y})\right] \text{ when } \boldsymbol{\theta_X} \precsim_{maj} \boldsymbol{\theta_Y} \text{ s.t. expectations exist.} \end{aligned}$$

5.3.2. Schur-convexity: Compound Distributions

Since aggregate claim rv is modeled with compound distributions, it is practical to obtain the stochastic majorization conditions in terms of parameters for a family of compound distributions. As a general case for compound distribution, we can define the family of compound distributions $\{H_\lambda, \lambda \in B\}$ obtained from the families $\{F_\theta, \theta \in A\}$ and $\{G_\lambda, \lambda \in B\}$ as

$$H_\lambda(x) = \int_\theta F_\theta(x)\,\mathrm{d}G_\lambda(\theta).$$

If we need to associate the Schur-convexity conditions with compound distributions, we could consider two examples of function sets, $\mathscr{C}_1$ and $\mathscr{C}_2$. ϕ is Borel-measurable Schur-convex in the first class; and it is continuous, symmetric, and convex in the second class. Here, it should be reminded that ϕ is Schur-convex if it is symmetric and convex. So, these examples consist of different sets of Schur-convex functions.

Using the conditions given in Proposition 5.3.4 and Remark 5.3.5, the stochastic majorization conditions for the family $\{H_\lambda, \lambda \in B\}$ of compound distributions is obtained by [31] and it is given in the following table.

Table 5.1: The stochastic majorization conditions for compound distributions

Assumptions on	$\{F_\theta, \theta \in A\}$	$\mathbb{E}_1^*$	$\mathbb{E}_2^*$	$\mathbb{P}_2^*$
	$\{G_\lambda, \lambda \in B\}$	$\mathbb{E}_1^*$	$\mathbb{E}_1^*$	$\mathbb{E}_1^*$
Conclusions of	$\{H_\lambda, \lambda \in B\}$	$\mathbb{E}_1^*$	$\mathbb{E}_2^*$	$\mathbb{P}_2^*$

In Table 5.1, the conditions needed for the compound distribution family H_λ are represented under the assumptions on the conditions for distribution families F_θ and G_λ. As it can be seen from the table, the stochastic majorization conditions for compound distributions are represented in terms of the conditions for primary and secondary distributions. This table shows us that it is possible to obtain Schur-convexity for compound distributions from the properties of compounding ones.

5.3.3. Schur-Convexity for Families of Distributions

"The families of multivariate distributions parameterized by a vector $\boldsymbol{\theta}$ can hold the property that expectations of Schur-convex functions generate Schur-convex functions of $\boldsymbol{\theta}$".

Definition 5.3.6. *"The family of distributions $\{F_{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \mathcal{A} \subset \mathbb{R}^n\}$ defined on $\mathbb{R}^n$ is called "parameterized to preserve Schur-convexity" if*

$$\psi(\boldsymbol{\theta}) = \mathbb{E}_{\boldsymbol{\theta}}\left[\phi(\boldsymbol{X})\right] = \int \phi(\boldsymbol{x}) dF_{\boldsymbol{\theta}}(\boldsymbol{x})$$

is Schur-convex in $\boldsymbol{\theta} \in \mathcal{A}$ for all Schur-convex functions ϕ s.t. expectations exist".

Nevius et al. [94] prove that there are several conditions which are equivalent to the condition that Schur-convexity is preserved. These conditions are given in the following proposition.

Proposition 5.3.7. *"The following conditions are equivalent:*

(i) $\psi(\boldsymbol{\theta}) = \mathbb{E}_{\boldsymbol{\theta}}\left[\phi(\boldsymbol{X})\right]$ *is Schur-convex in $\boldsymbol{\theta}$ for all Schur-convex functions ϕ on $\mathbb{R}^n$ s.t. expectations exist,*

(ii) $\mathbb{P}_{\boldsymbol{\theta}}\left(\phi(\boldsymbol{X}) > t\right) = \int_{\{\phi(\boldsymbol{x})>t\}} dF_{\boldsymbol{\theta}}(\boldsymbol{x})$ *is Schur-convex in $\boldsymbol{\theta}$ for every Borel-measurable Schur-convex functions ϕ on $\mathbb{R}^n$,*

(iii) $\mathbb{P}_{\boldsymbol{\theta}}(\mathcal{B}) = \int_{\mathcal{B}} dF_{\boldsymbol{\theta}}(\boldsymbol{x})$ *is Schur-convex in $\boldsymbol{\theta}$ for every Borel-measurable set $\mathcal{B}$ s.t. $\boldsymbol{x} \in \mathcal{B}$, $\boldsymbol{x} \precsim_{maj} \boldsymbol{y} \Rightarrow \boldsymbol{y} \in \mathcal{B}$".*

5.4. Ordering of Agricultural Claim Data

By classifying individual claims of an agricultural insurance portfolio according to the hazard regions and crop types, we arrange the aggregate claim vectors for the hazard class $i = 1, 2, \ldots, m$ and crop class $j = 1, 2, \ldots, p_i$ with regard to our setting $\boldsymbol{S}^{(i)} = \left(S_1^{(i)}, S_2^{(i)}, \ldots, S_{p_i}^{(i)}\right)'$.

In order to compare the riskiness of the aggregate claim classes using the majorization relation, we need both a risk measure and observations represented as vectors having the majorization relation. Once we have a Schur-convex function being taken as a risk measure, we need to check if the majorization relation exists between vectors. After that, we could order the functional values of the risk measure and could infer that the riskiness of the classes can also be ordered similarly.

To prioritize the aggregate claim vectors, we firstly check if the properties for the majorization given in Definition 5.2.1 are satisfied. Then, we use a risk measure which fulfils the conditions of Schur-convexity provided in Theorem 5.2.4.

Risk measures are classified into two types as (i) safety risk measures evaluating wealth under risk and (ii) dispersion measures assessing the uncertainty level [95]. Since our aim is to associate ordering notion with “decision under uncertainty” in this thesis, we choose the second class of risk measures.

In the first phase, we use the sample variance and the sample coefficient of variation as risk measures, which is one of the main dispersion measures. These measures are dispersion measures used for ordering the aggregate claim vectors of m hazard classes through majorization relation. In order to do that, we set the prioritization of the aggregate claim vectors as given in the following proposition.

Proposition 5.4.1. *Let $\boldsymbol{S}^{(k)}$ and $\boldsymbol{S}^{(l)}$ be aggregate claim vectors and $\mathbb{V}\left(\boldsymbol{S}^{(k)}\right)$ and $\mathbb{V}\left(\boldsymbol{S}^{(l)}\right)$ be variance vectors of k-th and l-th hazard classes, respectively. The majorization relation between these two hazard classes is given as follows:*

$$\boldsymbol{S}^{(k)} \precsim_{maj} \boldsymbol{S}^{(l)} \Leftrightarrow \phi\left(\mathbb{V}\left(\boldsymbol{S}^{(k)}\right)\right) \leq \phi\left(\mathbb{V}\left(\boldsymbol{S}^{(l)}\right)\right) \tag{5.9}$$

where ϕ is a Schur-convex function.

We order the aggregate claim vectors of the k-th and l-th hazard classes considering an ordering relation between the function values of their variance vectors.

5.4.1. Rearrangement of the aggregate claim vectors

According to our main setting in Equations (5.3) and (5.9), we rewrite the majorization definition as follows:

$$\boldsymbol{S}^{(k)} \precsim_{\text{maj}} \boldsymbol{S}^{(l)} \text{ if } \begin{cases} \sum_{t=1}^{j} S_{[t]}^{(k)} \leq \sum_{t=1}^{j} S_{[t]}^{(l)}; \quad j = 1, \ldots, p_i - 1 \\ \\ \sum_{t=1}^{p_i} S_{[t]}^{(k)} = \sum_{t=1}^{p_i} S_{[t]}^{(l)} \end{cases} \tag{5.10}$$

In Equation (5.10), we arrange the aggregate claim vectors by sorting them into descending order as

$$\boldsymbol{S}^{(k)} \downarrow = \left(S_{[1]}^{(k)}, S_{[2]}^{(k)}, \ldots, S_{[p_i]}^{(k)} \right),$$

where $S_{[1]}^{(k)} \geq S_{[2]}^{(k)} \geq \ldots \geq S_{[p_i]}^{(k)}$. Here, $S_{[t]}^{(k)}$ denotes the t-th element of the decreasing rearrangement of $\boldsymbol{S}^{(k)}$.

The aggregate claim vector $\boldsymbol{S}^{(k)}$ for the k-th hazard class is majorized by the aggregate claim vector $\boldsymbol{S}^{(l)}$ for the l-th hazard class if two conditions are satisfied. It is obvious that the condition $\sum_{t=1}^{p_i} S_{[t]}^{(k)} = \sum_{t=1}^{p_i} S_{[t]}^{(l)}$ is very unlikely to be fulfilled because aggregate claims are continuous rvs. Thus, we suggest to redefine the majorization relation in order to overcome this problem as follows:

$$\frac{\boldsymbol{S}^{(k)}}{\sum_{t=1}^{p_i} S_{[t]}^{(k)}} \precsim_{\text{maj}} \frac{\boldsymbol{S}^{(l)}}{\sum_{t=1}^{p_i} S_{[t]}^{(l)}} \text{ if } \begin{cases} \dfrac{\sum_{t=1}^{j} S_{[t]}^{(k)}}{\sum_{t=1}^{p_i} S_{[t]}^{(k)}} \leq \dfrac{\sum_{t=1}^{j} S_{[t]}^{(l)}}{\sum_{t=1}^{p_i} S_{[t]}^{(l)}}; \quad j = 1, \ldots, p_i - 1 \\ \\ \dfrac{\sum_{t=1}^{p_i} S_{[t]}^{(k)}}{\sum_{t=1}^{p_i} S_{[t]}^{(k)}} = \dfrac{\sum_{t=1}^{p_i} S_{[t]}^{(l)}}{\sum_{t=1}^{p_i} S_{[t]}^{(l)}} \end{cases} \tag{5.11}$$

To obtain Equation (5.11), we divide all the elements in Equation (5.10) by the summations. Now, we have two conditions to be checked in order to show that $\boldsymbol{S}^{(k)}$ is majorized by $\boldsymbol{S}^{(l)}$. These conditions can be given in the following remark.

Remark 5.4.2. ***(Majorization for aggregate claim vectors)*** *The aggregate claim vector $\boldsymbol{S}^{(k)}$ for the k-th hazard class is majorized by the aggregate claim vector $\boldsymbol{S}^{(l)}$ for the l-th hazard class if the following conditions are fulfilled:*

(i) $\dfrac{\sum_{t=1}^{j} S_{[t]}^{(k)}}{\sum_{t=1}^{p_i} S_{[t]}^{(k)}} \leq \dfrac{\sum_{t=1}^{j} S_{[t]}^{(l)}}{\sum_{t=1}^{p_i} S_{[t]}^{(l)}}; j = 1, \ldots, p_i - 1$

(ii) $\sum_{t=1}^{p_i} S_{[t]}^{(k)} \geq \sum_{t=1}^{p_i} S_{[t]}^{(l)}$

5.4.2. Schur-convexity of a risk measure

Since our data set is big enough, we use the sample variance and the sample coefficient of variation reflecting the variability of the data. Marshall et al. [31] show that the variance function $\phi_1(\boldsymbol{x})$ and the coefficient of variation $\phi_2(\boldsymbol{x})$ is strictly Schur-convex as given in the following propositions.

Proposition 5.4.3. *The sample variance defined as*

$$\phi_1(\boldsymbol{x}) = \phi_1(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

is strictly Schur-convex wrt $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$.

For the proof, see Appendix A.4.1

Proposition 5.4.4. *The sample coefficient of variation defined as*

$$\phi_2(\boldsymbol{x}) = \phi_2(x_1, x_2, \ldots, x_n) = \frac{[\phi_1(\boldsymbol{x})]^{1/2}}{\bar{x}}$$

is strictly Schur-convex wrt $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^+$.

For the proof, see Appendix A.4.2

5.5. Case Study: Ordering Existing Hazard Classes through Majorization

According to 2016's annual report of Turkish Agricultural Insurance Pool (TARSİM), which is a government-backed agricultural insurance system, Turkey has first and seventh biggest agronomy in the EU and the world, respectively. In terms of external economy, Turkey also exports a huge variety of agricultural products which makes the country have a competitive agronomy on a large international scale [96]. On the other hand, the agricultural industry contributes to economy with a rate of 6.1% in the Turkey's gross domestic product. Economically, taking into account all of above mentioned, agricultural functions could remain effective only through a sufficient risk management. For these reasons, we use agricultural insurance data in our thesis. We work with a unique agricultural claims data set covering more than 100 products for 5 different hazard types provided by TARSİM.

In order to do the case study, we first organize the data set according to our multivariate model setting given in Equation (1.1). Hazard regions are already determined based on the villages of the policyholders and the crop sensitivity classes are determined according to each hazard type. After obtaining aggregate claim vectors, we first examine the hazard classes in terms of their fulfilment of the majorization conditions. When we apply the majorization relation in the case study, we show that the Schur-convexity of the sample variance as a risk measure is useful for our risk prioritization purposes.

5.5.1. Data

TARSİM is a governmental institution taking the responsibility for the development of agricultural insurance in Turkey. TARSİM has the insurance lines such as crop insurance, district based drought yield insurance, greenhouse insurance, cattle insurance, sheep and goat insurance, poultry insurance, aquaculture insurance and beehive insurance. Since 94.8% of the policies of agricultural products are crop insurance policies in 2014, we focus on crop insurance [97]. Crop insurance covers the products exposed to various sources of risk such as hail, frost, storm and flood. Within the crop insurance products, 42.8% and 50.4% of the causes of the paid losess are hail and frost, respectively. Also, the frost hazard is covered together with the hail hazard. Thus, we study crop-hail insurance in order to analyze risks using a data set which is big enough.

After we omit the cancelled policies from the recorded data set, we have 975,971 crop policies (including zero claims) caused by hail in 2014. Since the information about the policyholders is not shared by TARSİM due to the privacy terms, we need to obtain the individual claims through village codes and lot-block codes, together. After merging the same village codes and lot-block codes, the number of policies decreased to 831,325. Among these policies, the number of the recorded claims arised from hail hazard (including zero claim amounts) is 54,113.

There are 23 hazard classes and 18 crop classes. We arrange the aggregate claim vectors according to the setting $\boldsymbol{S}^{(i)} = \left(S_1^{(i)}, S_2^{(i)}, \ldots, S_{p_i}^{(i)}\right)'$ for $i = 1, 2, \ldots, 23$ and $p_i = 18$. In order to check if these hazard classes are suitable for our risk prioritization purposes through majorization, we firstly examine the hazard classes in terms of their fulfillment of the majorization conditions. It is seen from the results that the existing hazard classification is not acceptable. Only 4 classes among all hazard classes seem to be majorized. In order to test our model setting, we do this case study using these classes which are suitable for the definition of the majorization relation. It is obtained that there is a strict majorization relation among the 4 classes.

At first, the majorization relation is checked among these classes. According to Remark 5.4.2, only 4 classes out of 23 hazard classes are suitable for our case study. The following results are obtained:

(i) $\dfrac{\sum_{t=1}^{j} S_{[t]}^{(8)}}{\sum_{t=1}^{18} S_{[t]}^{(8)}} < \dfrac{\sum_{t=1}^{j} S_{[t]}^{(9)}}{\sum_{t=1}^{18} S_{[t]}^{(9)}} < \dfrac{\sum_{t=1}^{j} S_{[t]}^{(19)}}{\sum_{t=1}^{18} S_{[t]}^{(19)}} > \dfrac{\sum_{t=1}^{j} S_{[t]}^{(20)}}{\sum_{t=1}^{18} S_{[t]}^{(20)}};\ j = 1, 2, \ldots, p_i - 1$

(ii) $\sum_{t=1}^{18} S_{[t]}^{(8)} > \sum_{t=1}^{18} S_{[t]}^{(9)} > \sum_{t=1}^{18} S_{[t]}^{(19)} > \sum_{t=1}^{18} S_{[t]}^{(20)}$

$$\Rightarrow \boldsymbol{S}^{(8)} \prec_{\text{maj}} \boldsymbol{S}^{(9)} \prec_{\text{maj}} \boldsymbol{S}^{(19)} \prec_{\text{maj}} \boldsymbol{S}^{(20)}$$

After that, we need to calculate the risk measure values related to sample variances for each hazard class. If same ordering exists among the values $\phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(k)}\right)\right)$, then we can conclude our case study by stating;

i. Our majorization setting for aggregate claims given in (5.9) to apply the Schur-convexity of the sample variance is accurate.

ii. The majorization ordering among the aggregate claim r.vectors related to their hazard regions exists for the riskiness of these classes, as well.

The risk measure values related to the sample variance of the classes that we choose for the case study are given in Table 6.2.

Table 5.2: The sample variance values of the variance of aggregate claim vectors

Risk Class	$\phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(k)}\right)\right)(*10^{13})$
k=8	0.008659
k=9	0.015409
k=19	0.027544
k=20	0.047684

It can be seen from Table 5.2 that the same ordering direction exists among hazard classes. Therefore, we can conclude that the 8-th class is the least risky and the 20-th class is the most risky class. The results from the least risky class to the most risky class can be given as $RC_8 \rightarrow RC_9 \rightarrow RC_{19} \rightarrow RC_{20}$ where RC_i denotes the riskiness of the risk class i.

It is seen from the results of all hazard classes that the existing classification is not acceptable. In order to prioritize the agricultural claims accurately, we should cluster actuarial risks of the agricultural insurance according to spatial and temporal characteristics of hazard regions. In order to do that, the dates of crop-hail insurance claims and meteorological data related to claim dates are obtained in addition to this data set.

5.6. Interim Conclusion: Majorization Analysis on Agricultural Claims Insurance

In this chapter, we aim to order aggregate claim vectors related to hazard classes. Using the data set of an agricultural insurance portfolio, hazard classes are ordered by taking the advantage of majorization properties and the use of Schur-convexity.

We propose a rearrangement of aggregate claim vectors in order to construct these vectors according to majorization conditions. If these conditions are fulfilled, we only need a Schur-convex function which could be taken as a risk measure. In our case study, this risk measure is sample variance. Even though all hazard classes could not be ordered, the case study provides us a significant inference. The results show that our multivariate model setting is convenient for majorization purposes, and it is possible to conduct an entire prioritization of the portfolio's actuarial risks with an accurate clustering which is discussed in the following chapter.

6. RISK CLUSTERING THROUGH SPATIOTEMPORAL INTERPOLATION

6.1. Introduction

Environmental risk management is different from actuarial and statistical inference of risks. Weather-related hazards have more influence on agricultural claims compared to other lines of non-life insurance. Therefore, it is necessary to consider the geographical characteristics of the portfolio. In this sense, the meteorological features of agricultural claims should be taken into account during risk assessment. Agricultural risk clustering is very useful when we need to group claims according to environmental characteristics especially for big data. In this study, since we have no meteorological information on the exact location of the claim, we estimate the approximate values of unknown variables using observed values through efficient interpolation techniques. For measuring the impacts of their environmental attributes, spatial interpolation techniques are very functional and appropriate in GIS applications [98–101]. As an extension of spatial interpolation techniques, STI techniques use 2-dimensional location and time points. Most commonly-used STI algorithms are kriging and inverse distance weighting (IDW) method.

Agricultural risk assessment is a specific area which evaluates risks in geographical framework. In order to prioritize weather-related risks, we cluster agricultural claims according to spatial and temporal characteristics of hazard regions by using *model-based clustering* techniques. For this aim, *IDW method with reduction approach* is used to estimate meteorological values related to the location and the time of the reported agricultural claims. Since height has a very significant influence on meteorological variables, altitude values are used to decide the optimal sample set. In order to find the optimal sample set for STI technique, a stochastic optimization algorithm is considered. We extend stochastic *DE optimization algorithm* changing one-dimensional distance-based approach to a multidimensional setting which suggests an angular-based distance computation. The closeness of estimated and actual altitudes determines the optimal location pairs. Moreover, we propose a solution to choose the population size of DE algorithm for the initialization step.

DE algorithm has some superior features such as straightforward execution, providing dependable, effective and robust results [102], hence it is preferred in many studies such as solvency requirements in insurance [103], mechanical and chemical engineering [104, 105], real-life applications [106], geophysics [107], artificial neural networks [108] and so on. On the other hand, in model-based clustering approach, the data is viewed as being generated from a mixture of probability distributions, each representing a different cluster [109].

The chapter is organized as follows: In Section 6.2, we introduce the extended DE algorithm and IDW method with reduction technique. We also explain our contributions in this section. We present the risk clustering approach in Section 6.3. In Section 6.4, we introduce the data set and we explain what we have done to reorganize the data in addition to the case study in Section 5.5.1. We also present the results of STI through DE optimization, risk clustering and stochastic majorization in this section. Finally in Section 6.5, we give concluding remarks of this chapter.

6.2. Spatiotemporal Techniques

Many spatial interpolation methods are summarized and grouped as IDW and extensions of IDW, clustering assisted regression, kriging and extensions of kriging, splines and extensions of splines, random forest and mixtures of these methods etc. [32]. The common assumption of these techniques is that sample points near the location to be estimated result in strong influence on estimation [110, 111].

As a method estimating the unknown value of a variable through its location and time, STI has become very popular in GIS applications as an extension of spatial interpolation techniques [112–116]. Therefore, it is a very useful approach to estimate meteorological quantities of a claim. It is claimed that considering both location and time provides more reliable estimations [117].

Even though kriging appears as the most appropriate technique among STI algorithms, it does not perform well computationally. For this reason, IDW is preferred [32]. Li and Heap [118] and Burrough and McDonnell [119] claim that any spatial interpolation method performs better according to the sample design, form and features of the data or factors affecting the sampling [120]. As suggested in [32], we choose to use IDW method with reduction technique in our thesis for the interpolation of unknown meteorological values related to the reported agricultural claims.

6.2.1. Optimization Problem

In this chapter, we explain our extension for DE algorithm after giving the optimization problem. In comparison with DE algorithm, random optimization methods have a drawback of choosing the sample points randomly which causes that the location of selected points might converge each other, so that the interpolation does not yield reasonable results especially for small samples. However, DE algorithm overcomes this disadvantage by selecting the sample points with an optimization criterion instead of just randomizing [32]. When one deals with the choice of the variable in a sample for optimization purposes specifically in agricultural insurance, geographical factors have high impact on estimation process. According to Capozzi et al. [121], mid-altitudes affect the probability of the occurance of hail hazard. Hence, we use altitude data to calculate the fitness value which is used to obtain the optimal sample set according to the distance between estimated and actual values.

The optimization problem is defined as maximizing the leave-one-out cross-validation (LOOCV) error of the $(k-1)$-dimensional sample when k-th element is left out, so that k-th element is the best estimator [32]. In our study, LOOCV error, which represents the total absolute error when k-th element is left out, is defined as

$$\mathrm{e}_k^{\text{abs}}\left(\boldsymbol{x}_{-k}\right) = \sum_{i \neq k} \left|\hat{x}_i - o_i\right|, \tag{6.1}$$

where e_k^{abs} is the total absolute error, $\hat{x}_i$ denotes STI's estimated value at location x_i using the remaining set $\boldsymbol{x}_{-k} = \boldsymbol{x} \backslash \{x_k\}$, and o_i is the observed value at location x_i.

IDW method with reduction technique needs 4 sample points to interpolate the unknown value. The set of remaining $(n-4)$ points is the initial set of DE algorithm where the entire sample set is $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ for univariate case.

In our case, we first choose m closest sample locations $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(m)}$ and then we obtain all possible 4-subsets out of the set of m distinct sample points. We aim to find the optimal subset which minimizes the value of the cost function given in Equation (6.3).

Let $\boldsymbol{X}$ be the input of the optimization including $N = \binom{m}{4}$ elements, i.e. $\boldsymbol{X} = [\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_N]$. Here, $\{\boldsymbol{X}_k;\ k = 1, 2, \ldots, N\}$ represents the k-th combination from the set $\left\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(m)}\right\}$ and it is defined as

$$\boldsymbol{X}_k = [\boldsymbol{x}_{k,1}; \boldsymbol{x}_{k,2}; \boldsymbol{x}_{k,3}; \boldsymbol{x}_{k,4}]\,;\ k = 1, 2, \ldots, N$$

where $[\boldsymbol{x}_{k,1}; \boldsymbol{x}_{k,2}; \boldsymbol{x}_{k,3}; \boldsymbol{x}_{k,4}] = \left[\boldsymbol{x}^{(j_{k,1})}; \boldsymbol{x}^{(j_{k,2})}; \boldsymbol{x}^{(j_{k,3})}; \boldsymbol{x}^{(j_{k,4})}\right]$ with $j_{i,D} \in \{1, 2, \ldots, m\}$ for $D = 1, 2, 3, 4$. Here, $\boldsymbol{X}_k$ is the (4×2)-dimensional matrix consisting of locations $\boldsymbol{x}_{k,j}$ having both latitudes $\left(\text{lat}_{\boldsymbol{x}_{k,j}}\right)$ and longitudes $\left(\text{lon}_{\boldsymbol{x}_{k,j}}\right)$ in each row.

In addition to LOOCV error, Susanto et al. [32] use another measure called "sparsity" within the cost function. The sparsity is actually the coefficient of variation (Coef.Var.) of pair-wise distances of m closest sample points. Let pair-wise distance vector be

$$\Delta_{\boldsymbol{X}^{\text{pw}}} = \left(\Delta_{\boldsymbol{x}^{(1)};\boldsymbol{x}^{(2)}}, \Delta_{\boldsymbol{x}^{(1)};\boldsymbol{x}^{(3)}}, \ldots, \Delta_{\boldsymbol{x}^{(1)};\boldsymbol{x}^{(m)}}, \ldots, \Delta_{\boldsymbol{x}^{(m-1)};\boldsymbol{x}^{(m)}}\right)$$

where $\left\{\Delta_{\boldsymbol{x}^{(j)};\boldsymbol{x}^{(j')}};\ j \neq j' \in 1, 2, \ldots, m\right\}$ represents the Euclidean distance between the location $\boldsymbol{x}^{(j)} = (\text{lat}_{\boldsymbol{x}^{(j)}}, \text{lon}_{\boldsymbol{x}^{(j)}})$ and $\boldsymbol{x}^{(j')} = (\text{lat}_{\boldsymbol{x}^{(j')}}, \text{lon}_{\boldsymbol{x}^{(j')}})$. The Coef.Var. is calculated as

$$\mathbb{CV}\left(\Delta_{\boldsymbol{X}^{\text{pw}}}\right) = \sigma\left(\Delta_{\boldsymbol{X}^{\text{pw}}}\right) / \mu\left(\Delta_{\boldsymbol{X}^{\text{pw}}}\right), \tag{6.2}$$

where $\sigma\left(\Delta_{\boldsymbol{X}^{\text{pw}}}\right)$ and $\mu\left(\Delta_{\boldsymbol{X}^{\text{pw}}}\right)$ are the standart deviation and the mean of the vector $\Delta_{\boldsymbol{X}^{\text{pw}}}$, respectively. Therefore, the optimization problem turns out to be a minimization problem as

$$\text{cost}_k = \left[\text{e}_k^{\text{abs}}\left(\boldsymbol{X}_{-k}\right)\right]^{-1} \mathbb{CV}\left(\Delta_{\boldsymbol{X}^{\text{pw}}}\right);\ k = 1, 2, \ldots, N. \tag{6.3}$$

where $\text{e}_k^{\text{abs}}\left(\boldsymbol{X}_{-k}\right)$ is the LOOCV error of the $(k-1)$-dimensional sample when k-th element is left out, and $\boldsymbol{X}_{-k} = \boldsymbol{X} \setminus \{\boldsymbol{X}_k\}$ is the "leave-one-out" set used as the input for DE algorithm. To sum up, our purpose is to minimize cost_k, so that k-th sample set is the optimal sample set for the interpolation.

6.2.2. Differential Evolution Algorithm: An Extension

The minimization problem defined in Equation (6.3) is solved by DE algorithm. This algorithm helps us to optimize non-linear and non-differentiable continuous space functions. According to Storn and Price [33], an optimization method must

(i) be able to achieve to optimize non-linear, non-differentiable and multimodal cost functions efficiently;

(ii) be suitable for parallel computing when it is difficult to calculate cost function fast;

(iii) be practical (not having too many control variables);

(iv) yield fast and consistent convergence to the global minimum.

As a result of these conditions, Storn and Price [33] propose DE algorithm to fulfil these necessities.

The most useful part of DE algorithm is that the gradient of the cost function is not a requisite since it is a direct search method [122, 123]. Direct search optimization algorithms work for finding a set around the point related to unknown value. Within this set of points, the one that yields a lower cost function value compared to others appears to be the optimal point. However, other optimization methods require first or higher derivatives of the cost function. Hence, direct search methods are more convenient when the cost function is non-differentiable [124]. Moreover, this algorithm is a stochastic parallel method using several processors to calculate the optimization problem fast.

In univariate case, the optimal vector is chosen among M vectors. M is the population size and $\boldsymbol{x}_{i,G}; i = 1, 2, \ldots, M$ is the D-dimensional target vector and it is used as G-th generation's population vector. The initial D-dimensional parameter population is determined randomly from M vectors for each generation, but the population size M remains the same [33].

The main steps of DE algorithm are initialization, mutation, crossover, and selection:

(i) In the "initialization" step, three parameter populations are determined randomly from Uniform $[1, M]$.

(ii) A transformed vector $\boldsymbol{v}_{i,G+1}$ is obtained in the "mutation" step. In univariate case, this vector is mutated by adding a weighted difference between two random population vectors multiplied to random base vector. As an extension of classical DE algorithm, where the elements in the parameter vector are one-dimensional, we adapt this calculation to the multidimensional case, i.e. the elements in the parameter vector are two-dimensional, which makes the parameter vector appear as a "parameter matrix".

(iii) In the "crossover" step, the *trial vector* $\boldsymbol{u}_{i,G+1}$ is formed by comparing pre-arranged *target vector* $\boldsymbol{x}_{i,G}$ and the transformed vector $\boldsymbol{v}_{i,G+1}$ wrt to a criterion given in Equation 6.7.

(iv) For the last "selection" step, the algorithm selects the trial vector or the target vector according to their cost function values.

DE algorithm's procedure is given in detail as follows:

- **Initialization and Mutation:** In univariate case, a vector is mutated for each D-dimensional target vector $\boldsymbol{x}_{i,G};\ i = 1, 2, \ldots, M$ of the generation G as

$$\boldsymbol{v}_{i,G+1} = \boldsymbol{x}_{r_1,G} + (\boldsymbol{x}_{r_2,G} - \boldsymbol{x}_{r_3,G})\, F \tag{6.4}$$

where $\boldsymbol{x}_{r_1,G}$ is the random base vector, $\boldsymbol{x}_{r_2,G}$ and $\boldsymbol{x}_{r_3,G}$ are the random parameter vectors with random integers r_1, r_2, r_3 and $F \in \mathbb{R}$ is a constant. Here, $r_k \in \{1, 2, \ldots, M\}\, ; k = 1, 2, 3$ are different from each other and also from i, so that $M \geq 4$. $F \in [0, 2]$ determines the amount for the increase of the difference $(\boldsymbol{x}_{r_2,G} - \boldsymbol{x}_{r_3,G})$. The choice of the values of M and F is discussed in detail in Section 6.2.3.

For the sake of simplicity, the "leave-one-out" set $\boldsymbol{X}_{-k} = \boldsymbol{X} \setminus \{\boldsymbol{X}_k\}$ is redefined as a population set given as

$$\boldsymbol{X}_{\text{pop}} = \{\boldsymbol{X}_{1,G}, \boldsymbol{X}_{2,G}, \ldots, \boldsymbol{X}_{M,G}\}$$

where $\boldsymbol{X}_{i,G};\ i = 1, 2, \ldots, M = N - 1$ is the i-th target matrix of the generation G. Here, $N = \binom{m}{4}$ is the dimension of combination subset of m closest points.

The first multidimensional extension of our study is in the initialization step. 3 parameter matrices are chosen from the population set $\boldsymbol{X}_{\text{pop}}$. The initialization set consisting of three $(D \times 2)$-dimensional $\boldsymbol{X}_{r_k,G}$ parameter matrices all of which includes $\boldsymbol{x}_{r_k,j,G}$ pairs for $j = 1, 2, \ldots, D$ is obtained as

$$\boldsymbol{X}_{i,G}^{(\text{init})} = \{\text{The set of } \boldsymbol{X}_{r_k,G};\ r_k \in \{1, 2, \ldots, M\} \text{ for } k = 1, 2, 3\}$$

where $r_1 \neq r_2 \neq r_3 \neq i$ are random integers are Uniform $[1, M]$-distributed. Here, $\boldsymbol{X}_{r_k,G}$ is the r_k-th element of the population set $\boldsymbol{X}_{\text{pop}} = \boldsymbol{X}_{-k} = \boldsymbol{X} \setminus \{\boldsymbol{X}_k\}$. Also, $\boldsymbol{X}_{i,G} = [\boldsymbol{x}_{i,1,G}; \boldsymbol{x}_{i,2,G}; \ldots \boldsymbol{x}_{i,D,G}]$ has D location pairs consisting of latitude $\left(\text{lat}_{\boldsymbol{x}_{i,j,G}}\right)$ and longitude $\left(\text{lon}_{\boldsymbol{x}_{i,j,G}}\right)$ values.

Adding a weigted Euclidean distance between the nodes $\boldsymbol{x}_{r_2,j,G}$ and $\boldsymbol{x}_{r_3,j,G}$, which is indicated as $\Delta_{\boldsymbol{x}_{r_2,j,G};\boldsymbol{x}_{r_3,j,G}}$, to $\boldsymbol{x}_{r_1,j,G}$ for each $j = 1, 2, \ldots, D$, we adapt the distance calculation from the univariate case of DE algorithm shown in Equation (6.4) to 2-dimensional case. In other words, this angular-based extension can be expressed through $\boldsymbol{X}_{i,G}^{(\text{init})}$ where $\boldsymbol{X}_{r_1,G}$ is the random base matrix, $\boldsymbol{X}_{r_2,G}$ and $\boldsymbol{X}_{r_3,G}$ are the random parameter matrices with random integers r_1, r_2, r_3. Here, it is assumed that the distance is calculated linearly and also that α, the angle between x-axis and the line joining the origin $(0, 0)$ and the point $\boldsymbol{x}_{i,j,G} = \left(\text{lat}_{\boldsymbol{x}_{i,j,G}}, \text{lon}_{\boldsymbol{x}_{i,j,G}}\right)$ does not change for the next amplification. Therefore, the extended generation of the mutant matrix is obtained as follows:

$$\text{lat}_{\boldsymbol{v}_{i,j,G+1}} = \text{lat}_{\boldsymbol{x}_{r_1,j,G}} + \left(\sin\left[\arctan\left(\frac{\text{lat}_{\boldsymbol{x}_{i,j,G}}}{\text{lon}_{\boldsymbol{x}_{i,j,G}}}\right)\right] \Delta_{\boldsymbol{x}_{r_2,j,G};\boldsymbol{x}_{r_3,j,G}}\right) F \tag{6.5}$$

for the latitude of $\boldsymbol{v}_{i,G+1}$, and

$$\text{lon}_{\boldsymbol{v}_{i,j,G+1}} = \text{lon}_{\boldsymbol{x}_{r_1,j,G}} + \left(\cos\left[\arctan\left(\frac{\text{lat}_{\boldsymbol{x}_{i,j,G}}}{\text{lon}_{\boldsymbol{x}_{i,j,G}}}\right)\right] \Delta_{\boldsymbol{x}_{r_2,j,G};\boldsymbol{x}_{r_3,j,G}}\right) F \tag{6.6}$$

for the longitude of $\boldsymbol{v}_{i,j,G+1}$. Therefore, the revised form of the mutant matrix is

$$\boldsymbol{v}_{i,G+1} = \left[\left(\text{lat}_{\boldsymbol{v}_{i,1,G+1}}, \text{lon}_{\boldsymbol{v}_{i,1,G+1}}\right); \left(\text{lat}_{\boldsymbol{v}_{i,2,G+1}}, \text{lon}_{\boldsymbol{v}_{i,2,G+1}}\right); \ldots; \left(\text{lat}_{\boldsymbol{v}_{i,D,G+1}}, \text{lon}_{\boldsymbol{v}_{i,D,G+1}}\right)\right]$$

as a $(D \times 2)$-dimensional matrix. For the j-th element of the target matrix $\boldsymbol{x}_{i,j,G}$, this operation is shown in Figure 6.1.

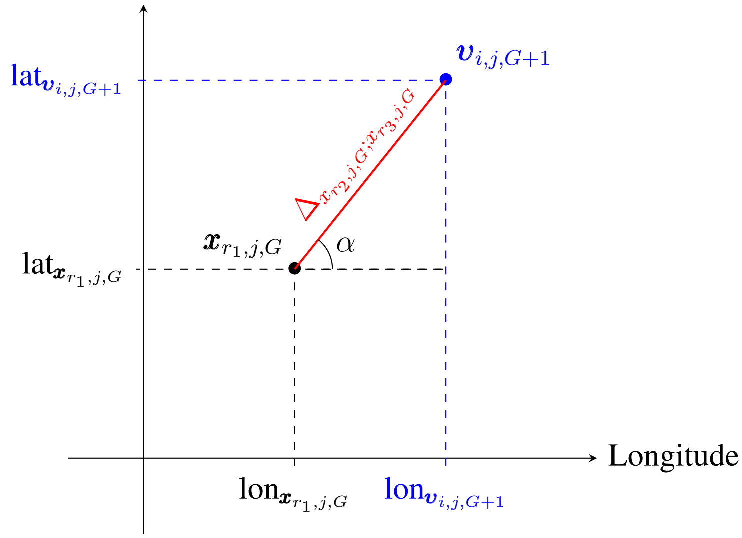


Figure 6.1: Mutation of $\boldsymbol{v}_{i,j,G+1}$ for the target matrix $\boldsymbol{X}_{i,G}$

- **Crossover:** The crossover step is necessary to ensure that both the initialization and mutation processes are heterogeneous and random [33]. The j-th element of trial matrix $\boldsymbol{u}_{i,G+1}$ is obtained from

$$\boldsymbol{u}_{i,j,G+1} = \begin{cases} \boldsymbol{v}_{i,j,G+1} & \text{if } \{u_j \leq \text{CR}\} \vee \{j = r_i\} \\ \boldsymbol{x}_{i,j,G} & \text{if } \{u_j > \text{CR}\} \wedge \{j \neq r_i\} \end{cases} ; j = 1, 2, \ldots, D \qquad (6.7)$$

where $u_j \in [0, 1]$ is a uniform random number generated for each j and CR $\in [0, 1]$ is a constant chosen by the user to determine the crossover form. Here, the trial matrix is also formed from latitudes and longitudes as $\boldsymbol{u}_{i,j,G+1} = \left(\text{lat}_{\boldsymbol{u}_{i,j,G+1}}, \text{lon}_{\boldsymbol{u}_{i,j,G+1}}\right)$ for each $j = 1, 2, \ldots, D$. The choice of the values for CR is discussed in detail in Section 6.2.3. r_i is a randomly chosen index in $\{1, 2, \ldots, D\}$ which ensures that $\boldsymbol{u}_{i,G+1} =$

$(\boldsymbol{u}_{i,1,G+1}; \boldsymbol{u}_{i,2,G+1}; \ldots; \boldsymbol{u}_{i,D,G+1})$ takes at least one element from $\boldsymbol{v}_{i,G+1}$.

- **Selection:** In this step, we determine the $(G+1)$-th generation's target matrix. For this aim, the fitness function values of both the trial matrix $\boldsymbol{u}_{i,G+1}$ and the target matrix $\boldsymbol{x}_{i,G}$ are compared. Naming these function values as "score value" and "cost value", score $(\boldsymbol{u}_{i,G+1})$ and cost $(\boldsymbol{x}_{i,G})$ are calculated by Equation (6.3). The next generation's target matrix is obtained as

$$\boldsymbol{x}_{i,j,G+1} = \begin{cases} \boldsymbol{u}_{i,j,G+1} & ;\text{ if score}\,(\boldsymbol{u}_{i,G+1}) \leq \text{cost}\,(\boldsymbol{x}_{i,G}) \\ \boldsymbol{x}_{i,j,G} & ;\text{ otherwise} \end{cases} \quad ; j = 1, 2, \ldots, D. \tag{6.8}$$

This equation shows that the trial matrix replaces the target matrix in the next generation if score $(\boldsymbol{u}_{i,G+1}) \leq$ cost $(\boldsymbol{x}_{i,G})$. For each element $\boldsymbol{X}_{i,G}$ of the population set $\boldsymbol{X}_{\text{pop}}$, this procedure is repeated, thus M iterations are run for each generation.

In order to summarize DE Algorithm explained in the previous subsections, the following flowchart is presented.

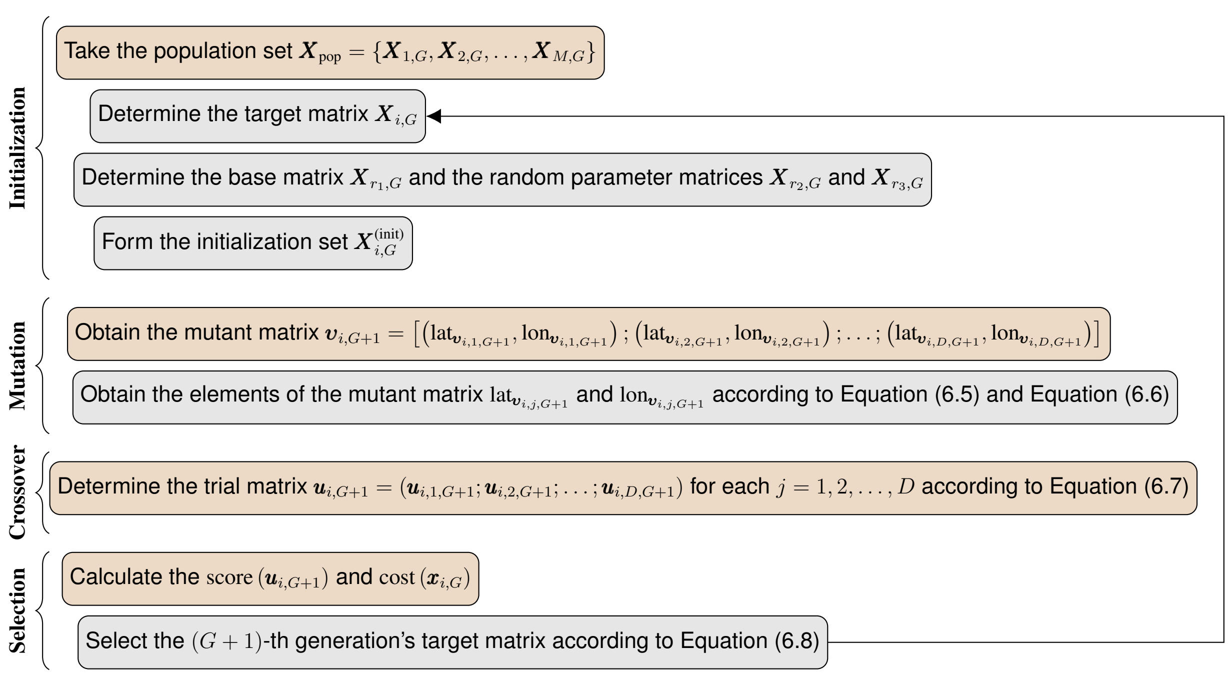


Figure 6.2: Flowchart for the extension of DE algorithm

6.2.3. Criteria for DE algorithm's control variables

The population size must be $M \geq 4$ since one base matrix and two parameter matrices are taken from the population randomly different from i, so that at least 4 elements are needed. Thus, different initialization sets $\boldsymbol{X}_{i,G}^{(\text{init})}$ is determined for each generation. Storn and Price [33] restrict this criterion as $5D \leq M \leq 10D$ where D is the number of the elements (locations in this study) included in the parameter matrix.

F adjusts the amount of the difference between two random parameter matrices added to the base matrix in the mutation step, so-called "mutation scaling factor" [125]. Since $F \in [0, 2]$ is a constant, the choice of this factor is discussed in several studies. Storn and Price [33] suggest that $F = 0.5$ can be preferred for the first trial. It is also recommended in this study that F can be increased if convergence is mature. If it converges too early, it is called premature convergence. Moreover, it is found in this study that the optimization is not effective when $F < 0.4$ or $F > 1$. Therefore, the most effective interval for mutation scaling factor is $F \in [0.4, 1]$.

Another control variable of DE algorithm is the constant CR $\in [0, 1]$ value, which is called as "crossover rate" [125]. Since higher values of CR usually provide fast convergence, the values CR $= 0.9$ or CR $= 1.0$ could be candidates for this factor [33].

Although various criteria appear in existing studies, the conditions may vary according to the structure of the data and the objective function. Our parameter choice for this study is explained in the application.

6.2.4. IDW Method: Reduction Technique

Susanto et al. [32] propose an extensive alteration of the IDW-based spatiotemporal algorithm, so called "reduction approach". The idea behind this technique [126] claims that temporal distance should be taken independently from spatial distance.

In DE algorithm part, the optimal set of 4 locations is determined as the sample input of the IDW method. The meteorological values for different variables shown in Table 6.1 are available at these locations at different times. Consider that $\boldsymbol{X} = [\boldsymbol{x}_1; \boldsymbol{x}_2; \boldsymbol{x}_3; \boldsymbol{x}_4]$ is the optimal input set. Here $\boldsymbol{x}_i$s are inputs used for estimating the meteorological value related to an agricultural claim located at the point x. Define $o_{i,t}$ as the observed meteorological value at the location $\boldsymbol{x}_i = (\text{lat}_{\boldsymbol{x}_i}, \text{lon}_{\boldsymbol{x}_i})$ and at time $\boldsymbol{t}_{\boldsymbol{x}_i}$. Here, $\boldsymbol{t}_{\boldsymbol{x}_i}$ is the vector of times when the meteorological value at the location $\boldsymbol{x}_i$ is measured and let $\boldsymbol{t}_{\boldsymbol{x}_i} \in [t_{\text{start}}, t_{\text{end}}]$. Hence, the values t_{start} and t_{end} vary according to $\boldsymbol{x}_i$ for each meteorological variable. In addition, consider that the agricultural claim occurs at the location $x = (\text{lat}_x, \text{lon}_x)$ and at time t_x.

According to this technique, the unknown meteorological value $\hat{x}$ related to the location and the time of the agricultural claim is estimated as follows:

$$\hat{x} = \sum_{t_{\text{start}}}^{t_{\text{end}}} \sum_{i=1}^{4} w_{t,i} o_{t,i} \tag{6.9}$$

where $o_{t,i}$ is the observed point and $w_{t,i}$ is the weight related to each observed point. The weights are calculated wrt the traditional IDW technique using the inverse of 2-dimensional spatial distance and 1-dimensional temporal distance as

$$w_{t,i} = \frac{\Delta_{s_i}^{-u_s} \Delta_{t_i}^{-u_t}}{\sum_{t_{\text{start}}}^{t_{\text{end}}} \sum_{i=1}^{4} \Delta_{s_i}^{-u_s} \Delta_{t_i}^{-u_t}}. \tag{6.10}$$

In this equation, the 2-dimensional spatial distance is the Euclidean distance between the i-th known location $\boldsymbol{x}_i = (\text{lat}_{\boldsymbol{x}_i}, \text{lon}_{\boldsymbol{x}_i})$ and the unknown point's location $x = (\text{lat}_x, \text{lon}_x)$ and it is indicated as $\Delta_{s_i} = \Delta_{\boldsymbol{x}_i, x}$. The temporal distance is the time difference between the elements of $\boldsymbol{t}_{\boldsymbol{x}_i}$ and t_x denoted as $\Delta_{t_i} = \Delta_{\boldsymbol{t}_{\boldsymbol{x}_i}, t_x}$. In order to consider the seasonality in the data set simply, we compute the temporal distances based on $(mod\, 365)$. Here, the constants u_s and u_t are determined by user and they are called "spatial distance-decay factor" and "temporal distance-decay factor", respectively [32].

The calculation of the weight $w_{t,i}$ in Equation (6.10) shows that it might yield "division by zero" error if Δ_{s_i} and/or Δ_{t_i} is equal to zero. In order to eliminate this problem, Susanto et al. [32] recommend to adjust these values to 1 since it is the smallest value that can be assigned to both spatial and temporal distances. In addition, this drawback is discussed by De Mesnard [127] and it is suggested that the sample value can be taken directly since the distance between unknown point and known point is zero. As recommended, we also replace the location of the unknown value with the sample point in our study when the distance is zero. As a result, we determine the distances as follows:

(i) If $\{\Delta_{t_i} = 0\} \wedge \{\Delta_{s_i} = 0\}$, then $w_{t,i} = 1$, so that $\hat{x} = o_{t,i}$ which means that the estimated value is simply the sample point as in [127].

(ii) If $\{\Delta_{t_i} = 0\} \wedge \{\Delta_{s_i} \neq 0\}$, then Δ_{t_i} is adjusted as $\Delta_{t_i} = 1$ as in [32] and Δ_{s_i} is calculated as an Euclidean distance.

(iii) If $\{\Delta_{t_i} \neq 0\} \wedge \{\Delta_{s_i} = 0\}$, then Δ_{s_i} is adjusted as $\Delta_{s_i} = 1$ as in [32] and Δ_{t_i} is calculated as the absolute difference between dates.

(iv) If $\{\Delta_{t_i} \neq 0\} \wedge \{\Delta_{s_i} \neq 0\}$, then Δ_{s_i} is calculated as an Euclidean distance and Δ_{t_i} is calculated as the absolute difference between dates.

The optimization and interpolation results for our crop-hail insurance data set are presented in Section 6.4.

6.3. Risk Clustering

According to our main model setting for the prioritization of aggregate claim vectors, we use cluster analysis which is grouping a set of objects by gathering the ones having more similar characteristics. Determination of the structure for the data to be clustered is not very easy when no information apart from observed values is available. Clustering should be distinguished from discriminant analysis. In discriminant analysis, known groupings of some observations are used to categorize others and it helps to infer the structure of entire data. Once we estimate the meteorological information related to the agricultural insurance claims, we can cluster risks wrt their environmental features.

Clustering is one of the main analysis steps that has been studied by many researchers [128]. Studies on machine learning and data mining show that traditional clustering methods are vulnerable in the presence of complex characteristics of recent data sets. These data sets also force researchers to question if existing clustering methods are scalable and suitable for visualization [129].

Clustering approaches could be categorized as parametric methods [130, 131] and non-parametric models [132, 133]. Since the performance of a clustering method depends on the structure of data, an optimal clustering approach does not exist. However, paramtetric methods appear to be more informative. Clustering algorithms using the parametric model framework apply Expectation-Maximization (EM) algorithm [134, 135] on statistical parametric model for the data. If data is formed as vectors, multivariate Gaussian mixtures, which are popular model-based clustering methods, are comparable under the most commonly used types of distance such as Euclidean or Mahalanobis distances [136].

6.3.1. Model-Based Clustering

Clustering methodology used in this study is based on multivariate normal mixtures. In order to select the best model, Bayesian Information Criterion (BIC) is used [137, 138]. In the mixture model, the number of components and the related probability densities are determined by EM algorithm under a hierarchical procedure.

The mixture density of $\boldsymbol{y} = (y_1, \ldots, y_n)$ is given as

$$f(\boldsymbol{y}) = \prod_{i=1}^{n} \sum_{k=1}^{m} \tau_k f_k\left(y_i \,|\boldsymbol{\theta}_k\right), \tag{6.11}$$

where $f_k\left(y_i \,|\boldsymbol{\theta}_k\right)$ is a pdf with parameter vector $\boldsymbol{\theta}_k$ and τ_k is the probability of belonging to the k-th component. Here, n is the data size and m is the number of clusters. Since τ_k is the mixing probability of the k-th cluster, the summation of τ_ks should be 1, i.e. $\sum_{k=1}^{m} \tau_k = 1$.

The probability density function f_k is usually assumed to belong multivariate normal distribution family where the parameters are the mean vector $\boldsymbol{\mu}_k$ and the covariance matrix $\boldsymbol{\Sigma}_k$. Thus, the parameter vector $\boldsymbol{\theta}_k$ includes the multivariate normal distribution parameters $\boldsymbol{\mu}_k$ and $\boldsymbol{\Sigma}_k$.

The multivariate normal distributed variable y_i has the pdf defined below:

$$f_k\left(y_i\left|\boldsymbol{\theta}_k\right.\right) = f_k\left(y_i\left|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\right.\right) = \left|2\pi\boldsymbol{\Sigma}_k\right|^{-1/2}\exp\left\{-\frac{1}{2}\left(y_i-\boldsymbol{\mu}_k\right)'\boldsymbol{\Sigma}_k^{-1}\left(y_i-\boldsymbol{\mu}_k\right)\right\}$$

where $\boldsymbol{\theta}_k = (\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$.

To estimate the parameters in model-based clustering, EM algorithm is used to maximize the likelihood [134, 135]. Two steps (E-step and M-step) of each EM iteration is as follows:

(1) Consider that the estimates of j-th component ($\boldsymbol{\mu}_j$, $\boldsymbol{\Sigma}_j$ and τ_j) are given, the conditional probability z_{ik} is obtained in the E-step as follows:

$$z_{ik} = \frac{\tau_k f_k\left(y_i\left|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\right.\right)}{\sum_{j=1}^{m}\tau_k f_k\left(y_i\left|\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j\right.\right)}. \tag{6.12}$$

In this equation, z_{ik} indicates the probability that object i belongs to the k-th component.

(2) According to the estimates of z_{ik}, parameters are estimated from the data [139].

The risk clustering and stochastic majorization results are given in the following section.

6.4. Application

Our aim in this chapter is to obtain the estimated meteorological information of the agricultural claim data, to cluster aggregate claims according to this information, and finally to prioritize aggregate claim vectors of these risk clusters. For this aim, we first select the optimal sample data set, which is used as an input for IDW method with reduction technique, in the extended DE optimization algorithm. In order to estimate the unknown meteorological values for the points related to the reported claims, we use this input set in STI method. We solve the optimization problem, obtain the estimations and prioritize actuarial risks using Matlab. For displaying the graphs and obtaining some of the numerical results, we use the tools of ArcGIS software. For the clustering part, we use R software.

6.4.1. Data

Clustering of agricultural claims has two main steps. The first step is related to the claim data including locations (latitudes, longitudes and altitudes) and dates whereas the second step is to obtain meteorological data considering temporal and spatial information about claim data.

The claim dates were not given related to each claim in the claim data set. The claim amounts (*Data Set 1*) and the claim dates (*Data Set 2*) were taken separately. The problem was that these two data sets, which were needed to be merged, were given at different times. Therefore, an amount recorded as a “paid claim” in *Data Set 1* might be recorded as an “outstanding claim” in *Data Set 2* since *Data Set 2* is older than *Data Set 1*. On the other hand, the values of these amounts are not always the same due to the contradictions, delays, law courts etc.. Therefore, we have obtained the dates of individual claims checking the consistency of paid and outstanding claim amounts and replacing more recent data when the values are different. We have also checked if the claim amount exceeded the insurance amount for each claim. Here, we have replaced the claim amount with the insurance amount if the claim size is greater than the insurance amount.

In addition to the dates, we also needed the information about the locations where the claims occurred. The village codes, where the policies are written, are available in the data set, but TARSİM does not provide the spatial information of these villages. Therefore, latitudes, longitudes and altitudes of 43,090 villages are found using online “batch geo-coding” tools of “ArcGIS Geocode Addresses”.

On the other hand, the meteorological data, which is available from the date 01.01.1980 to 31.12.2016, is given by General Directorate of Meteorology (MGM). This data, which is used as a sample data set, is recorded at 415 weather stations at various times. The spatial information of the weather stations, which are latitudes, longitudes and altitudes of the stations, are also obtained by batch geo-coding.

As the last step before clustering the aggregate claims, we merge the claim data and the related meteorological information. For the unknown values in the claim data, one of the common STI methods is used. By the help of IDW technique with reduction approach, the meteorological quantities of claims could be estimated related to its location and time.

In the following map, the brown circles indicate the location of 43,090 villages where claims are reported whereas the yellow ones are for the locations of 415 meteorological stations.

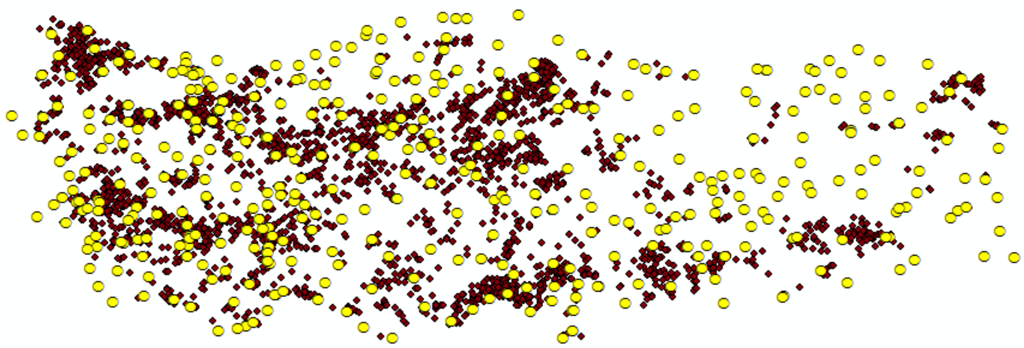


Figure 6.3: The locations of the claims (brown) and the meteorological stations (yellow)

As it can be seen from the map, the claims are mostly centred around the middle, the west and the south of the country due to the fact that agricultural activity is higher in these areas. Another reason of this localization is that economic conditions, education and insurance culture is more improved in these regions. Hence, in order to estimate the unknown meteorological value of the brown circles through IDW technique with reduction approach, we use the known information of the optimal set of yellow circles chosen among the ones surrounding the brown ones using extended DE algorithm. The meteorological variable interpolated for the claims are presented in the following table.

Table 6.1: The information of the meteorological variables which are used in the application

No	Variable Name	Unit	Abbreviation	Size
1	Daily Average Cloud	%	avcloud	98,510
2	Daily Average Humidity	%	avhum	927,254
3	Daily Average Pressure	hPa	avpres	828,340
4	Daily Average 5 cm Soil Temperature	°C	avsoiltemp5	839,143
5	Daily Average 10 cm Soil Temperature	°C	avsoiltemp10	836,883
6	Daily Average 20 cm Soil Temperature	°C	avsoiltemp20	833,734
7	Daily Average 50 cm Soil Temperature	°C	avsoiltemp50	821,938
8	Daily Average Temperature	°C	avtemp	944,670
9	Daily Average Vapor Pressure	hPa	avvappres	804,448
10	Daily Insolation Force	cal/cm^2	insoforce	169,071
11	Daily Maximum Temperature	°C	maxtemp	957,491
12	Daily Minimum Surface Temperature	°C	minsurftemp	848,233
13	Daily Minimum Temperature	°C	mintemp	959,525
14	Daily Local 100 cm Soil Temperature	°C	soiltemp100	826,804
15	Daily Total Evaporation	mm	toteva	226,467
16	Daily Total Global Solar Radiation	cal/cm^2	totglsolrad	169,077
17	Daily Total Insolation Duration	hour	totinso	443,246
18	Daily Total Precipitation	mm	totprec	490,522

6.4.2. Results of STI through Optimization

It is explained in Section 6.2.3 that the criteria for DE algorithm's control variables are acceptable when they are chosen as: (i) the population size, $5D \leq M \leq 10D$; (ii) the crossover ratio, CR $= 0.9$ or CR $= 1.0$ for the fast convergence; and (iii) the scaling factor, $F = 0.5$. Therefore, we determine these factors as $D = 4$, $M = 35$, CR $= 0.9$ and F $= 0.5$. Since we use initial vectors including 4 sample points to interpolate the unknown value as it is explained in Section 6.2.1, the condition of $5D \leq M \leq 10D \equiv 20 \leq M \leq 40$ results in $M = 35$. As mentioned before, the population matrix is obtained from the combination of the m closest points. Thus, for determining M as $20 \leq M = \binom{m}{4} \leq 40$, M can only be equal to 7 since $\binom{7}{4} = 35$, not $\binom{6}{4} = 15$ or $\binom{8}{4} = 70$. As a result, we first choose 7 closest sample locations to the unknown point and we test all combinations of these locations in order to find the best 4-dimensional set of sample points for IDW method.

In addition to DE algorithm control variables, the choice of the spatial and temporal distance-decay factors u_s and u_t have a significant influence on the weights determining the IDW interpolation estimation. As it is inferred from Equation (6.10), the weights $w_{t,i}$ are proportional to the power of the inverse distances, i.e. $\Delta_{s_i}^{-u_s}$ and $\Delta_{t_i}^{-u_t}$. Each weight $w_{t,i}$ will be the same if $u_s = u_t = 0$. If these power values are very high, only a few closest sample points will have impact on the estimation. In the Geostatistical Analyst tool of the software ArcGIS, the power functions are taken as greater than 1. We choose $u_s = u_t = 2$ as a special power value which is known as inverse distance squared weighted interpolation.

The following table displays the results of DE algorithm for the optimal sample points obtained for the last element of unique claim location pairs. In this table, the error term $\mathrm{e}_k^{\mathrm{abs}}\left(\boldsymbol{X}_{-k}\right)$, the coefficient of variation $\mathbb{CV}\left(\Delta_{\boldsymbol{X}^{\mathrm{pw}}}\right)$ and the cost value cost_k are calculated from Equation (6.1), Equation (6.2) and Equation (6.3), respectively.

Table 6.2: The results for the optimal sample set obtained with DE algorithm

Variable	$\left[\mathrm{e}_k^{\mathrm{abs}}\left(\boldsymbol{X}_{-k}\right)\right]^{-1}$	$\mathbb{CV}\left(\Delta_{\boldsymbol{X}^{\mathrm{pw}}}\right)$	cost_k
avcloud	0.001138952164009	0.733469869622734	0.000835387095242
avhum	0.005434783647316	0.087039863472736	0.000473042826666
avpres	0.005434783131625	0.087039863472736	0.000473042781781
avsoiltemp5	0.005434783211462	0.087039863472736	0.000473042788730
avsoiltemp10	0.005434782759576	0.087039863472736	0.000473042749398
avsoiltemp20	0.005434783100120	0.087039863472736	0.000473042779038
avsoiltemp50	0.005434782710980	0.087039863472736	0.000473042745168
avtemp	0.005434782667254	0.087039863472736	0.000473042741362
avvappres	0.005434783938753	0.087039863472736	0.000473042852033
insoforce	0.005263157894737	0.376221217837329	0.001980111672828
maxtemp	0.005434782662232	0.087039863472736	0.000473042740925
minsurftemp	0.005434788197136	0.087039863472736	0.000473043222682
mintemp	0.005434788197136	0.087039863472736	0.000473043222682
soiltemp100	0.005434782843587	0.087039863472736	0.000473042756710
toteva	0.005434782608696	0.156829118621673	0.000852332166422
totglsolrad	0.005263157894737	0.376221217837329	0.001980111672828
totinso	0.005434782608696	0.148721987392154	0.000808271670610
totprec	0.005434784027935	0.087039863472736	0.000473042859795

In Table 6.2, the reason of identical $\mathbb{CV}\left(\Delta_{\boldsymbol{X}^{\mathrm{pw}}}\right)$ is that the coefficient of variation is calculated from the pair-wise distances of the entire sample, not from the leave-one-out sample. Since most of the meteorological variables are measured at the same stations, these results are not surprising. As it is seen from Table 6.1, the sizes of meteorological variables that have same $\mathbb{CV}\,(\Delta_{\boldsymbol{X}^{\text{pw}}})$ are high which means that these variables are measured at all meteorological stations.

After we find the optimal initial input set of sample points with DE algorithm, we estimate the meteorological values using IDW reduction technique. The basic descriptive statistics (minimum, maximum, mean, standard deviation and coefficient of variation) of the estimated values for each meteorological variable are presented in the following table.

Table 6.3: The numerical results of the estimated meteorological variables obtained with STI

Variable	Minimum	Maximum	Mean	Std.Dev.	Coef.Var.
avcloud	0.112387	8.212468	8.212468	1.081829	0.131730
avhum	5.418993	85.829103	53.007691	10.615359	0.200261
avpres	99.680297	1011.476578	818.432825	143.390523	0.175201
avsoiltemp5	1.638861	39.848590	20.665928	5.961296	0.288460
avsoiltemp10	1.887984	37.653357	20.015126	5.789628	0.289263
avsoiltemp20	1.630357	36.708751	19.386182	5.649834	0.291436
avsoiltemp50	1.689714	34.431422	17.885948	5.144025	0.287601
avtemp	1.590005	31.636077	17.631339	4.640650	0.263205
avvappres	0.015913	26.939887	11.846548	4.059395	0.342665
insoforce	23.655901	665.757626	353.390889	154.450782	0.437054
maxtemp	0.000894	39.117486	23.452979	5.536492	0.236068
minsurftemp	-2.686932	26.446437	10.793316	4.502987	0.417201
mintemp	-1.659817	27.273697	12.074831	4.435626	0.367345
soiltemp100	1.569048	32.764589	16.000496	4.586364	0.286639
toteva	0.006840	11.015796	3.744394	1.493412	0.398839
totglsolrad	5.797273	672.247079	366.594729	144.470776	0.394089
totinso	0.003298	12.184676	5.677811	2.374340	0.418179
totprec	0.007241	14.015522	1.021892	0.890617	0.871537

For an explanatory interpretation of the choropleth maps and the intervals specified for the varying colors in the choropleth maps represented in Figure 6.4, Figure 6.5 and Figure 6.6; the statistics given in Table 6.3 and the histogram graphs in Appendix A.5.1 obtained with ArcGIS software are necessary.

(a) avcloud (b) avhum

(c) avpres (d) avsoiltemp5

(e) avsoiltemp10 (f) avsoiltemp20

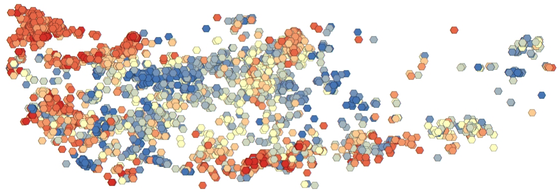
Figure 6.4: Choropleth maps for the estimated meteorological values related to the reported agricultural claims using IDW reduction technique: (a)-(f)

(g) avsoiltemp50 (h) avtemp

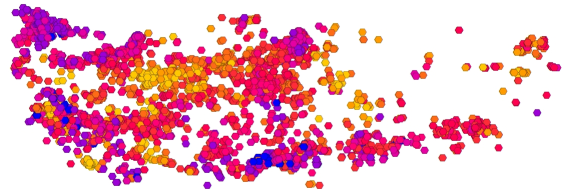
(i) avvappres (j) insoforce

(k) maxtemp (l) minsurftemp

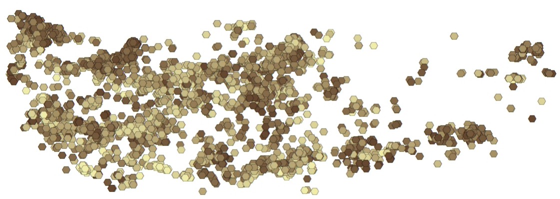
Figure 6.5: Choropleth maps for the estimated meteorological values related to the reported agricultural claims using IDW reduction technique: (g)-(l)



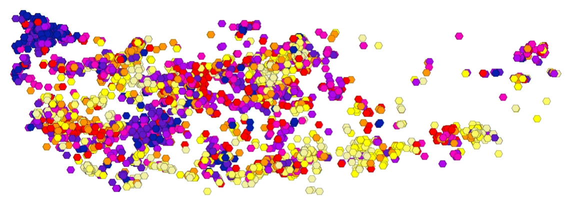
(m) mintemp



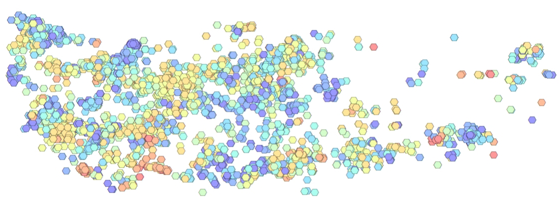
(n) soiltemp100



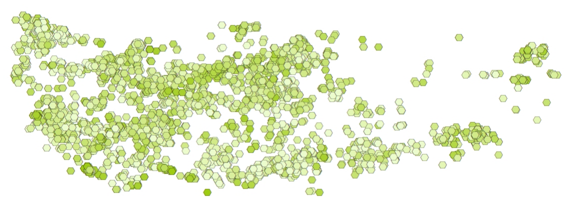
(o) toteva



(p) totglsolrad



(q) totinso



(r) totprec

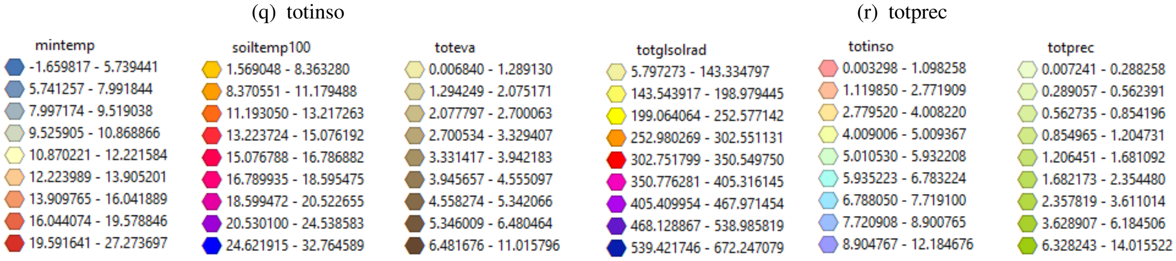


Figure 6.6: Choropleth maps for the estimated meteorological values related to the reported agricultural claims using IDW reduction technique: (m)-(r)

We choose 9 color classes to obtain Figure 6.4, Figure 6.5 and Figure 6.6 for the sake of consistency with the model-based clustering results. According to these figures, the colors scatter wrt the environmental characteristics of the regions. Therefore, the ones gathering together in a region determines one cluster. We examine the analytic part of clustering in the following section.

6.4.3. Results of Risk Clustering

After we obtain the interpolated values of meteorological variables related to each claim, we cluster the claims according to their environmental characteristics using these estimated values. The results of risk clustering are represented in the following tables.

Table 6.4: Model-based clustering results of the best model

Log-likelihood	n	m	**df**	**BIC**
2,617,353	54,113	9	1,565	-5,251,763

According to Table 6.4, the minimum BIC belongs to the clustering based on 9 clusters. Table 6.5 shows the number of the data in each class and the mixing weights of these classes.

Table 6.5: Sample sizes and mixing probabilities of risk clusters (RC)

RC	**Sample sizes (n_i)**	**Mixing Probabilities (τ_i)**
i=1	3,063	τ_1=0.05735849
i=2	18,344	τ_2=0.33459850
i=3	5,615	τ_3=0.10445179
i=4	2,520	τ_4=0.04752900
i=5	5,852	τ_5=0.10807725
i=6	12,986	τ_6=0.24360421
i=7	1,097	τ_7=0.02026161
i=8	594	τ_8=0.01233283
i=9	4,042	τ_9=0.07178633
Total	54,113	1.00000000

As it can be seen from Table 6.5, the second class which has 18,344 claims has the highest mixing probability that is approximately 33.46 %. The detailed results such as means and covariance matrices of 18 meteorological variables are displayed for each risk cluster in Appendix A.5.2.

6.4.4. Results of Stochastic Majorization Ordering

As we mention in Section 5.5.1, the number of the recorded claims arised from hail hazard (including zero claim amounts) is 54,113. There are 23 crop classes and we obtain that there exist 9 risk clusters. Therefore, we organize individual claims wrt the crop classes and risk clusters, and we arrange the aggregate claim vectors according to the setting $\boldsymbol{S}^{(i)} = \left(S_1^{(i)}, S_2^{(i)}, \ldots, S_{p_i}^{(i)}\right)'$ for $i = 1, 2, \ldots, 9$ and $p_i = 18$.

In this section, we order the risk clusters determined in Section 6.4.3 through stochastic majorization relation. In order to do that, we implement the procedure explained in Chapter 5. We first check if two conditions in Remark 5.4.2 are fulfilled. After that, we order the aggregate claim vectors of risk clusters according to our majorization ordering setting given in Equation (5.9).

Firstly, we find that Condition (i) and Condition (ii) are fulfilled for majorization relation. For Condition (i), the rearranged aggregate claim vectors are given as rows in the following table. For $i = 1, 2, \ldots, 9$ and $j = 1, 2, \ldots, 18 = p_i$, the aggregate claim vectors are given in entire (9×18)-dimensional matrix in which rows (columns) represent risk clusters (crop classes).

Table 6.6: The rearrangement of the aggregate claim vectors of the risk clusters (RC) according to Condition (i) in Remark 5.4.2

RC	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
$i=2$	0.272804	0.446156	0.553132	0.657484	0.750398	0.829882	0.900794	0.934624	0.962297
$i=6$	0.323349	0.525459	0.621250	0.696938	0.766751	0.835965	0.902375	0.944223	0.963547
$i=3$	0.351716	0.554672	0.714374	0.801337	0.865665	0.901373	0.934207	0.965199	0.980348
$i=4$	0.555706	0.700582	0.765794	0.817928	0.878269	0.921183	0.948115	0.965558	0.983617
$i=5$	0.577045	0.770220	0.841609	0.885400	0.913437	0.936396	0.955604	0.972400	0.984556
$i=9$	0.664807	0.782018	0.842155	0.900122	0.951110	0.970283	0.979057	0.986269	0.991676
$i=1$	0.695275	0.829866	0.896246	0.932562	0.964168	0.983704	0.991187	0.994250	0.996227
$i=7$	0.762466	0.829874	0.917053	0.948840	0.971371	0.985755	0.991198	0.995076	0.997939
$i=8$	0.880705	0.909133	0.931771	0.953525	0.978049	0.987398	0.996605	0.999416	0.999745

RC	$j=10$	$j=11$	$j=12$	$j=13$	$j=14$	$j=15$	$j=16$	$j=17$	$j=18$
$i=2$	0.978663	0.987315	0.991110	0.994783	0.997564	0.999133	0.999783	0.999927	1.000000
$i=6$	0.980092	0.990892	0.994985	0.997820	0.998916	0.999447	0.999790	0.999931	1.000000
$i=3$	0.991748	0.995718	0.998445	0.999277	0.999646	0.999867	0.999956	1.000000	1.000000
$i=4$	0.992792	0.996480	0.998541	0.999373	0.999727	0.999881	1.000000	1.000000	1.000000
$i=5$	0.995921	0.997638	0.999026	0.999583	0.999729	0.999939	1.000000	1.000000	1.000000
$i=9$	0.997376	0.998680	0.999431	0.999643	0.999809	0.999944	1.000000	1.000000	1.000000
$i=1$	0.997938	0.999297	0.999676	0.999858	1.000000	1.000000	1.000000	1.000000	1.000000
$i=7$	0.999938	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
$i=8$	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Let $\text{Cond}_{(\text{I})}^{(i)}$ denote $\dfrac{\sum_{t=1}^{j} S_{[t]}^{(i)}}{\sum_{t=1}^{18} S_{[t]}^{(i)}}$. Then, the following inequality is obtained from Table 6.6:

$$\text{Cond}_{(\text{I})}^{(2)} < \text{Cond}_{(\text{I})}^{(6)} < \text{Cond}_{(\text{I})}^{(3)} < \text{Cond}_{(\text{I})}^{(4)} < \text{Cond}_{(\text{I})}^{(5)} < \text{Cond}_{(\text{I})}^{(9)} < \text{Cond}_{(\text{I})}^{(1)} < \text{Cond}_{(\text{I})}^{(7)} < \text{Cond}_{(\text{I})}^{(8)}$$

which proves that

$$\frac{\sum_{t=1}^{j} S_{[t]}^{(2)}}{\sum_{t=1}^{18} S_{[t]}^{(2)}} < \frac{\sum_{t=1}^{j} S_{[t]}^{(6)}}{\sum_{t=1}^{18} S_{[t]}^{(6)}} < \frac{\sum_{t=1}^{j} S_{[t]}^{(3)}}{\sum_{t=1}^{18} S_{[t]}^{(3)}} < \frac{\sum_{t=1}^{j} S_{[t]}^{(4)}}{\sum_{t=1}^{18} S_{[t]}^{(4)}} < \frac{\sum_{t=1}^{j} S_{[t]}^{(5)}}{\sum_{t=1}^{18} S_{[t]}^{(5)}} <$$

$$\frac{\sum_{t=1}^{j} S_{[t]}^{(9)}}{\sum_{t=1}^{18} S_{[t]}^{(9)}} < \frac{\sum_{t=1}^{j} S_{[t]}^{(1)}}{\sum_{t=1}^{18} S_{[t]}^{(1)}} < \frac{\sum_{t=1}^{j} S_{[t]}^{(7)}}{\sum_{t=1}^{18} S_{[t]}^{(7)}} < \frac{\sum_{t=1}^{j} S_{[t]}^{(8)}}{\sum_{t=1}^{18} S_{[t]}^{(8)}}$$

for each $j = 1, 2, \ldots, p_i - 1 = 17$. Therefore, Condition (i) is fulfilled for the risk clustering obtained in Section 6.4.3.

Moreover, for Condition (i), the last elements of the decreasing rearrangement of aggregate claim vectors for each risk cluster are given in following table.

Table 6.7: The sum of the decreasing rearrangement of aggregate claim vectors of the risk clusters (RC) according to Condition (ii) in Remark 5.4.2

RC	$\sum_{t=1}^{18} S_{[t]}^{(i)}$
$i = 2$	51,998,805.49
$i = 6$	42,230,639.03
$i = 3$	34,835,236.40
$i = 4$	22,423,303.41
$i = 5$	15,892,955.21
$i = 9$	15,837,720.50
$i = 1$	13,064,498.98
$i = 7$	4,603,761.61
$i = 8$	1,412,315.24

Let $\text{Cond}_{(\text{II})}^{(i)}$ indicate $\sum_{t=1}^{18} S_{[t]}^{(i)}$. Then, the following inequality is obtained from Table 6.7:

$$\text{Cond}_{(\text{II})}^{(2)} > \text{Cond}_{(\text{II})}^{(6)} > \text{Cond}_{(\text{II})}^{(3)} > \text{Cond}_{(\text{II})}^{(4)} > \text{Cond}_{(\text{I})}^{(5)} > \text{Cond}_{(\text{II})}^{(9)} > \text{Cond}_{(\text{II})}^{(1)} > \text{Cond}_{(\text{I})}^{(7)} > \text{Cond}_{(\text{I})}^{(8)}$$

which proves that

$$\sum_{t=1}^{18} S_{[t]}^{(2)} > \sum_{t=1}^{18} S_{[t]}^{(6)} > \sum_{t=1}^{18} S_{[t]}^{(3)} > \sum_{t=1}^{18} S_{[t]}^{(4)} > \sum_{t=1}^{18} S_{[t]}^{(5)} >$$

$$\sum_{t=1}^{18} S_{[t]}^{(9)} > \sum_{t=1}^{18} S_{[t]}^{(1)} > \sum_{t=1}^{18} S_{[t]}^{(7)} > \sum_{t=1}^{18} S_{[t]}^{(8)}.$$

Therefore, Condition (ii) is fulfilled for the risk clustering obtained in Section 6.4.3. As a result, there exist a majorization relation among all risk clusters. According to these results, we expect that Schur-convex function values are also ordered identically. It means that we do not need to calculate the sample variances or sample coefficient of variations of risk clusters since we know that the ordering is same wrt Condition (i) and Condition (ii) once we prove that the sample variance or the sample coefficient of variation is Schur-convex. Therefore, the majorization results for sample variances and sample coefficient of variations of these risk clusters, which are given in the following table, are only a verification of the results given in Table 6.6 and Table 6.7.

Table 6.8: The sample variance and sample coefficient of variation values of the variance of aggregate claim vectors of the risk clusters (RC)

RC	$\phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(j)}\right)\right)$ $(*10^8)$	$\phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(j)}\right)\right)$
$\boldsymbol{i=2}$	0.031924	1.290789
$\boldsymbol{i=6}$	0.091042	1.469259
$\boldsymbol{i=3}$	0.123261	1.846709
$\boldsymbol{i=4}$	0.161089	1.895143
$\boldsymbol{i=5}$	0.163895	1.956126
$\boldsymbol{i=9}$	0.170576	1.980451
$\boldsymbol{i=1}$	0.520934	2.316822
$\boldsymbol{i=7}$	0.837824	2.327122
$\boldsymbol{i=8}$	3.932997	3.695403

Since our setting is proven to be accurate in the case study in Section 5.5, the values of risk measure is ordered as in majorization relation. The ordering inequality is given as follows:

$$\phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(2)}\right)\right) \leq \phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(6)}\right)\right) \leq \phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(3)}\right)\right) \leq \phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(4)}\right)\right) \leq \phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(5)}\right)\right) \leq \phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(9)}\right)\right) \leq \phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(1)}\right)\right) \leq \phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(7)}\right)\right) \leq \phi_1\left(\mathbb{V}\left(\boldsymbol{S}^{(8)}\right)\right)$$

or

$$\phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(2)}\right)\right) \leq \phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(6)}\right)\right) \leq \phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(3)}\right)\right) \leq \phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(4)}\right)\right) \leq \phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(5)}\right)\right) \leq \phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(9)}\right)\right) \leq \phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(1)}\right)\right) \leq \phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(7)}\right)\right) \leq \phi_2\left(\mathbb{V}\left(\boldsymbol{S}^{(8)}\right)\right)$$

$$\Rightarrow \boldsymbol{S}^{(2)} \prec_{\text{maj}} \boldsymbol{S}^{(6)} \prec_{\text{maj}} \boldsymbol{S}^{(3)} \prec_{\text{maj}} \boldsymbol{S}^{(4)} \prec_{\text{maj}} \boldsymbol{S}^{(5)} \prec_{\text{maj}} \boldsymbol{S}^{(9)} \prec_{\text{maj}} \boldsymbol{S}^{(1)} \prec_{\text{maj}} \boldsymbol{S}^{(7)} \prec_{\text{maj}} \boldsymbol{S}^{(8)}$$

These results show us that the stochastic majorization relation is very useful in the context of partial ordering when the risk clustering is accurate.

6.5. Interim Conclusion: Risk Clustering through STI

Agricultural insurance risks cannot be analyzed only by actuarial and statistical modeling ignoring the weather-related characteristics of the insured areas. As it is very difficult to measure the exact values of weather-related variables at the locations where claims are reported, we seek for an approach which could estimate the missing data associated with environmental risks accurately. Spatiotemporal techniques are effective tools as more accurate alternatives to spatial methods for this purpose. We suggest an extended method to obtain an optimal sample set for IDW reduction technique. Another interesting issue that we deal with is the clustering of risks. Since the number of objects to be ordered is high, they need to be collected in groups in terms of some criteria. Parametric clustering methods are efficient in this context.

In this chapter, we extend DE algorithm offering a computation that measures the distance with an angular-based formula. We also claim that there exist a basic solution to determine the population size in initialization step of the optimization procedure. Having proposed an extension for DE algorithm, we take into account the impact of altitudes using 2-dimensional locations all of which are very significant determinants of weather-related risks. Lastly, we cluster claims according to a parametric model-based method.

As we mention before, analytic tools are not always sufficient for environmental risk management. For this reason, we use GIS applications for this part of our study. The results of DE optimization, STI estimations and risk clustering are represented by both statistical measures and choropleth maps as GIS tools.

7. CONCLUSION AND FURTHER STUDY

We aim to provide a summary of the main findings of our thesis as well as some suggestions for further research in this chapter.

7.1. Concluding Remarks

The major contribution of this thesis is providing a perspective for the effects of risk perception on prioritization of risks. The work is original as it constructs a multivariate stop-loss ordering relation under CPT for the first time. In this thesis, we concentrate on impacts of risk perception and various concepts of risk handling the tendencies and bias of individuals. In this sense, CPT sufficiently captures the inconsistency in decision making. Another significant extension of this work is to optimize multivariate sets for estimation characteristics of environmental risks.

Having provided our multivariate model setting in Chapter 1, we discuss the concept of risk and various preference approaches as a resulf of this discussion in Chapter 2. The impacts of the preferences of a DM steer us to the "utility" and "prospect" terms. Therefore, we give notations and definitions to explain the approaches for modeling preferences such as EUT, distorted expectation theory, expected utility with non-additive probabilities and prospect theory.

In Chapter 3, POT and various ordering relations within the frame of POT are explained. In addition to technical ordering of risk, the perspective provided by the use of GIS has an important influence on this work since insurance products which covers the environmental risks are specific in this context. GIS appears as a tool to assess risk and offers a variety to our study.

Impacts of risk perception on prioritization of risks are taken into account in Chapter 4 through CPT. We obtain stop-loss premium solutions under CPT for three different value functions using zero-utility premium principle, which is original compared to preivous studies.

In Chapter 5, we modify the axioms of majorization theory for our multivariate setting of aggregate claim vectors. As a result of a case study, we show the drawbacks of classification when one does not consider the spatial and temporal features of risk especially for environmental insurance products.

Lastly in Chapter 6, we introduce the data and explain how we overcome the problems in this data set. We estimate the meteorological information related to insurance claims and use weather-related variables to cluster agricultural claims. We introduce IDW method with reduction approach as an STI technique and propose a multivariate extension of DE optimization algorithm to estimate the input sample set used in STI method. The results show that an accurate clustering is very fundamental for stochastic majorization ordering. Stochastic majorization needs an order-preserving function for practical purposes, thus in application part, we use variance and coefficient of variation as risk measures which are proven to be Schur-convex.

7.2. Suggestions for Further Research

For future studies, different risk measures can be investigated for ordering the aggregate claims using the stochastic majorization relation if they are proven to be Schur-convex (or Schur-concave) functions.

Moreover, a risk-based clustering as an alternative to model-based clustering, for instance which considers a regression modeling, could be investigated. As a suggestion for environmental risk evaluation, in addition to Euclidean distance, different distance measures such as Mahalanobis distance, Minkowski distance or cosine distance might be examined and compared for spatial interpolation.

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A. APPENDIX

A.1. Appendix (Chapter 2): Decision Making Under Uncertainty

A.1.1. Comparison of the EUT and the YDT

We derive the Equation (2.4) with using the method of integration by parts as follows:

$$\begin{aligned}
\mathbb{E}(X) &= \textstyle\int_{-\infty}^{+\infty} x f_X(x)\,\mathrm{d}x = \int_{-\infty}^{0} x f_X(x)\,\mathrm{d}x + \int_{0}^{+\infty} x f_X(x)\,\mathrm{d}x \\
&= I_1 + I_2. \\
\xrightarrow[\mathrm{d}v=f_X(x)\,\mathrm{d}x,\, v=F_X(x)]{u=x,\,\mathrm{d}u=\mathrm{d}x} I_1 &= x F_X(x)|_{-\infty}^{0} - \textstyle\int_{-\infty}^{0} F_X(x)\,\mathrm{d}x = -\int_{-\infty}^{0} F_X(x)\,\mathrm{d}x, \\
\xrightarrow[\mathrm{d}v=f_X(x)\,\mathrm{d}x,\, v=-(1-F_X(x))]{u=x,\,\mathrm{d}u=\mathrm{d}x} I_2 &= -x[1-F_X(x)]|_{0}^{+\infty} - \textstyle\int_{0}^{+\infty} -[1-F_X(x)]\,\mathrm{d}x \\
&= \textstyle\int_{0}^{+\infty} [1-F_X(x)]\,\mathrm{d}x, \\
\xrightarrow{S_X(x)=1-F_X(x)} \mathbb{E}(X) &= -\textstyle\int_{-\infty}^{0} [1-S_X(x)]\,\mathrm{d}x + \int_{0}^{+\infty} S_X(x)\,\mathrm{d}x.
\end{aligned}$$

For non-negative X rv, this equation can be written as

$$\mathbb{E}(X) = \int_0^{\infty} (1 - F_X(x))\,\mathrm{d}x \tag{A.1}$$

Denuit et al. [1] compare the EUT and the YDT in terms of expectation formulas derived by the integrations of VaRs. We provide derivations of these expectations given in Equations (2.8) and (2.9) as follows:

- The derivation of the Equation (2.8) can be given as follows:

$$\mathbb{E}[u(X)] = \int_0^{+\infty} u(x) f_X(x)\,\mathrm{d}x,$$

Here we need to transform the variable x as

$$x = \mathrm{VaR}[X;p] = F_X^{-1}(p)$$

$$\Rightarrow \mathrm{d}x = \mathrm{d}F_X^{-1}(p) = \left[\frac{\mathrm{d}}{\mathrm{d}p} F_X^{-1}(p)\right] \mathrm{d}p = \left[\frac{\mathrm{d}}{\mathrm{d}x} F_X(x)\right]^{-1} \mathrm{d}p = [f_X(x)]^{-1}\,\mathrm{d}p$$

Lower bound: $x = 0 \Rightarrow \mathrm{VaR}[X;p] = F_X^{-1}(p) = 0 \Rightarrow p = 0$

Upper bound: $x = +\infty \Rightarrow \mathrm{VaR}[X;p] = F_X^{-1}(p) = +\infty \Rightarrow p = 1$

By using this transformation, the integral transforms as

$$\mathbb{E}[u(X)] = \int_0^1 u\left(\mathrm{VaR}[X;p]\right) f_X(x) \left[f_X(x)\right]^{-1} \mathrm{d}p = \int_0^1 u\left(\mathrm{VaR}[X;p]\right) \mathrm{d}p.$$

- The derivation of the Equation (2.9) is as follows:

$$\begin{aligned}
\xrightarrow{\text{Equation 2.6}} \mathbb{E}_g(X) &= \textstyle\int_0^{+\infty} g\left(S_X(x)\right) \mathrm{d}x \\
\xrightarrow[\mathrm{d}v=\mathrm{d}x\Rightarrow v=x]{u=g(S_X(x))\Rightarrow \mathrm{d}u=\mathrm{d}g(S_X(x))} \quad &= xg\left(S_X(x)\right)|_0^{+\infty} - \textstyle\int_0^{+\infty} x\,\mathrm{d}g\left(S_X(x)\right) \\
&= (0-0) - \textstyle\int_0^{+\infty} x\,\mathrm{d}g\left(S_X(x)\right) = -\textstyle\int_0^{+\infty} x\,\mathrm{d}g\left(S_X(x)\right)
\end{aligned}$$

Here we need to transform the variable x as

$$x = \text{VaR}[X; 1-p] = F_X^{-1}(1-p)$$

$$\Rightarrow S_X(x) = 1 - F_X(x) = 1 - F_X\left[F_X^{-1}(1-p)\right] = 1 - (1-p) = p$$

Lower bound: $x = 0 \Rightarrow \text{VaR}[X; 1-p] = F_X^{-1}(1-p) = 0 \Rightarrow 1-p = 0 \Rightarrow p = 1$

Upper bound: $x = +\infty \Rightarrow \text{VaR}[X; 1-p] = F_X^{-1}(1-p) = +\infty \Rightarrow 1-p = 1 \Rightarrow p = 0$

By using this transformation, the integral transforms as

$$\begin{aligned} \mathbb{E}_g(X) &= -\int_1^0 \text{VaR}[X; 1-p]\,\mathrm{d}g(p) = \int_0^1 \text{VaR}[X; 1-p]\,\mathrm{d}g(p) \\ \xrightarrow{p=1-p} &= \int_0^1 \text{VaR}[X; p]\,\mathrm{d}g(1-p). \end{aligned}$$

A.2. Appendix (Chapter 3): Stochastic Ordering Relations for Risk Prioritization

A.2.1. Antisymmetry for zeta matrix: Axiom (iii) in Proposition 3.2.3 and Remark 3.2.4

The antisymmetry axiom of POT is controlled through the condition ($x \leq y$ and $y \leq x \Rightarrow x = y$) for two objects x and y in the object set X. According to the definition of zeta matrix given in Equation (3.2), this axiom should be checked by controlling entries of the diagonal of $\zeta_{x,y}$ and $\zeta_{y,x}$ simultaneously. For an object set to be partially ordered; if $x \neq y$, the condition $(\{\zeta_{x,y} = 1\} \wedge \{\zeta_{y,x} = 1\})$ must be false in order to satisfy antisymmetry. Therefore, this condition can be satisfied if one of the following equations is true:

i. $\zeta * \zeta^T = \boldsymbol{I}_{n\times n}$, or

ii. $\eta * \eta^T = \mathbf{0}_{n\times n}$

However, Patil and Taillie [25] suggest the following equations for this axiom:

i'. $\zeta * \zeta = \boldsymbol{I}_{n\times n}$, or

ii'. $\eta * \eta = \mathbf{0}_{n\times n}$

which are proven to be false in our thesis. In order to eliminate this contradiction, we give a poset example used in Section 3.2.3.

We consider Poset I given in Figure 3.4. For the zeta matrix of Poset I obtained as

$$\zeta_I = \begin{array}{c|cccccc} & x & y & z & v & w & u \\ \hline x & 1 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 1 & 0 & 0 & 0 & 0 \\ z & 1 & 0 & 1 & 0 & 0 & 0 \\ v & 0 & 1 & 0 & 1 & 0 & 0 \\ w & 1 & 0 & 1 & 0 & 1 & 0 \\ u & 1 & 1 & 1 & 1 & 0 & 1 \end{array}$$

the component-wise multiplication given in (i') is calculated as

$$\zeta_I * \zeta_I = \begin{array}{c|cccccc} & x & y & z & v & w & u \\ \hline x & 1 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 1 & 0 & 0 & 0 & 0 \\ z & 1 & 0 & 1 & 0 & 0 & 0 \\ v & 0 & 1 & 0 & 1 & 0 & 0 \\ w & 1 & 0 & 1 & 0 & 1 & 0 \\ u & 1 & 1 & 1 & 1 & 0 & 1 \end{array}$$

where $*$ indicates the component-wise multiplication of two same dimensional matrices. As a result, the condition given in (i') is not true since $\zeta_I * \zeta_I \neq \boldsymbol{I}_{n\times n}$.

However, for the transpose of zeta matrix obtained as

$$\zeta_I^T = \begin{array}{c|cccccc} & x & y & z & v & w & u \\ \hline x & 1 & 0 & 1 & 0 & 1 & 1 \\ y & 0 & 1 & 0 & 1 & 0 & 1 \\ z & 0 & 0 & 1 & 0 & 1 & 1 \\ v & 0 & 0 & 0 & 1 & 0 & 1 \\ w & 0 & 0 & 0 & 0 & 1 & 0 \\ u & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \Rightarrow \zeta_I * \zeta_I^T = \begin{array}{c|cccccc} & x & y & z & v & w & u \\ \hline x & 1 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 1 & 0 & 0 & 0 & 0 \\ z & 0 & 0 & 1 & 0 & 0 & 0 \\ v & 0 & 0 & 0 & 1 & 0 & 0 \\ w & 0 & 0 & 0 & 0 & 1 & 0 \\ u & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

which proves that the condition given in (i) as $\zeta_I * \zeta_I^T = \boldsymbol{I}_{n\times n}$ is true as we suggest.

Similarly, using the transformation $\eta_I = \zeta_I - \boldsymbol{I}_{n\times n}$;

$$\eta_I = \begin{array}{c|cccccc} & x & y & z & v & w & u \\ \hline x & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 & 0 & 0 \\ z & 1 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & 1 & 0 & 0 & 0 & 0 \\ w & 1 & 0 & 1 & 0 & 0 & 0 \\ u & 1 & 1 & 1 & 1 & 0 & 0 \end{array}$$

the component-wise multiplication given in (ii') is calculated as

$$\eta_I * \eta_I = \begin{array}{c|cccccc} & x & y & z & v & w & u \\ \hline x & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 & 0 & 0 \\ z & 1 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & 1 & 0 & 0 & 0 & 0 \\ w & 1 & 0 & 1 & 0 & 0 & 0 \\ u & 1 & 1 & 1 & 1 & 0 & 0 \end{array}$$

As a result, the condition given in (ii') is not true since $\eta_I * \eta_I \neq \mathbf{0}_{n\times n}$.

However, for the transpose of eta matrix obtained as

$$\eta_I^T = \begin{array}{c|cccccc} & x & y & z & v & w & u \\ \hline x & 0 & 0 & 1 & 0 & 1 & 1 \\ y & 0 & 0 & 0 & 1 & 0 & 1 \\ z & 0 & 0 & 0 & 0 & 1 & 1 \\ v & 0 & 0 & 0 & 0 & 0 & 1 \\ w & 0 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \Rightarrow \eta_I * \eta_I^T = \begin{array}{c|cccccc} & x & y & z & v & w & u \\ \hline x & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & 0 & 0 & 0 & 0 & 0 \\ w & 0 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

which proves that the condition given in (ii) as $\eta_I * \eta_I^T = \mathbf{0}_{n\times n}$ is true as we suggest.

A.2.2. Stochastic Dominance and Other Ordering Relations for Actuarial Applications

- The net premium π_Y for risk Y can be indicated as a transition to a premium calculated according to expected value principle as

$$\begin{aligned}
\pi_Y &= \mathbb{E}(Y) \xrightarrow{Equation\ (A.1)} = \int_0^\infty (1 - F_Y(x))\,\mathrm{d}x \\
&= \int_0^\infty (1 - F_Y(x))\,\mathrm{d}x + \int_0^\infty (F_X(x) - F_X(x))\,\mathrm{d}x \\
&= \left[\int_0^\infty (1 - F_X(x))\,\mathrm{d}x + \int_0^\infty (F_X(x) - F_Y(x))\,\mathrm{d}x\right] \frac{\mathbb{E}(X)}{\mathbb{E}(X)} \\
\xrightarrow{Equation\ (A.1)} &= \left[1 + \frac{\int_0^\infty (F_X(x) - F_Y(x))\,\mathrm{d}x}{\int_0^\infty (1 - F_X(x))\,\mathrm{d}x}\right] \mathbb{E}(X)
\end{aligned} \tag{A.2}$$

- Ordering of expected values wrt the relation ($\precsim_{\text{sl}}$): $X \precsim_{\text{sl}} Y \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y)$ can be proven as

$$\begin{aligned}
X \precsim_{\text{sl}} Y :\Leftrightarrow &\quad \mathbb{E}\left[(X-d)_+\right] \leq \mathbb{E}\left[(Y-d)_+\right] \\
\xrightarrow{Equation\ (3.11)} &\quad \int_d^\infty (1 - F_X(x))\,\mathrm{d}x \leq \int_d^\infty (1 - F_Y(x))\,\mathrm{d}x \\
\Rightarrow &\quad 1 - F_X(x) \leq 1 - F_Y(x) \\
\Rightarrow &\quad \int_0^\infty (1 - F_X(x))\,\mathrm{d}x \leq \int_0^\infty (1 - F_Y(x))\,\mathrm{d}x \\
\xrightarrow{Equation\ (A.1)} &\quad \mathbb{E}(X) \leq \mathbb{E}(Y)
\end{aligned} \tag{A.3}$$

- Ordering of variances wrt the relation ($\precsim_{\text{sl}}$): $\{X \precsim_{\text{sl}} Y\} \wedge \{\mathbb{E}(X) = \mathbb{E}(Y)\} \Rightarrow \mathbb{V}(X) \leq \mathbb{V}(Y)$ can be proven as

$$
\begin{aligned}
\textstyle\int_0^\infty \mathbb{E}\left[(X-t)_+\right] \mathrm{d}t \quad &= \textstyle\int_0^\infty \int_t^\infty [1 - F_X(x)] \, \mathrm{d}x \, \mathrm{d}t \\
\xrightarrow[\mathrm{d}v = \mathrm{d}t]{u = \int_t^\infty [1 - F_X(x)] \, \mathrm{d}x} \quad &= \{t \textstyle\int_t^\infty [1 - F_X(x)] \, \mathrm{d}x\}|_0^\infty + \textstyle\int_0^\infty t[1 - F_X(t)] \, \mathrm{d}t \\
\xrightarrow[\mathrm{d}v = \mathrm{d}x]{u = [1 - F_X(x)]} \quad &= t\{x[1 - F(x)]|_t^\infty + \textstyle\int_t^\infty x f(x) \, \mathrm{d}x\}|_0^\infty + \textstyle\int_0^\infty t[1 - F_X(t)] \, \mathrm{d}t \\
\xrightarrow[t\{x(1-F(x))|_t^\infty + \int_t^\infty x f(x) \, \mathrm{d}x\} = 0]{t = 0 \ \& \ t = \infty} \quad &= \textstyle\int_0^\infty t[1 - F_X(t)] \, \mathrm{d}t = A \\
\xrightarrow{u = t[1 - F_X(t)] \, \mathrm{d}t \ , \ \mathrm{d}v = \mathrm{d}t} \quad &= t^2[1 - F_X(t)]|_0^\infty - \left\{\textstyle\int_0^\infty -t^2 f(t) \, \mathrm{d}t + \textstyle\int_0^\infty t[1 - F_X(t)] \, \mathrm{d}t\right\} \\
\xrightarrow{A = \int_0^\infty t[1 - F_X(t)] \, \mathrm{d}t} \quad &\Rightarrow A = 0 + \mathbb{E}(X^2) - A \\
&\Rightarrow A = \textstyle\int_0^\infty t[1 - F_X(t)] \, \mathrm{d}t = \frac{1}{2}\mathbb{E}(X^2)
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
\xrightarrow{\textit{Equation (A.4)}} \quad & \textstyle\int_0^\infty \mathbb{E}\left[(X-t)_+\right] \mathrm{d}t \leq \int_0^\infty \mathbb{E}\left[(Y-t)_+\right] \mathrm{d}t \Rightarrow \frac{1}{2}\mathbb{E}(X^2) \leq \frac{1}{2}\mathbb{E}(Y^2) \\
& \mathbb{E}\left[(X-t)_+\right] \leq \mathbb{E}\left[(Y-t)_+\right] \Rightarrow \mathbb{E}(X^2) \leq \mathbb{E}(Y^2) \\
\xrightarrow{\mathbb{E}(X) = \mathbb{E}(Y)} \quad & \mathbb{V}(X) \leq \mathbb{V}(Y)
\end{aligned}
\tag{A.5}
$$

- Ordering of variances wrt the relation ($\precsim_\ell$): $\{X \precsim_\ell Y\} \wedge \{\mathbb{E}(X) = \mathbb{E}(Y)\} \Rightarrow \mathbb{V}(X) \leq \mathbb{V}(Y)$ can be proven as

$$\begin{aligned}
\mathbb{E}(\widetilde{X}) &= \textstyle\int_0^\infty \ell_X(t)\,\mathrm{d}t \\
\xrightarrow{\textit{Equation (3.13)}} &= \textstyle\int_0^\infty \frac{\mathbb{E}\left[(X-t)_+\right]}{\mathbb{E}(X)}\,\mathrm{d}t = \frac{1}{\mathbb{E}(X)} \int_0^\infty \mathbb{E}\left[(X-t)_+\right]\mathrm{d}t \\
\xrightarrow{\textit{Equation (A.4)}} &= \frac{1}{\mathbb{E}(X)}[\frac{1}{2}\mathbb{E}(X^2)](**) \\
X \precsim_\ell Y &\Rightarrow \ell_X(x) \leq \ell_Y(x) \Rightarrow \textstyle\int_0^\infty \ell_X(t)\,\mathrm{d}t \leq \int_0^\infty \ell_Y(t)\,\mathrm{d}t \Rightarrow \mathbb{E}(\widetilde{X}) \leq \mathbb{E}(\widetilde{Y}) \\
\xrightarrow{(**)} &\ \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)} \leq \frac{\mathbb{E}(Y^2)}{2\mathbb{E}(Y)} \xrightarrow{\mathbb{E}(X)=\mathbb{E}(Y)} \mathbb{V}(X) \leq \mathbb{V}(Y)
\end{aligned} \tag{A.6}$$

- Relationship between mean residual life of X and hazard function of $\widetilde{X}$ can be proven as

$$
\begin{aligned}
h_{\widetilde{X}}(x) &= \frac{f_{\widetilde{X}}(x)}{S_{\widetilde{X}}(x)} = [\frac{1-F_X(x)}{\mathbb{E}(X)}]/[1-\widetilde{F}_X(x)] \\
&= [\frac{1-F_X(x)}{\mathbb{E}(X)}]/[1-\int_0^x \tilde{f}_X(t)\,\mathrm{d}t] \\
&= [\frac{1-F_X(x)}{\mathbb{E}(X)}]/[1-\int_0^x \frac{1-F_X(t)}{\mathbb{E}(X)}\,\mathrm{d}t] \\
&= [\frac{1-F_X(x)}{\mathbb{E}(X)}]/[\frac{\mathbb{E}(X)-\int_0^x[1-F_X(t)]\,\mathrm{d}t}{\mathbb{E}(X)}] \\
&= [1-F_X(x)]/[\mathbb{E}(X)-\int_0^\infty[1-F_X(t)]\,\mathrm{d}t+\int_x^\infty[1-F_X(t)]\,\mathrm{d}t] \\
&= [1-F_X(x)]/[\mathbb{E}(X)-\mathbb{E}(X)+\int_x^\infty[1-F_X(t)]\,\mathrm{d}t] \\
&= [1-F_X(x)]/[\int_x^\infty[1-F_X(t)]\,\mathrm{d}t]
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
\mathbb{E}(X_P) &= \mathbb{E}(X-t|X>t) = \frac{\mathbb{E}[(X-t)I_{X>t}]}{\mathbb{P}(X>t)} \\
&= \frac{\int_t^\infty (x-t)f_X(x)\,\mathrm{d}x}{1-F_X(t)} \\
&= \frac{\{-(x-t)[1-F_X(x)]|_t^\infty\}+\int_t^\infty[1-F_X(x)]\,\mathrm{d}x}{1-F_X(t)} \\
&= \frac{\int_t^\infty[1-F_X(x)]\,\mathrm{d}x}{1-F_X(t)}
\end{aligned}
\tag{A.8}
$$

- The survival function of $\breve{X}$ is obtained as

$$F_{\hat{X}}(x) = \mathbb{P}(\hat{X} \leq x) \xrightarrow{\hat{X}=\dfrac{X}{\mathbb{E}(X)}} = \mathbb{P}\left(\frac{X}{\mathbb{E}(X)} \leq x\right) = \mathbb{P}(X \leq x\mathbb{E}(X)) = F_X(x\mathbb{E}(X)) \tag{A.9}$$

$$\begin{aligned}
k_X(x) = S_{\breve{X}}(x) = 1 - F_{\breve{X}}(x) &= \textstyle\int_x^\infty f_{\breve{X}}(t)\,\mathrm{d}t = \int_x^\infty [1 - F_{\hat{X}}(t)]\,\mathrm{d}t \\
\xrightarrow{Equation\ (A.9)} &= \textstyle\int_x^\infty [1 - F_X(t\mathbb{E}(X))]\,\mathrm{d}t \\
\xrightarrow{u=t\mathbb{E}(X),\ du=\mathbb{E}(X)\,\mathrm{d}t(***)} &= \textstyle\int_{x\mathbb{E}(X)}^\infty [1 - F_X(u)] \dfrac{\mathrm{d}u}{\mathbb{E}(X)} = \dfrac{1}{\mathbb{E}(X)} \int_{x\mathbb{E}(X)}^\infty [1 - F_X(t)]\,\mathrm{d}t \\
\xrightarrow{Equation\ (3.13)} &= \ell_X(x\mathbb{E}(X)) = \frac{\mathbb{E}\left[(X - x\mathbb{E}(X))_+\right]}{\mathbb{E}(X)}
\end{aligned} \tag{A.10}$$

- Ordering of coefficient of variations wrt the relation ($\precsim_k$): $X \precsim_k Y \Rightarrow \mathbb{CV}(X) \leq \mathbb{CV}(Y)$ can be proven as

$$\begin{aligned}
&\textstyle\int_0^\infty k_X(t)\,\mathrm{d}t = \int_0^\infty \ell_X(t\mathbb{E}(X))\,\mathrm{d}t \xrightarrow{(***)} = \dfrac{1}{\mathbb{E}(X)} \int_0^\infty \ell_X(u)\,\mathrm{d}u \\
&\xrightarrow{(**)} = \frac{1}{\mathbb{E}(X)}\left(\frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}\right) = \frac{1}{2}\left(\frac{\mathbb{V}(X) + [\mathbb{E}(X)]^2}{[\mathbb{E}(X)]^2}\right) = \frac{1}{2}\left([\mathbb{CV}(X)]^2 + 1\right)
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
&X \precsim_k Y \Rightarrow k_X(x) \leq k_Y(x) \Rightarrow \textstyle\int_0^\infty k_X(t)\,\mathrm{d}t \leq \int_0^\infty k_Y(t)\,\mathrm{d}t \\
&\xrightarrow{Equation\ (A.11)} \frac{1}{2}\left([\mathbb{CV}(X)]^2 + 1\right) \leq \frac{1}{2}\left([\mathbb{CV}(Y)]^2 + 1\right) \Rightarrow \mathbb{CV}(X) \leq \mathbb{CV}(Y)
\end{aligned} \tag{A.12}$$

- Ordering of variances wrt the relation ($\precsim_k$): $\{X \precsim_k Y\} \wedge \{\mathbb{E}(X) = \mathbb{E}(Y)\} \Rightarrow \mathbb{V}(X) \leq \mathbb{V}(Y)$ can be proven as

$$\begin{aligned} X \precsim_k Y \quad & \Rightarrow \mathbb{CV}(X) \leq \mathbb{CV}(Y) \\ \xrightarrow{\text{Equation (A.11)}} \quad & \frac{1}{2}\frac{\mathbb{V}(X) + [\mathbb{E}(X)]^2}{[\mathbb{E}(X)]^2} \leq \frac{1}{2}\frac{\mathbb{V}(Y) + [\mathbb{E}(Y)]^2}{[\mathbb{E}(Y)]^2} \\ \xrightarrow{\mathbb{E}(X)=\mathbb{E}(Y)} \quad & \mathbb{V}(X) \leq \mathbb{V}(Y) \end{aligned} \tag{A.13}$$

A.2.3. Surplus process in classical risk theory

In the classical risk process, the surplus process of insurer at a fixed time $t > 0$, denoted as $\{U(t)\}_{t\geq 0}$, is described by Dickson [140] as follows:

$$U(t) = u + ct - S(t). \tag{A.14}$$

Here, u is the amount of initial surplus ($t = 0$); c denotes the premium rate per time unit, so ct is the premium amount paid by insureds in the time interval $[0, t]$; and $\{S(t)\}_{t\geq 0}$ is the aggregate claims process indicating the amount paid by the insurer in $[0, t]$. Aggregate claims process is given as

$$S(t) = \sum_{i=1}^{N(t)} X_i. \tag{A.15}$$

Here, individual claim amounts are iid rvs $\{X_i\}_{i=1}^{\infty}$, where X_i is the amount of the ith claim having df F_X. $\{N(t)\}_{t\geq 0}$ which is the number of claims in the time interval $[0, t]$ is assumed as a Poisson process. The distribution of this process is Poisson with parameter λt. Hence, $\{S(t)\}_{t\geq 0}$ is said to be a compound Poisson process denoted by $S(t) \sim$ Compound Poisson$(\lambda t, F_X)$. For this ordering, Heilmann and Schröter [24] assume that the premium rate is calculated according to the expected value principle, i.e. $c = (1 + \theta)\lambda\mathbb{E}(X),\ \theta \geq 0$.

To see how the relation $\precsim_\ell$ defined in Equation (3.12) is applied, T is defined in [24] as the time at which the surplus falls below the initial surplus for the first time

$$T = \inf\{t > 0 : U(t) < u\}.$$

Then,

$$\mathbb{P}(T < \infty, a < u - U(t) \leq b) = \frac{1}{(1+\theta)\mathbb{E}(X)} \int_a^b [1 - F_X(x)]\,\mathrm{d}x.$$

A.3. Appendix (Chapter 4): Risk Prioritization through Stop-Loss Dominance under CPT

A.3.1. Solution of CPT stop-loss premium for Value Function 1

Proof. Let the value function $u(w) = cw$. The minimum premium that the reinsurer accept to cover the risk $(S-d)_+$ is determined by using the Equation (4.14). We obtain $\pi_S(d)$ as follows.

$$\upsilon(w) = \mathbb{E}_{gh}\left[\upsilon\left(w + \pi_S(d) - (S-d)_+\right)\right]$$

$$\xrightarrow{Prop.\ 4.3.2,\ (C_2)} \quad cw = c\mathbb{E}_{gh}\left[\left(w + \pi_S(d) - (S-d)_+\right)\right]$$

$$\xrightarrow[(S-d)_+=Y,\ c=w+\pi_S(d)]{Equation\ (4.8)} \quad w = \mathbb{E}_{gh}(-Y) + (w + \pi_S(d)) + \int_0^{w+\pi_S(d)} \left[h\left(\mathbb{P}(Y > s)\right) - \bar{g}\left(\mathbb{P}(Y > s)\right)\right] \mathrm{d}s$$

$$\xrightarrow{Prop.\ 4.3.2,\ (C_3)} \quad w = -\mathbb{E}_{hg}(Y) + (w + \pi_S(d)) + \int_0^{w+\pi_S(d)} \left[h\left(\mathbb{P}(Y > s)\right) - \bar{g}\left(\mathbb{P}(Y > s)\right)\right] \mathrm{d}s$$

Now divide the equation into the following integral parts and take $Y = (S-d)_+$ again.

$$w = I_1 + (w + \pi_S(d)) + I_2 \tag{A.16}$$

where $I_1 = -\mathbb{E}_{hg}\left[(S-d)_+\right]$ and $I_2 = \int_0^{w+\pi_S(d)} \left\{h\left[\mathbb{P}\left((S-d)_+ > s\right)\right] - \bar{g}\left[\mathbb{P}\left((S-d)_+ > s\right)\right]\right\} \mathrm{d}s$.

At first, we will simplify I_1:

$$I_1 = -\mathbb{E}_{hg}\left[(S-d)_+\right] \xrightarrow{(S-d)_+\geq 0} = -\mathbb{E}_{h}\left[(S-d)_+\right] = -\int_0^{\infty} h\left[\mathbb{P}\left((S-d)_+ > t\right)\right] \mathrm{d}t$$

$$= -\int_0^{\infty} h\left[\mathbb{P}\left((S-d)_+ > t\right)_{\{S\leq d\}}\right] \mathrm{d}t - \int_0^{\infty} h\left[\mathbb{P}\left((S-d)_+ > t\right)_{\{S>d\}}\right] \mathrm{d}t$$

$$\xrightarrow[\Rightarrow \mathbb{P}(0>t)=0;\ 0\leq t<\infty]{S\leq d \Rightarrow (S-d)_+=0} \quad = -\int_0^{\infty} h(0)\, \mathrm{d}t - \int_0^{\infty} h\left[\mathbb{P}\left(S-d > t\right)\right] \mathrm{d}t \xrightarrow{h(0)=0} = -\int_0^{\infty} h\left[\mathbb{P}\left(S > d+t\right)\right] \mathrm{d}t$$

$$\Rightarrow I_1 = -\int_0^{\infty} h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t \tag{A.17}$$

Now, we will simplify I_2:

$$\begin{aligned}
I_2 &= \int_0^{w+\pi_S(d)} \left\{h\left[\mathbb{P}\left((S-d)_+ > s\right)\right] - \bar{g}\left[\mathbb{P}\left((S-d)_+ > s\right)\right]\right\}\mathrm{d}s \\
&= \int_0^{w+\pi_S(d)} \left\{h\left[\mathbb{P}\left((S-d)_+ > s\right)_{\{S\le d\}}\right] - \bar{g}\left[\mathbb{P}\left((S-d)_+ > s\right)_{\{S\le d\}}\right]\right\}\mathrm{d}s \\
&\quad + \int_0^{w+\pi_S(d)} \left\{h\left[\mathbb{P}\left((S-d)_+ > s\right)_{\{S> d\}}\right] - \bar{g}\left[\mathbb{P}\left((S-d)_+ > s\right)_{\{S> d\}}\right]\right\}\mathrm{d}s \\
\xrightarrow[\Rightarrow \mathbb{P}(0>s)=0;\ 0\le s\le w+\pi_S(d)]{S\le d\Rightarrow (S-d)_+=0} &= \int_0^{w+\pi_S(d)} \left\{h(0) - \bar{g}(0)\right\}\mathrm{d}s \\
&\quad + \int_0^{w+\pi_S(d)} \left\{h\left[\mathbb{P}\left((S-d)_+ > s\right)_{\{S> d\}}\right] - \bar{g}\left[\mathbb{P}\left((S-d)_+ > s\right)_{\{S> d\}}\right]\right\}\mathrm{d}s \\
\xrightarrow[\bar{g}(0)=1-g(1)=0]{h(0)=0} &= \int_0^{w+\pi_S(d)} \left\{h\left[\mathbb{P}\left(S-d > s\right)\right] - \bar{g}\left[\mathbb{P}\left(S-d > s\right)\right]\right\}\mathrm{d}s \\
\xrightarrow{t=s} &= \int_0^{w+\pi_S(d)} h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t - \int_0^{w+\pi_S(d)} \bar{g}\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t \\
\xrightarrow{\bar{g}(p)=1-g(1-p)} &= \int_0^{w+\pi_S(d)} h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t - \int_0^{w+\pi_S(d)} \left\{1 - g\left[\mathbb{P}\left(S \le d+t\right)\right]\right\}\mathrm{d}t \\
&= \int_0^{w+\pi_S(d)} h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t - (w+\pi_S(d)) + \int_0^{w+\pi_S(d)} g\left[\mathbb{P}\left(S \le d+t\right)\right]\mathrm{d}t
\end{aligned}$$

$$\Rightarrow I_2 = \int_0^{w+\pi_S(d)} h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t - (w+\pi_S(d)) + \int_0^{w+\pi_S(d)} g\left[\mathbb{P}\left(S \le d+t\right)\right]\mathrm{d}t \tag{A.18}$$

Lastly, if we take the simplified Equation (A.17) and Equation (A.18) , the Equation (A.16) can be rewritten as

$$w = -\int_0^\infty h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t + (w + \pi_S(d))$$

$$+\int_0^{w+\pi_S(d)} h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t - (w+\pi_S(d)) + \int_0^{w+\pi_S(d)} g\left[\mathbb{P}\left(S \le d+t\right)\right]\mathrm{d}t$$

$$\Rightarrow w = -\int_0^\infty h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t + \int_0^{w+\pi_S(d)} \left\{h\left[\mathbb{P}\left(S > d+t\right)\right] + g\left[\mathbb{P}\left(S \le d+t\right)\right]\right\}\mathrm{d}t$$

In order to find $\pi_S(d)$, we define $\varphi(x) = \int_0^x \left\{h\left[\mathbb{P}\left(S > d+t\right)\right] + g\left[\mathbb{P}\left(S \le d+t\right)\right]\right\}\mathrm{d}t$.

$$\xrightarrow[\Rightarrow\varphi(x)\text{ is invertible}]{Appendix\ A.3.2} w = -\int_0^\infty h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t + \varphi\left(w+\pi_S(d)\right)$$

$$w + \int_0^\infty h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t = \varphi\left(w+\pi_S(d)\right)$$

$$\varphi^{-1}\left(w + \int_0^\infty h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t\right) = w + \pi_S(d)$$

$$\pi_S(d) = \varphi^{-1}\left(w + \int_0^\infty h\left[\mathbb{P}\left(S > d+t\right)\right]\mathrm{d}t\right) - w \tag{A.19}$$

□

A.3.2. Proof for invertibility of $\varphi(x)$ in Equation (4.16)

Proof. In order to prove that "$\varphi(x)$ is invertible", we need to show that $\varphi(x)$ is either strictly increasing or decreasing. Thus, we check $\varphi'(x) > 0$ or $\varphi'(x) < 0$ as follows.

$$\begin{aligned}
\varphi'(x) &= \frac{d}{dx}\int_0^x \left\{h\left[\mathbb{P}\left(S > d+t\right)\right] + g\left[\mathbb{P}\left(S \le d+t\right)\right]\right\}dt \\
&= h\left[\mathbb{P}\left(S > d+x\right)\right] + g\left[\mathbb{P}\left(S \le d+x\right)\right] \\
\xrightarrow{h,g:[0,1]\to[0,1]} &\ge 0
\end{aligned}$$

Here, if $h\left[\mathbb{P}\left(S > d+x\right)\right] + g\left[\mathbb{P}\left(S \leq d+x\right)\right] = 0$, then both $h\left[\mathbb{P}\left(S > d+x\right)\right] = 0$ and $g\left[\mathbb{P}\left(S \leq d+x\right)\right] = 0$, i.e. both $\mathbb{P}\left(S > d+x\right) = 0$ and $\mathbb{P}\left(S \leq d+x\right) = 0$ must be true.

However, if $\mathbb{P}\left(S > d+x\right) = 0$, then $\mathbb{P}\left(S \leq d+x\right) = 1$; or if $\mathbb{P}\left(S \leq d+x\right) = 0$, then $\mathbb{P}\left(S > d+x\right) = 1$. Thus, $h\left[\mathbb{P}\left(S > d+x\right)\right] + g\left[\mathbb{P}\left(S \leq d+x\right)\right] \neq 0$.

So we obtain that $h\left[\mathbb{P}\left(S > d+x\right)\right] + g\left[\mathbb{P}\left(S \leq d+x\right)\right] > 0$.

We conclude the proof as $\varphi'(x) > 0$, thus $\varphi(x)$ is invertible. □

A.3.3. Solution of CPT stop-loss premium for Value Function 2

Proof. Let the value function $\upsilon(w) = \dfrac{1-e^{-bw}}{a}$. The minimum premium that the reinsurer accept to cover the risk $(S-d)_+$ is determined by using the Equation (4.14). We obtain $\pi_S(d)$ as follows.

$$\upsilon(w) = \mathbb{E}_{gh}\left[\upsilon\left(w + \pi_S(d) - (S-d)_+\right)\right]$$

$$\xrightarrow{Prop.\ 4.3.2,\ (C_2)} \frac{1-e^{-bw}}{a} = \frac{1}{a}\mathbb{E}_{gh}\left[1 - e^{-b(w+\pi_S(d)-(S-d)_+)}\right]$$

Take $c = 1$ and $-e^{-b(w+\pi_S(d)-(S-d)_+)} = Y$; then

$$\xrightarrow{Equation\ (4.8)} \quad 1-e^{-bw} = \mathbb{E}_{gh}(Y) + 1 + \textstyle\int_0^1 \left[h\left(\mathbb{P}(-Y > s)\right) - \bar{g}\left(\mathbb{P}(-Y > s)\right)\right] \mathrm{d}s$$

$$\xrightarrow{Prop.\ 4.3.2,\ (C_3)} \quad 1-e^{-bw} = -\mathbb{E}_{hg}\left[e^{-b(w+\pi_S(d)-(S-d)_+)}\right] + 1$$

$$+ \textstyle\int_0^1 \left[h\left(\mathbb{P}(e^{-b(w+\pi_S(d)-(S-d)_+)} > s)\right) - \bar{g}\left(\mathbb{P}(e^{-b(w+\pi_S(d)-(S-d)_+)} > s)\right)\right] \mathrm{d}s$$

$$\xrightarrow{se^{b\left(w+\pi_S(d)\right)}=t} \quad -e^{-bw} = -e^{-b(w+\pi_S(d))}\mathbb{E}_{hg}\left[e^{b(S-d)_+}\right]$$

$$+\int_0^{e^{b\left(w+\pi_S(d)\right)}} \left\{ \begin{array}{c} h\left[\mathbb{P}\left(e^{-b(w+\pi_S(d))}e^{b(S-d)_+} > \dfrac{t}{e^{b(w+\pi_S(d))}}\right)\right] \\ \\ -\bar{g}\left[\mathbb{P}\left(e^{-b(w+\pi_S(d))}e^{b(S-d)_+} > \dfrac{t}{e^{b(w+\pi_S(d))}}\right)\right] \end{array} \right\} \frac{\mathrm{d}t}{e^{b(w+\pi_S(d))}}$$

$$\Rightarrow \quad -e^{-bw} = -e^{-b(w+\pi_S(d))}\mathbb{E}_{hg}\left[e^{b(S-d)_+}\right]$$

$$+e^{-b(w+\pi_S(d))}\int_0^{e^{b\left(w+\pi_S(d)\right)}} \left\{h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right]\right\}\mathrm{d}t$$

$$\Rightarrow \quad -e^{-bw} = -e^{-bw}e^{-b\pi_S(d)}\mathbb{E}_{hg}\left[e^{b(S-d)_+}\right]$$

$$+e^{-bw}e^{-b\pi_S(d)}\int_0^{e^{b\left(w+\pi_S(d)\right)}} \left\{h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right]\right\}\mathrm{d}t$$

$$\Rightarrow \quad 1 = e^{-b\pi_S(d)}\mathbb{E}_{hg}\left[e^{b(S-d)_+}\right]$$

$$-e^{-b\pi_S(d)}\int_0^{e^{b\left(w+\pi_S(d)\right)}} \left\{h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right]\right\}\mathrm{d}t$$

$$\Rightarrow \quad e^{b\pi_S(d)} = \mathbb{E}_{hg}\left[e^{b(S-d)_+}\right] - \int_0^{e^{b\left(w+\pi_S(d)\right)}} \left\{h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right]\right\}\mathrm{d}t$$

Now divide the equation into the following integral parts.

$$e^{b\pi_S(d)} = I_3 - I_4 \tag{A.20}$$

where $I_3 = \mathbb{E}_{hg}\left[e^{b(S-d)_+}\right]$ and $I_4 = \int_0^{e^{b\left(w+\pi_S(d)\right)}} \left\{h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right]\right\}\mathrm{d}t$.

At first, we will simplify I_3:

$$I_3 = \mathbb{E}_{hg}\left[e^{b(S-d)_+}\right] \xrightarrow{e^{b(S-d)_+}\geq 0} = \mathbb{E}_h\left[e^{b(S-d)_+}\right] = \int_0^\infty h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right]\mathrm{d}t$$

$$= \int_0^\infty h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)_{\{S\leq d\}}\right]\mathrm{d}t + \int_0^\infty h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)_{\{S>d\}}\right]\mathrm{d}t$$

$$\xrightarrow[\substack{\Rightarrow \mathbb{P}(e^0=1>t)=1;\ 0\leq t<1 \\ \Rightarrow \mathbb{P}(e^0=1>t)=0;\ 1\leq t<\infty}]{S\leq d \Rightarrow (S-d)_+=0} = \int_0^1 h(1)\,\mathrm{d}t + \int_1^\infty h(0)\,\mathrm{d}t + \int_0^\infty h\left[\mathbb{P}\left(e^{b(S-d)} > t\right)\right]\mathrm{d}t$$

$$\xrightarrow{h(0)=0,\ h(1)=1} = 1 + \int_0^\infty h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right]\mathrm{d}t$$

$$\Rightarrow I_3 = 1 + \int_0^\infty h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right]\mathrm{d}t \tag{A.21}$$

Now, we will simplify I_4:

$$I_4 = \int_0^{e^{b\left(w+\pi_S(d)\right)}}\left\{h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)\right]\right\}\mathrm{d}t$$

$$= \int_0^{e^{b\left(w+\pi_S(d)\right)}}\left\{h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)_{\{S\leq d\}}\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)_{\{S\leq d\}}\right]\right\}\mathrm{d}t$$

$$+ \int_0^{e^{b\left(w+\pi_S(d)\right)}}\left\{h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)_{\{S>d\}}\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)_{\{S>d\}}\right]\right\}\mathrm{d}t$$

Since $S \leq d \Rightarrow (S-d)_+ = 0$, we can write $\mathbb{P}(e^0 = 1 > t) = 1$ for $0 \leq t < 1$ and $\mathbb{P}(e^0 = 1 > t) = 0$ for $1 \leq t \leq e^{b(w+\pi_S(d))}$. Thus,

$$= \int_0^1\left\{h(1) - \bar{g}(1)\right\}\mathrm{d}t + \int_1^{e^{b\left(w+\pi_S(d)\right)}}\left\{h(0) - \bar{g}(0)\right\}\mathrm{d}t$$

$$+ \int_0^{e^{b\left(w+\pi_S(d)\right)}}\left\{h\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)_{\{S>d\}}\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)_+} > t\right)_{\{S>d\}}\right]\right\}\mathrm{d}t$$

$$\xrightarrow[h(1)=1,\ \bar{g}(1)=1-g(0)=1]{h(0)=0,\ \bar{g}(0)=1-g(1)=0} \quad = \int_0^{e^{b\left(w+\pi_S(d)\right)}} \left\{ h\left[\mathbb{P}\left(e^{b(S-d)} > t\right)\right] - \bar{g}\left[\mathbb{P}\left(e^{b(S-d)} > t\right)\right] \right\} \mathrm{d}t$$

$$= \int_0^{e^{b\left(w+\pi_S(d)\right)}} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t - \int_0^{e^{b\left(w+\pi_S(d)\right)}} \bar{g}\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t$$

$$\xrightarrow{\bar{g}(p)=1-g(1-p)} \quad = \int_0^{e^{b\left(w+\pi_S(d)\right)}} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t$$

$$- \int_0^{e^{b\left(w+\pi_S(d)\right)}} \left\{1 - g\left[\mathbb{P}\left(S \leq \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t\right\}$$

$$= \int_0^{e^{b\left(w+\pi_S(d)\right)}} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t$$

$$- \left(e^{b(w+\pi_S(d))}\right) + \int_0^{e^{b\left(w+\pi_S(d)\right)}} g\left[\mathbb{P}\left(S \leq \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t$$

$$\Rightarrow I_4 = \int_0^{e^{b\left(w+\pi_S(d)\right)}} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t - \left(e^{b(w+\pi_S(d))}\right) + \int_0^{e^{b\left(w+\pi_S(d)\right)}} g\left[\mathbb{P}\left(S \leq \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t \tag{A.22}$$

Lastly, if we take the simplified Equation (A.21) and Equation (A.22) , the Equation (A.20) can be rewritten as

$$e^{b\pi_S(d)} = 1 + \int_0^{\infty} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t - \int_0^{e^{b\left(w+\pi_S(d)\right)}} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t + \left(e^{b(w+\pi_S(d))}\right)$$

$$- \int_0^{e^{b\left(w+\pi_S(d)\right)}} g\left[\mathbb{P}\left(S \leq \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t$$

$$\Rightarrow e^{b\pi_S(d)} = 1 + \int_{e^{b\left(w+\pi_S(d)\right)}}^{\infty} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t + \left(e^{b(w+\pi_S(d))}\right)$$

$$- \int_0^{e^{b\left(w+\pi_S(d)\right)}} g\left[\mathbb{P}\left(S \leq \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t$$

$$\Rightarrow 1 = \left[e^{b(w+\pi_S(d))}\left(e^{-bw}-1\right)\right] - \int_{e^{b(w+\pi_S(d))}}^{\infty} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t$$

$$+ \int_0^{e^{b(w+\pi_S(d))}} g\left[\mathbb{P}\left(S \leq \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t$$

In order to find $\pi_S(d)$, we define

$$\varphi(x) = x\left(e^{-bw}-1\right) + \int_0^x g\left[\mathbb{P}\left(S \leq \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t - \int_x^{\infty} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t.$$

$$\xrightarrow[\Rightarrow \varphi(x) \text{ is invertible}]{Appendix\ A.3.4} 1 = \varphi\left(e^{b(w+\pi_S(d))}\right)$$

$$\Rightarrow \varphi^{-1}(1) = e^{b(w+\pi_S(d))} \Rightarrow \ln\left[\varphi^{-1}(1)\right] = b\left(w + \pi_S(d)\right)$$

$$\pi_S(d) = \frac{\ln\left[\varphi^{-1}(1)\right]}{b} - w \tag{A.23}$$

□

A.3.4. Proof for invertibility of $\varphi(x)$ in Equation (4.18)

Proof. In order to prove that "$\varphi(x)$ is invertible", we need to show that $\varphi(x)$ is either strictly increasing or decreasing. Thus, we check $\varphi'(x) > 0$ or $\varphi'(x) < 0$ as follows.

$$\varphi'(x) = \frac{d}{dx}\left\{x\left(e^{-bw}-1\right) + \int_0^x g\left[\mathbb{P}\left(S \leq \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t \right.$$

$$\left. - \int_x^{\infty} h\left[\mathbb{P}\left(S > \frac{\ln t + bd}{b}\right)\right] \mathrm{d}t\right\}$$

$$= \left(e^{-bw}-1\right) + g\left[\mathbb{P}\left(S \leq \frac{\ln x + bd}{b}\right)\right] + h\left[\mathbb{P}\left(S > \frac{\ln x + bd}{b}\right)\right]$$

$$\xrightarrow[\lim_{w\to\infty} e^{-bw}=0]{g(p)+h(1-p)<1;\ p\neq 0,1 [27]} \varphi'(x) < 0$$

Here, we need to assume $1 - e^{-bw} \neq h\left[\mathbb{P}\left(S > \frac{\ln x + bd}{b}\right)\right] + g\left[\mathbb{P}\left(S \leq \frac{\ln x + bd}{b}\right)\right]$ for $\varphi'(x) \neq 0$. □

A.3.5. Solution of CPT stop-loss premium for Value Function 3

Proof. Let the value function $\upsilon(w) = \begin{cases} w^{\alpha} & , w \geq 0 \\ -\lambda(-w)^{\beta} & , w < 0 \end{cases}$. The minimum premium that the reinsurer accept to cover the risk $(S-d)_+$ is determined by using the Equation (4.14). We obtain $\pi_S(d)$ as follows.

$$\upsilon(w) = \mathbb{E}_{gh}\left[\upsilon\left(w + \pi_S(d) - (S-d)_+\right)\right]$$

$$\xrightarrow{w \geq 0} w^{\alpha} = \mathbb{E}_{gh}\left[\upsilon\left(w + \pi_S(d) - (S-d)_+\right)\right]$$

For $(S-d)_+ \leq w + \pi_S(d)$, the amount of $w + \pi_S(d) - (S-d)_+$ is seen as a gain and we use the value and probability weighting functions for gains. Otherwise, we use these functions for losses.

$$\Rightarrow w^{\alpha} = \mathbb{E}_g\left[\left(w + \pi_S(d) - (S-d)_+\right)^{\alpha}\right] - \mathbb{E}_h\left[-\lambda\left((S-d)_+ - w - \pi_S(d)\right)^{\beta}\right]$$

$$\xrightarrow{Equation\ (4.6)} = \int_{-\infty}^{0}\left\{g\left(\mathbb{P}\left[\left(w + \pi_S(d) - (S-d)_+\right)^{\alpha} > t\right]\right) - 1\right\}\mathrm{d}t$$

$$+ \int_{0}^{\infty} g\left(\mathbb{P}\left[\left(w + \pi_S(d) - (S-d)_+\right)^{\alpha} > t\right]\right)\mathrm{d}t$$

$$- \int_{-\infty}^{0}\left\{h\left(\mathbb{P}\left[-\lambda\left((S-d)_+ - w - \pi_S(d)\right)^{\beta} > t\right]\right) - 1\right\}\mathrm{d}t$$

$$- \int_{0}^{\infty} h\left(\mathbb{P}\left[-\lambda\left((S-d)_+ - w - \pi_S(d)\right)^{\beta} > t\right]\right)\mathrm{d}t$$

If we represent each of the four integrals in the above equation as $I_5,\ I_6,\ I_7$ and I_8, respectively, then we have the following equation.

$$w^{\alpha} = I_5 + I_6 - I_7 - I_8. \tag{A.24}$$

At first, we will simplify I_5:

$$\begin{aligned} I_5 &= \int_{-\infty}^{0} \{g\left(\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^{\alpha} > t\right]\right) - 1\}\, \mathrm{d}t \\ &= \int_{-\infty}^{0} \left\{g\left(\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^{\alpha} > t\right]_{\{S\le d\}}\right) - 1\right\} \mathrm{d}t \\ &+ \int_{-\infty}^{0} \left\{g\left(\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^{\alpha} > t\right]_{\{S> d\}}\right) - 1\right\} \mathrm{d}t \end{aligned}$$

(i) If $S \le d \Rightarrow (S-d)_+ = 0$ and since $w, \pi_S(d) \ge 0$, then we can write $\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^{\alpha} > t\right] = \mathbb{P}\left[(w+\pi_S(d))^{\alpha} > t\right] = 1$ for $-\infty < t < 0$,

(ii) If $S > d \Rightarrow (S-d)_+ = S-d$ and since $(S-d)_+ \le w+\pi_S(d) \Rightarrow w+\pi_S(d)-(S-d)_+ \ge 0$, then we can write $\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^{\alpha} > t\right] = 1$ for $-\infty < t < 0$.

Thus, we obtain I_5 as follows.

$$\Rightarrow I_5 = \int_{-\infty}^{0} \{g(1)-1\}_{\{S\le d\}}\, \mathrm{d}t + \int_{-\infty}^{0} \{g(1)-1\}_{\{S>d\}}\, \mathrm{d}t \xrightarrow{g(1)=1} I_5 = 0. \tag{A.25}$$

Now, we will simplify I_6:

$$\begin{aligned} I_6 &= \int_{0}^{\infty} g\left(\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^{\alpha} > t\right]\right) \mathrm{d}t \\ &= \int_{0}^{\infty} g\left(\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^{\alpha} > t\right]_{\{S\le d\}}\right) \mathrm{d}t \\ &+ \int_{0}^{\infty} g\left(\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^{\alpha} > t\right]_{\{S> d\}}\right) \mathrm{d}t \end{aligned}$$

(i) If $S \le d \Rightarrow (S-d)_+ = 0$, then

$$\int_0^\infty g\left(\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^\alpha > t\right]_{\{S\le d\}}\right)\mathrm{d}t = \int_0^\infty g\left(\mathbb{P}\left[(w+\pi_S(d))^\alpha > t\right]\right)\mathrm{d}t$$

$$= \int_0^{(w+\pi_S(d))^\alpha} g\left(\mathbb{P}\left[(w+\pi_S(d))^\alpha > t\right]\right)\mathrm{d}t$$

$$+ \int_{(w+\pi_S(d))^\alpha}^\infty g\left(\mathbb{P}\left[(w+\pi_S(d))^\alpha > t\right]\right)\mathrm{d}t$$

$$= \int_0^{(w+\pi_S(d))^\alpha} g(1)\,\mathrm{d}t + \int_{(w+\pi_S(d))^\alpha}^\infty g(0)\,\mathrm{d}t$$

$$\xrightarrow{g(0)=0,\ g(1)=1} = (w+\pi_S(d))^\alpha .$$

(ii) If $S > d \Rightarrow (S-d)_+ = S-d$ and since $(S-d)_+ \le w + \pi_S(d)$, then

$$\int_0^\infty g\left(\mathbb{P}\left[(w+\pi_S(d)-(S-d)_+)^\alpha > t\right]_{\{S> d\}}\right)\mathrm{d}t = \int_0^\infty g\left(\mathbb{P}\left[(w+\pi_S(d)-S+d)^\alpha > t\right]\right)\mathrm{d}t$$

$$= \int_0^\infty g\left(\mathbb{P}\left[w+\pi_S(d)-S+d > t^{1/\alpha}\right]\right)\mathrm{d}t$$

$$= \int_0^\infty g\left(\mathbb{P}\left[S < -t^{1/\alpha}+w+\pi_S(d)+d\right]\right)\mathrm{d}t$$

$$\xrightarrow[\substack{0<t<\infty \\ \Rightarrow -\infty < s < (w+\pi_S(d)+d)^\alpha}]{\substack{-t^{1/\alpha}+w+\pi_S(d)+d = s^{1/\alpha} \\ \Rightarrow -\frac{1}{\alpha}t^{\left(\frac{1}{\alpha}-1\right)}\mathrm{d}t = \frac{1}{\alpha}s^{\left(\frac{1}{\alpha}-1\right)}\mathrm{d}s}} = \int_{-\infty}^{(w+\pi_S(d)+d)^\alpha} g\left(\mathbb{P}\left[S < s^{1/\alpha}\right]\right)\mathrm{d}s.$$

Thus, we obtain I_6 as follows.

$$\Rightarrow I_6 = (w+\pi_S(d))^\alpha + \int_{-\infty}^{(w+\pi_S(d)+d)^\alpha} g\left(\mathbb{P}\left[S < s^{1/\alpha}\right]\right)\mathrm{d}s. \tag{A.26}$$

Now, we will simplify I_7:

$$\begin{aligned} I_7 &= \int_{-\infty}^{0} \left\{ h\left(\mathbb{P}\left[-\lambda\left((S-d)_+ - w - \pi_S(d) \right)^\beta > t \right] \right) - 1 \right\} \mathrm{d}t \\ &= \int_{-\infty}^{0} \left\{ h\left(\mathbb{P}\left[-\lambda\left((S-d)_+ - w - \pi_S(d) \right)^\beta > t \right]_{\{S\le d\}} \right) - 1 \right\} \mathrm{d}t \\ &+ \int_{-\infty}^{0} \left\{ h\left(\mathbb{P}\left[-\lambda\left((S-d)_+ - w - \pi_S(d) \right)^\beta > t \right]_{\{S> d\}} \right) - 1 \right\} \mathrm{d}t \end{aligned}$$

(i) If $S \le d \Rightarrow (S-d)_+ = 0$, and since $(S-d)_+ > w + \pi_S(d) \Rightarrow w + \pi_S(d) < 0$ which is impossible where $w,\ \pi_S(d) \ge 0$, then

$$\int_{-\infty}^{0} \left\{ h\left(\mathbb{P}\left[-\lambda\left((S-d)_+ - w - \pi_S(d) \right)^\beta > t \right]_{\{S\le d\}} \right) - 1 \right\} \mathrm{d}t = 0.$$

(ii) If $S > d \Rightarrow (S-d)_+ = S - d$ and since $(S-d)_+ > w + \pi_S(d)$, then

$$\begin{aligned} &\int_{-\infty}^{0} \left\{ h\left(\mathbb{P}\left[-\lambda\left((S-d)_+ - w - \pi_S(d) \right)^\beta > t \right]_{\{S> d\}} \right) - 1 \right\} \mathrm{d}t \\ &\quad = \int_{-\infty}^{0} \left\{ h\left(\mathbb{P}\left[-\lambda\left(S-d - w - \pi_S(d) \right)^\beta > t \right] \right) - 1 \right\} \mathrm{d}t \\ &\quad = \int_{-\infty}^{0} \left\{ h\left(\mathbb{P}\left[\left(S-d - w - \pi_S(d) \right)^\beta < -\frac{t}{\lambda} \right] \right) - 1 \right\} \mathrm{d}t \\ &\quad = \int_{-\infty}^{0} \left\{ h\left(\mathbb{P}\left[S-d - w - \pi_S(d) < \left(-\frac{t}{\lambda} \right)^{1/\beta} \right] \right) - 1 \right\} \mathrm{d}t \\ &\quad = \int_{-\infty}^{0} \left\{ h\left(\mathbb{P}\left[S < \left(-\frac{t}{\lambda} \right)^{1/\beta} + d + w + \pi_S(d) \right] \right) - 1 \right\} \mathrm{d}t \end{aligned}$$

$$\underset{-\infty<t<0\Rightarrow-\infty<s<-\lambda(d+w+\pi_S(d))^\beta}{\xrightarrow[]{\substack{\left(-\frac{t}{\lambda}\right)^{1/\beta}+d+w+\pi_S(d)=\left(-\frac{s}{\lambda}\right)^{1/\beta} \\ \Rightarrow\frac{1}{\beta}\left(-\frac{t}{\lambda}\right)^{\left(\frac{1}{\beta}-1\right)}\frac{-1}{\lambda}\,\mathrm{d}t=\frac{1}{\beta}\left(-\frac{s}{\lambda}\right)^{\left(\frac{1}{\beta}-1\right)}\frac{-1}{\lambda}\,\mathrm{d}s}}}= \int_{-\infty}^{-\lambda(d+w+\pi_S(d))^\beta}\left\{h\left(\mathbb{P}\left[S<\left(-\frac{s}{\lambda}\right)^{1/\beta}\right]\right)-1\right\}\mathrm{d}s.$$

Thus, we obtain I_7 as follows.

$$\Rightarrow I_7=\int_{-\infty}^{-\lambda(d+w+\pi_S(d))^\beta}\left\{h\left(\mathbb{P}\left[S<\left(-\frac{s}{\lambda}\right)^{1/\beta}\right]\right)-1\right\}\mathrm{d}s. \tag{A.27}$$

Now, we will simplify I_8:

$$\begin{aligned} I_8 &= \int_0^\infty h\left(\mathbb{P}\left[-\lambda\left((S-d)_+-w-\pi_S(d)\right)^\beta>t\right]\right)\mathrm{d}t \\ &= \int_0^\infty h\left(\mathbb{P}\left[-\lambda\left((S-d)_+-w-\pi_S(d)\right)^\beta>t\right]_{\{S\le d\}}\right)\mathrm{d}t \\ &\quad+\int_0^\infty h\left(\mathbb{P}\left[-\lambda\left((S-d)_+-w-\pi_S(d)\right)^\beta>t\right]_{\{S>d\}}\right)\mathrm{d}t \end{aligned}$$

(i) If $S\le d\Rightarrow (S-d)_+=0$, and since $(S-d)_+>w+\pi_S(d)\Rightarrow w+\pi_S(d)<0$ which is impossible where $w,\ \pi_S(d)\ge 0$, then

$$\int_0^\infty h\left(\mathbb{P}\left[-\lambda\left((S-d)_+-w-\pi_S(d)\right)^\beta>t\right]_{\{S\le d\}}\right)\mathrm{d}t=0.$$

(ii) If $S>d\Rightarrow (S-d)_+=S-d$ and since $(S-d)_+>w+\pi_S(d)\Rightarrow -\lambda\left(S-d-w-\pi_S(d)\right)^\beta<0$, then we can write $\mathbb{P}\left[-\lambda\left(S-d-w-\pi_S(d)\right)^\beta>t\right]=0$ for $0\le t<\infty$.

Thus, we obtain I_8 as follows.

$$\Rightarrow I_8=\int_0^\infty h(0)\,\mathrm{d}t \xrightarrow{h(0)=0} I_8=0. \tag{A.28}$$

Lastly, if we take the simplified Equation (A.25), Equation (A.26), Equation (A.27) and Equation (A.28) , the Equation (A.24) can be rewritten as

$$\begin{aligned} w^{\alpha} = \; & (w+\pi_S(d))^{\alpha} + \int_{-\infty}^{(w+\pi_S(d)+d)^{\alpha}} g\left(\mathbb{P}\left[S < s^{1/\alpha}\right]\right) \mathrm{d}s \\ & - \int_{-\infty}^{-\lambda(d+w+\pi_S(d))^{\beta}} \left\{ h\left(\mathbb{P}\left[S < \left(-\frac{s}{\lambda}\right)^{1/\beta}\right]\right) - 1 \right\} \mathrm{d}s \end{aligned}$$

In order to find $\pi_S(d)$, we define

$$\varphi(x) = (x-d)^{\alpha} + \int_{-\infty}^{x^{\alpha}} g\left(\mathbb{P}\left[S < s^{1/\alpha}\right]\right) \mathrm{d}s - \int_{-\infty}^{-\lambda x^{\beta}} \left\{ h\left(\mathbb{P}\left[S < \left(-\frac{s}{\lambda}\right)^{1/\beta}\right]\right) - 1 \right\} \mathrm{d}s.$$

$$\xrightarrow[\Rightarrow \varphi(x) \text{ is invertible}]{Appendix\ A.3.6} w^{\alpha} = \varphi\left(w + \pi_S(d) + d\right)$$

$$\Rightarrow \varphi^{-1}\left(w^{\alpha}\right) = w + \pi_S(d) + d$$

$$\pi_S(d) = \varphi^{-1}\left(w^{\alpha}\right) - w - d \tag{A.29}$$

□

A.3.6. Proof for invertibility of $\varphi(x)$ in Equation (4.20)

Proof. In order to prove that "$\varphi(x)$ is invertible", we need to show that $\varphi(x)$ is either strictly increasing or decreasing. Thus, we check $\varphi'(x) > 0$ or $\varphi'(x) < 0$ as follows.

$$\begin{aligned} \varphi'(x) \; & = \frac{d}{dx}\Big[(x-d)^{\alpha} + \int_{-\infty}^{x^{\alpha}} g\left(\mathbb{P}\left[S < s^{1/\alpha}\right]\right) \mathrm{d}s \\ & - \int_{-\infty}^{-\lambda x^{\beta}} \left\{ h\left(\mathbb{P}\left[S < \left(-\frac{s}{\lambda}\right)^{1/\beta}\right]\right) - 1 \right\} \mathrm{d}s \Big] \end{aligned}$$

$$= \alpha (x-d)^{\alpha-1} + g \left(\mathbb{P} \left[S < (x^{\alpha})^{1/\alpha} \right] \right) (\alpha x^{\alpha-1})$$

$$- \left\{ h \left(\mathbb{P} \left[S < \left(-\frac{-\lambda x^{\beta}}{\lambda} \right)^{1/\beta} \right] \right) - 1 \right\} \left(-\lambda \beta x^{\beta-1} \right)$$

$$= \alpha (x-d)^{\alpha-1} + g \left[\mathbb{P} (S < x) \right] (\alpha x^{\alpha-1}) - \left\{ h \left[\mathbb{P} (S < x) \right] - 1 \right\} \left(-\lambda \beta x^{\beta-1} \right)$$

$$\xrightarrow{\alpha=\beta} \quad = \alpha (x-d)^{\alpha-1} + (\alpha x^{\alpha-1}) \left(g \left[\mathbb{P} (S < x) \right] + \lambda h \left[\mathbb{P} (S < x) \right] - \lambda \right)$$

$\Rightarrow < 0$ or > 0

Here, we need to assume that $g \left[\mathbb{P} (S < d) \right] + \lambda h \left[\mathbb{P} (S < d) \right] \neq \lambda$ when $x = d$.

□

A.4. Appendix (Chapter 5): Risk Prioritization Through Stochastic Majorization

A.4.1. Proof of Schur-convexity of Sample Variance

Proposition A.4.1. *The sample variance defined as*

$$\phi_1(\boldsymbol{x}) = \phi_1(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

is strictly Schur-convex wrt $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$.

Proof. In order to show that ϕ_1 fulfills the Schur's conditions, we firstly check whether it is symmetric or not. Since this function gives the same values for all permutations of $\boldsymbol{x}$, it is symmetric. For all $(x_1, x_2, \ldots, x_n)$, the function $\phi_1(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is same since $(x_i - \bar{x})^2$ is same as the result of that $\bar{x}$ is same for all $i = 1, 2, \ldots, n$.

We also need to show that $(x_i - x_j)\left(\frac{\partial \phi_1}{\partial x_i} - \frac{\partial \phi_1}{\partial x_j}\right) > 0$ for all $1 \leq i, j \leq n$ in order to prove $\phi_1(\boldsymbol{x})$'s being strictly Schur-convex. We use Schur's Condition given by Theorem 5.2.4.

Let $x_1 > x_2$. Then,

$$\frac{\partial \phi_1}{\partial x_1} = \frac{1}{n}\left[\begin{array}{l} 2\left(x_1 - \frac{x_1 + \ldots + x_n}{n}\right)\left(1 - \frac{1}{n}\right) + 2\left(x_2 - \frac{x_1 + \ldots + x_n}{n}\right)\left(-\frac{1}{n}\right) \\ \\ + \ldots + 2\left(x_n - \frac{x_1 + \ldots + x_n}{n}\right)\left(-\frac{1}{n}\right) \end{array}\right]$$

and

$$\frac{\partial \phi_1}{\partial x_2} = \frac{1}{n}\left[\begin{array}{l} 2\left(x_1 - \frac{x_1 + \ldots + x_n}{n}\right)\left(-\frac{1}{n}\right) + 2\left(x_2 - \frac{x_1 + \ldots + x_n}{n}\right)\left(1 - \frac{1}{n}\right) \\ \\ + \ldots + 2\left(x_n - \frac{x_1 + \ldots + x_n}{n}\right)\left(-\frac{1}{n}\right) \end{array}\right]$$

Thus,

$$\frac{\partial \phi_1}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2} = \frac{2}{n}\left[\left(x_1 - \frac{x_1 + \ldots + x_n}{n}\right) - \left(x_2 - \frac{x_1 + \ldots + x_n}{n}\right)\right]$$

$$= \frac{2}{n}(x_1 - x_2)$$

Since $x_1 > x_2 \Rightarrow x_1 - x_2 > 0$, $\frac{\partial \phi_1}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2} = \frac{2}{n}(x_1 - x_2) > 0$.

This is true for all $1 \leq i, j \leq n$. Therefore, $(x_i - x_j)\left(\frac{\partial \phi_1}{\partial x_i} - \frac{\partial \phi_1}{\partial x_j}\right) > 0$ is true for all $1 \leq i, j \leq n$. $\phi_1(\boldsymbol{x})$ is strictly Schur-convex wrt $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. □

A.4.2. Proof of Schur-convexity of Sample Coefficient of Variation

Proposition A.4.2. *The sample variance defined as*

$$\phi_2(\boldsymbol{x}) = \phi_2(x_1, x_2, \ldots, x_n) = \frac{[\phi_1(\boldsymbol{x})]^{1/2}}{\bar{x}}$$

is strictly Schur-convex wrt $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ $in\mathbb{R}^+$.

Proof. In order to show that ϕ_2 fulfills the Schur's conditions, we firstly check whether it is symmetric or not. Since this function gives the same values for all permutations of $\boldsymbol{x}$, it is symmetric. For all $(x_1, x_2, \ldots, x_n)$, the function $\phi_2(x_1, x_2, \ldots, x_n)$ is same since $\phi_1(x_1, x_2, \ldots, x_n)$ is same (by Proof A.4.1) and $\bar{x}$ is same for all $i = 1, 2, \ldots, n$.

We also need to show that $(x_i - x_j)\left(\frac{\partial \phi_2}{\partial x_i} - \frac{\partial \phi_2}{\partial x_j}\right) > 0$ for all $1 \leq i, j \leq n$ in order to prove that $\phi_2(\boldsymbol{x})$ is strictly Schur-convex. We use Schur's Condition given by Theorem 5.2.4.

Let $x_1 > x_2$. Then,

$$\frac{\partial \phi_2}{\partial x_1} = \left(\frac{1}{\bar{x}}\right)\left(\frac{1}{2}\right)[\phi_1(\boldsymbol{x})]^{-1/2}\left(\frac{\partial \phi_1}{\partial x_1}\right)$$

$$\xrightarrow{Proof\ A.4.1} = \frac{[\phi_1(\boldsymbol{x})]^{-1/2}}{2n\bar{x}}\left[\begin{array}{l} 2\left(x_1 - \frac{x_1+\ldots+x_n}{n}\right)\left(1-\frac{1}{n}\right) + 2\left(x_2 - \frac{x_1+\ldots+x_n}{n}\right)\left(-\frac{1}{n}\right) \\ + \ldots + 2\left(x_n - \frac{x_1+\ldots+x_n}{n}\right)\left(-\frac{1}{n}\right) \end{array}\right]$$

and

$$\frac{\partial \phi_2}{\partial x_2} = \left(\frac{1}{\bar{x}}\right)\left(\frac{1}{2}\right)[\phi_1(\boldsymbol{x})]^{-1/2}\left(\frac{\partial \phi_1}{\partial x_2}\right)$$

$$\xrightarrow{Proof\ A.4.1} = \frac{[\phi_1(\boldsymbol{x})]^{-1/2}}{2n\bar{x}}\left[\begin{array}{l} 2\left(x_1 - \frac{x_1+\ldots+x_n}{n}\right)\left(-\frac{1}{n}\right) + 2\left(x_2 - \frac{x_1+\ldots+x_n}{n}\right)\left(1-\frac{1}{n}\right) \\ + \ldots + 2\left(x_n - \frac{x_1+\ldots+x_n}{n}\right)\left(-\frac{1}{n}\right) \end{array}\right]$$

Thus,

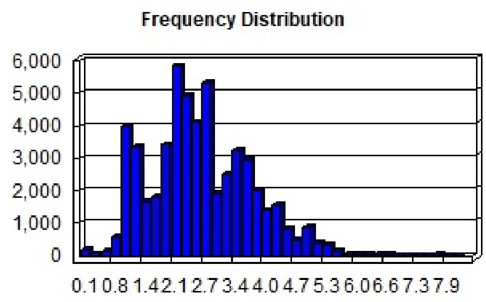
$$\begin{aligned}\frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_2}{\partial x_2} &= \frac{[\phi_1(\boldsymbol{x})]^{-1/2}}{n\bar{x}}\left[\left(x_1 - \frac{x_1+\ldots+x_n}{n}\right) - \left(x_2 - \frac{x_1+\ldots+x_n}{n}\right)\right] \\ &= \frac{[\phi_1(\boldsymbol{x})]^{-1/2}}{n\bar{x}}(x_1 - x_2)\end{aligned}$$

Since $x_1 > x_2 \Rightarrow x_1 - x_2 > 0$ and $\boldsymbol{x} \in \mathbb{R}^+ \Rightarrow \bar{x} > 0$, $\frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_2}{\partial x_2} = \frac{[\phi_1(\boldsymbol{x})]^{-1/2}}{n\bar{x}}(x_1 - x_2) > 0$.

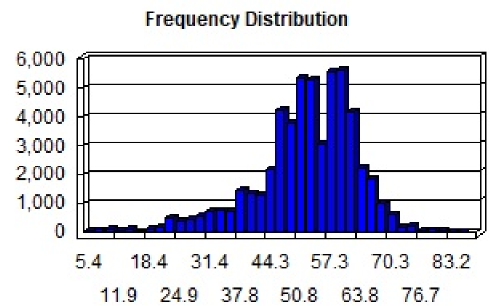
This is true for all $1 \leq i, j \leq n$. Therefore, $(x_i - x_j)\left(\frac{\partial \phi_2}{\partial x_i} - \frac{\partial \phi_2}{\partial x_j}\right) > 0$ is true for all $1 \leq i, j \leq n$. $\phi_2(\boldsymbol{x})$ is strictly Schur-convex wrt $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^+$. □

A.5. Appendix (Chapter 6): Risk Clustering Through STI

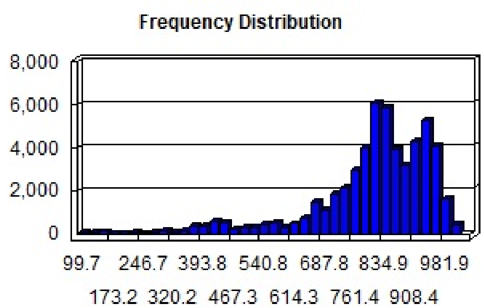
A.5.1. Histogram graphs of the estimated meteorological variables obtained with STI



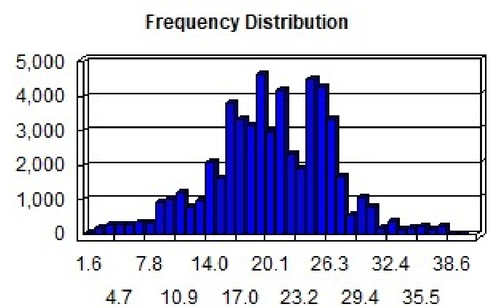
(a) avcloud



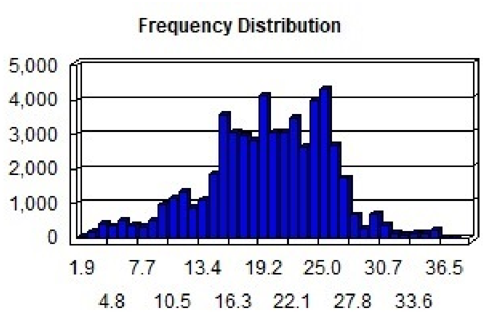
(b) avhum



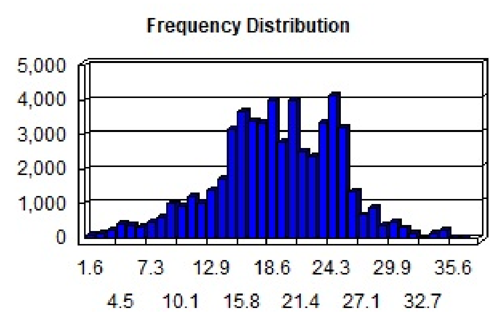
(c) avpres



(d) avsoiltemp5



(e) avsoiltemp10



(f) avsoiltemp20

Figure A.1: The histograms of the estimated meteorological values: (a)-(f)

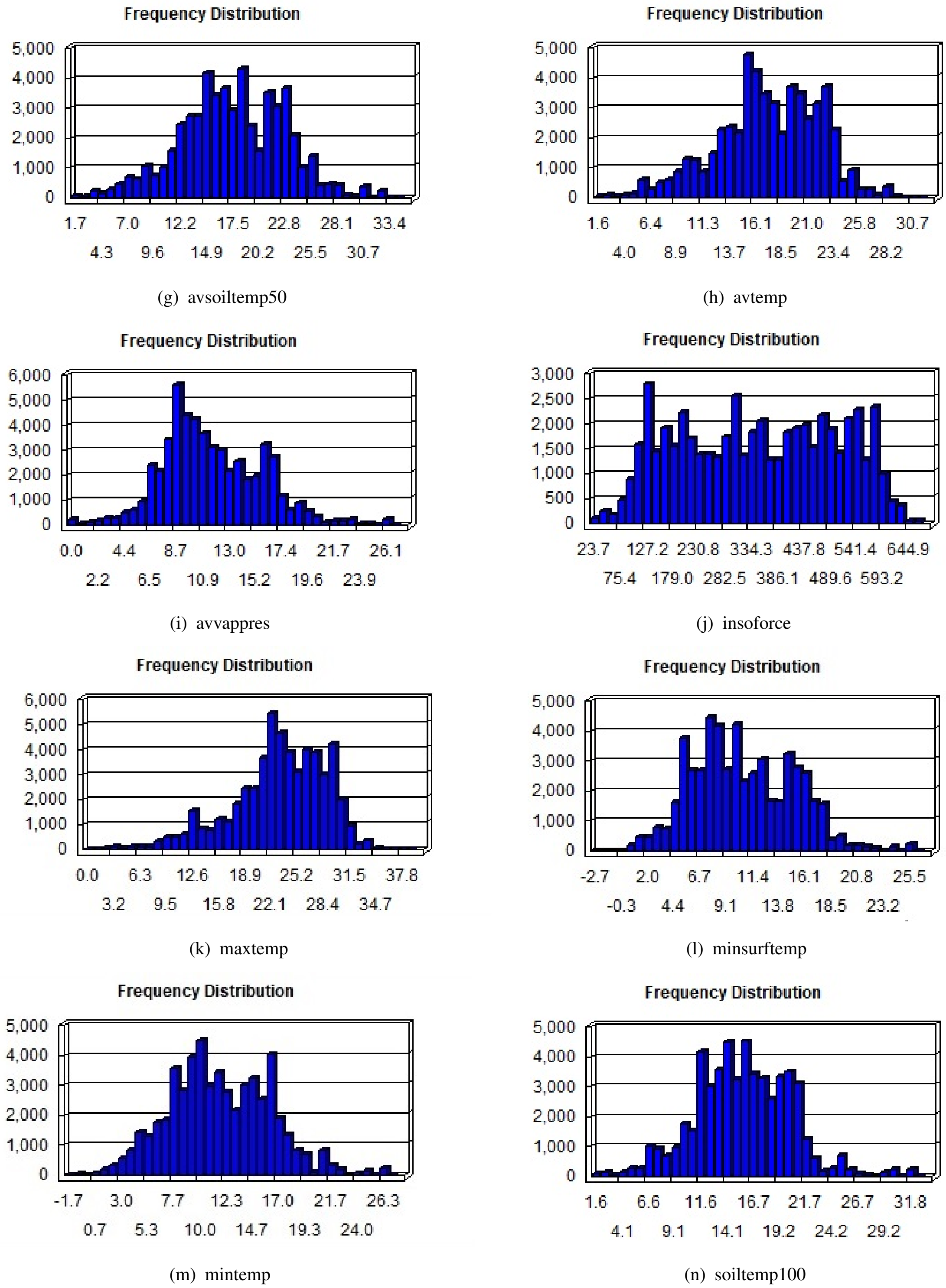
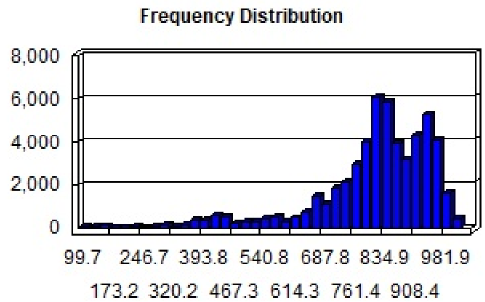
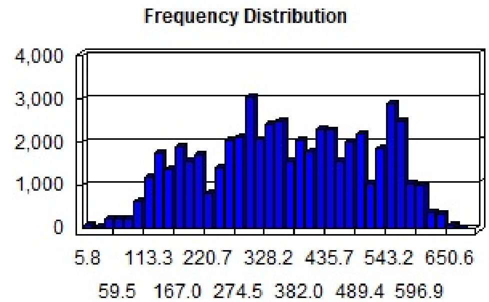


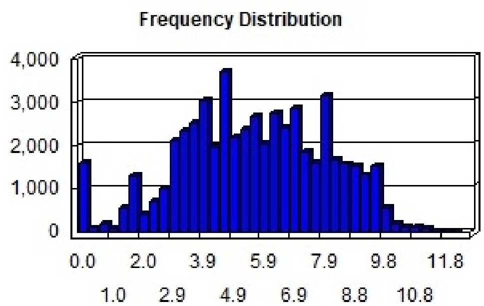
Figure A.2: The histograms of the estimated meteorological values: (g)-(n)



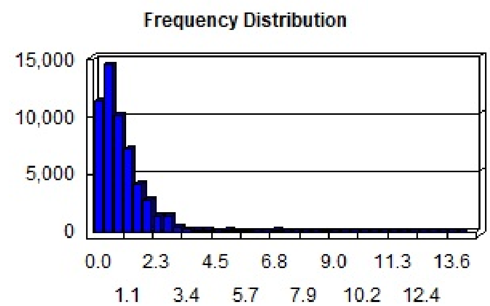
(o) toteva



(p) totglsolrad



(q) totinso



(r) totprec

Figure A.3: The histograms of the estimated meteorological values: (o)-(r)

A.5.2. Means and covariances of meteorological variables for each risk cluster

Table A.1: The means of meteorological variables for 1st, 2nd, 3rd risk cluster

Variable	RC1	RC2	RC3
minsurf	8.788189	8.8281923	15.5461123
totprec	1.466348	0.8865548	0.8872827
totinso	6.178126	5.7099397	7.1767608
totglsolrad	289.731508	307.7783323	338.9304482
toteva	3.068682	3.3232559	4.7150127
avtemp	13.798889	16.5387931	21.1974264
avhum	41.514418	54.6180217	52.6388427
avvappres	9.801051	10.6689119	15.1958562
avpres	791.458346	823.2496504	875.31934
avsoiltemp50	13.911024	16.368625	22.729601
avsoiltemp20	15.78924	17.8993169	24.0480312
avsoiltemp10	14.403511	18.6605555	24.6804167
avsoiltemp5	14.996502	19.2629557	27.2255031
mintemp	10.087009	10.4409772	17.2292644
maxtemp	20.251147	22.8620844	25.9921597
insoforce	254.278476	320.5589598	302.1036239
soiltemp100	12.016795	14.731008	21.0769166
avcloud	2.640343	3.0701164	2.4202439

Table A.2: The means of meteorological variables for 4th, 5th, 6th risk cluster

Variable	RC4	RC5	RC6
minsurf	12.906555	6.311327	13.7424415
totprec	0.9866787	1.752387	0.9829862
totinso	6.2008081	4.333961	6.5982279
totglsolrad	364.4457881	305.893582	456.2283794
toteva	4.220361	2.982059	4.4107307
avtemp	18.6997178	11.514396	21.0517734
avhum	56.8489691	40.274581	58.8870503
avvappres	13.8301822	7.685148	14.6175528
avpres	856.3475145	576.922714	903.8997313
avsoiltemp50	19.0927162	11.263783	21.2369373
avsoiltemp20	20.9221512	11.901326	22.9611222
avsoiltemp10	19.9474618	12.435957	23.8501352
avsoiltemp5	21.3304141	13.393678	24.3040616
mintemp	12.5258116	6.883572	14.8792172
maxtemp	22.5190031	16.073289	27.7582911
insoforce	364.4166012	290.325946	430.1956694
soiltemp100	16.5696259	10.322108	18.8862896
avcloud	2.5656042	2.863458	2.2291128

Table A.3: The means of meteorological variables for 7th, 8th, 9th risk cluster

Variable	RC7	RC8	RC9
minsurf	8.3695278	12.4512379	10.3796518
totprec	0.5853526	0.7639731	0.7165093
totinso	4.9917168	4.7881601	1.8473166
totglsolrad	336.6498463	557.623263	506.6833009
toteva	4.3196837	3.6460996	3.2608636
avtemp	19.1343661	20.1227974	16.639702
avhum	50.394717	50.5318629	53.0603058
avvappres	6.9156867	12.044096	11.0028778
avpres	664.4551099	848.1502853	821.5887622
avsoiltemp50	18.4187536	19.2824201	18.4957785
avsoiltemp20	20.3680573	21.5912591	19.8717833
avsoiltemp10	21.8090424	23.1745981	21.4168935
avsoiltemp5	0.9016968	22.9437831	19.895738
mintemp	11.7445774	13.2020035	11.6787432
maxtemp	14.3023841	26.9272529	24.1756238
insoforce	162.0148217	555.1631569	506.6030765
soiltemp100	14.5687959	17.331259	16.2691584
avcloud	2.6674196	2.6863729	2.6383672

Table A.4: The covariance matrix of meteorological variables for 1st (above), 2nd (center) and 3rd (below) risk cluster

	minsurf	totprec	totinso	totglsolrad	toteva	avtemp	avhum	avvappres	avpres	avsoiltemp50	avsoiltemp20	avsoiltemp10	avsoiltemp5	mintemp	maxtemp	insoforce	soiltemp100	avcloud
minsurf	6.9	0	0.71	15.22	0.58	3.06	6.51	5.58	110.76	7.86	5.23	4.97	3.27	3.24	1.81	-93.43	3.39	-0.28
totprec	0	0.43	0.07	5.71	-0.14	-0.01	0.33	-0.15	-15.95	0.26	-0.45	0.06	-0.33	-0.04	-0.23	-4.55	-0.01	-0.05
totinso	0.71	0.07	1.36	9.11	0.09	0.28	-0.93	0.52	61.25	0.3	2.31	1.1	1.41	0.71	0.82	24.62	0.93	-0.1
totglsolrad	15.22	5.71	9.11	7609.51	-20.18	-17.41	51.15	-25.13	-1472.33	35.03	-126.64	-33.38	79.81	19.63	14.93	2216.91	14.68	4.3
toteva	0.58	-0.14	0.09	-20.18	1.06	0.77	-1.22	0.2	-49.8	0.91	1.9	1.25	1.65	0.59	-0.53	10.3	1.04	-0.16
avtemp	3.06	-0.01	0.28	-17.41	0.77	7.62	10.63	4.09	98.19	3.43	6.37	5.3	4.93	2.78	2.61	-33.31	4.4	-0.11
avhum	6.51	0.33	-0.93	51.15	-1.22	10.63	92.68	14.61	581.92	3.98	8.82	3.03	9.92	3.99	10.81	-258.26	4.58	1.25
avvappres	5.58	-0.15	0.52	-25.13	0.2	4.09	14.61	7.16	199.65	5.85	6.13	4.63	2.91	2.94	2.98	-114.91	3.1	-0.07
avpres	110.76	-15.95	61.25	-1472.33	-49.8	98.19	581.92	199.65	28883.85	-35.3	362.86	-6.19	99.95	137.09	241.6	-2167.66	10.45	-6.83
avsoiltemp50	7.86	0.26	0.3	35.03	0.91	3.43	3.98	5.85	-35.3	12.41	4.07	6.36	2.33	3.23	-0.19	-137.83	4.37	-0.31
avsoiltemp20	5.23	-0.45	2.31	-126.64	1.9	6.37	8.82	6.13	362.86	4.07	23.07	10.46	13.2	4.99	4.59	97.4	8.25	-0.69
avsoiltemp10	4.97	0.06	1.1	-33.38	1.25	5.3	3.03	4.63	-6.19	6.36	10.46	14.82	6.27	1.82	3.51	37.87	7.73	-0.41
avsoiltemp5	3.27	-0.33	1.41	79.81	1.65	4.93	9.92	2.91	99.95	2.33	13.2	6.27	18.51	4.48	4.1	182.53	7.2	-0.55
mintemp	3.24	-0.04	0.71	19.63	0.59	2.78	3.99	2.94	137.09	3.23	4.99	1.82	4.48	5.58	1.44	-7.45	3.22	-0.33
maxtemp	1.81	-0.23	0.82	14.93	-0.53	2.61	10.81	2.98	241.6	-0.19	4.59	3.51	4.1	1.44	11.12	32.24	2.55	-0.21
insoforce	-93.43	-4.55	24.62	2216.91	10.3	-33.31	-258.26	-114.91	-2167.66	-137.83	97.4	37.87	182.53	-7.45	32.24	7971.76	64.55	-5.26
soiltemp100	3.39	-0.01	0.93	14.68	1.04	4.4	4.58	3.1	10.45	4.37	8.25	7.73	7.2	3.22	2.55	64.55	8.01	-0.35
avcloud	-0.28	-0.05	-0.1	4.3	-0.16	-0.11	1.25	-0.07	-6.83	-0.31	-0.69	-0.41	-0.55	-0.33	-0.21	-5.26	-0.35	0.42
	minsurf	**totprec**	**totinso**	**totglsolrad**	**toteva**	**avtemp**	**avhum**	**avvappres**	**avpres**	**avsoiltemp50**	**avsoiltemp20**	**avsoiltemp10**	**avsoiltemp5**	**mintemp**	**maxtemp**	**insoforce**	**soiltemp100**	**avcloud**
minsurf	10.98	-0.15	0.05	-67.72	1.84	6.81	1.03	6.01	37.36	8.23	8.88	9.55	9	8.24	6.19	-4.14	7.13	-0.38
totprec	-0.15	0.24	-0.05	-4.74	0.02	-0.22	0.72	0.07	5.56	-0.2	-0.22	-0.17	-0.27	-0.2	-0.28	-2.72	-0.22	0.1
totinso	0.05	-0.05	5.73	-14.03	1.3	1.84	-1.33	0.67	2.76	-0.16	0.18	0.4	0.62	1.01	1.75	-7.72	-0.65	-0.46
totglsolrad	-67.72	-4.74	-14.03	18978.19	-13.72	-35.96	-45.18	-71.96	-2508.27	-93.3	-75.6	-78.85	-109.98	-100.51	-14.03	10133.29	-86.87	-23.18
toteva	1.84	0.02	1.3	-13.72	1.92	1.76	-1.11	0.98	-7.64	1.59	1.82	2.38	1.91	1.29	1.82	-8.28	0.92	-0.17
avtemp	6.81	-0.22	1.84	-35.96	1.76	10.16	-1.23	5.79	12.9	7.36	8.39	8.84	9.55	7.7	10.3	26.55	6.04	-1
avhum	1.03	0.72	-1.33	-45.18	-1.11	-1.23	51.69	7.85	236.9	-0.95	-0.83	-2.78	-2.04	0.48	-2.92	-25.88	-1.16	1.46
avvappres	6.01	0.07	0.67	-71.96	0.98	5.79	7.85	7.9	67.98	5.09	5.65	5.71	6.07	6.38	4.9	-9.62	4.47	0.03
avpres	37.36	5.56	2.76	-2508.27	-7.64	12.9	236.9	67.98	6561.97	44.17	42.98	34.03	34.37	46.87	-11.38	-165.16	39.27	16.73
avsoiltemp50	8.23	-0.2	-0.16	-93.3	1.59	7.36	-0.95	5.09	44.17	11.08	11.41	10.44	11.18	7.18	7.52	-5.16	9.46	-0.3
avsoiltemp20	8.88	-0.22	0.18	-75.6	1.82	8.39	-0.83	5.65	42.98	11.41	13.08	11.65	12.3	7.91	8.62	-7.35	9.72	-0.46
avsoiltemp10	9.55	-0.17	0.4	-78.85	2.38	8.84	-2.78	5.71	34.03	10.44	11.65	16.08	12.7	8.26	9.04	4.31	9.28	-1.2
avsoiltemp5	9	-0.27	0.62	-109.98	1.91	9.55	-2.04	6.07	34.37	11.18	12.3	12.7	14.68	8.63	10.3	-35.49	9.44	-1.07
mintemp	8.24	-0.2	1.01	-100.51	1.29	7.7	0.48	6.38	46.87	7.18	7.91	8.26	8.63	11.22	6.74	-21.8	6.67	-0.22
maxtemp	6.19	-0.28	1.75	-14.03	1.82	10.3	-2.92	4.9	-11.38	7.52	8.62	9.04	10.3	6.74	12.58	23.94	6.06	-1.57
insoforce	-4.14	-2.72	-7.72	10133.29	-8.28	26.55	-25.88	-9.62	-165.16	-5.16	-7.35	4.31	-35.49	-21.8	23.94	18958.11	-6.48	-9.52
soiltemp100	7.13	-0.22	-0.65	-86.87	0.92	6.04	-1.16	4.47	39.27	9.46	9.72	9.28	9.44	6.67	6.06	-6.48	9.63	-0.06
avcloud	-0.38	0.1	-0.46	-23.18	-0.17	-1	1.46	0.03	16.73	-0.3	-0.46	-1.2	-1.07	-0.22	-1.57	-9.52	-0.06	1.06
	minsurf	**totprec**	**totinso**	**totglsolrad**	**toteva**	**avtemp**	**avhum**	**avvappres**	**avpres**	**avsoiltemp50**	**avsoiltemp20**	**avsoiltemp10**	**avsoiltemp5**	**mintemp**	**maxtemp**	**insoforce**	**soiltemp100**	**avcloud**
minsurf	17.26	-0.96	5.27	111	1.49	12.39	15.65	16.4	248.68	19.67	18.58	18.55	19.16	16	12.35	-327.24	17.85	0.12
totprec	-0.96	0.47	-0.59	-7.82	-0.17	-0.97	-0.26	-0.99	-12.38	-0.99	-1.15	-1.5	-1.49	-0.9	-1.05	10.89	-0.64	0.02
totinso	5.27	-0.59	5.42	-65.17	0	6.13	1.15	6.31	149.03	6.46	4.47	6.68	7.55	5.56	8.39	-259.49	5.84	0.52
totglsolrad	111	-7.82	-65.17	9650.69	33.95	-36.66	466.32	103.67	-172.81	100.77	161.59	125.08	85.89	135.48	-139.23	634.68	86.24	1.77
toteva	1.49	-0.17	0	33.95	1.4	0.77	1.09	1	18.65	1.87	2.2	1.86	1.96	1.39	1.14	26.14	1.33	-0.25
avtemp	12.39	-0.97	6.13	-36.66	0.77	15.83	5.71	13.85	283.84	16.15	16	17.58	16.07	12.13	15.4	-371.97	13.8	0.47
avhum	15.65	-0.26	1.15	466.32	1.09	5.71	75.39	17.07	275.69	14.66	19.91	18.19	13.81	15.52	-1.65	-452.19	14.36	3.81
avvappres	16.4	-0.99	6.31	103.67	1	13.85	17.07	19.15	277.19	19.28	18.69	19.89	19.44	15.7	14.58	-400.94	17.52	0.94
avpres	248.68	-12.38	149.03	-172.81	18.65	283.84	275.69	277.19	9569.47	344.8	277.83	316.75	363.6	280.59	317.13	-8862.38	286.69	15.46
avsoiltemp50	19.67	-0.99	6.46	100.77	1.87	16.15	14.66	19.28	344.8	27.36	25.21	25.24	24.82	19.91	17.11	-421.72	22.43	-0.06
avsoiltemp20	18.58	-1.15	4.47	161.59	2.2	16	19.91	18.69	277.83	25.21	30.2	27.37	23.56	18.35	14.26	-335.32	20.73	0.52
avsoiltemp10	18.55	-1.5	6.68	125.08	1.86	17.58	18.19	19.89	316.75	25.24	27.37	29.31	24.16	18.96	16.34	-407.24	20.79	0.86
avsoiltemp5	19.16	-1.49	7.55	85.89	1.96	16.07	13.81	19.44	363.6	24.82	23.56	24.16	29.88	21.35	19.3	-428.07	22.89	0.17
mintemp	16	-0.9	5.56	135.48	1.39	12.13	15.52	15.7	280.59	19.91	18.35	18.96	21.35	18.37	13.12	-366.15	19.02	0.01
maxtemp	12.35	-1.05	8.39	-139.23	1.14	15.4	-1.65	14.58	317.13	17.11	14.26	16.34	19.3	13.12	25.33	-475.56	16.3	0.4
insoforce	-327.24	10.89	-259.49	634.68	26.14	-371.97	-452.19	-400.94	-8862.38	-421.72	-335.32	-407.24	-428.07	-366.15	-475.56	25135.78	-412.69	-56.03
soiltemp100	17.85	-0.64	5.84	86.24	1.33	13.8	14.36	17.52	286.69	22.43	20.73	20.79	22.89	19.02	16.3	-412.69	24.05	0.13
avcloud	0.12	0.02	0.52	1.77	-0.25	0.47	3.81	0.94	15.46	-0.06	0.52	0.86	0.17	0.01	0.4	-56.03	0.13	0.94

Table A.5: The covariance matrix of meteorological variables for 4th (above), 5th (center) and 6th (below) risk cluster

	minsurf	totprec	totinso	totglsolrad	toteva	avtemp	avhum	avvappres	avpres	avsoiltemp50	avsoiltemp20	avsoiltemp10	avsoiltemp5	mintemp	maxtemp	insoforce	soiltemp100	avcloud
minsurf	542.38	-32.32	55.9	184.37	91.38	526.32	-94.49	438.94	45.28	481.29	647.73	659.31	616.58	498.28	569.48	179.79	441.92	-49.12
totprec	-32.32	2.51	-3.21	1.67	-5.11	-32	6.96	-26.01	4.58	-28.34	-38.1	-39.57	-37.27	-30.5	-35.92	1.95	-27.08	3.01
totinso	55.9	-3.21	9.94	88.05	10.48	54.48	-14.33	45.46	19.8	52.4	68.96	65.59	62.25	48.19	55.5	87.58	42.88	-5.73
totglsolrad	184.37	1.67	88.05	14313.75	139.22	103.98	214.82	162.81	2368.8	302.77	296.76	217.04	222.36	75.8	7.66	14316.22	44.43	-57.97
toteva	91.38	-5.11	10.48	139.22	17.2	87.5	-11.87	74.97	46.75	83.53	111.1	110.99	104.01	82.26	92.53	138.45	72.71	-8.52
avtemp	526.32	-32	54.48	103.98	87.5	517.61	-104.29	425.3	29.18	465.22	627.68	640.78	599.35	487.67	559.83	99.6	432.65	-48.47
avhum	-94.49	6.96	-14.33	214.82	-11.87	-104.29	105.8	-62.36	335.81	-76	-101.12	-99.02	-97.34	-84.32	-101.57	215.53	-81.98	11.47
avvappres	438.94	-26.01	45.46	162.81	74.97	425.3	-62.36	362.35	184.27	393.63	528.93	535.7	500.85	403.8	458.91	159.11	356.24	-39.09
avpres	45.28	4.58	19.8	2368.8	46.75	29.18	335.81	184.27	10249.84	156.45	150.8	101.69	126.89	49.5	37.45	2368.86	1.47	-7.03
avsoiltemp50	481.29	-28.34	52.4	302.77	83.53	465.22	-76	393.63	156.45	435.75	580.9	585.83	548.61	439.72	498.42	298.7	388.07	-43.99
avsoiltemp20	647.73	-38.1	68.96	296.76	111.1	627.68	-101.12	528.93	150.8	580.9	785.3	792.6	739.14	595.13	675.68	291.24	524.79	-58.84
avsoiltemp10	659.31	-39.57	65.59	217.04	110.99	640.78	-99.02	535.7	101.69	585.83	792.6	816.27	758.91	615.66	703.48	211.47	542.81	-59.87
avsoiltemp5	616.58	-37.27	62.25	222.36	104.01	599.35	-97.34	500.85	126.89	548.61	739.14	758.91	712.53	573.7	658.77	217.14	505.99	-56.41
mintemp	498.28	-30.5	48.19	75.8	82.26	487.67	-84.32	403.8	49.5	439.72	595.13	615.66	573.7	470.12	537.27	71.61	412.52	-45.53
maxtemp	569.48	-35.92	55.5	7.66	92.53	559.83	-101.57	458.91	37.45	498.42	675.68	703.48	658.77	537.27	640.2	2.92	477.24	-53.03
insoforce	179.79	1.95	87.58	14316.22	138.45	99.6	215.53	159.11	2368.86	298.7	291.24	211.47	217.14	71.61	2.92	14319.15	40.78	-57.58
soiltemp100	441.92	-27.08	42.88	44.43	72.71	432.65	-81.98	356.24	1.47	388.07	524.79	542.81	505.99	412.52	477.24	40.78	370.08	-40.36
avcloud	-49.12	3.01	-5.73	-57.97	-8.52	-48.47	11.47	-39.09	-7.03	-43.99	-58.84	-59.87	-56.41	-45.53	-53.03	-57.58	-40.36	5.5
	minsurf	**totprec**	**totinso**	**totglsolrad**	**toteva**	**avtemp**	**avhum**	**avvappres**	**avpres**	**avsoiltemp50**	**avsoiltemp20**	**avsoiltemp10**	**avsoiltemp5**	**mintemp**	**maxtemp**	**insoforce**	**soiltemp100**	**avcloud**
minsurf	5.41	-0.69	0.68	25.8	0.84	4.48	3.54	3.39	104.52	4.96	4.63	4.99	6	2.61	5.22	22.1	4.48	-0.66
totprec	-0.69	1.5	-0.61	-39.95	-0.48	-1.19	3.4	0.19	40.18	-0.84	-1.08	-0.92	-1.52	-0.63	-0.99	-36.37	-0.24	0.31
totinso	0.68	-0.61	1.26	30.98	0.64	1.15	-1.94	-0.3	-33.62	0.73	1.18	1.02	1.31	0.65	1.04	42.7	0.31	-0.36
totglsolrad	25.8	-39.95	30.98	7759.71	30.24	58	-155.56	-11.7	-3371.89	43.57	54.13	75.99	92.98	35.02	86.15	2571.05	15.19	-13.65
toteva	0.84	-0.48	0.64	30.24	1.15	0.81	-1.72	0	-14.75	0.75	1.02	1.16	1.51	0.38	0.88	34.36	0.48	-0.27
avtemp	4.48	-1.19	1.15	58	0.81	9.22	7.66	2.09	107.53	6.79	6.02	5.71	7.91	2.69	8.02	71.33	5.97	-0.91
avhum	3.54	3.4	-1.94	-155.56	-1.72	7.66	100.14	5.51	739.15	8.13	11.3	9.7	5.89	4.5	8.81	37.31	8.96	0.68
avvappres	3.39	0.19	-0.3	-11.7	0	2.09	5.51	4.62	152.92	3.09	2.66	3.18	3.04	1.49	3.74	-12.18	3.06	-0.1
avpres	104.52	40.18	-33.62	-3371.89	-14.75	107.53	739.15	152.92	28078.37	159.15	135.45	20.08	83.17	52.86	59.87	-3175.79	197.53	-2.12
avsoiltemp50	4.96	-0.84	0.73	43.57	0.75	6.79	8.13	3.09	159.15	9.05	5.69	6.27	8.47	2.16	7.15	37.06	7.77	-0.84
avsoiltemp20	4.63	-1.08	1.18	54.13	1.02	6.02	11.3	2.66	135.45	5.69	11.41	7.88	8.14	3.99	9.17	60.23	5.15	-0.81
avsoiltemp10	4.99	-0.92	1.02	75.99	1.16	5.71	9.7	3.18	20.08	6.27	7.88	13.11	10.03	2.27	7.64	102.79	5.35	-0.72
avsoiltemp5	6	-1.52	1.31	92.98	1.51	7.91	5.89	3.04	83.17	8.47	8.14	10.03	13.75	2.59	8.12	92.46	7.15	-1.03
mintemp	2.61	-0.63	0.65	35.02	0.38	2.69	4.5	1.49	52.86	2.16	3.99	2.27	2.59	4.51	4.34	28.98	1.72	-0.4
maxtemp	5.22	-0.99	1.04	86.15	0.88	8.02	8.81	3.74	59.87	7.15	9.17	7.64	8.12	4.34	18.33	90.12	6.35	-0.72
insoforce	22.1	-36.37	42.7	2571.05	34.36	71.33	37.31	-12.18	-3175.79	37.06	60.23	102.79	92.46	28.98	90.12	8636.62	4.99	-21.13
soiltemp100	4.48	-0.24	0.31	15.19	0.48	5.97	8.96	3.06	197.53	7.77	5.15	5.35	7.15	1.72	6.35	4.99	7.7	-0.74
avcloud	-0.66	0.31	-0.36	-13.65	-0.27	-0.91	0.68	-0.1	-2.12	-0.84	-0.81	-0.72	-1.03	-0.4	-0.72	-21.13	-0.74	0.44
	minsurf	**totprec**	**totinso**	**totglsolrad**	**toteva**	**avtemp**	**avhum**	**avvappres**	**avpres**	**avsoiltemp50**	**avsoiltemp20**	**avsoiltemp10**	**avsoiltemp5**	**mintemp**	**maxtemp**	**insoforce**	**soiltemp100**	**avcloud**
minsurf	14.87	-1.37	2.08	170.77	2.58	11.68	10.25	11.77	220.3	10.71	12.73	11.97	11.92	12.68	10.68	192.75	9.19	-4.01
totprec	-1.37	1.11	0.17	-31.67	-0.19	-1.4	-0.49	-1.28	-17.12	-1.5	-1.73	-1.82	-1.71	-1.29	-1.57	-21.93	-1.18	0.6
totinso	2.08	0.17	10.15	74.33	2.39	2.86	-1.45	1.96	36	3.27	2.65	3.13	3.35	2.59	1.6	67.81	1.36	-1.04
totglsolrad	170.77	-31.67	74.33	16728.41	21.45	149.27	245.48	169.85	3110.27	158.98	176.2	176.04	155.52	127.53	130.95	8969.12	115.15	-49.77
toteva	2.58	-0.19	2.39	21.45	3.89	3.02	-2.26	1.96	29.3	2.08	2.52	2.61	2.98	2.47	2.67	70.3	1.5	-1.21
avtemp	11.68	-1.4	2.86	149.27	3.02	11.52	2.32	9.37	180.61	9.89	11.7	11.61	12.14	10.49	11.17	160.33	8.28	-3.68
avhum	10.25	-0.49	-1.45	245.48	-2.26	2.32	87.8	17.95	324.13	3.3	5.37	3.02	2.52	4.75	-1.73	335.36	3.36	-0.58
avvappres	11.77	-1.28	1.96	169.85	1.96	9.37	17.95	13.26	221.81	9.01	10.23	10.01	10	10.1	8.61	188.92	7.4	-2.89
avpres	220.3	-17.12	36	3110.27	29.3	180.61	324.13	221.81	6631.68	178.16	211.82	197.7	206.8	187.84	183.77	3902.9	174.49	-63.62
avsoiltemp50	10.71	-1.5	3.27	158.98	2.08	9.89	3.3	9.01	178.16	14.07	12.73	11.98	11.92	9.92	10.09	145.05	10.48	-3.48
avsoiltemp20	12.73	-1.73	2.65	176.2	2.52	11.7	5.37	10.23	211.82	12.73	16.34	14.24	14.78	11.29	11.58	154	10.61	-3.76
avsoiltemp10	11.97	-1.82	3.13	176.04	2.61	11.61	3.02	10.01	197.7	11.98	14.24	15.02	15.1	11.19	11.17	151.45	9.68	-3.53
avsoiltemp5	11.92	-1.71	3.35	155.52	2.98	12.14	2.52	10	206.8	11.92	14.78	15.1	16.39	11.3	11.53	136.11	10.16	-3.52
mintemp	12.68	-1.29	2.59	127.53	2.47	10.49	4.75	10.1	187.84	9.92	11.29	11.19	11.3	11.79	9.78	137.07	8.29	-3.47
maxtemp	10.68	-1.57	1.6	130.95	2.67	11.17	-1.73	8.61	183.77	10.09	11.58	11.17	11.53	9.78	14.66	159.81	8.75	-4.06
insoforce	192.75	-21.93	67.81	8969.12	70.3	160.33	335.36	188.92	3902.9	145.05	154	151.45	136.11	137.07	159.81	21072.35	104.46	-70.2
soiltemp100	9.19	-1.18	1.36	115.15	1.5	8.28	3.36	7.4	174.49	10.48	10.61	9.68	10.16	8.29	8.75	104.46	10.94	-3.07
avcloud	-4.01	0.6	-1.04	-49.77	-1.21	-3.68	-0.58	-2.89	-63.62	-3.48	-3.76	-3.53	-3.52	-3.47	-4.06	-70.2	-3.07	2.05

Table A.6: The covariance matrix of meteorological variables for 7th (above), 8th (center) and 9th (below) risk cluster

	minsurf	totprec	totinso	totglsolrad	toteva	avtemp	avhum	avvappres	avpres	avsoiltemp50	avsoiltemp20	avsoiltemp10	avsoiltemp5	mintemp	maxtemp	insoforce	soiltemp100	avcloud
minsurf	6.46	-0.19	-1.14	-41.69	2.23	5.37	22.7	-2.79	-75.87	7.42	0.03	10.07	5.89	8.75	-11.44	-210.37	3.99	-0.76
totprec	-0.19	3.51	4.17	-68.98	-2.9	-12.05	11.73	2.28	-59.76	-2.01	-5.93	-10.41	-1.89	-5.99	11.05	45.13	0.75	0.46
totinso	-1.14	4.17	13.89	-123.09	-7.63	-28.02	18.35	7.9	-60.4	-9.73	-12.18	-27.08	-4.49	-18.15	30.47	271.61	1.02	0.53
totglsolrad	-41.69	-68.98	-123.09	8426.2	-31.43	126.52	-951.28	37.79	-58.14	-152.72	33.33	-104.02	-563.62	76.12	160.45	7282.19	-206.88	16.31
toteva	2.23	-2.9	-7.63	-31.43	7.92	20.83	-1.32	-5.82	73.19	10.46	11.31	23.33	15.25	11.8	-24.5	-283.77	3.71	-1.45
avtemp	5.37	-12.05	-28.02	126.52	20.83	80.57	-44.31	-14.7	413.79	17.97	36.67	76.9	31.66	41.65	-83.22	-739.05	2.16	-2.8
avhum	22.7	11.73	18.35	-951.28	-1.32	-44.31	296.9	-23.23	-1020.17	68.81	-35.27	15.02	54.25	20.66	-31.33	-1463.93	31.96	2.71
avvappres	-2.79	2.28	7.9	37.79	-5.82	-14.7	-23.23	11.93	131.4	-17.29	-3.42	-23.95	-10.51	-17.3	31.05	447.34	-2.03	0.4
avpres	-75.87	-59.76	-60.4	-58.14	73.19	413.79	-1020.17	131.4	9739.09	-279.15	311.18	171.65	214.66	-101.19	-97.31	3572.42	-15.71	-18.77
avsoiltemp50	7.42	-2.01	-9.73	-152.72	10.46	17.97	68.81	-17.29	-279.15	40.25	6.04	37.18	26.72	26.32	-44.75	-755.81	10.33	-0.82
avsoiltemp20	0.03	-5.93	-12.18	33.33	11.31	36.67	-35.27	-3.42	311.18	6.04	28.1	30.89	22.31	10.93	-24	-139.49	4.13	-2.03
avsoiltemp10	10.07	-10.41	-27.08	-104.02	23.33	76.9	15.02	-23.95	171.65	37.18	30.89	90.69	48.43	50.15	-101.68	-1212.85	9.41	-2.64
avsoiltemp5	5.89	-1.89	-4.49	-563.62	15.25	31.66	54.25	-10.51	214.66	26.72	22.31	48.43	68.17	13.64	-50.81	-855.74	20.87	-2.31
mintemp	8.75	-5.99	-18.15	76.12	11.8	41.65	20.66	-17.3	-101.19	26.32	10.93	50.15	13.64	39.14	-62.28	-754.55	2.35	-1.14
maxtemp	-11.44	11.05	30.47	160.45	-24.5	-83.22	-31.33	31.05	-97.31	-44.75	-24	-101.68	-50.81	-62.28	130.64	1575.72	-8.25	2.73
insoforce	-210.37	45.13	271.61	7282.19	-283.77	-739.05	-1463.93	447.34	3572.42	-755.81	-139.49	-1212.85	-855.74	-754.55	1575.72	25679.72	-245.77	43.95
soiltemp100	3.99	0.75	1.02	-206.88	3.71	2.16	31.96	-2.03	-15.71	10.33	4.13	9.41	20.87	2.35	-8.25	-245.77	10.9	-1.19
avcloud	-0.76	0.46	0.53	16.31	-1.45	-2.8	2.71	0.4	-18.77	-0.82	-2.03	-2.64	-2.31	-1.14	2.73	43.95	-1.19	2.22

	minsurf	totprec	totinso	totglsolrad	toteva	avtemp	avhum	avvappres	avpres	avsoiltemp50	avsoiltemp20	avsoiltemp10	avsoiltemp5	mintemp	maxtemp	insoforce	soiltemp100	avcloud
minsurf	36.61	-7.33	-35.67	165.67	-15.14	29.83	-11.6	19.4	-191.57	35.36	26.19	13.73	30.97	36.05	16.73	-278.23	17.74	-5.01
totprec	-7.33	4.47	12.39	-158.62	4.62	-10.44	14.14	-3.12	24.12	-8.37	-8.37	-7.93	-11.33	-9.87	-7.18	-61.67	-6.19	2.57
totinso	-35.67	12.39	71.7	-732.28	22.56	-42.99	61.86	-15.34	49.95	-45.78	-39.46	-32.98	-54.33	-47.88	-31.45	-221.73	-27.14	13.83
totglsolrad	165.67	-158.62	-732.28	15819.53	-135.51	488.75	-1190.37	13.88	664.06	467.89	473.56	538.52	645.52	445.15	458.29	11653.85	335.72	-178.04
toteva	-15.14	4.62	22.56	-135.51	21.31	-24.99	8.23	-11.4	-27.17	-13.86	-7.9	-4.29	-21.91	-17.09	-12.26	242.05	-7.63	-0.07
avtemp	29.83	-10.44	-42.99	488.75	-24.99	47.16	-38.8	21.98	-148.37	32.77	24.87	24.22	47.35	40.73	27.25	-79.32	24.4	-3.6
avhum	-11.6	14.14	61.86	-1190.37	8.23	-38.8	128.15	-5.94	192.62	-25.52	-33.76	-66.97	-71.97	-44.44	-35.35	-1200.34	-38.38	15.77
avvappres	19.4	-3.12	-15.34	13.88	-11.4	21.98	-5.94	21.85	-312.99	13.53	6.92	15.82	28.81	26.38	8.7	-230.05	18.23	1.67
avpres	-191.57	24.12	49.95	664.06	-27.17	-148.37	192.62	-312.99	9370.43	-39.72	-9.78	-429.34	-483.78	-394.68	-51.99	-712.31	-362.73	-28.84
avsoiltemp50	35.36	-8.37	-45.78	467.89	-13.86	32.77	-25.52	13.53	-39.72	46.9	36.02	13.86	29.34	36.18	22.63	-74.83	17.07	-8.17
avsoiltemp20	26.19	-8.37	-39.46	473.56	-7.9	24.87	-33.76	6.92	-9.78	36.02	32.53	16.76	24.47	28.41	19.84	133.46	14.55	-9.51
avsoiltemp10	13.73	-7.93	-32.98	538.52	-4.29	24.22	-66.97	15.82	-429.34	13.86	16.76	53.15	54.52	36.64	20.12	596.03	33.53	-6.92
avsoiltemp5	30.97	-11.33	-54.33	645.52	-21.91	47.35	-71.97	28.81	-483.78	29.34	24.47	54.52	78.39	57.25	31.01	340.03	42.24	-6.93
mintemp	36.05	-9.87	-47.88	445.15	-17.09	40.73	-44.44	26.38	-394.68	36.18	28.41	36.64	57.25	51.83	24.67	41.43	33.1	-6.25
maxtemp	16.73	-7.18	-31.45	458.29	-12.26	27.25	-35.35	8.7	-51.99	22.63	19.84	20.12	31.01	24.67	21.44	163.67	16.12	-5.9
insoforce	-278.23	-61.67	-221.73	11653.85	242.05	-79.32	-1200.34	-230.05	-712.31	-74.83	133.46	596.03	340.03	41.43	163.67	18837.33	209.3	-160.23
soiltemp100	17.74	-6.19	-27.14	335.72	-7.63	24.4	-38.38	18.23	-362.73	17.07	14.55	33.53	42.24	33.1	16.12	209.3	27.18	-3.95
avcloud	-5.01	2.57	13.83	-178.04	-0.07	-3.6	15.77	1.67	-28.84	-8.17	-9.51	-6.92	-6.93	-6.25	-5.9	-160.23	-3.95	6.32

	minsurf	totprec	totinso	totglsolrad	toteva	avtemp	avhum	avvappres	avpres	avsoiltemp50	avsoiltemp20	avsoiltemp10	avsoiltemp5	mintemp	maxtemp	insoforce	soiltemp100	avcloud
minsurf	262.53	-7.63	72.08	6.08	27.51	188.42	-881.32	132.39	201.89	263.82	233.32	228.55	348.45	277.87	226.26	1.28	207.9	-72.11
totprec	-7.63	2.04	0.44	-39.84	-0.37	-6.47	20	-3.61	-31.7	-6.43	-7.4	-7.64	-9.49	-8.09	-5.97	-39.61	-4.68	1.76
totinso	72.08	0.44	30.36	-161.26	9.03	47.36	-262.49	39.81	-62.1	70.37	57.74	62.05	97.74	80.37	60.32	-163.07	58.89	-20.97
totglsolrad	6.08	-39.84	-161.26	14609.55	27.24	246.76	3.72	-68.93	199.83	42.93	184.13	117.78	211.56	4.87	135.84	14621.37	-3.3	-4.34
toteva	27.51	-0.37	9.03	27.24	5.09	18.05	-101.53	13.66	4.35	28.47	23.93	24.27	37.63	29.83	24.49	26.66	23.59	-8.73
avtemp	188.42	-6.47	47.36	246.76	18.05	146.16	-639.67	94.09	114.74	195.78	175.02	168.34	258.52	200.37	168.09	243.74	152.36	-52.59
avhum	-881.32	20	-262.49	3.72	-101.53	-639.67	3302.63	-448.49	110.01	-941.79	-802.95	-785	-1232.09	-964.1	-792.3	22.33	-750.47	268.48
avvappres	132.39	-3.61	39.81	-68.93	13.66	94.09	-448.49	73.79	-7.04	131.6	114.62	117.91	181.49	145.08	111.88	-71.89	106.11	-36.33
avpres	201.89	-31.7	-62.1	199.83	4.35	114.74	110.01	-7.04	9661.86	19.98	138.98	156.83	-140.46	102.3	66.98	198.5	-38.36	7.95
avsoiltemp50	263.82	-6.43	70.37	42.93	28.47	195.78	-941.79	131.6	19.98	294.9	251.63	230.91	370.79	280.01	246.82	39.58	231.55	-79.64
avsoiltemp20	233.32	-7.4	57.74	184.13	23.93	175.02	-802.95	114.62	138.98	251.63	224.71	203.91	321.86	244.23	214.81	181.07	196.09	-67
avsoiltemp10	228.55	-7.64	62.05	117.78	24.27	168.34	-785	117.91	156.83	230.91	203.91	208.17	310	248.37	197.77	112.85	182.54	-63.95
avsoiltemp5	348.45	-9.49	97.74	211.56	37.63	258.52	-1232.09	181.49	-140.46	370.79	321.86	310	495.72	377.29	313.02	205.56	293.94	-101.84
mintemp	277.87	-8.09	80.37	4.87	29.83	200.37	-964.1	145.08	102.3	280.01	244.23	248.37	377.29	303.18	239.11	-1.23	222.89	-78.4
maxtemp	226.26	-5.97	60.32	135.84	24.49	168.09	-792.3	111.88	66.98	246.82	214.81	197.77	313.02	239.11	212.21	132.83	194.03	-66.29
insoforce	1.28	-39.61	-163.07	14621.37	26.66	243.74	22.33	-71.89	198.5	39.58	181.07	112.85	205.56	-1.23	132.83	14633.95	-6.16	-3.11
soiltemp100	207.9	-4.68	58.89	-3.3	23.59	152.36	-750.47	106.11	-38.36	231.55	196.09	182.54	293.94	222.89	194.03	-6.16	185.95	-63.54
avcloud	-72.11	1.76	-20.97	-4.34	-8.73	-52.59	268.48	-36.33	7.95	-79.64	-67	-63.95	-101.84	-78.4	-66.29	-3.11	-63.54	23.49

CURRICULUM VITAE

Credentials

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Evaluation of the Efficiency of Agricultural Insurance and Management of Natural Disasters in Agriculture (FUK-2015-6321)

A Study of Risk Prioritization and Evaluation in Agricultural Insurance Portfolio (FHD-2018-16686)

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Multivariate Stochastic Prioritization of Dependent Actuarial Risks in Agricultural Insurance, ASTIN and AFIR/ERM Colloquia 2017, Panama City, 20-24 August **2017**.

Risk Classification in Agricultural Insurance under Dependency Assumption, 3rd International Researchers, Statisticians and Young Statisticians Congress, Konya, 24-26 May **2017**.

Prioritization of Dependent Actuarial Risks: Stochastic Majorization, Innovations in Insurance, Risk- and Asset Management Conference, Munich, 5-7 April **2017**.

Stochastic Prioritization of Dependent Actuarial Risks: Preferences Among Prospects, 18th Int. Conf. on Actuarial Science and Quantitative Finance, Amsterdam, 1-2 December **2016**.

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THESIS/DISSERTATION ORIGINALITY REPORT

HACETTEPE UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING
TO THE DEPARTMENT OF ACTUARIAL SCIENCES

Date: 27/07/2018

Thesis Title: Multivariate Stochastic Prioritization of Dependent Actuarial Risks under Uncertainty

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ADVISOR APPROVAL

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