INTEGRALS OF MOTION IN CURVED SPACE-TIME

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ABSTRACT

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Integrals of motion are the quantities that remain constant during the motion of a point particle that allow to determine various important properties without solving the equations of motion. In this thesis, a systematic analysis for the motion of a relativistic particle in curved space-time is given and the relation of the integrals of motion to the Killing vectors and the Killing tensors of the space-time in which the particle moves is explained. As examples, motion on the Schwarzschild, the Kerr and the generalized Lense-Thirring space-times are studied.

Keywords: general relativity, black holes, integrals of motion, Killing tensors.

ÖZET

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Hareket integralleri noktasal bir parçacığın hareketi sırasında sabit kalan ve hareketin farklı önemli özelliklerini hareket denklemini çözmeden elde etmeye yarayan niceliklerdir. Bu tezde göreli bir parçacığın hareketinin sistematik bir analizi verilmekte ve hareket integralleri ile parçacığın üzerinde hareket ettiği uzay-zamanın Killing vektörleri ve Killing tensörleri arasındaki ilişki açıklanmaktadır. Örnek olarak Schwarzschild, Kerr ve genelleştirilmiş Lense-Thirring uzay-zamanı üzerindeki hareket çalışılmıştır.

Keywords: genel görelilik, kara delikler, hareket sabitleri, Killing tensörler.

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CONTENTS

1. INTRODUCTION

For the analysis of the motion of a point particle, the most straightforward way is to solve the equations of motion. However, apart from the practical difficulties of solving such a problem in a generic case, the importance and usefulness of conserved quantities is very well-known. In the Newtonian mechanics, arguably the simplest example, such a consideration leads to the concepts of energy and momentum, which help us to not only obtain the path of the particle easily, but also provide a better understanding of the underlying structure. These conserved quantities that remain constant along the path of the particle are also called the constants of motion or the integrals of motion.

With the photographing of the black hole at the center of the M87 galaxy in 2019 and the 2020 Nobel physics awards finding the winners, the importance of black hole researches in the scientific world has been registered. However, very interestingly, the theoretical awareness of black holes goes back to the years when solutions to Einstein's general relativity were studied. In this thesis, the particle motion in Schwarzschild, Kerr and a generalization of Lense-Thirring space-times, which are solutions of Einstein's field equations, will be examined. These space-times represent black hole solutions under suitable conditions.

After a brief review of some concepts of differential geometry that are useful for our purposes in Chapter 3, we will introduce alternative formulations of particle mechanics and their relation to the Killing vectors and Killing tensors of the space-time in which the particle moves. According to the findings of Eisenhart in [1], there exists a connection between Killing tensors and the separability of the Hamilton-Jacobi equation. This association will be frequently employed when examining the space-times under study.

The rest of the thesis is devoted to the examples. In Chapter 4, the motion in the Schwarzschild space-time, the static, spherically symmetric vacuum solution of general relativity [2, 3], is analyzed from the perspective of integrals of motion.

In Chapter 5, we first move our attention to the Kerr space-time, the stationary vacuum solution of general relativity [4], and give a derivation of the Carter constant [5]. We also present its relation to a Killing tensor. In 1918, two years following Karl Schwarzschild's resolution of Einstein's field equations, Hans Thirring and Josef Lense discovered a solution in [6] for the Einstein's field equations in a vacuum that is non-exact but applicable for large distances. After examining the motion in this space-time, we will study the motion in the generalized Lense-Thirring space-time, which was recently proposed for the description of slowly rotating black holes [7]. The reason behind this proposal is the occurrence of an additional singularity in the standard Lense-Thirring space-time, which has been studied for numerous years. And we see that in the generalized Lense-Thirring space-time, this singularity disappears.

We end the thesis with a summary in Chapter 6.

2. REVIEW OF DIFFERENTIAL GEOMETRY

In this chapter, some concepts of differential geometry that are relevant in the study of particle motion in curved space-time will be reviewed. Until section 2.6, the book by Cosimo Bambi [8] will be used as the reference.

2.1. Scalar, Vector and Tensor

Considering two points with coordinates x^μ and $x^\mu + dx^\mu$, the line element that is used to find the infinitesimal distance between these points is defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor and its signature is $(-, +, +, +)$. Note that the infinitesimal distance is invariant under coordinate transformations.

One can classify all the physical quantities according to their behaviour under coordinate transformations and the simplest example is a scalar. If, for example, $\phi(x)$ is a scalar, then under a coordinate transformation

$$x^\mu \rightarrow x^{\mu'} = x^{\mu'}(x), \tag{2}$$

it remains invariant as follows

$$\phi'(x') = \phi(x). \tag{3}$$

A vector is an element of a vector space. Thus, they must satisfy certain axioms. However, since these axioms are quite obvious, they will not be included here. For our purposes, the transformation properties of a vector V with components V^μ under a coordinate transformation (2) is enough and it is as follows

$$V^{\mu'}(x') = \frac{\partial x^{\mu'}}{\partial x^\nu} V^\nu(x). \tag{4}$$

For the dual vectors with components V_μ, the transformation rule is given by

$$V_{\mu'}(x') = \frac{\partial x^\nu}{\partial x^{\mu'}} V_\nu(x). \tag{5}$$

Tensors can be thought of the generalization of vectors and dual vectors, and they may have multi-indices. Vectors are type $(1, 0)$ tensors. Let T be a tensor of type (r, s). Its components transform as

$$T^{\mu_1'\mu_2'...\mu_r'}_{\nu_1'\nu_2'...\nu_s'}(x') = \frac{\partial x^{\mu_1'}}{\partial x^{\rho_1}} \frac{\partial x^{\mu_2'}}{\partial x^{\rho_2}} ... \frac{\partial x^{\mu_r'}}{\partial x^{\rho_r}} \frac{\partial x^{\sigma_1}}{\partial x^{\nu_1'}} \frac{\partial x^{\sigma_2}}{\partial x^{\nu_2'}} ... \frac{\partial x^{\sigma_s}}{\partial x^{\nu_s'}} T^{\rho_1\rho_2...\rho_r}_{\sigma_1\sigma_2...\sigma_s}(x). \tag{6}$$

According to this classification, while the scalars are tensors of type $(0, 0)$, the dual vectors are type $(0, 1)$. One can change the type of a tensor by raising and lowering the indices using the metric tensor. The contraction of indices also changes the type. Some examples are given below

$$\begin{aligned} T^{\mu\nu\rho} =& g^{\mu\alpha} g^{\nu\beta} g^{\rho\sigma} T_{\alpha\beta\sigma}, \\ T^{\mu}_{\ \nu\rho} =& g^{\mu\alpha} g_{\alpha\beta} T^{\beta}_{\ \nu\rho,} \\ g^{\mu\alpha} T_{\alpha\nu} =& T^{\mu}_{\ \nu.} \end{aligned} \tag{7}$$

2.2. Covariant Derivative

For various applications, the derivatives of physical quantities are surely needed, and therefore, their behaviour under coordinate transformations should also be studied.

If ϕ is a scalar, its partial derivative with respect to the coordinates x^μ is a dual vector, since it transforms as a dual vector:

$$\frac{\partial \phi'}{\partial x^{\mu'}} = \frac{\partial x^\nu}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x^\nu}. \tag{8}$$

Let us study the derivative of the components of a vector V^μ.

$$\partial'_\nu V^{\mu'} = \frac{\partial V^{\mu'}}{\partial x^{\nu'}} = \frac{\partial x^\rho}{\partial x^{\nu'}}\frac{\partial}{\partial x^\rho}\left(\frac{\partial x^{\mu'}}{\partial x^\sigma}V^\sigma\right) = \frac{\partial x^\rho}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^\sigma}\frac{\partial V^\rho}{\partial x^\rho} + \frac{\partial x^\rho}{\partial x^{\nu'}}\frac{\partial^2 x^{\mu'}}{\partial x^\rho \partial x^\sigma}V^\sigma, \quad (9)$$

which is not the transformation rule of a type $(1,1)$ tensor. The reason is simple. Although the infinitesimal displacement dx^μ, which is defined as

$$dx^\mu = \lim_{\Delta x^\mu \to 0} \Delta x^\mu, \quad (10)$$

is a tensor since it transforms as

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} dx^\nu, \quad (11)$$

dV^μ is not. Therefore, the partial derivative of the components of a tensor cannot yield a tensor.

Let us show it explicitly. The difference of the values of two vectors at different points is as follows

$$\begin{aligned} dV^{\mu'} &= V^{\mu'}(x' + dx') - V^{\mu'}(x'), \\ &= \left(\frac{\partial x^{\mu'}}{\partial x^\alpha}\right)_{x+dx} V^\alpha(x + dx) - \left(\frac{\partial x^{\mu'}}{\partial x^\alpha}\right)_x V^\alpha(x), \end{aligned} \quad (12)$$

from which we obtain

$$\begin{aligned} dV^{\mu'} &= \left(\frac{\partial x^{\mu'}}{\partial x^\alpha}\right)_x \left[V^\alpha(x+dx) - V^\alpha(x)\right] + \left(\frac{\partial^2 x^{\mu\prime}}{\partial x^\alpha \partial x^\beta}\right)_x V^\alpha(x+dx)dx^\beta, \\ &= \left(\frac{\partial x^{\mu'}}{\partial x^\alpha}\right)_x dV^\alpha + \left(\frac{\partial^2 x^{\mu\prime}}{\partial x^\alpha \partial x^\beta}\right) V^\alpha(x+dx)dx^\beta + ..., \end{aligned} \quad (13)$$

which is not the transformation rule of a tensor of type $(1,0)$ due to the 2nd term. As a result, we see that an infinitesimal change of the components of the vector V^μ that transforms as a tensor is needed to obtain a covariant derivative which transforms properly. In order to find it, let us consider a massive particle for which $ds^2 = -d\tau^2 < 0$ where τ is the proper time that is invariant under coordinate transformations by definition. As we will derive in the next chapter, the shortest distance between two points in space-time is given by the geodesic equation

$$\dot{u^\mu} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0, \tag{14}$$

where $\dot{} = \frac{d}{d\tau}$, $u^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity of the particle that follows the geodesics, and the Christoffel symbols ${\Gamma^\mu}_{\alpha\beta}$ are given by

$${\Gamma^\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} \left(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\alpha\rho} - \partial_\rho g_{\alpha\beta} \right). \tag{15}$$

Writing the geodesic equation as $\frac{Du^\mu}{d\tau} = 0$ gives rise to the following infinitesimal variation

$$Du^\mu = du^\mu + {\Gamma^\mu}_{\alpha\beta} u^\alpha dx^\beta. \tag{16}$$

It is straightforward to show that Du^μ transforms as a tensor, i.e., $Du^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\alpha} Du^\alpha$.

We can apply this result to a vector with components V^μ and write

$$DV^\mu = dV^\mu + {\Gamma^\mu}_{\alpha\beta} V^\alpha dx^\beta, \tag{17}$$

from which a covariant derivative of a type $(1, 1)$ tensor can be obtained as follows:

$$DV^\mu = \nabla_\nu V^\mu dx^\nu, \tag{18}$$

where

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\alpha} V^\alpha. \tag{19}$$

Using the fact that $V^\mu W_\mu$ is a scalar and defining $\nabla_\mu \phi = \partial_\mu \phi$, one can show that the covariant derivative of a dual vector is given as

$$\nabla_\mu W_\mu = \partial_\nu W_\mu - \Gamma^\alpha_{\mu\nu} W_\alpha. \tag{20}$$

The generalization to a tensor of type (p, q) is obvious.

2.3. Riemann Tensor

Generally, covariant derivatives do not commute. This leads to the definition of a new tensor, which is very important in characterizing the curvature of space-time. The Riemann tensor is defined as follows:

$$\left[\nabla_\mu, \nabla_\nu\right] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho = R^\rho{}_{\alpha\mu\nu} V^\alpha. \tag{21}$$

From this, the components of it can be found to be

$$R^\mu{}_{\rho\nu\sigma} = \partial_\nu \Gamma^\mu{}_{\sigma\rho} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\nu\alpha}\Gamma^\alpha{}_{\sigma\rho} - \Gamma^\mu{}_{\sigma\alpha}\Gamma^\alpha{}_{\nu\rho}. \tag{22}$$

Not all the components of the Riemann tensor are independent. It can be easily seen that

$$R^\mu{}_{\sigma\nu\rho} = -R^\mu{}_{\sigma\rho\nu}. \tag{23}$$

When the first index is lowered, one obtains

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\rho\nu} + \partial_\nu\partial_\rho g_{\mu\sigma} - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\mu\rho}\right) + \Gamma_{\alpha\mu\sigma}\Gamma^\alpha{}_{\nu\rho} - \Gamma_{\alpha\mu\rho}\Gamma^\alpha{}_{\nu\sigma}. \tag{24}$$

Using this form, one can also show

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} = R_{\rho\sigma\mu\nu}. \tag{25}$$

Obviously, the components of the Riemann tensor vanish for Minkowski space-time in Cartesian coordinates. By the "zero-component lemma," which states that if all the components of a tensor vanish in a certain coordinate system, they vanish in all the admissible coordinate systems, we can conclude that a space-time is flat if and only if $R^\mu{}_{\nu\rho\sigma} = 0$.

2.4. Ricci Tensor and Ricci Scalar

From the contractions of the Riemann tensor, a type $(0,2)$ tensor and a scalar quantity that are related to the curvature of the space-time can be naturally obtained. The Ricci tensor is defined as

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}. \tag{26}$$

Since $R_{\alpha\mu\beta\nu} = R_{\beta\nu\alpha\mu}$, we have

$$R_{\mu\nu} = g^{\alpha\beta} R_{\beta\nu\alpha\mu} R_{\nu\mu} = R_{\nu\mu}, \tag{27}$$

where we have used the symmetries of the inverse metric tensor to interchange α and β indices in the Riemann tensor. As a result, the Ricci tensor is symmetric.

The Ricci scalar, or the curvature scalar is obtained from the contraction of the Ricci tensor as follows

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}. \tag{28}$$

As will be seen soon, they form the basis of Einstein's formulation of the general theory of relativity.

2.5. Bianchi Identities

The Bianchi identities are two important identities satisfied by the Riemann tensor. The first Bianchi identity reads

$$0 = R^{\mu}{}_{\nu\rho\sigma} + R^{\mu}{}_{\rho\sigma\nu} + R^{\mu}{}_{\sigma\nu\rho}, \tag{29}$$

which can be shown to be true by using the definition of the Riemann tensor (22) as follows

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} + R^{\mu}{}_{\rho\sigma\nu} + R^{\mu}{}_{\sigma\nu\rho} &= \partial_\rho \Gamma^{\mu}{}_{\sigma\nu} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\alpha}{}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\nu}{}_{\rho\nu} + \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} \\ +\Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\alpha}{}_{\nu\rho} - \partial_\nu \Gamma^{\mu}{}_{\sigma\rho} - \Gamma^{\mu}{}_{\nu\alpha}\Gamma^{\alpha}{}_{\sigma\rho} &+ \partial_\nu \Gamma^{\mu}{}_{\rho\sigma} + \Gamma^{\mu}{}_{\nu\alpha}\Gamma^{\alpha}{}_{\rho\sigma} - \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\alpha}{}_{\nu\sigma} = 0. \end{aligned} \tag{30}$$

The second Bianchi identity relates the components of the covariant derivative of the Riemann Tensor:

$$0 = \nabla_\mu R^\alpha{}_{\beta\nu\rho} + \nabla_\nu R^\alpha{}_{\beta\rho\mu} + \nabla_\rho R^\alpha{}_{\beta\mu\nu}. \tag{31}$$

Contracting the indices ν and α yields

$$\begin{aligned} 0 =& \nabla_\mu R_{\beta\rho} + \nabla_\alpha R^\alpha{}_{\beta\rho\mu} + \nabla_\rho R^\alpha{}_{\beta\mu\alpha}, \\ =& \nabla_\mu R_{\beta\rho} + \nabla_\alpha R^\alpha{}_{\beta\rho\mu} - \nabla_\rho R_{\beta\mu}. \end{aligned} \tag{32}$$

The contraction of the indices β and μ gives

$$\begin{aligned} 0 &= \nabla_\mu R^\mu{}_\rho + \nabla_\alpha (g^{\beta\mu} R^\alpha{}_{\beta\rho\mu}) - \nabla_\rho (g^{\beta\mu} R_{\beta\mu}), \\ 0 &= \nabla_\mu R^\mu{}_\rho + \nabla_\alpha R^\alpha{}_\rho - \nabla_\rho R, \\ &= 2\nabla_\mu R^\mu{}_\rho - \nabla_\rho R, \end{aligned} \tag{33}$$

which gives

$$\nabla_\mu R^\mu{}_\rho = \frac{1}{2} \nabla_\rho R. \tag{34}$$

This expression can be manipulated to obtain a covariantly conserved tensor:

$$\begin{aligned} 0 &= \nabla_\mu R^\mu{}_\nu - \frac{1}{2} \nabla_\nu R, \\ &= \nabla_\mu (R^\mu{}_\nu - \frac{1}{2} \delta^\mu{}_\nu R), \end{aligned} \tag{35}$$

from which we define the Einstein tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{36}$$

which is symmetric and covariantly conserved:

$$G_{\mu\nu} = G_{\nu\mu}, \quad \nabla_\nu G^{\nu\mu} = 0. \tag{37}$$

Since the energy-momentum conservation in curved space-time is expressed by a symmetric and covariantly conserved energy-momentum tensor as follows

$$T_{\mu\nu} = T_{\nu\mu}, \quad \nabla_\nu T^{\nu\mu} = 0. \tag{38}$$

Einstein proposed that the relation between space-time curvature and the energy-momentum should be in the form $G_{\mu\nu} \propto T_{\mu\nu}$. The proportionality constant can be determined from the non-relativistic limit, which yields

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{39}$$

which are the Einstein's equations (G: Newton's constant, $c = 1$).

2.6. Lie Derivative

In this section, a new type of derivative will be defined by following Poisson's book [9] .

Consider a curve γ parameterized by $x^\alpha(\lambda)$ and a vector field V^α in a neighbourhood of γ. Under the coordinate transformation

$$x^{\alpha'} = x^\alpha + dx^\alpha = x^\alpha + \frac{dx^\alpha}{d\lambda}, \tag{40}$$

where $\frac{dx^\alpha}{d\lambda}$ is the tangent vector to the curve γ, the vector V^α becomes:

$$V^{\alpha'}(x') = \frac{\partial x^{\alpha'}}{\partial x^\beta} V^\beta(x) = \left(\delta^\alpha_\beta + \partial_\beta \frac{dx^\alpha}{d\lambda} d\lambda\right) V^\beta(x) = V^\alpha(x) + \partial_\beta \frac{dx^\alpha}{d\lambda} V^\beta(x)\, d\lambda, \tag{41}$$

which can be written as

$$V^{\alpha'}(x + dx) = V^\alpha(x) + \partial_\beta \frac{dx^\alpha}{d\lambda} V^\beta(x) d\lambda. \tag{42}$$

Examining the left hand side, one has

$$V^\alpha(x+dx) = V^\alpha(x) + \partial_\beta V^\alpha(x)dx^\beta = V^\alpha(x) + \frac{dx^\beta}{d\lambda}\partial_\beta V^\alpha(x)d\lambda. \tag{43}$$

Thus; we see that $V^{\alpha'}(x+dx)$ and $V^\alpha(x+dx)$ are not equal to each other, in general. The definition of the Lie derivative arises from this fact. The Lie derivative of a vector field V^α along the curve γ with the tangent vector $u^\alpha = \frac{dx^\alpha}{d\lambda}$ is defined as

$$\pounds_u V^\alpha(x) = \frac{V^\alpha(x+dx) - V^{\alpha'}(x+dx)}{d\lambda}. \tag{44}$$

Using (43) it can be written as

$$\pounds_u V^\alpha(x) = \partial_\beta V^\alpha u^\beta - \partial_\beta u^\alpha V^\beta. \tag{45}$$

Since the Lie derivative can be written with the covariant derivatives as

$$\pounds_u V^\alpha = \nabla_\beta V^\alpha u^\beta - \nabla_\beta u^\alpha V^\beta, \tag{46}$$

the result is obviously a tensor. Repeating the same steps, one can extend the definition to different type of tensors as follows

$$\begin{aligned} \pounds_u \phi &= u^\alpha \partial_\alpha \phi, \\ \pounds_u T^\alpha_{\ \beta} &= u^\mu \nabla_\mu T^\alpha_{\ \beta} - T^\mu_{\ \beta}\nabla_\mu u^\alpha + T^\alpha_{\ \mu}\nabla_\beta u^\mu. \end{aligned} \tag{47}$$

A tensor field $T^{\mu\cdots}_{\nu\cdots}$ is said to be Lie transported along a curve γ with the tangent vector $u^\alpha = \frac{dx^\alpha}{d\lambda}$ if its Lie derivative along the curve vanishes

$$\pounds_u T^{\mu\cdots}_{\nu\cdots} = 0. \tag{48}$$

Suppose we choose coordinates such that $x^1 = x^2 = x^3 =$ constant on γ and $x^0 = \lambda$ changes only on γ. In this coordinate system,

$$u^\alpha = \frac{dx^\alpha}{d\lambda} \stackrel{*}{=} \delta^\alpha_{\ 0}, \quad \partial_\beta u^\alpha \stackrel{*}{=} 0, \tag{49}$$

such that the Lie derivative becomes

$$\pounds_u T^{\alpha\dots}_{\beta\dots} \stackrel{*}{=} u^\mu \partial_\mu T^{\alpha\dots}_{\beta\dots} \stackrel{*}{=} \frac{\partial}{\partial x^0} T^{\alpha\dots}_{\beta\dots}. \tag{50}$$

From this, we infer two main results:

1) If $\pounds_u T^{\alpha\dots}_{\beta\dots} = 0$, then one can construct a coordinate system such that $u^\alpha \stackrel{*}{=} \delta^\alpha_{\ 0}$ and $\partial_0 T^{\alpha\dots}_{\beta\dots} = 0$.

2) If in a given coordinate system, the components of a tensor do not depend on a particular coordinate, say $x^\star$, then $\pounds_u T^{\alpha\dots}_{\beta\dots} = 0$ with $u^\alpha = \delta^\alpha_{\ \star}$.

2.7. Killing Vector and Killing Tensor

We are now in a position to define an object which will be crucially important in the study of the motion of relativistic particles, the Killing vector, and its generalization, i.e., the Killing tensor. In this section, we additionally follow Padmanabhan's book [10].

The definition of a Killing vector is quite simple. If the metric is Lie transported along a vector ξ^μ, it is called a Killing vector. Studying the Lie derivative of the metric tensor along a vector ξ^μ,

$$\begin{aligned} \pounds_\xi g_{\mu\nu} &= \xi^\alpha \nabla_\alpha g_{\mu\nu} + \nabla_\mu \xi^\alpha g_{\alpha\nu} + \nabla_\nu \xi^\alpha g_{\mu\alpha}, \\ &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \end{aligned} \tag{51}$$

where we have used $\nabla_\alpha g_{\mu\nu} = 0$, one obtains the Killing equation

$$0 = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{52}$$

which is the defining equation for Killing vectors. By solving this equation, one can find directions along which the metric tensor is Lie transported and they are called $isometries$ of the space-time under consideration.

The simplest Killing vectors can be obtained by checking whether there are any coordinates of which the components of the metric tensor are independent. From the results of the previous section, for such a coordinate, say $x^\star$, $\xi^\mu = \delta^\mu_\star$ is a Killing vector. Such Killing vectors are called translational Killing vectors.

Killing tensors are natural generalizations of Killing vectors. If a tensor of type $(0, n)$ which is completely symmetric in its indices, $K_{\mu_1 \ldots \mu_n} = K_{(\mu_1 \ldots \mu_n)}$ and

$$\nabla_{(\mu} K_{\mu_1 \ldots \mu_n)} = 0, \tag{53}$$

is called a Killing tensor. Later we will see that we can generate constants of motion from the contraction of a Killing (vector) tensor with particles 4-momentum as follows

$$K = p^{\mu_1} \ldots p^{\mu_n} K_{\mu_1 \ldots \mu_n}. \tag{54}$$

3. RELATIVISTIC PARTICLE MOTION

In this chapter, we aim to introduce different formalisms used to study the motion of a relativistic particle in curved space-time, which are the Lagrangian, the Hamiltonian, and the Hamilton-Jacobi formalisms. In section 3.1, we follow Bambi's book [8]. For the remainder of the chapter, the reference is the book by Landau on non-relativistic classical mechanics [11] where we make necessary modifications to make the results appropriate for relativistic applications with the help of [12].

3.1. Lagrangian Formalism

In the Lagrangian formalism, we parametrize the path of the particle as $x^\mu = x^\mu(\lambda)$ and consider a Lagrangian of the form $L = L(x, \dot{x}, \lambda)$ where $\dot{} = \frac{d}{d\lambda}$ denotes the derivative with respect to the parameter λ. Although it is possible to consider more general cases, this form of the Lagrangian covers most of the physically interesting cases. The action is defined as

$$S = \int d\lambda \, L(x, \dot{x}, \lambda). \tag{55}$$

According to the action principle, when one considers the variations of the path $x^\mu \to x^\mu + \delta x^\mu$ such that $\delta x^\mu \big|_{\lambda_i, \lambda_f} = 0$, then the equations of motion follow from the condition $\delta S = 0$, i.e., the action is extremized for the actual path of the particle. Calculating the variation of the action

$$\delta S = \int d\lambda \left[\frac{\partial L}{\partial x^\mu} \delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu \right] = \int d\lambda \left[\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} \right] \delta x^\mu + \left(\frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu \right) \Bigg|_{\lambda=\lambda_i}^{\lambda=\lambda_f}, \tag{56}$$

one obtains the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} = 0. \tag{57}$$

In the above equation, one can define a generalized force and a generalized momentum as follows

$$f_\mu = \frac{\partial L}{\partial x^\mu}, \qquad p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}, \tag{58}$$

such that the force is equal to the change of momentum: $f_\mu = \frac{dp_\mu}{d\lambda}$. If the Lagrangian L does not depend on a particular coordinate $x^\star$, i.e. $\frac{\partial L}{\partial x^\star} = 0$, it is said to be a cyclic coordinate whose corresponding generalized force is zero ($f_\star = 0$), and therefore the momentum conjugate to this coordinate is conserved ($p_\star = \text{constant}$).

Consider the derivative of the Lagrangian L with respect to the parameter λ

$$\begin{aligned} \frac{dL}{d\lambda} &= \frac{\partial L}{\partial x^\mu}\dot{x}^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\ddot{x}^\mu + \frac{\partial L}{\partial \lambda}, \\ &= \left[\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu}\right]\dot{x}^\mu + \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\dot{x}^\mu\right) + \frac{\partial L}{\partial \lambda}. \end{aligned} \tag{59}$$

Note that the first term vanishes when the Euler-Lagrange equations are satisfied. Defining

$$Q = \frac{\partial L}{\partial \dot{x}^\mu}\dot{x}^\mu - L, \tag{60}$$

we obtain

$$\frac{dQ}{d\lambda} = -\frac{\partial L}{\partial \lambda}, \tag{61}$$

on-shell, i.e., when the Euler-Lagrange equations are satisfied.

Therefore, when the Lagrangian has no explicit dependence on the parameter λ, $\frac{\partial L}{\partial \lambda} = 0$, Q is a constant of motion.

For a non-relativistic particle with the coordinate $q = q(t)$ parametrized by the Newtonian time, this just coincides with the familiar expression for the energy as follows

$$S = \int dt\, L(q, \dot{q}), \quad L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q), \tag{62}$$

$$Q = \frac{\partial L}{\partial \dot{q}}\dot{q} - V(q) = \frac{1}{2}m\dot{q}^2 + V(q) = E, \tag{63}$$

and

$$\frac{dE}{dt} = \frac{\partial L}{\partial t} = 0. \tag{64}$$

For a relativistic particle moving in curved space-time, we consider a Lagrangian of the form

$$L = \frac{1}{2} g_{\mu\nu} \dot{x^\mu} \dot{x}^\nu, \tag{65}$$

which can be considered as a generalized kinetic term. The action describing the motion is given by

$$S = \frac{1}{2} \int d\lambda \; g_{\mu\nu} \dot{x^\mu} \dot{x}^\nu. \tag{66}$$

One can easily show that the Euler-Lagrange equations following from this action is the same as the ones obtained from $\delta \int ds = 0$. Therefore, we find the paths which are the shortest distances between two points, i.e., geodesics. Working with the Lagrangian (65), which is in the form of a generalized kinetic term, is just a useful way of studying the geodesics of the space-time and there is no need to introduce any potential. All the gravitational effects are contained in the metric $g_{\mu\nu}$.

Let us assume λ is an affine parameter which is linearly related to the proper time and τ is the proper time of a massive particle. The line element reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -d\tau^2 = -m^2 d\lambda^2, \tag{67}$$

which gives

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -m^2 \leq 0. \tag{68}$$

The Euler-Lagrange equations corresponding to the action (66)

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0, \tag{69}$$

where the Christoffel symbols are given in (15). Since $\frac{\partial L}{\partial \lambda} = 0$, we have the following conserved quantity,

$$Q = \frac{\partial L}{\partial \dot{x}^\mu}\dot{x}^\mu - L = \frac{\dot{x}^2}{2} = -\frac{m^2}{2}, \tag{70}$$

which expresses just the conservation of mass.

Let us calculate the momenta.

$$\begin{aligned} p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} &= \frac{\partial}{\partial \dot{x}^\mu}\left(\frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right), \\ &= \frac{1}{2}g_{\alpha\beta}\left(\delta^\alpha_\mu\dot{x}^\beta + \dot{x}^\alpha\delta^\beta_\mu\right), \\ &= \frac{1}{2}\left(g_{\alpha\beta}\delta^\alpha_\mu\dot{x}^\beta + g_{\alpha\beta}\dot{x}^\alpha\delta^\beta_\mu\right), \\ &= g_{\mu\nu}\dot{x}^\nu. \end{aligned} \tag{71}$$

Raising the indices, we also find

$$p^\mu = g^{\mu\nu}p_\nu = g^{\mu\nu}g_{\nu\alpha}\dot{x}^\alpha = \dot{x^\mu}. \tag{72}$$

Let us also calculate

$$p^2 = g_{\mu\nu}p^\mu p^\nu = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -m^2 = \text{constant}. \tag{73}$$

Where we have used (68). This will be the equation that we will use to impose the conservation of mass.

3.2. Hamiltonian Formalism

As an alternative, one can derive the equations of motion by taking the coordinates x^μ and momenta p_μ as independent variables. This is the basis of the Hamiltonian formalism. As we will show, it has the advantage that the relation between constants of motion and the Killing (vectors) tensors of the space-time can be made explicit.

In order to develop the Hamiltonian formulation, we start by studying the infinitesimal changes of the Lagrangian $L = L(x, \dot{x}, \lambda)$.

$$dL = \frac{\partial L}{\partial x^\mu} dx^\mu + \frac{\partial L}{\partial \dot{x^\mu}} d\dot{x^\mu} + \frac{\partial L}{\partial \lambda} d\lambda, \tag{74}$$

which may be written as

$$\begin{aligned} dL &= \dot{p}_\mu dx^\mu + p_\mu d\dot{x}^\mu + \frac{\partial L}{\partial \lambda} d\lambda, \\ &= \dot{p}_\mu dx^\mu + d\left(p_\mu \dot{x}^\mu\right) - \dot{x}^\mu dp_\mu + \frac{\partial L}{\partial \lambda} d\lambda. \end{aligned} \tag{75}$$

Collecting the total differentials, we obtain

$$d\left(p_\mu \dot{x}^\mu - L\right) = \dot{x}^\mu dp_\mu - \dot{p}_\mu dx^\mu - \frac{\partial L}{\partial \lambda} d\lambda. \tag{76}$$

If we define the Hamiltonian as

$$H = H(x, p) = p_\mu \dot{x}^\mu - L, \tag{77}$$

we obtain the Hamilton's equations of motion as follows

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu}, \qquad \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu}, \qquad \frac{\partial H}{\partial \lambda} = -\frac{\partial L}{\partial \lambda}. \tag{78}$$

Using (77), one can also obtain the Hamiltonian form of the action as

$$S = \int d\lambda \, (p_\mu \dot{x}^\mu - H). \tag{79}$$

It is easy to show that the Hamilton equations follow from $\delta S = 0$ when the coordinates x^μ and momenta p_μ are varied independently, and assuming $\delta x^\mu \big|_{\lambda_i, \lambda_f} = 0$, $\delta p_\mu \big|_{\lambda_i, \lambda_f} = 0$.

The derivative of the Hamiltonian is

$$\frac{dH}{d\lambda} = \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial x^\mu}\dot{x}^\mu + \frac{\partial H}{\partial p_\mu}\dot{p}_\mu. \tag{80}$$

Using the Hamilton's equations (78), we get

$$\frac{dH}{d\lambda} = \frac{\partial H}{\partial \lambda}. \tag{81}$$

As a result, if the Hamiltonian H does not depend on the parameter λ, then, it is conserved, i.e., $\frac{dH}{d\lambda} = 0$. From (78), we also have

$$\frac{dH}{d\lambda} = \frac{\partial H}{\partial \lambda} = -\frac{\partial L}{\partial \lambda}. \tag{82}$$

Therefore, the conservation of the Hamiltonian can also be seen by checking if the Lagrangian has any dependence on λ.

For our Lagrangian given in (65), the definition of the Hamiltonian (77) gives

$$H = \frac{1}{2}g_{\mu\nu}p^\mu p^\nu. \tag{83}$$

Therefore, its conservation is just the conservation of mass.

As an alternative formulation, the Hamilton's equations (78) should agree with the Euler-Lagrange equations. Now, we will show it explicitly. From $\dot{p}_\mu$ in (78) , we get

$$\frac{d}{d\lambda}\left(g_{\mu\alpha}\dot{x}^\alpha\right) = -\frac{1}{2}\partial_\mu g_{\alpha\beta}p^\alpha p^\beta. \tag{84}$$

Taking the derivative at the left hand side leads to

$$g_{\mu\alpha}\ddot{x}^\alpha + \partial_\beta g_{\mu\alpha}\dot{x}^\alpha\dot{x}^\beta = -\frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta, \tag{85}$$

$$g_{\mu\alpha}\ddot{x}^\alpha = \frac{1}{2}\left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right)\dot{x}^\alpha\dot{x}^\beta, \tag{86}$$

which is just the Euler-Lagrange equations presented in (69).

3.3. Poisson Brackets

The Hamiltonian formalism offers the possibility of studying the conserved quantities in a more systematic form. Consider a quantity f which is a function of the coordinates x^μ , momenta p_μ and the parameter λ. Taking its derivative gives

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial \lambda} + \frac{\partial f}{\partial x^\mu}\dot{x}^\mu + \frac{\partial f}{\partial p_\mu}\dot{p}_\mu, \tag{87}$$

which takes the following form if the Hamilton's equations (78) are used

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial \lambda} + \frac{\partial f}{\partial x^\mu}\frac{\partial H}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial H}{\partial x^\mu}. \tag{88}$$

For simplicity, we write (88) in the following form

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial \lambda} + \{f, H\}, \tag{89}$$

where the second term at the right hand side is called the Poisson bracket of f with the Hamiltonian H given as

$$\{f, H\} = \frac{\partial f}{\partial x^\mu}\frac{\partial H}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial H}{\partial x^\mu}. \tag{90}$$

Similarly, we can define the Poisson bracket of two quantities $f = f(x, p, \lambda)$ and $g = g(x, p, \lambda)$ as follows

$$\{f, g\} = \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu} - \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu}. \tag{91}$$

There are some useful properties of Poisson brackets

$$\begin{aligned}\{f,g\} &= -\{g,f\},\\ \{f+g,h\} &= \{f,h\}+\{g,h\},\\ \{fg,h\} &= f\{g,h\}+\{f,h\}g.\end{aligned} \tag{92}$$

The last property is known as Leibniz rule of Poisson bracket, and it leads to a very important identity, called the Jacobi identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0. \tag{93}$$

Additionally, the Poisson bracket of a constant c with any function is zero:

$$\{f,c\}=0. \tag{94}$$

If one of the quantities in the Poisson bracket is the coordinates x^μ or momenta p_μ, then the result takes a particularly simple form as follows

$$\{f,x^\mu\}=\frac{\partial f}{\partial p_\mu}, \tag{95}$$

$$\{f,p_\mu\}=-\frac{\partial f}{\partial x^\mu}. \tag{96}$$

For the coordinates and momenta, the fundamental Poisson brackets take the following form

$$\{x^\mu,x^\nu\}=0, \qquad \{p_\nu,p_\mu\}=0, \qquad \{x^\mu,p_\nu\}=\delta^\mu{}_\nu. \tag{97}$$

3.4. Integrals of The Motion

We are now ready for a more systematic analysis of the conserved quantities, which are also called constants of motion or the integrals of the motion. If a quantity f is an integral of motion,

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial \lambda} + \{f, H\} = 0. \tag{98}$$

If it has no explicit dependence on the parameter λ, then its Poisson bracket with the Hamiltonian H should vanish:

$$\frac{d}{d\lambda} f(x,p) = 0 \iff \{f, H\} = 0. \tag{99}$$

According to the Poisson theorem, the Poisson bracket of two constants of motion is also a constant of motion. The proof is very easy: Assume that C_1 and C_2 are constants of motion, and examine the Jacobi identity

$$\{C_1, \{C_2, H\}\} + \{C_2, \{H, C_1\}\} + \{H, \{C_1, C_2\}\} = 0. \tag{100}$$

Because $\{C_1, H\} = \{C_2, H\} = 0$, $\{C_1, \{C_2, H\}\} = 0$ and $\{C_2, \{H, C_1\}\} = 0$. Therefore, $\{H, \{C_1, C_2\}\} = 0$, i.e., $\{C_1, C_2\}$ is an integral of motion.

Let us assume that an integral of motion is a linear combination of momenta, $K = K^\mu p_\mu$.

$$\begin{aligned}
\frac{dK}{d\lambda} &= \{K, H\}, \\
&= \frac{\partial K}{\partial x^\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial K}{\partial p_\alpha}\frac{\partial H}{\partial x^\alpha}, \\
&= \partial_\alpha K^\mu p_\mu p^\alpha - \frac{1}{2} p^\alpha \partial_\alpha g_{\mu\nu} p^\mu p^\nu, \\
&= \nabla_\alpha K^\mu p_\mu p^\alpha - \Gamma^\mu_{\alpha\beta} K^\beta p_\mu p^\alpha - \frac{1}{2} K^\alpha \nabla_\alpha g_{\mu\nu} p^\mu p^\nu + \frac{1}{2} K^\alpha p^\mu p^\nu \left(\Gamma^\beta{}_{\alpha\mu} g_{\beta\nu} + \Gamma^\beta{}_{\alpha\nu} g_{\mu\beta} \right), \\
&= \nabla_\mu K_\nu p^\mu p^\nu, \\
&= \nabla_{(\mu} K_{\nu)} p^\mu p^\nu.
\end{aligned} \tag{101}$$

We see that it is a constant of motion $\left(\frac{dK}{d\lambda}=0\right)$ if and only if the vector K_μ is a Killing vector, i.e., $\nabla_{(\mu}K_{\nu)}=0$. Taking integrals of motion that are of higher-orders in momenta p_μ such as $K=K^{\mu_1\cdots\mu_n}p_{\mu_1}...p_{\mu_n}$, one can similarly show that the tensor $K_{\mu_1\cdots\mu_n}$ should be a Killing tensor of type $(0,n)$, i.e., $K_{\mu_1\cdots\mu_n}=K_{(\mu_1\cdots\mu_n)}$ and $\nabla_{(\alpha}K_{\mu_1\cdots\mu_n)}=0$.

From these results, we see that the integrals of motion are intimately related to the properties of the space-time. The integrals of motion that are linear in momenta arise due to the isometries of the space-time and they can be easily found from the translational Killing vectors. However, finding the Killing tensors of a space-time is a highly complicated task and it is, when possible, much easier to find the higher-order integrals of motion directly, where the Hamilton-Jacobi formalism will be our main tool in this thesis. From the higher-order integrals of motion, the components of the Killing tensors can be read off easily.

3.5. Canonical Transformations

Having obtained the Hamilton's equations (78), we can ask a natural question: Is it possible to define new coordinates and momenta such that the corresponding Hamilton's equations describe the same dynamics? As we will see in the next section, the answer to this question helps us to find a third formulation of mechanics.

Let us assume that our new coordinates X^μ and new momenta P_μ are functions of the old ones and parameter λ:

$$X^\mu = X^\mu(x,p,\lambda), \qquad P_\mu = P_\mu(x,p,\lambda), \tag{102}$$

and they satisfy the Hamilton's equations with a new Hamiltonian $H' = H'(X,P,\lambda)$ as follows

$$\dot{X}^\mu = \frac{\partial H'}{\partial P_\mu}, \qquad \dot{P}_\mu = -\frac{\partial H'}{\partial X^\mu}. \tag{103}$$

In order to find the relations between new and old variables, we can make use of the fact that the Hamilton's equations follow from the extremization of the Hamiltonian form of the action given in (79). We first write it as

$$S = \int \big(p_\mu dx^\mu - H d\lambda\big)\,. \tag{104}$$

For our new coordinates and momenta, we have

$$S = \int \big(P_\mu dX^\mu - H' d\lambda\big)\,. \tag{105}$$

If these two actions will lead to the same dynamics, they can only differ by a total differential (since its variation do not contribute to the equations of motion). Therefore, we can write

$$p_\mu dx^\mu - H d\lambda = P_\mu dX^\mu - H' d\lambda + dF, \tag{106}$$

where F is a function of the old coordinates x^μ, new coordinates X^μ and the parameter λ. We call it a generating function of first kind and solving for its differential gives

$$dF_1 = p_\mu dx^\mu - P_\mu dX^\mu + (H' - H) d\lambda, \tag{107}$$

from which we obtain

$$p_\mu = \frac{\partial F_1}{\partial x^\mu}, \qquad P_\mu = -\frac{\partial F_1}{\partial X^\mu}, \qquad H' = H + \frac{\partial F_1}{\partial \lambda}. \tag{108}$$

Manipulating the second term at the right-hand side of (107), we can obtain a generating function with a different functional dependence:

$$dF_1 = p_\mu dx^\mu - d\left(P_\mu X^\mu\right) + X^\mu dP_\mu + (H' - H) d\lambda, \tag{109}$$

$$d\left(F_1 + P_\mu X^\mu\right) = p_\mu dx^\mu + X^\mu dP_\mu + (H' - H) d\lambda. \tag{110}$$

Defining $F_2 = P_\mu X^\mu + F_1 = F_2(x, P, \lambda)$, which is a function of the old coordinates x^μ, new momenta P_μ and the parameter λ, we get a generating function of the second kind that satisfies

$$p_\mu = \frac{\partial F_2}{\partial x^\mu}, \qquad X^\mu = \frac{\partial F_2}{\partial P_\mu}, \qquad H' = H + \frac{\partial F_2}{\partial \lambda}. \tag{111}$$

By similar manipulations, we can also find a generating function of third kind $F_3 = F_3\left(p^\mu, P_\mu, \lambda\right)$ and fourth kind $F_4 = F_4\left(p_\mu, X_\mu, \lambda\right)$. These transformations from old coordinates and momenta to the new ones that preserve the dynamics are called the canonical transformations and $F_i, i = 1, 2, 3, 4$ are the generating functions of these transformations. For our purposes, a particular generating function of the second kind will play a very important role as we will see now.

3.6. Hamilton-Jacobi Formalism

Our third formulation of particle mechanics, the Hamilton-Jacobi formalism, is based the complete integrability or the Louville integrability, whose proof can be found in [13].

A system with n degrees of freedom can be solved by integration if one is able to find n independent integrals of motion with vanishing Poisson brackets $\left(\{K_i, K_j\} = 0, i = 1, ..., n\right)$. In order to find them, we first consider the Hamiltonian form of our action and from (104) we write

$$dS = p_\mu dx^\mu - H d\lambda, \tag{112}$$

which tells us that if we consider the action S as a function of the coordinates x^μ and the parameter λ, $S = S\left(x, \lambda\right)$, then we have

$$\frac{\partial S}{\partial x^\mu} = p_\mu, \qquad \frac{\partial S}{\partial \lambda} = -H. \tag{113}$$

From these equations, we obtain

$$\frac{\partial}{\partial \lambda} S\left(x, \lambda\right) + H\left(x, \frac{\partial S}{\partial x}\right) = 0, \tag{114}$$

which is the Hamilton-Jacobi equation.

Note that, we use $p_\mu = \frac{\partial S}{\partial x^\mu}$ $\left(x^\mu = x^0, x^1, ..., x^{n-1}\right)$ in the Hamiltonian. In this equation, the independent variables are the coordinates x^μ and the parameter λ. Therefore, a complete integral of this equation must have $n+1$ constants. Since the action S appears only with the partial derivatives in the Hamilton-Jacobi equation, one of the constants should be an integration constant and we can write the solution as

$$S\left(x^\mu, \lambda\right) = f\left(x^\mu, \alpha_\mu, \lambda\right) + c, \tag{115}$$

where α_μ are the remaining arbitrary constants. We will assume that the action S is a generating function of the second kind, i.e., $F_2\left(x^\mu, P_\mu, \lambda\right) = S\left(x^\mu, \lambda\right) = f\left(x^\mu, \alpha_\mu, \lambda\right)$ such that the constants α_μ correspond to the new momenta P_μ and check the consequences.

$$\frac{\partial F_2}{\partial x^\mu} = p_\mu \to \frac{\partial S}{\partial x^\mu} = p_\mu, \tag{116}$$

$$\frac{\partial F_2}{\partial P_\mu} = X^\mu \to \frac{\partial S}{\partial \alpha_\mu} = \beta^\mu, \tag{117}$$

$$\frac{\partial F_2}{\partial \lambda} = H' - H \to \frac{\partial S}{\partial \lambda} = H' - H. \tag{118}$$

Equation (116) shows that our assumption is consistent with the defining properties of a generating function of second kind given in (111) . In equation (117) we just renamed the new coordinates $X^\mu = \beta^\mu$. The most crucial information follows from (118) : When the Hamilton-Jacobi equation (114) is used, we obtain $H' = 0$. Therefore, when we use the action as a generating function of second kind, the new Hamiltonian H' vanishes. As a result, the new canonical variables are constant $\left(X^\mu, P_\mu\right) = \left(\beta^\mu, \alpha_\mu\right) =$ constant. Both α_μ and β^μ form a set of integrals of motion with vanishing Poisson brackets and we achieve complete integrability.

Let us now focus on the Hamilton-Jacobi equation and try to understand how we can solve it. First, we express it as

$$\Psi = \Psi\left(x^\mu, \lambda, \frac{\partial S}{\partial x^\mu}, \frac{\partial S}{\partial \lambda}\right) = 0, \tag{119}$$

and assume that the coordinate x^1 and $\frac{\partial S}{\partial x^1}$ appear in the equation only in the form $\phi\left(x^1, \frac{\partial S}{\partial x^1}\right)$. Then, we can write

$$\Psi = \Psi\left(x^\nu, \lambda, \frac{\partial S}{\partial x^\nu}, \frac{\partial S}{\partial \lambda}, \phi\left(x^1, \frac{\partial S}{\partial x^1}\right)\right) = 0, \tag{120}$$

where x^ν are all the coordinates except x^1. If we try to look for a solution of the form,

$$S = S + S'\left(x^\nu, \lambda\right) + S_1\left(x^1\right), \tag{121}$$

we get

$$\Psi\left(x^\nu, \lambda, \frac{\partial S}{\partial x^\nu}, \frac{\partial S}{\partial \lambda}, \phi\left(x^1, \frac{dS}{dx^1}\right)\right) = 0, \tag{122}$$

which should be satisfied for all values of x^1 and this is only possible when $\phi\left(x^1, \frac{dS}{dx^1}\right) = \alpha_1 =$ constant. If this separation process is possible for all coordinates, we would obtain

$$S = \sum_\mu S_\mu\left(x^\mu, \alpha_1, ..., \alpha_n\right) - H\left(\alpha_1, ..., \alpha_n\right)\lambda, \tag{123}$$

where we have used $\frac{\partial S}{\partial \lambda} = -H =$ constant.

Cyclic variables are extremely useful to separate the action. If, for example, the coordinate x^1 is a cyclic coordinate $\left(\frac{\partial H}{\partial x^1} = 0\right)$, then it does not appear in the Hamilton-Jacobi equation. Therefore, we have

$$f\left(x^1, \frac{\partial S}{\partial x^1}\right) = \frac{\partial S}{\partial x^1} = \alpha_1, \tag{124}$$

and we can just write

$$S = S'\left(x^\nu, \lambda\right) + \alpha_1 x^1. \tag{125}$$

Let us now apply our formalism to the action for a relativistic particle moving in curved space-time given in (66), for which the Hamilton-Jacobi equation $\left(\frac{\partial S}{\partial \lambda} + H = 0\right)$ takes the following form

$$\frac{\partial S}{\partial \lambda} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0, \tag{126}$$

since $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$. Having already known that $H = -\frac{1}{2} m^2 =$ constant, we can write

$$S\left(x^\mu, \lambda\right) = S\left(x^\mu\right) - \frac{1}{2} m^2 \lambda, \tag{127}$$

whose substitution into (126) gives

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m^2 = 0. \tag{128}$$

This is the form that we will use in the rest of the thesis. Whenever there are cyclic coordinates, say x^1 and x^2 , one can make use of it by using the following ansatz

$$S = \sum_\nu S_\nu\left(x^\nu\right) + p_1 x^1 + p_2 x^2, \tag{129}$$

where ν denotes all the coordinates other than x^1 and x^2, and try to solve the equation (128). Note that if the coordinates x^1 and x^2 are cyclic, this means $\partial_1 g_{\mu\nu} = 0$ and $\partial_2 g_{\mu\nu} = 0$. Therefore, according to our results in section 2.7, we have Killing vectors $\xi^\mu{}_{(1)} = \delta^\mu{}_1$ and $\xi^\mu{}_{(2)} = \delta^\mu{}_2$ and p_1 and p_2 are constants of motions generated by these Killing vectors.

We will see that when we try to solve the Hamilton Jacobi equation (114) for a rotating black hole with an ansatz similar to (129), there will appear an additional constant of motion that is called the Carter constant. As we have seen before, this is connected to the existence of a Killing tensor of type $(0, 2)$, or rank 2.

4. PARTICLE MOTION IN SCHWARZSCHILD SPACE-TIME

In this chapter, we give an analysis of the particle motion in Schwarzschild space-time, which is the static, spherically symmetric vacuum solution of general relativity and Einstein field equations that describes the gravitational field outside a spherical mass. Starting from the following form of the line element of a static, spherically symmetric metric [2, 3],

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{130}$$

one can easily show that the vacuum field equations $G_{\mu\nu} = 0$ is solved for the metric function $f(r) = 1 - \frac{r_s}{r}$, where the Schwarzschild radius r_s is an integration constant. From the non-relativistic limit, one finds $r_s = 2M$ (in units $G = 1, c = 1$) where M is the mass. In this context, the mass M is not necessarily associated with a black hole, because the solution of the Einstein field equations is valid for any mass M.

4.1. Lagrangian Formulation

Now, we will study the Euler-Lagrange equations by following Bambi's book [8]. For the line element given in (130) , the Lagrangian reads

$$L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}\left(-f\dot{t}^2 + \frac{1}{f}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right). \tag{131}$$

Due to the spherical symmetry of the problem, we can work on the equatorial plane $\left(\theta = \frac{\pi}{2}, \dot{\theta} = 0\right)$, which leads to the following simplified form

$$L = \frac{1}{2}\left(-f\dot{t}^2 + \frac{1}{f}\dot{r}^2 + r^2\dot{\phi}^2\right). \tag{132}$$

We have 3 dynamical variables (t, r, ϕ) and 2 cyclic variables (t, ϕ) whose momenta are conserved. Taking our Killing vectors as $\xi^{\mu}_{(t)} = -\delta^{\mu}_{t}$ and $\xi^{\mu}_{(\phi)} = \delta^{\mu}_{\phi}$, we have $p_t = -E =$ constant and $p_\phi = L =$ constant. The Euler-Lagrange equations corresponding to t and ϕ coordinates are as follows:

$$t : \qquad -f\dot{t} = E, \tag{133}$$

$$\phi : \qquad r^2\dot{\phi} = L. \tag{134}$$

Instead of the Euler-Lagrange equation for the coordinate, we can make use of the conservation of mass, which is a constant of motion due to the trivial Killing tensor, $K_{\mu\nu} = g_{\mu\nu}$, given as

$$K = K^{\mu\nu}p_\mu p_\nu = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -m^2 = \text{constant}. \tag{135}$$

Using the expressions for $\dot{t}$ and $\dot{r}$ from equations (133) and (134), we obtain

$$-m^2 f = -E^2 + \dot{r}^2 + \frac{L^2 f}{r^2}. \tag{136}$$

Inserting the explicit form of the metric function $f(r) = 1 - \frac{2M}{r}$, we finally obtain

$$\frac{\dot{r}^2}{2} = \frac{E^2 - m^2}{2} - V_{\text{eff}}, \tag{137}$$

where the effective potential is

$$V_{\text{eff}} = -\frac{m^2 M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}. \tag{138}$$

Note that the slightly unusual form of the effective potential is due to our usual normalization of the 4-velocity $\left(\dot{x}^2 = -m^2\right)$. For $\dot{x}^2 = -k \, (k = 1 : \text{massive}, k = 0 : \text{massless})$, we would obtain

$$V_{\text{eff}} = -k\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^3}{r^3}, \tag{139}$$

where the first two terms are the usual Newtonian terms and the last term is the relativistic correction.

4.2. Hamilton-Jacobi Formalism

We now will try to obtain the equations of motion from the Hamilton-Jacobi formalism. The inverse metric is given by

$$g^{\mu\nu} = \text{diag}\left(-\frac{1}{f(r)}, f(r), \frac{1}{r^2}, \frac{1}{r^2\sin\theta^2}\right), \tag{140}$$

with which, the Hamilton Jacobi equation (128) becomes

$$m^2 = \frac{1}{f}\left(\frac{\partial S}{\partial t}\right)^2 - f\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2. \tag{141}$$

Let us write the following ansatz using our conserved momenta $\big(p_t = -E, p_\phi = L\big)$

$$S = -Et + L\phi + S_r(r), \tag{142}$$

whose substitution into (141) yields

$$m^2 = \frac{E^2}{f(r)} - f(r)\left(\frac{dS}{dr}\right)^2 - \frac{L^2}{r^2}, \tag{143}$$

and

$$\frac{dS_r}{dt} = \sqrt{\frac{E^2}{f(r)} - \frac{m^2}{f(r)} - \frac{L^2}{r^2 f(r)}}. \tag{144}$$

Therefore, the action is given by

$$S = -Et + L\phi + \int dr\sqrt{\frac{E^2}{f(r)} - \left(m^2 + \frac{L^2}{r^2}\right)f(r)}. \tag{145}$$

According to Hamilton-Jacobi formalism, the equations of motion should follow form

$$\frac{\partial S}{\partial E} = \text{constant}, \qquad \frac{\partial S}{\partial L} = \text{constant}. \tag{146}$$

Let us study the first one.

$$\frac{\partial S}{\partial E} = -t + \frac{E}{m}\int dr \frac{1}{f(r)}\left[\left(\frac{E}{m}\right)^2 - \left(1+\frac{L^2}{m^2r^2}\right)f(r)\right]^{-\frac{1}{2}} = -t_0 = \text{constant},$$

$$t - t_0 = \int_{t_0}^{t} dt = \frac{E}{m}\int_{r_0}^{r} dr \frac{1}{f(r)}\left[\left(\frac{E}{m}\right)^2 - \left(1+\frac{L^2}{m^2r^2}\right)f(r)\right]^{-\frac{1}{2}}. \tag{147}$$

From this equation, we obtain

$$\frac{1}{f}\frac{dr}{dt} = \frac{m}{E}\left[\left(\frac{E}{m}\right)^2 - \left(1+\frac{L^2}{m^2r}\right)f\right]^{\frac{1}{2}}. \tag{148}$$

Using the chain rule,

$$\frac{dr}{d\lambda} = \frac{dr}{dt}\frac{dt}{d\lambda} = \frac{E}{f}\frac{dr}{dt}, \tag{149}$$

we can also write our result as

$$\frac{\dot{r}^2}{2} = \frac{E^2 - m^2}{2} - V_{\text{eff}}, \tag{150}$$

where

$$V_{\text{eff}} = -\frac{m^2 M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}, \tag{151}$$

which is, of course, the same result (138) that we obtain from the Lagrangian formulation.

From $\frac{\partial S}{\partial L} = \text{constant}$, we obtain

$$\frac{\partial S}{\partial L} = \phi - \int dr \frac{L}{r^2}\left(E^2 - \left(m^2 + \frac{L^2}{r^2}\right)f\right)^{-\frac{1}{2}} = \phi_0 = \text{constant},$$

$$\int_{\phi_0}^{\phi} d\phi = \int_{r_0}^{r} dr \frac{L}{r^2}\left(E^2 - \left(m^2 + \frac{L^2}{r^2}\right)f\right)^{-\frac{1}{2}}, \tag{152}$$

which gives

$$\frac{dr}{d\phi} = \frac{r^2}{L}\left(E^2 - \left(m^2 + \frac{L^2}{r^2}\right) f\right)^{\frac{1}{2}} . \tag{153}$$

After this simple application, we are now ready to study the motion in stationary spacetimes where the separation of the Hamilton-Jacobi equation will lead to a non-trivial Killing tensor.

5. STATIONARY SPACE-TIMES

In this chapter, we will analyze the integrals of motion for two stationary spacetimes. The crucial difference will be that in addition to the conserved momenta corresponding to the translational Killing vectors in the t- and ϕ- directions, we will obtain an integral of motion that is quadratic in momenta.

5.1. Particle Motion in Kerr Space-time

In this section, we follow Padmanabhan's book [10] and start with the most important example which has enormous importance in astrophysical applications: The Kerr space-time, which is the axially symmetric vacuum solution of general relativity that describes a gravitational field outside of an object with mass M and angular momentum J. The line element is given by

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 \\ &+\rho^2 d\theta^2 + \left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2, \end{aligned} \tag{154}$$

where $\rho^2 = r^2 + a^2\cos^2\theta,\ \Delta = r^2 - 2Mr + a^2$ and $a = \frac{J}{M}$ represents angular momentum per unit mass. When $a = 0$, one obtains the Schwarzschild black hole. In the $M \to 0$ limit, we get

$$ds^2 = -dt^2 + \frac{\rho^2}{r^2+a^2}dr^2 + \rho^2 d\theta^2 + \left(r^2+a^2\right)\sin^2\theta d\phi^2, \tag{155}$$

which is just the line element of the Minkowski spacetime. It can be converted to the Cartesian form by the following coordinate transformations

$$x = \sqrt{r^2+a^2}\sin\theta\cos\phi, \qquad y = \sqrt{r^2+a^2}\sin\theta\sin\phi, \qquad z = r\cos\theta. \tag{156}$$

From this, we see that $(\theta = 0, r = 0)$ corresponds to a disc of radius a, and therefore the r-coordinate is quite different then the usual radial coordinate of static space-times.

An alternative form of the line element that is commonly used in computations is as follows:

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sum^2 \sin^2\theta}{\rho^2} d\phi^2, \quad (157)$$

where $\sum^2 = \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta$.

The independent components of the inverse metric that we will need in the Hamilton-Jacobi equation are as follows

$$g^{rr} = \frac{\Delta}{\rho^2}, \quad g^{\theta\theta} = \frac{1}{\rho^2}, \quad g^{tt} = -\frac{\sum^2}{\rho^2 \Delta}, \quad g^{t\phi} = -\frac{2Mar}{\rho^2 \Delta}, \quad g^{\phi\phi} = \frac{a^2 \sin^2\theta - \Delta}{\rho^2 \Delta \sin^2\theta}. \quad (158)$$

Due to the Killing vectors $\xi^{\mu}_{\ (t)} = -\delta^{\mu}_{\ t}$ and $\xi^{\mu}_{\ (\phi)} = \delta^{\mu}_{\ \phi}$, we have the following conserved momenta

$$\begin{aligned} p_t = -E = g_{t\nu}\dot{x}^{\nu} = g_{tt}\dot{t} + g_{t\phi}\dot{\phi} = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}\dot{t} - \frac{2Mar\sin^2\theta}{\rho^2}\dot{\phi}, \\ p_{\phi} = L = g_{\phi\nu}\dot{x}^{\nu} = g_{\phi\phi}\dot{\phi} + g_{\phi t}\dot{t} = \frac{\sum^2\sin^2\theta}{\rho^2}\dot{\phi} - \frac{2Mar\sin^2\theta}{\rho^2}\dot{t}. \end{aligned} \quad (159)$$

The conservation of mass, which is the result of the trivial Killing tensor $K_{\mu\nu} = g_{\mu\nu}$, yields

$$\begin{aligned} -m^2 &= g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2 + 2g_{t\phi}\dot{t}\dot{\phi}, \\ &= -\frac{\Delta - a^2\sin^2\theta}{\rho^2}\dot{t}^2 + \frac{\rho^2}{\Delta}\dot{r}^2 + \rho^2\dot{\theta}^2 + \frac{\sum^2\sin^2\theta}{\rho^2}\dot{\phi}^2 - \frac{4Mar\sin^2\theta}{\rho^2}\dot{t}\dot{\phi}. \end{aligned} \quad (160)$$

To see the existence of another constant of motion, let us study the Hamilton-Jacobi equation (128):

$$g^{tt}\left(\frac{\partial S}{\partial t}\right)^2 + g^{rr}\left(\frac{\partial S}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial S}{\partial \theta}\right)^2 + g^{\phi\phi}\left(\frac{\partial S}{\partial \phi}\right)^2 + 2g^{t\phi}\frac{\partial S}{\partial t}\frac{\partial S}{\partial \phi} = -m^2. \quad (161)$$

Using the ansatz

$$S = -Et + L\phi + S_r(r) + S_{\theta}(\theta), \quad (162)$$

one obtains

$$-\frac{\sum^2}{\rho^2\Delta}E^2+\frac{\Delta}{\rho^2}\left(\frac{dS_r}{dr}\right)^2+\frac{1}{\rho^2}\left(\frac{dS_\theta}{d\theta}\right)^2+\frac{a^2\sin^2\theta-\Delta}{\rho^2\Delta\sin^2\theta}L^2+2\frac{2Mar}{\rho^2\Delta}EL=-m^2. \quad (163)$$

Multiplying this equation by ρ^2, there appears two parts depending on r and θ coordinates separately, and therefore they can only be a constant. Denoting the separation constant K, we get

$$\left(\frac{dS_\theta}{d\theta}\right)^2+\left(aE\sin\theta+\frac{L}{\sin\theta}\right)^2+a^2m^2\cos^2\theta=K, \quad (164)$$

$$\left(\frac{dS_r}{dr}\right)^2+\frac{1}{\Delta}\left[\left(r^2+a^2\right)E-aL\right]^2+m^2r^2=-K. \quad (165)$$

One can solve for S_r and S_θ to obtain the complete form of the action. However, we have already reached our aim and obtain our integral of motion. Let us write it in terms of momenta

$$K=p_\theta^2+a^2\sin^2\theta p_t^2+\frac{1}{\sin^2\theta}p_\phi^2-2ap_tp_\phi-a^2p^2\cos^2\theta, \quad (166)$$

$$-K=p_r^2+\frac{1}{\Delta}\left[\left(r^2+a^2\right)-p_t-ap_\phi\right]^2-p^2r^2, \quad (167)$$

from which we can find two Killing tensors but they are, of course, not independent. Using the first one, we obtain the following components of a Killing tensor

$$\begin{aligned}
K^{\theta\theta}&=1-\frac{a^2\cos^2\theta}{\rho^2},\\
K^{tt}&=a^2\sin^2\theta-a^2\cos^2\theta g^{tt}=a^2\sin^2\theta+a^2\cos^2\theta\frac{\Sigma^2}{\rho^2\Delta},\\
K^{t\phi}&=-2a-a^2\cos^2\theta g^{t\phi}=-2a+a^2\cos^2\theta\frac{2Mar}{\rho^2\Delta},\\
K^{\phi\phi}&=\frac{1}{\sin^2\theta}-a^2\cos^2\theta g^{\phi\phi}=\frac{1}{\sin^2\theta}-a^2\cos^2\theta\frac{a^2\sin^2\theta-\Delta}{\rho^2\Delta\sin^2\theta}.
\end{aligned} \quad (168)$$

In matrix form, we have

$$K^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^2\sin^2\theta + a^2\cos^2\theta\frac{\Sigma^2}{\rho^2\Delta} & 0 & -2a + a^2\cos^2\theta\frac{2Mar}{\rho^2\Delta} \\ 0 & 0 & a^2\sin^2\theta + a^2\cos^2\theta\frac{\Sigma^2}{\rho^2\Delta} & 0 \\ 0 & -2a + a^2\cos^2\theta\frac{2Mar}{\rho^2\Delta} & 0 & \frac{1}{\sin^2\theta} - a^2\cos^2\theta\frac{a^2\sin^2\theta - \Delta}{\rho^2\Delta\sin^2\theta} \end{pmatrix}. \tag{169}$$

This additional integral of motion that is quadratic in momenta and the result of a non-trivial Killing tensor is called the Carter constant [5].

In the next section, we will derive it for a space-time that has been introduced recently [7, 14] to get rid of some problems in the slow rotation limit of the Kerr space-time.

5.2. Particle Motion in Generalized Lense-Thirring Space-time

We will now study the slow rotation limit of the Kerr space-time, which is studied in different contexts in the literature. When we expand the line element of Kerr space-time (154) , we find

$$ds^2 = -fdt^2 + \frac{dr^2}{f} + 2a\sin^2\theta(f-1)dtd\phi + r^2\sin^2\theta d\phi^2 + r^2d\theta^2 + O(a^2), \tag{170}$$

where $f(r) = 1 - \frac{2M}{r}$. As $a \to 0$, we get the line element of the Schwarzschild space-time. The space-time up to $O(a)$ is called the Lense-Thirring space-time [6] and can be used as the slow rotation limit of the Kerr space-time. It should be noted that the Lense-Thirring space-time, which has been known for many years, can be clearly seen from its relation to the Kerr metric that, it represents a non-exact and slow-rotation solution for the Einstein field equations.

In the linear form, the metric is well-defined. But when it comes to the second order terms in the rotation parameter a, an inconvenience appears in the form of a singularity. To discuss the singularity structure, calculating the Kretschmann scalar, which is very important for the curvature of space-time, we find

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \mathrm{Kr} = \mathrm{Kr}_{(0)} + \mathrm{Kr}_{(1)}a + \mathrm{Kr}_{(2)}a^2 + O(a^3), \tag{171}$$

where

$$\begin{aligned} \mathrm{Kr}_0 &= \frac{48M^2}{r^6}, \\ \mathrm{Kr}_1 &= 0, \\ \mathrm{Kr}_2 &= -\frac{\psi}{r^9(r-2M)^2}, \end{aligned} \tag{172}$$

and

$$\begin{aligned} \psi = 8M^2(&\cos^2\theta(26M^2(8r+3) - Mr(195r+92) + 9r^2(5r+3)) \\ &+ 26M^2(8r+3) - Mr(199r+88) + r^2(47r+25)). \end{aligned} \tag{173}$$

The zeroth order term in the rotation parameter Kr_0 corresponds to the Kretschmann scalar of the Schwarzschild spacetime and indicates a singularity at $r = 0$. From Kr_2, we see that there is an additional singularity at $r = 2M$. To remedy this problem, in [14], the following line element for the slowly rotating black holes is suggested by completing the rotation term to a square

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2\sin^2\theta\left(d\phi + \frac{a(f-1)}{r^2}dt\right)^2 + r^2 d\theta^2. \tag{174}$$

The Kretschmann scalar now reads

$$\mathrm{Kr} = \frac{48M^2}{r^6} - \frac{144(M^2(-3M + 2r + (3M + r)\cos^2\theta))a^2}{r^9} + O(a^3), \tag{175}$$

and the problem of additional singularity at $r = 2M$ is solved.

Let us study the constants of motion in this generalized Lense-Thirring spacetime. The conserved momenta corresponding to the translational Killing vectors are

$$
\begin{aligned}
p_t = g_{t\nu}\dot{x}^\nu = g_{tt}\dot{t} + g_{t\phi}\dot{\phi} = -f\dot{t} + a\sin^2\theta(f-1)\dot{\phi} = -E = \text{constant}, \\
p_\phi = g_{\phi\nu}\dot{x}^\nu = g_{\phi\phi}\dot{\phi} + g_{\phi t}\dot{t} = r^2\sin^2\theta\dot{\phi} + a\sin^2\theta(f-1)\dot{t} = L = \text{constant}.
\end{aligned} \tag{176}
$$

From the trivial Killing tensor $K_{\mu\nu} = g_{\mu\nu}$, we get

$$
\begin{aligned}
K^{\mu\nu}p_\mu p_\nu &= K_{\mu\nu}p^\mu p^\nu = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2 + 2g_{t\phi}\dot{t}\dot{\phi}, \\
&= -f\dot{t}^2 + \frac{1}{f}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 + 2a\sin^2\theta(f-1)\dot{t}\dot{\phi}, \\
&= -m^2 = \text{constant}.
\end{aligned} \tag{177}
$$

Using the components of the inverse metric

$$
\begin{aligned}
g^{tt} &= \frac{r}{2M-r}, \\
g^{rr} &= 1 - \frac{2M}{r}, \\
g^{\theta\theta} &= \frac{1}{r^2}, \\
g^{\phi\phi} &= \frac{4a^2M^2}{(2M-r)r^5} + \frac{1}{r^2\sin^2\theta}, \\
g^{t\phi} &= g^{\phi t} = \frac{2aM}{(2m-r)r^2},
\end{aligned} \tag{178}
$$

the Hamilton-Jacobi equation can be obtained as follows:

$$
\begin{aligned}
& g^{tt}\left(\frac{\partial S}{\partial t}\right)^2 + g^{rr}\left(\frac{\partial S}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial S}{\partial \theta}\right)^2 + g^{\phi\phi}\left(\frac{\partial S}{\partial \phi}\right)^2 + 2g^{t\phi}\frac{\partial S}{\partial t}\frac{\partial S}{\partial \phi} = -m^2. \\
& \frac{r}{2M-r}E^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{dS}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS}{d\theta}\right)^2 \\
& + \left(\frac{4a^2M^2}{(2M-r)r^5} + \frac{1}{r^2\sin^2\theta}\right)L^2 + \frac{4aM}{(2m-r)r^2}EL = -m^2.
\end{aligned} \tag{179}
$$

Multiplying both sides by r^2, we get

$$\frac{r^3}{2M-r}E^2 + (r^2-2Mr)\left(\frac{dS}{dr}\right)^2 + \left(\frac{dS}{d\theta}\right)^2 \\ + \left(\frac{4a^2M^2}{(2M-r)r^3} + \frac{1}{\sin^2\theta}\right)L^2 + \frac{4aM}{2M-r}EL = -m^2r^2. \quad (180)$$

The separation of this equation gives

$$\begin{aligned} K &= p_\theta^2 + \frac{L^2}{\sin^2\theta}, \\ -K &= \frac{r^3}{2M-r}E^2 + (r^2-2Mr)p_r^2 + L^2\left(\frac{4a^2M^2}{(2M-r)r^3}\right) + \frac{4aM}{2M-r}EL + m^2r^2. \end{aligned} \quad (181)$$

The Carter constant K is a consequence of the Killing tensor whose components are

$$K^{\mu\nu} = \mathrm{diag}\left(0, 0, 1, \frac{1}{\sin^2(\theta)}\right). \quad (182)$$

More details can be found in [15].

6. SUMMARY

In this thesis, an analysis of the integrals of motion for a relativistic particle moving in curved space-time was given. The integrals of motion are directly related to the Killing vectors and the Killing tensors of the space-time that we have defined in Chapter 2.

The simplest example of an integral of motion is the conserved momentum corresponding to a cyclic coordinate, which is already apparent in the Lagrangian formulation of mechanics. In the Hamiltonian and Hamilton-Jacobi formalisms, one can understand how the integrals of motion that are higher order in momenta can be constructed with the help of Killing tensors. All the details were discussed in Chapter 3.

In Chapter 4, the integrals of motion for a particle moving in the Schwarzschild space-time was presented. When the motion in a stationary space-time is considered, a crucial difference arises.

In Chapter 5, the most famous example, the Kerr space-time and a space-time which has attracted lots of attention in the literature, the Lense-Thirring space-time were considered. We showed that an integral of motion which is a consequence of a non-trivial Killing tensor exists in these space-times.

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