f-SUPPLEMENTED LATTICES

f-TÜMLENMİŞ KAFESLER

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ABSTRACT

f-SUPPLEMENTED LATTICES

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The main purpose of this thesis is to generalize some known results about F-supplemented modules to lattices. Let L be a complete modular lattice with smallest element 0 and greatest element 1. A homomorphic image of an f-small element under a lattice homomorphism need not be f-small unlike the module case. For compactly generated compact lattices f-supplement elements are compact. For compactly generated lattices, f-radical is the join of all f-small elements. Moreover for compact lattices, f-radical itself is an f-small element. Let L be a compactly generated compact lattice. If L is f-supplemented, then the quotient sublattice $1/\operatorname{rad}_f(L)$ of L is semiatomic. A compact lattice L is f-supplemented if and only if every maximal element m of L with $f \leq m$ has an f-supplement in L. A join of f-supplemented lattices containing f is f-supplemented. Let L be a compact lattice and $f \leq a$ be an element of L. If $a/0$ is f-supplemented and $1/a$ has no maximal element, then L is f-supplemented. If a lattice L is amply f-supplemented, then the quotient sublattice $1/a$ is amply $(f \vee a)$-supplemented for every element a of L and the sublattice $a/0$ is amply f-supplemented for every f-supplement element a of L. L is amply f-supplemented if and only if for every element a of L, there is an element $x \leq a$ such that the sublattice $x/0$ is f-supplemented and the inequality $x \leq a$ is f-cosmall in L.

Keywords: f-small elements, f-radical f-supplemented lattices, amply f-supplemented lattices, compact lattices, compactly generated lattices.

ÖZET

f-TÜMLENMİŞ KAFESLER

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Bu tezde temel olarak f-tümlenmis modüller hakkında bilinen sonuçların kafes teorisine genelleştirilmesi üzerine çalışılması amaçlanmıştır. L, en büyük elemanı 1 en küçük elemanı 0 olan tam modüler bir kafes olsun. Bir modülün herhangi bir F-küçük alt modülünün bir modül homomorfizması altındaki görüntüsü de F-küçük alt modüldür. Bu özellik kafeslerde her zaman doğru değildir. Kompakt üretilmiş kompakt kafeslerde f-tümleyen elemanlar kompakttır. Kompakt üretilmiş kafeslerin f-radikali f-küçük elemanlarının supremumuna eşittir. Dahası kompakt kafeslerin f-radikalinin kendisi f-küçük elemandır. Bir L kafesi f-tümlenmiş kafes ise $1/\operatorname{rad}_f(L)$ bölüm alt kafesi yarı atomiktir. Bir kompakt L kafesinin f-tümlenmiş olması için gerek ve yeter koşul L kafesinin $f \leq m$ koşulunu sağlayan her maksimal elemanının L'de bir f-tümleyeninin var olmasıdır. f-tümlenmiş kafeslerin supremumu da f-tümlenmiştir. L bir kompakt kafes ve $f \leq a \in L$ olsun. $a/0$ alt kafesi f-tümlenmiş ve $1/a$ bölüm alt kafesinin maksimal elemanı yok ise L kafesi de f-tümlenmiştir. Bol f-tümlenmiş bir L kafesinin her a elemanı için $1/a$ bölüm alt kafesi $(f \vee a)$-tümlenmiş ve her f-tümleyen a elemanı için $a/0$ alt kafesi f-tümlenmiştir. Bir L kafesinin bol f-tümlenmiş olması için gerek ve yeter koşul her $a \in L$ için, $x/0$ alt kafesi f-tümlenmiş ve $x \leq a$ içermesi f-eşküçük içerme olacak şekilde L'nin bir $x \leq a$ elemanının var olmasıdır.

Keywords: f-küçük elemanlar, f-tümlenmiş kafesler, bol f-tümlenmiş kafesler, kompakt kafesler, kompakt üretilmiş kafesler.

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ABBREVIATIONS

L	A lattice
P	A set
P°	The dual of P
$\operatorname{Max} P$	The set of maximal elements of P
$\operatorname{Min} P$	The set of minimal elements of P
S^u	The set of all upper bounds
S^l	The set of all lower bounds
$\leq$	A partial order
$\leq_P$	A partial order over P
$\subseteq$	subset
$x < y$	$x \leq y$ and $x \neq y$
$x \vert y$	x divides y
y/x	quotient sublattice
$x \prec y$	y covers x
$x \vee y = \sup\{x, y\}$	least upper bound of x, y called x join y
$x \wedge y = \inf\{x, y\}$	greatest lower bound of x, y called x meet y
$\bigvee X$	least upper bound of X
$\bigwedge X$	greatest lower bound of X
$\cong$	isomorphism
$\cup$	union
$\cap$	intersection
$A(L)$	The set of all atoms in L
$\ll_f$	f-small
$\operatorname{rad}_f(L)$	f-radical of L
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers

$\mathbb{R}$	The set of real numbers
$\mathcal{P}(X)$	The set of the subsets of X
$R(A)$	a section of a relaiton R

1. INTRODUCTION

An element b of a complete lattice L is said to be pseudo-complement of an element a, if b is maximal with respect to the property $a \wedge b = 0$. If L is a lattice of submodules of a module M, then a submodule N of M is called a complement submodule if N is a pseudo-complement in L. Dually, if N is a pseudo-complement in L°, then N is said to be a supplement submodule of M. Since the lattice of submodules of a module is upper continuous, complement submodules always exist. However, since dual of the lattice of submodules of a module need not be upper continuous, the existence of supplement submodules can not be guaranteed.

Recently several authors have studied different generalizations of supplemented modules. F-supplemented modules are one of them which have been introduced and studied by Özdemir in [1]. Namely, a submodule $V \subseteq M$ is called an F-supplement of $U \subseteq M$ in M if V is minimal in the set $\{L \subseteq M \mid U + L = M \text{ and } F \subseteq L\}$.

Also it has become quite popular to generalize module theoretic concepts into modular lattices since the 1970s. Many properties of F-supplemented modules are true for any lattice and sometimes their proofs can be obtained by arranging their proofs in modules. In this thesis, generalization of some known properties and results about F-supplemented and amply F-supplemented modules to modular lattices have been studied. Examples showing that not every generalization is possible are given. In this study, it was mostly preferred to present results whose proofs might be different from those in modules. Some proven results for lattices give new results for modules. In addition, proofs of some results for lattices are easier to obtain than known proofs of these results for modules.

In Chapter 2 some preliminary information which will be needed in the next sections of the thesis are recalled.

In Chapter 3 some properties of f-small elements are presented. An example showing that a homomorphic image of an f-small element under a lattice homomorphism need not be f-small unlike the module case is given (see Example 3.1.3).

In Chapter 4 f-supplemented lattices are investigated. It is shown for a compactly generated lattice L that if $a \leq b$ and a is an f-supplement in L, then a is an f-supplement in $b/0$. Moreover if b is an f-supplement in L, then a is an f-supplement in L if and only if a is an f-supplement in $b/0$ (see Theorem 4.1.4). It is proved that if b is an f-supplement of c in a compactly generated lattice L, then for $a \leq c$, $(b \vee a)$ is an $(f \vee a)$-supplement of c in $1/a$ (see Proposition 4.1.5). Let $a \leq b$ be elements of a compactly generated lattice L. If a is an f-supplement in L and b is an $(f \vee a)$-supplement in $1/a$, then b is an f-supplement in L (see Proposition 4.1.8). Let L be a compactly generated lattice and c be an f-supplement of b in L. It is shown that if $a \leq b$ and $a \vee c = 1$, then c is an f-supplement of a (see Proposition 4.2.2). It is obtained that an f-supplement element in a compactly generated lattice L is compact (see Proposition 4.2.3). If c is an f-supplement of b and $a \ll_f L$ in a compactly generated lattice L, then c is an f-supplement of $a \vee b$ in L (see Proposition 4.2.4). It is proved that if L is a compactly generated lattice, then f-radical of L is the join of all f-small elements of L (see Theorem 4.2.9). If L is a compact lattice, then f-radical of L is an f-small element of L (see Proposition 4.2.10). It is shown for a compactly generated compact lattice L that if c is an f-supplement of b in L and if $a \ll_f L$, then $a \wedge c \ll_f c/0$ and $\operatorname{rad}_f(c/0) = c \wedge \operatorname{rad}_f(L)$ (see Proposition 4.2.11). It is obtained for a compactly generated lattice L that L is f-supplemented if and only if the quotient sublattice $1/f$ is supplemented (see Corollary 4.2.13). It is proved that the finite join of f-supplemented principal ideals is also f-supplemented (see Proposition 4.2.15). It is also proved that if a compactly generated compact lattice L is f-supplemented, then the quotient sublattice $1/\operatorname{rad}_f(L)$ of L is semiatomic (see Proposition 4.2.18). It is shown for a compact lattice L that L is f-supplemented if and only if every maximal element m of L with $f \leq m$ has an f-supplement in L (see Theorem 4.2.28). It is also shown for a compact lattice L that if each sublattice in a collection $\{a_i/0\}_{i \in I}$ of sublattices of L with $1 = \bigvee_{i \in I} a_i$ such that $f \leq a_i$ for each $i \in I$ is f-supplemented, then L is also f-supplemented (see Theorem 4.2.29).

In Chapter 5 amply f-supplemented lattices are investigated. It is proved that if a lattice L is amply f-supplemented, then the quotient sublattice $1/a$ is amply $(f \vee a)$-supplemented for every element a of L (see Proposition 5.1.2). It is shown that if L is an amply f-supplemented lattice, then for every f-supplement a in L, $a/0$ is amply f-supplemented (see Proposition 5.1.3). Let a, b be elements of a lattice L with $a \vee b = 1$. It is proved that if a and b have ample f-supplements in L, then $a \wedge b$ has ample f-supplements in L (see Proposition 5.1.5). Let a, b be elements of a lattice L such that $b \ll_f L$. It is shown that if $a \vee b$ has ample f-supplements in L, then a has also ample f-supplements in L (see Proposition 5.1.6). It is proved that L is amply f-supplemented if and only if every element a of L is of the form $a = x \vee y$ with $x/0$ is f-supplemented and $y \ll_f L$ if and only if for every element a of L, there is an element $x \leq a$ such that the sublattice $x/0$ is f-supplemented and the inequality $x \leq a$ is f-cosmall in L (see Theorem 5.1.8). It is obtained that if the sublattice $a/0$ is f-supplemented for every element a of a lattice L, then L is amply f-supplemented (see Corollary 5.1.9). Finally a new result for modules is obtained. Namely if every submodule of a left R-module M is F-supplemented, then M is amply F-supplemented (see Corollary 5.1.10).

2. PRELIMINARIES

In this chapter, some preliminary information which will be needed is given. Definitions, examples, propositions and theorems which are not cited here can be found in [2], [3], [4], [5], [6], [7], [8], [9], [10] and [11].

2.1 Ordered Sets

Definition 2.1.1. Let P be a set. A *partial order* on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,

(i) $x \leq x$;

(ii) $x \leq y$ and $y \leq x$ imply $x = y$;

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$

These conditions are referred to, respectively, as *reflexivity*, *antisymmetry* and *transitivity*.

Definition 2.1.2. A set P equipped with an order relation $\leq$ is said to be a *partially ordered set* or shortly *poset* and when it is necessary to specify the order relation it is denoted by $(P, \leq)$. Usually it is simply said that ‘‘let P be an ordered set”.

Definition 2.1.3. Let P be a partially ordered set. Then P is a *chain* if, for all $x, y \in P$, either $x \leq y$ or $y \leq x$.

Definition 2.1.4. A subset P' of a poset $(P, \leq_P)$ is called a *subposet* if it is partially ordered by restriction (i.e., $x \leq_{P'} x'$ by definition if and only if $x, x' \in P$ and $x \leq_P x'$).

Definition 2.1.5. Let P be a partially ordered set and $x, y \in P$. If $x \leq y$ and $x \neq y$, then we write $x < y$.

Definition 2.1.6. Let P be a partially ordered set. If for all $x, y \in P$, $x < y$ and $x \leq z < y$ means $x = z$; or equivalently, if $x < y$ in a poset P and there is no element $z \in P$ such that $x < z < y$, then we say that x is *covered by* y (or y *covers* x), which is denoted by $x \prec y$.

Definition 2.1.7. Given any partially ordered set $(P, \leq)$ we can form a new partially ordered set P° by defining $x \leq_{P^\circ} y$ if and only if $y \leq x$. The set P° is called the *dual* of P.

Definition 2.1.8. Let P be a partially ordered set and let $Q \subseteq P$. Then $a \in Q$ is a *maximal element* of Q if $a \leq x$ and $x \in Q$ imply $a = x$. The set of maximal elements of Q is denoted by $\operatorname{Max} Q$. Suppose that Q *takes an order relation from* P: if $x, y \in Q$, then $x \leq_Q y$ if and only if $x \leq_P y$. In this case, Q has an order relation with the order inherited from P. If there exists an element T of Q such that $x \leq \top$ with the order inherited from P for all $x \in Q$, then $\operatorname{Max} Q = \{\top\}$. In this case $\top$ is called the *greatest* (or *maximum*) element of Q.

Definition 2.1.9. Let P be a partially ordered set and let $Q \subseteq P$. Then $a \in Q$ is a *minimal element* of Q if $a \leq x$ and $x \in Q$ imply $a = x$. The set of minimal elements of Q is denoted by $\operatorname{Min} Q$. Suppose that Q *takes an order relation from* P: if $x, y \in Q$, then $x \leq_Q y$ if and only if $x \leq_P y$. In this case, Q has an order relation with the order inherited from P. If there exists an element $\perp$ of Q satisfying the condition $\perp \leq x$ with the order inherited from P for all $x \in Q$, then $\operatorname{Min} Q = \{\perp\}$. In this case $\perp$ is called the *least* (or *minimum*) element of Q.

Remark 2.1.10. Maximal elements need not exist. For example in the subset Q of $\mathcal{P}(\mathbb{N})$ consisting of all subsets of $\mathbb{N}$ other than $\mathbb{N}$ itself, there is no top element, but $\mathbb{N}\backslash\{n\} \in \operatorname{Max} Q$ for each $n \in \mathbb{N}$. The subset of $\mathcal{P}(\mathbb{N})$ consisting of all finite subsets on $\mathbb{N}$ has no maximal elements.

2.2 Lattices and Complete Lattices

Definition 2.2.1. Let P be a partially ordered set and let $S \subseteq P$. An element $x \in P$ is an *upper bound* of S if $s \leq x$ for all $s \in S$. Similarly, an element $x \in P$ is a *lower bound* of S if $x \leq s$ for all $s \in S$. The set of all upper bounds of S is denoted by S^u and the set of all lower bounds by S^l:

$$S^u = \{x \in P \mid s \leq x, \forall s \in S\};$$

$$S^l = \{x \in P \mid s \geq x, \forall s \in S\}.$$

Definition 2.2.2. Let P be a partially ordered set and let $S \subseteq P$. If S^u has a least element x, then x is called the *least upper bound* of S and denoted by $\sup S$=$\bigvee S$. Equivalently, x is the least upper bound of S if

(i) x is an upper bound of S, and

(ii) $x \leq y$ for all upper bounds y of S.

If $S = \{x, y\}$, then $\sup\{x, y\}$ is denoted by $x \vee y$.

Definition 2.2.3. Let P be an partially ordered set and let $S \subseteq P$. If S^l has a greatest element x, then x is called the *greatest lower bound* of S and denoted by $\inf S$=$\bigwedge S$. Equivalently, x is the least upper bound of S if

(i) x is an lower bound of S, and

(ii) $y \leq x$ for all lower bounds y of S.

If $S = \{x, y\}$, then $\inf\{x, y\}$ is denoted by $x \wedge y$.

Definition 2.2.4. Let P be a non-empty ordered set.

(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a *lattice*.

(ii) If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a *complete lattice*.

Lemma 2.2.5. *[6, 2.8] Let L be a lattice and let $a, b \in L$. Then the following are equivalent:*

(i) $a \leq b$

(ii) $a \vee b = b$

(iii) $a \wedge b = a$

Theorem 2.2.6. *[3, Proposition 3.1.1] Let L be a lattice. Then for any $a, b, c \in L$ the following are satisfied.*

(i) $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;

(ii) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$;

(iii) $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$;

(iv) $a \vee a = a$, $a \wedge a = a$.

Conversely, if L is a set with two binary operators $\vee, \wedge$ satisfying (i)-(iii), then (iv) also holds and a partial ordering may be defined on L by the rule

$$a \leq b \text{ if and only if } a \vee b = b.$$

Relative to this ordering L is a lattice such that the join of a, b is $a \vee b = \sup\{a, b\}$ and the meet is $a \wedge b = \inf\{a, b\}$.

Proof. (ii) Since $\sup\{a, b\} = a \vee b$, we can write $a \vee b$ as $b \vee a$. A similar remark applies to $\inf\{a, b\} = a \wedge b$.

(i) Likewise the supremum of $\{a, b, c\}$ can be written as $a \vee (b \vee c)$ or $(a \vee b) \vee c$.

(iii) Since $a \wedge b \leq a \leq a \vee b$, (iii) holds by Lemma 2.2.5.

(iv) This is a trivial consequence of the definition. Also if we replace b by $a \wedge a$ in $a \wedge (a \vee b) = a$, we get

$$a = a \wedge (a \vee (a \wedge a)) = a \wedge a \text{ by } a \vee (a \wedge b) = a.$$

Similarly, we get $a \vee a = a$.

Now let L be a set with two operators $\vee, \wedge$ satisfying (i)-(iii). Then the operators $\vee, \wedge$ also satisfy (iv) as we just seen in the proof of (iv). Furthermore, $a \vee b = b$ if and only if $a \wedge b = a$. If $a \vee b = b$, then by (iii)

$$a \wedge b = a \wedge (a \vee b) = a$$

and if $a \wedge b = a$, again by (iii)

$$a \vee b = (a \wedge a) \vee b = b$$

holds. If we define the relation '$\leq$' by $a \leq b$ if and only if $a \vee b = b$, then the below property holds:

$$a \leq b \text{ and } b \leq c \Rightarrow a \vee b = b \text{ and } b \vee c = c;$$

hence according to (i)

$$c = b \vee c = (a \vee b) \vee c = a \vee (b \vee c) = a \vee c$$

therefore $a \leq c$. Moreover $a \leq a$ by (iv). If $a \leq b$, $b \leq a$, then

$$b = a \vee b = b \vee a = a$$

by (ii).

□

Theorem 2.2.7. *[2, Theorem 1.2] A poset $(L, \leq)$ is a complete lattice if and only if each subset of L, including the empty subset, has a meet.*

Proof. ($\Rightarrow$) It is clear by the definition of complete lattice.
($\Leftarrow$) Let B be an arbitrary subset of L and

$$C = \{x \in L \mid \forall b \in B, b \leq x\}$$

be the subset of all the upper bounds of B in L. If we take $u = \inf C$, then $u \leq c$ for all $c \in C$ and if $d \leq c$ for all elements c of C and an element l of L, then $d \leq u$. Therefore $b \leq u$ for all $b \in B$ by definition of C. If $e \in L$ is an upper bound of B, then by definition $e \in C$ and $u \leq e$. Thus $u = \sup B$. □

The dual of this theorem is as follows:

Theorem 2.2.8. *A poset L is a complete lattice if and only if every subset of L, including the empty subset, has a supremum.*

In any lattice L, each finite (nonempty) subset F has a $\sup$ and an $\inf$, as an easy induction shows. Explicitly the $\sup$ and $\inf$ of $\{a_1, \ldots, a_n\}$ are given by $a_1 \vee a_2 \vee \ldots \vee a_n$ and $a_1 \wedge a_2 \wedge \ldots \wedge a_n$, respectively. Here by Theorem 2.2.6 (i), we may omit brackets, by (iv), omit repetitions, and order of the factors is immaterial, commutativity is given by (ii). Consequently, $\bigvee F$ and $\bigwedge F$ can be defined for a finite nonempty subset F of a lattice.

Definition 2.2.9. Let L be a lattice. We say L has a *one* if there exists $1 \in L$ such that $a \wedge 1 = a$ for all $a \in L$. Dually, L is said to have a *zero* if there exists $0 \in L$ such that $a \vee 0 = a$ for all $a \in L$. A lattice possessing 0 and 1 is called a *bounded lattice*.

For any subset S of a poset L is a poset. But even if L is a lattice, S need not be a lattice.

Definition 2.2.10. If a subset S of a lattice L is closed under the operators $\vee$, $\wedge$ in L, in other words if given any $a, b \in S$, $a \vee b, a \wedge b \in S$, then S is called a *sublattice* of L. In this case, S is also a lattice (see [3, p.54] or [2, p.6]).

2.3 Lattice Homomorphisms

Definition 2.3.1. Let P and Q be two partially ordered sets. A map $\varphi : P \to Q$ is said to be

(i) *order-preserving* (or *monotone*) if $x \leq y$ in P implies $\varphi(x) \leq \varphi(y)$ in Q for all $x, y \in P$;

(ii) an *order-embedding* if for all $x, y \in P$, $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q;

(iii) an *order-isomorphism* if it is an order-embedding which maps P onto Q. If there is an order-isomorphism defined on P onto Q, P and Q are called *ordered isomorphic sets*.

Theorem 2.3.2. *[6, 1.17] Let P and Q be finite ordered sets and $\varphi : P \longrightarrow Q$ be a bijective map. Then the following are equivalent:*

(i) *φ is an order-isomorphism;*

(ii) *$x < y$ in P if and only if $\varphi(x) < \varphi(y)$ in Q;*

(iii) *$x \prec y$ in P if and only if $\varphi(x) \prec \varphi(y)$ in Q.*

Proof. $(i) \Leftrightarrow (ii)$ Clear by definition.
$(ii) \Rightarrow (iii)$ Let $x \prec y \in P$. Then $x < y$ and therefore $\varphi(x) < \varphi(y) \in Q$ by (ii). Now assume that there exists $w \in Q$ such that $\varphi(x) < w < \varphi(y)$. Since φ is onto, there exists $u \in P$ such that $w = \varphi(u)$. So $x < u < y$ by (ii). This is a contradiction. Therefore $\varphi(x) \prec \varphi(y) \in Q$. Suppose conversely $\varphi(x) \prec \varphi(y) \in Q$. Then $\varphi(x) < \varphi(y)$ and therefore $x < y \in P$ by (ii). Now suppose that there exists $u \in P$ such that $x < u < y$. Since φ is one-to-one, $\varphi(x) < \varphi(u) < \varphi(y)$. This contradicts with the fact that $\varphi(x) \prec \varphi(y)$. Thus $x \prec y$.
$(iii) \Rightarrow (ii)$ Let $x < y$ in P. Then there exist elements such that

$$x = x_0 \prec x_1 \prec \ldots \prec x_n = y$$

By (iii)

$$\varphi(x) = \varphi(x_0) \prec \varphi(x_1) \prec \ldots \prec \varphi(x_n) = \varphi(y).$$

Hence $\varphi(x) < \varphi(y)$. The other hand can be proved by using surjectivity of φ. □

Definition 2.3.3. Let L and K be lattices. A map $f : L \longrightarrow K$ is said to be a *lattice homomorphism* if f is join-preserving and meet-preserving, that is, for all $a, b \in L$ the following holds:

(i) $f(a \vee b) = f(a) \vee f(b)$

(ii) $f(a \wedge b) = f(a) \wedge f(b)$

A bijective lattice homomorphism is said to be a *lattice isomorphism*. If $f : L \longrightarrow K$ is a one-to-one homomorphism, then the sublattice $f(L)$ of K is isomorphic to L and we refer to f as an *embedding homomorphism* of L into K.

Proposition 2.3.4. *[2, Remark 1.3] The inverse of a lattice isomorphism is also a lattice isomorphism.*

Proof. Let $f : L \longrightarrow K$ be a lattice isomorphism and $a, b \in L$. Since

$$f(f^{-1}(a \vee b)) = a \vee b = f(f^{-1}(a)) \vee f(f^{-1}(b)) \text{ and}$$

$$f(f^{-1}(a \wedge b)) = a \wedge b = f(f^{-1}(a)) \wedge f(f^{-1}(b)) \text{ , then}$$

$f^{-1}(a \vee b) = f^{-1}(a) \vee f^{-1}(b)$ and $f^{-1}(a \wedge b) = f^{-1}(a) \wedge f^{-1}(b)$. □

Proposition 2.3.5. *[2, Remark 1.4] Each lattice homomorphism is an order homomorphism. The converse is not true.*

Proof. Let $f : L \longrightarrow K$ be a lattice homomorphism. Then for all $a, b \in L$ the following equalities hold:

$$f(a \vee b) = f(a) \vee f(b) \text{ and } f(a \wedge b) = f(a) \wedge f(b).$$

Now we know by Lemma 2.2.5 that $a \leq b$ if and only if $a \vee b = b$ and $a \vee b = b$ if and only if $a \wedge b = a$. Thus $f(b) = f(a \vee b) = f(a) \vee f(b)$ if and only if $f(a) \leq f(b)$. □

Example 2.3.6. For a nonempty set X and a relation $(X, X; R)$ consider $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ defined by $f(A) = R(A)$ where $R(A) = \{x \in X \mid \exists a \in A : (a, x) \in R\}$ (also called the section of R after A) for each subset $A \in \mathcal{P}(X)$. f is an order morphism but not a lattice morphism. Notice that

$$f(A \cup A') = R(A \cup A') = R(A) \cup R(A') = f(A) \cup f(A')$$

holds but only

$$f(A \cap A') = R(A \cap A') \subseteq R(A) \cap R(A') = f(A) \cap f(A')$$

holds in general. If we take $X = \mathbb{R}$ and $R = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. The middle equality fails for $A = (-1, 0]$ and $A' = [0, 1]$.

Proposition 2.3.7. *[2, Remark 1.5] A function between two lattices is an order-isomorphism if and only if it is a lattice isomorphism.*

Proof. ($\Rightarrow$) Let $f : L \longrightarrow K$ be a lattice isomorphism. Since $a \wedge b \leq a$ and $a \wedge b \leq b$,

$$f(a \wedge b) \leq f(a) \text{ and } f(a \wedge b) \leq f(b)$$

Therefore, $f(a \wedge b)$ is a lower bound for $f(a)$ and $f(b)$. If c' is another lower bound for $f(a)$ and $f(b)$, $c' \leq f(a)$ and $c' \leq f(b)$. Now suppose there exists $c \in L$ such that $f(c) = c'$. Since f^{-1} is also an order-isomorphism,

$$c = f^{-1}(c') \leq f^{-1}(f(a)) = a \text{ and } c = f^{-1}(c') \leq f^{-1}(f(b)) = b$$

therefore $a \wedge b \leq c$. Since f is an order-isomorphism,

$$f(a \wedge b) \leq f(c) = c'.$$

Hence

$$f(a \wedge b) = \inf\{f(a), f(b)\}.$$

Similarly, since $a \leq a \vee b$ and $b \leq a \vee b$,

$$f(a) \leq f(a \vee b) \text{ and } f(b) \leq f(a \vee b).$$

Therefore $f(a \vee b)$ is an upper bound for $f(a)$ and $f(b)$. If d' is another upper bound for $f(a)$ and $f(b)$, then $f(a) \leq d'$ and $f(b) \leq d'$. Now suppose there exists $d \in L$ such that

$f(d) = d'$. Since f^{-1} is also an order-isomorphism,

$$a = f^{-1}(a) \leq f^{-1}(d') = d \text{ and } b = f^{-1}(b) \leq f^{-1}(d') = d$$

thus $d \leq a \vee b$. Since f is an order-isomorphism,

$$d' = f(d) \leq f(a \vee b).$$

Hence

$$f(a \vee b) = \sup \{f(a), f(b)\}.$$

($\Leftarrow$) It is clear by Proposition 2.3.5. $\square$

2.4 Modular Lattices

Lemma 2.4.1. *[6, Lemma 4.1] Let L be a lattice and let $a, b, c \in L$. Then the following properties hold:*

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$*, and dually;*

(ii) $a \geq c$ *implies* $a \wedge (b \vee c) \geq (a \wedge b) \vee c$*, and dually;*

(iii) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

Proof. (i) $a \wedge b \leq b \vee c$ and $a \wedge c \leq b \vee c$. On the other hand $a \wedge b \leq a$ and $a \wedge c \leq a$. Hence the following inequalities can be written:

$$(a \wedge b) \vee (a \wedge c) \leq b \wedge c,$$

$$(a \wedge b) \vee (a \wedge c) \leq a.$$

Then by these two inequalities we get

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c).$$

(ii) Let $c \leq a$. Since $c \leq b \vee c$, $(a \wedge b) \leq a$ and $(a \wedge c) \leq b \leq b \vee c$, the following inequalities can be written:

$$c \leq a \wedge (b \vee c),$$

$$a \wedge b \leq a \wedge (b \vee c).$$

Then by these two inequalities we get

$$(a \wedge b) \vee c \leq a \wedge (b \vee c).$$

(iii) Since $a \wedge b \leq a \vee b$, $a \wedge b \leq b \vee c$ and $a \wedge b \leq a \vee c$, we have

$$a \wedge b \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

Since $b \wedge c \leq a \vee b$, $b \wedge c \leq b \vee c$ and $b \wedge c \leq a \vee c$, we have

$$b \wedge c \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

Similarly since $c \wedge a \leq a \vee b$, $c \wedge a \leq b \vee c$ and $c \wedge a \leq c \vee a$, we have

$$c \wedge a \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

Therefore by these three inequalities, we get

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

□

Definition 2.4.2. Let L be a lattice. If L satisfies the modular law, then L is called a *modular lattice*.

Modular Law: If for all $a, b, c \in L,\ a \geq c$ implies $a \wedge (b \vee c) = (a \wedge b) \vee c$.

Example 2.4.3. Let M be a left R-module and $S(M)$ be the set of all submodules of M. The set $(S(M), \leq)$ is a poset where $\leq$ is shown as being submodule. For $N, K \in S(M)$, $S(M)$ is a complete modular lattice defined by the operators $\vee$ and $\wedge$ with

$$N \vee K = N + K, N \wedge K = N \cap K.$$

It is known by Lemma 2.4.1 (i) that if $a \geq c$ for all $a, b, c \in L$, then the following inequality holds:

$$a \wedge (b \vee c) \geq (a \wedge b) \vee c.$$

Therefore if for every $a, b, c \in L$, $a \geq c$ implies $a \wedge (b \vee c) \leq (a \wedge b) \vee c$, then L is a modular lattice.

Proposition 2.4.4. *[8, Theorem 85 and Theorem 89] Let L be a modular lattice.*

(i) *Every sublattice of L is also a modular lattice.*

(ii) *The image of L under a lattice homomorphism is also a modular lattice.*

Proof. (i) Let S be a sublattice of a modular lattice L and $a, b, c \in S$ such that $b \leq a$. Since L is modular, the following equality holds:

$$a \wedge (b \vee c) = b \vee (a \wedge c).$$

Since S is the sublattice, for every element a, b, c taken from S

$$a \wedge (b \vee c) = b \vee (a \wedge c) \in S.$$

Therefore, modular law is also provided in S. Thus S is also a modular lattice.

(ii) Let K be the image of L under a lattice homomorphism f and $x, y, z \in K$. It is enough to show that

$$x \wedge (y \vee z) = (x \wedge z) \vee y$$

while $x \geq y$. Let $f(a) = x$ and $f(c) = z$ for some $a, c \in L$. Then by [8, Theorem 82], there exists at least one $b \in L$ such that $f(b) = y$ and $a \geq b$. Since L is modular, the following inequality holds:

$$a \wedge (b \vee c) \leq b \vee (a \wedge c).$$

Therefore

$$f(a \wedge (b \vee c)) = f((a \wedge c) \vee b).$$

Now

$$f(a \wedge (b \vee c)) = f(a) \wedge (f(b) \vee f(c)) \leq (f(a) \wedge f(c)) \vee f(b) = f((a \wedge c) \vee b).$$

This means that $x \wedge (y \vee z) \leq (x \wedge z) \vee y$. Hence K is also modular.

□

Lemma 2.4.5. *[2, Lemma 12.3] In any modular lattice*

$$[(c \vee d) \wedge b] \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)]$$

holds for every $b, c, d \in L$.

Proof. Since $c \wedge (b \vee d) \leq b \vee c$, the following equation can be written by modular law:

$$[c \wedge (b \vee d)] \vee [d \wedge (b \vee c)] = [(c \wedge (b \vee d)) \vee d] \wedge (b \vee c).$$

If we apply the modular law again using that $d \leq b \vee d$, we can write the equality

$$[(c \wedge (b \vee d)) \vee d] = (c \vee d) \wedge (b \vee d)$$

By the two equality above, we obtain

$$[c \wedge (b \vee d)] \vee [d \wedge (b \vee c)] = [(c \wedge (b \vee d)) \vee d] \wedge (b \vee c) = [(c \vee d) \wedge (b \vee d)] \wedge (b \vee c).$$

Since $b \leq (b \vee d) \wedge (b \vee c)$, the following inequality holds:

$$(c \vee d) \wedge b \leq [(c \vee d) \wedge (b \vee d)] \wedge (b \vee c).$$

Thus we obtain the conclusion that

$$[(c \vee d) \wedge b] \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)].$$

□

Definition 2.4.6. A sublattice $\{x \in L \mid a \leq x \leq b\}$ of a lattice L is called a *quotient sublattice* and it is denoted by b/a.

Theorem 2.4.7. *[2, Theorem 1.5] Let a and b be elements in a modular lattice L. Then the quotient sublattices $(a \vee b)/b$ and $a/(a \wedge b)$ are isomorphic.*

Proof. Let

$$f : (a \vee b)/b \to a/(a \wedge b)$$

be defined by $f(x) = x \wedge a$ for all $x \in (a \vee b)/b$ and

$$g : a/(a \wedge b) \to (a \vee b)/b$$

be defined by $g(y) = y \vee b$ for all $y \in a/(a \wedge b)$. In this case, f and g are mutually inverse lattice homomorphisms. Therefore

$$(a \vee b)/b \cong a/(a \wedge b).$$

□

2.5 Compactly Generated Lattices

Definition 2.5.1. If an element p of a complete lattice L covers 0, i.e. $(0 \prec p)$, p is called an *atom*. The set of all atoms in a lattice L is denoted by $A(L)$ (see [11]).

Definition 2.5.2. If 1 is the supremum of atoms in a complete lattice L, then L is called a *semiatomic lattice*.

Definition 2.5.3. [11] A complete lattice L is said to be *upper continuous*, if for every $a \in L$ and for every chain $C \subseteq L$,

$$a \vee (\bigvee_{x \in C} x) = \bigvee_{x \in C} (a \wedge x).$$

Definition 2.5.4. Let c be an element of a lattice L. If every subset S of L that satisfies the condition $c \leq \bigvee S$ has a finite subset F such that $c \leq \bigvee F$, then c is called a *compact element* (see [10]).

Lemma 2.5.5. *Let L be a complete lattice, c_1 and c_2 be two compact elements of L. Then $c_1 \vee c_2 \in L$ is also compact (see [7]).*

Proof. Let $c_1 \vee c_2 \leq \bigvee S$ for every subset S in L. Then $c_1 \leq \bigvee S$ and $c_2 \leq \bigvee S$. Since c_1 and c_2 are compact, there exist finite subsets F_1, F_2 of S such that $c_1 \leq \bigvee F_1$ and $c_2 \leq \bigvee F_2$. Therefore

$$c_1 \vee c_2 \leq \left(\bigvee F_1\right) \vee \left(\bigvee F_2\right) = \bigvee (F_1 \vee F_2).$$

Here clearly $F_1 \vee F_2 \subseteq S$ is finite. □

Corollary 2.5.6. *[7, Lemma 2.1] Each finite join of compact elements is a compact element.*

Definition 2.5.7. ([2, p.22]) If $1 \in L$ is compact in a complete lattice L, then L is said to be *compact*.

Lemma 2.5.8. *[2, Lemma 2.4] Let L be a compact lattice and $1 \neq a \in L$. Then the sublattice $1/a$ has at least one maximal element which is different from 1.*

Proof. $S = 1/a - \{1\} \neq$ by hypothesis. Moreover, since 1 is compact, S contains the joins of the chains in S and therefore Zorn's Lemma can be applied. If $C \subseteq S$ is any chain, then $\bigvee C = 1$. Assume that $\bigvee C = 1$. Therefore, since 1 is compact, there exists at least one element $c_0 \in C$ such that $1 \leq c_0$ or $1 = c_0$ and this is a contradiction. Thus there exists at least one maximal element in $1/a$ which is different from 1. □

Definition 2.5.9. If every element in a complete lattice L is a join of compact elements, then L is called a *compactly generated lattice* (see [10]).

Proposition 2.5.10. *[4, Lemma 2] Every compactly generated lattice is upper continuous.*

Proof. For any lattice L and for every chain C in L, the following inequality holds

$$\bigvee_{x \in C} (a \wedge x) \leq a \wedge \left(\bigvee C\right).$$

Let us obtain inverse of this inequality for compactly generated lattices by using special way: if for every compact element c and any two elements x, y in L, $c \leq x$ requires $c \leq y$, then $x \leq y$, because every element is a join of compact elements. Let c be a compact element satisfying the condition $c \leq a \wedge \left(\bigvee C\right)$ in L. Then $c \leq \bigvee C$ and thus for a proper $x_0 \in C$, $c \leq x_0$. Moreover, $c \leq a$. Hence

$$c \leq a \wedge x_0 \leq \bigvee_{x \in C} (a \wedge x).$$

□

Lemma 2.5.11. *[2, Exercise 2.6] Let a be an element of an upper continuous complete lattice L. Then the compact elements in $a/0$ are exactly the compact elements of L that belong to $a/0$.*

Proof. Let $c \in a/0$ be compact in L. Therefore it is also compact in $a/0$. Suppose now that $c \in a/0$ be compact only in $a/0$. If $c \leq \bigvee C$ for a chain $C \subseteq L$, then

$$c = a \wedge c \leq a \wedge \left(\bigvee C\right) = \bigvee_{x \in C} (a \wedge x).$$

So $\bigvee_{x \in C} (a \wedge x)$ is also a cover of c in $a/0$. Thus there exists a finite chain $F \subseteq C$ such that

$$c \leq a \wedge \bigvee_{x \in F} (a \wedge x).$$

Therefore

$$\bigvee_{x \in F} (a \wedge x) = a \wedge (\bigvee_{x \in F} x) = a \wedge x_0$$

for an $x_0 \in F$, i.e. $c \leq a \wedge x_0$. □

Lemma 2.5.12. *[2, Exercise 2.7] Let L be a compactly generated lattice and a be an element of L. Then $a/0$ is also compactly generated.*

Proof. Every compactly generated lattice is upper continuous by Proposition 2.5.10. Therefore, we know that for all $a \in L$, the compact elements in $a/0$ are exactly the compact elements of L that belong to $a/0$ by Lemma 2.5.11. Therefore $a/0$ is also compactly generated. □

Lemma 2.5.13. *[2, Exercise 2.9]*

(i) *If c is a compact element in a complete lattice L and $a \in L$, then $c \vee a$ is compact in $1/a$.*

(ii) *If L is a compact lattice and $a \in L$, then $1/a$ is also a compact lattice.*

(iii) *If L is a compactly generated lattice, then the quotient sublattice $1/a$ is also compactly generated for every element a of L.*

Proof. (i) Let $c \in L$ be compact and $c \vee a \leq \bigvee_{i \in I} a_i$ for $\{a_i\}_{i \in I} \subseteq 1/a$. Then $c \leq \bigvee_{i \in I} a_i$. Since c is compact in L, there exists a finite subset $F \subseteq I$ such that $c \leq \bigvee_{i \in F} a_i$. Therefore

$$c \vee a \leq (\bigvee_{i \in F} a_i) \vee a = \bigvee_{i \in F} (c_i \vee a).$$

(ii) Taking $c = 1$ it is the special case of (i).

(iii) Let $x \in 1/a$. Since L is a compactly generated lattice and $x \in L$, $x = \bigvee_{i \in I} c_i$ for some compact elements $\{c_i\}_{i \in I} \subseteq L$. Therefore, we can write

$$x = x \vee a = (\bigvee_{i \in I} c_i) \vee a = \bigvee_{i \in I} (c_i \vee a).$$

If c_i is compact in L for all $i \in I$, then the element $c_i \vee a$ is also compact in $1/a$ by (i).

□

3. SOME GENERALIZATIONS OF SMALL ELEMENTS

3.1 f-Small Elements

Throughout L will denote an arbitrary complete modular lattice with smallest element 0 and greatest element 1, f will denote an element of L such that $f \neq 1$ unless otherwise stated.

Definition 3.1.1. An element a of L is said to be *f-small* in L if $a \vee b \neq 1$ holds for every $f \leq b \neq 1$. It is denoted by $a \ll_f L$.

Remark 3.1.2. It is shown in [1, Lemma 1 (1)] that a homomorphic image of an F-small submodule under a module homomorphism is F-small. The following example shows that this fact need not be true for lattices. That is a homomorphic image of an f-small element under a lattice morphism need not be f-small.

Example 3.1.3. [12, Example 1.1] Let $A = \{1, 2, 3, 6, 12\}$ and $B = \{1, 2, 3, 6\}$. Consider the lattices $(A, |)$ and $(B, |)$ where $|$ is the divisibility relation: $x \mid y$ means x divides y. Consider the lattice morphism $g : (A, |) \to (B, |)$ defined by $g(k) = k$ for $k = 1, 2, 3, 6$ and $g(12) = 6$. Clearly, $3 \ll_2 A$ since $3 \vee x \neq 12$ for all $2 \leq x < 12$. But $g(3) = 3 \not\ll_2 B$ since $3 \vee 2 = 6$ whilst $2 \neq 6$.

Remark 3.1.4. We will write $a < b$ if $a \leq b$ and $a \neq b$.

Lemma 3.1.5. *[12, Lemma 1.2 (1)] Let $a < b$ be elements in L. If $a \ll_{(f \wedge b)} b/0$, then $a \ll_f L$.*

Proof. Let $a \vee x = 1$ for some $f \leq x \in L$. Then we have $b = b \wedge 1 = b \wedge (a \vee x) = a \vee (b \wedge x)$ by modular law which we have for modular lattices. Since $a \ll_{(f \wedge b)} b/0$ and $f \wedge b \leq b \wedge x$, $b \wedge x = b$ and therefore $x = a \vee x = 1$. □

Lemma 3.1.6. *[12, Lemma 1.2 (2)] Let $a < b$ be elements in L. If $a \ll_{(f \wedge b)} b/0$, then $a \vee c \ll_{[(f \wedge b) \vee c]} (b \vee c)/c$ for every c in L.*

Proof. Let $(a \vee c) \vee x = b \vee c$ for an element $x \in (b \vee c)/c$ with $(f \wedge b) \vee c \leq x$. Then we have $b = b \wedge (b \vee c) = b \wedge [a \vee (c \vee x)]$ and therefore $b = a \vee [b \wedge (c \vee x)] = a \vee (b \wedge x)$. Since $f \wedge b \leq (f \wedge b) \vee c \leq x$, $f \wedge b \leq b \wedge x$. Also since $a \ll_{(f \wedge b)} b/0$, $b \wedge x = b$. So $b \leq x$ and therefore $b \vee c = x$. □

Lemma 3.1.7. *[12, Lemma 1.2 (3)] Let $a < b$ elements in L. $b \ll_f L$ if and only if $a \ll_f L$ and $b \ll_{(f \vee a)} 1/a$.*

Proof. (⇒) Let $a \vee x = 1$ for some $f \leq x \in L$. Since $a < b$, $b \vee x = 1$ and since $b \ll_f L$, $x = 1$. Let $b \vee y = 1$ for some $f \vee a \leq y \in 1/a$. Since $b \ll_f L$ and $f \leq y \in L$, $y = 1$.
(⇐) Let $b \vee z = 1$ for some $f \leq z \in L$. Then $b \vee (a \vee z) = 1$ and $f \vee a \leq z \vee a \in 1/a$. Now since $b \ll_{(f \vee a)} 1/a$, $a \vee z = 1$ and therefore $z = 1$ since $a \ll_f L$. □

Lemma 3.1.8. *[12, Lemma 2.12] Let a and b be elements of a lattice L. $a \ll_f L$ and $b \ll_f L$ if and only if $a \vee b \ll_f L$.*

Proof. (⇒)Suppose that $a \ll_f L$ and $b \ll_f L$. Let $x \in L$ with $f \leq x$ and $(a \vee b) \vee x = 1$. Since $a \ll_f L$, $b \vee x = 1$. Since $b \ll_f L$, $x = 1$.
(⇐) Clear by Lemma 3.1.7 (3). □

Lemma 3.1.9. [13, Lemma 2.4] *Let a and b be elements of a lattice L such that $f \leq a$ and $f \leq b$. If $a' \ll_f a/0$ and $b' \ll_f b/0$, then $a' \vee b' \ll_f (a \vee b)/0$.*

Proof. Since $a' \ll_f a/0$, $a' \ll_f (a \vee b)/0$ by Lemma 3.1.5. Similarly since $b' \ll_f b/0$, $b' \ll_f (a \vee b)/0$ by Lemma 3.1.5. Therefore $a' \vee b' \ll_f (a \vee b)/0$ by Lemma 3.1.8. □

4. f-SUPPLEMENTED LATTICES

4.1 f-Supplement Elements

Definition 4.1.1. Let a, b be elements of L. b is said to be *supplement* of a in L if $a \vee b = 1$ and $a \wedge b \ll b/0$. L is called a *supplemented* lattice if every element of L has a supplement in L.

Definition 4.1.2. Let a, b be elements of L. b is said to be f*-supplement* of a in L if b is minimal in the set $\{x \in L \mid f \leq x$ and $a \vee x = 1\}$. Since b is an f-supplement of an element in L, b is called an f*-supplement element*. If every element of a lattice L has an f-supplement in L, then L is called an f*-supplemented lattice*.

Lemma 4.1.3. [12, Lemma 1.4] *Let a, b be elements of a lattice L. Then a is an f-supplement of b in L if and only if $f \leq a$ and $a \vee b = 1$ and $a \wedge b \ll_f a/0$.*

Proof. ($\Rightarrow$) Let c be an element of L with $f \leq c$ such that $(a \wedge b) \vee c = a$. Then $1 = a \vee b = [(a \wedge b) \vee c] \vee b = c \vee b$. Since a is minimal in the set $\{x \in L \mid f \leq x$ and $b \vee x = 1\}$, $c = a$.
($\Leftarrow$) Assume that $b \vee y = 1$ for some $f \leq y \leq a$. Then $a = 1 \wedge a = (b \vee y) \wedge a = y \vee (a \wedge b)$. Since $a \wedge b \ll_f a/0$, $y = a$. □

The following result generalizes [1, Theorem 1].

Theorem 4.1.4. *Let $a \leq b$ be elements of a compactly generated lattice L. Then the following properties hold:*

(1) If a is an f-supplement in L, then a is an f-supplement in $b/0$.

(2) If b is an f-supplement in L, then

(i) a is an f-supplement in L if and only if a is an f-supplement in $b/0$.

(ii) $a \ll_f L$ if and only if $a \ll_f b/0$.

Proof. (1) Let a be an f-supplement of x in L. Then by Lemma 4.1.3, $f \leq a$, $x \vee a = 1$ and $x \wedge a \ll_f a/0$. Now by modular law we have

$$b = b \wedge 1 = b \wedge (x \vee a) = a \vee (b \wedge x).$$

Also $a \wedge b \wedge x \leq a \wedge x \ll_f a/0$. Therefore a is an f-supplement of $b \wedge x$ in $b/0$.
(2) Suppose that b is an f-supplement of x in L. That is by Lemma 4.1.3,

$$f \leq b, x \vee b = 1 \text{ and } x \wedge b \ll_f b/0.$$

(i) ($\Rightarrow$) Clear by (1).
($\Leftarrow$) Let a be an f-supplement of y in $b/0$. Then by Lemma 4.1.3,

$$f \leq y, a \vee y = b \text{ and } a \wedge y \ll_f a/0.$$

Now we have $1 = x \vee b = x \vee (a \vee y)$. Assume $a' \vee (x \vee y) = 1$ for some $f \leq a' \leq a$. Since $f \leq a' \vee y$ and b is an f-supplement of x in L, $a' \vee y = b$ by minimality of b. Now $a' = a$ by minimality of a.
(ii)($\Rightarrow$) Assume that $a \vee z = b$ for some $z \in L$ with $f \leq z \leq b$. So $a \vee (z \vee x) = b \vee x = 1$. Since $f \leq z \vee x$ and $a \ll_f L$, $z \vee x = 1$. By modular law we have

$$b = b \wedge 1 = b \wedge (z \vee x) = z \vee (b \wedge x).$$

Since $b \wedge x \ll_f b/0$, it follows that $z = b$. Thus $a \ll_f b/0$.
($\Leftarrow$) Clear by Lemma 3.1.5. □

The following result generalizes [1, Proposition 4].

Proposition 4.1.5. *Let L be a compactly generated lattice. If b is an f-supplement of c in L, then for $a \leq c$, $(b \vee a)$ is an $(f \vee a)$-supplement of c in $1/a$.*

Proof. Since b is an f-supplement of c in L,

$$f \leq b, b \vee c = 1 \text{ and } b \wedge c \ll_f b/0$$

by Lemma 4.1.3. Since $f \leq b$, $f \vee a \leq b \vee a$ and since $a \leq c$, $1 = b \vee c = b \vee c \vee a$. Also

$$1 = (b \vee a) \wedge c = a \vee (b \wedge c)$$

by modular law. Since $b \wedge c \ll_f b/0$,

$$(b \wedge c) \vee a \ll_{(f \vee a)} (b \vee a)/a$$

by Lemma 3.1.6. □

The following lemmas generalize [14, 1.24 and 2.3 (1)] from modules to modular lattices.

Lemma 4.1.6. *Let a, b, c be elements of a lattice L. Assume that $a \vee b = 1$ and $(a \wedge b) \vee c = 1$. Then $a \vee (b \wedge c) = b \vee (a \wedge c) = 1$.*

Proof. By modular law we have

$$a \vee (b \wedge c) = a \vee (b \wedge a) \vee (b \wedge c) = a \vee [b \wedge [(b \wedge a) \vee c]] = a \vee (b \wedge 1) = a \vee b = 1$$

and

$$b \vee (a \wedge c) = b \vee (b \wedge a) \vee (a \wedge c) = b \vee [a \wedge [(b \wedge a) \vee c]] = b \vee a = 1.$$

□

Lemma 4.1.7. *Let a, b, c be elements of a lattice L. If $1 = a \vee b$, $b \leq c$ and $c \ll 1/b$, then $(a \wedge c) \ll 1/(a \wedge b)$.*

Proof. Let $(a \wedge c) \vee x = 1$ for some $x \in 1/(a \wedge b)$. Since $a \vee b = 1$, $a \vee c = 1$. Then $1 = c \vee (a \wedge x)$ by Lemma 4.1.6. Since $c \ll 1/b$ and $c \vee b \vee (a \wedge x) = 1$, $b \vee (a \wedge x) = 1$. Also $1 = x \vee (a \wedge b)$ by Lemma 4.1.6 and hence $x = 1$. Thus $(a \wedge c) \ll 1/(a \wedge b)$. □

The following result generalizes [1, Proposition 5].

Proposition 4.1.8. *Let $a \leq b$ be elements of a compactly generated lattice L. If a is an f-supplement in L and b is an $(f \vee a)$-supplement in $1/a$, then b is an f-supplement in L.*

Proof. Let a be an f-supplement of x in L. That is

$$f \leq a, a \vee x = 1 \text{ and } a \wedge x \ll_f a/0$$

by Lemma 4.1.3. Let b be an $(f \vee a)$-supplement of y in $1/a$. That is

$$a \leq y \leq 1, f \vee a \leq b, b \vee y = 1 \text{ and } b \wedge y \ll_{(f \vee a)} b/a$$

by Lemma 4.1.3. We want to show that b is an f-supplement of $x \wedge y$ in L. $a \leq b$ and $a \leq y$ implies $a \leq b \wedge y$ and therefore $1 = a \vee x = (b \wedge y) \vee x$. Also since $1 = b \vee y$, $1 = b \vee (x \wedge y)$ by Lemma 4.1.6. Since

$$b = b \wedge 1 = b \wedge (a \vee x) = a \vee (b \wedge x)$$

by modular law,

$$b \wedge (x \wedge y) \ll b/(a \wedge x)$$

by Lemma 4.1.7. So we have

$$b \wedge (x \wedge y) \ll_{[f \vee (a \wedge x)]} b/(a \wedge x).$$

Since $a \wedge x \ll_f a/0$, $a \wedge x \ll_f b/0$ by Lemma 3.1.5. So

$$b \wedge (x \wedge y) \ll_f b/0$$

by Lemma 3.1.6. This means that b is an f-supplement of $x \wedge y$ in L. $\square$

4.2 f-Supplemented Lattices

Recall that if $1 = \bigvee_{i \in I} x_i$ for some elements $x_i \geq a$ implies that $1 = \bigvee_{i \in F} x_i$ for some finite subset F of I, then L is said to be *compact* (see [2]). If each element of L is a join of compact elements, then L is said to be *compactly generated* (see [15]).

Lemma 4.2.1. *Let L be a compactly generated compact lattice, $f \leq a$, $f \leq b$ and $a \vee b = 1$. Then there are compact elements $f \leq a' \leq a$ and $f \leq b' \leq b$ such that $a' \vee b' = 1$.*

Proof. Since L is compactly generated, $a = \bigvee_{i \in I} a_i$ where $f \leq a_i$ for each $i \in I$ and $b = \bigvee_{j \in J} b_j$ where $f \leq b_j$ for each $j \in J$. Now

$$1 = a \vee b = (\bigvee_{i \in I} a_i) \vee (\bigvee_{j \in J} b_j).$$

Since L is compact, there exist finite subsets $F_1 \subseteq I$ and $F_2 \subseteq J$ such that

$$1 = a \vee b = (\bigvee_{i \in F_1} a_i) \vee (\bigvee_{j \in F_2} b_j)$$

where $a = \bigvee_{i \in F_1} a_i$ and $b = \bigvee_{j \in F_2} b_j$ are compact by Corollary 2.5.6.

□

The following result generalizes [2, Proposition 12.2 (1)].

Proposition 4.2.2. *Let L be a compactly generated lattice and c be an f-supplement of b in L. If $a \leq b$ and $a \vee c = 1$, then c is an f-supplement of a.*

Proof. Suppose $a \vee c' = 1$ for some $f \leq c' \leq c$. Since $a \leq b$, $b \vee c' = 1$. Since c is an f-supplement of b in L, it is minimal in the set $\{x \in L \mid f \leq x, b \vee x = 1\}$. Therefore $c' = c$. □

The following result generalizes [2, Proposition 12.2 (2)].

Proposition 4.2.3. *Let L be a compactly generated compact lattice and c be an f-supplement of b in L. Then c is compact.*

Proof. Since c is an f-supplement of b in L, c is minimal in the set $\{x \in L \mid f \leq x, b \vee x = 1\}$. By Lemma 4.2.1, there exists a compact element c' with $f \leq c' \leq c$ such that $b \vee c' = 1$. Therefore $c = c'$ by minimality of c. □

The following result generalizes [2, Proposition 12.2 (4)].

Proposition 4.2.4. *Let L be a compactly generated lattice and c be an f-supplement of b in L. If $a \ll_f L$, then c is an f-supplement of $a \vee b$ in L.*

Proof. Since c is an f-supplement of b in L, c is minimal in the set $\{x \in L \mid f \leq x, b \vee x = 1\}$. Suppose $(a \vee b) \vee c' = 1$ for some $f \leq c' \leq c$. Since $a \ll_f L$ and $b \vee c' = 1$. Therefore $c = c'$ by minimality of c. □

Definition 4.2.5. [12, Definition 11] The meet of all maximal elements $m \neq 1$ of L such that $f \leq m$ is called the *f-radical* of L. It is denoted by $\mathrm{rad}_f(L)$.

Remark 4.2.6. $a \ll_f L$ and m is a maximal element in L such that $f \leq m$, then $a \vee m \neq 1$ and therefore $a \vee m = m$. So $a \leq m$. This means that all f-small elements of L are less than $\mathrm{rad}_f(L)$.

Lemma 4.2.7. [16, Proposition 2.9] *Let a be an element of L. Then $a \ll_f L$ if and only if $a \vee f \ll 1/f$.*

Proof. ($\Rightarrow$) Let $f \leq x$ such that $(a \vee f) \vee x = 1$. Since $a \ll_f L$, $x = f \vee x = 1$.
($\Leftarrow$) Let $f \leq x$ such that $a \vee x = 1$. Then we have $1 = a \vee f \vee x$. Since $a \vee f \ll 1/f$, $x = 1$. □

Lemma 4.2.8. *Let L be a lattice. Then $\mathrm{rad}_f(L) = \mathrm{rad}(1/f)$.*

Proof.

$$\operatorname{rad}(1/f) = \bigwedge_{i \in I} m_i$$

where m_i is a maximal element of L with $f \leq m_i$ for all $i \in I$. Therefore

$$\operatorname{rad}_f(L) = \operatorname{rad}(1/f).$$

□

Theorem 4.2.9. *Let L be a compactly generated lattice. Then*

$$\operatorname{rad}_f(L) = \bigvee_{i \in I} \{c_i \in L \mid c_i \ll_f L\}.$$

Proof. We have

$$\operatorname{rad}(1/f) = \bigvee_{i \in I} \{c_i \in L \mid c_i \ll_f L\} = \bigvee_{i \in I} \{c_i \in L \mid f \leq c_i, c_i \ll_f L\} = \operatorname{rad}_f(L)$$

by Lemma 4.2.7 and Lemma 4.2.8. □

Proposition 4.2.10. *If L is a compact lattice, then $\operatorname{rad}_f(L) \ll_f L$.*

Proof. Let $\operatorname{rad}_f(L) \vee x = 1$ for some $f \leq x$. Therefore $\operatorname{rad}_f(1/x) = 1$, by Lemma 4.2.8. Since L is compact, $x = 1$ and therefore $\operatorname{rad}_f(L) \ll_f L$. □

The following result generalizes [1, Proposition 2].

Proposition 4.2.11. *Let L be a compactly generated compact lattice and c be an f-supplement of b in L. If $a \ll_f L$, then $a \wedge c \ll_f c/0$ and $\operatorname{rad}_f(c/0) = c \wedge \operatorname{rad}_f(L)$.*

Proof. Let $(a \wedge c) \vee c' = c$ for some $f \leq c' \in c/0$. Since $a \wedge c \leq a$ and $a \ll_f L$, $a \wedge c \ll_f L$ by Lemma 3.1.7. Then

$$1 = b \vee c = b \vee [(a \wedge c) \vee c'] = (a \wedge c) \vee b \vee c'$$

and $f \leq b \vee c'$. Therefore $1 = b \vee c'$. Thus $c = c'$. We know that

$$\text{rad}_f(c/0) \leq c \wedge \text{rad}_f(L)$$

is always true. Since L is compact, $\text{rad}_f(L) \ll_f L$ by Proposition 4.2.10. So c is an f-supplement of $b \vee \text{rad}_f(L)$ in L. That is

$$f \leq c, (b \vee \text{rad}_f(L)) \vee c = 1 \text{ and } (b \vee \text{rad}_f(L)) \wedge c \ll_f c/0$$

by Lemma 4.1.3. Since $\text{rad}_f(c/0)$ is a join of all f-small elements of $c/0$ by Theorem 4.2.9,

$$(b \vee \text{rad}_f(L)) \wedge c \leq \text{rad}_f(c/0).$$

Therefore

$$\text{rad}_f(L) \wedge c \leq (b \vee \text{rad}_f(L)) \wedge c \leq \text{rad}_f(c/0).$$

□

The following result generalizes [1, Proposition 9].

Proposition 4.2.12. *Let L be a compactly generated f-supplemented lattice. Then $1/a$ is $(f \vee a)$-supplemented for every element a of L.*

Proof. Let $b \in 1/a$. Since L is f-supplemented, there is an f-supplement c of b in L. That is

$$f \leq c, b \vee c = 1 \text{ and } b \wedge c \ll_f c/0$$

by Lemma 4.1.3. Now by modular law and Lemma 3.1.6

$$b \wedge (c \vee a) = (b \wedge c) \vee a \ll_{(f \vee a)} (c \vee a)/a.$$

□

The following result generalizes [1, Theorem 2].

Corollary 4.2.13. *Let L be a compactly generated lattice. L is f-supplemented if and only if the quotient sublattice $1/f$ is supplemented.*

Proof. ($\Rightarrow$) Suppose that L is f-supplemented. Then $1/f$ is supplemented by Proposition 4.2.12.
($\Leftarrow$) Let $a \in L$. Since $1/f$ is supplemented, $(f \vee a)$ has a supplement b in $1/f$. That is

$$(f \vee a) \vee b = 1 \text{ and } (f \vee a) \wedge b \ll b/f.$$

Then

$$1 = (a \vee f) \vee b = a \vee b.$$

Let $(a \wedge b) \vee x = b$ for some $f \leq x \leq b$ in L. Therefore

$$(a \wedge b) \vee x \vee f = b.$$

Since

$$f \vee (a \wedge b) = (f \vee a) \wedge b \ll b/f, x = b.$$

This means that $(a \wedge b) \ll_f b/0$. Thus b is an f-supplement of a in L. □

The following result generalizes [1, Proposition 6].

Lemma 4.2.14. *Let L be a compactly generated lattice and a, b be elements of L with $f \leq a$. If $a/0$ is f-supplemented and $a \vee b$ has an f-supplement in L, then b has an f-supplement in L.*

Proof. Let c be an f-supplement of $a \vee b$ in L and d be an f-supplement of $a \wedge (b \vee c)$ in $a/0$. Then

$$f \leq c, (a \vee b) \vee c = 1 \text{ and } (a \vee b) \wedge c \ll_f c/0;$$

$$f \leq d, [a \wedge (b \vee c)] \vee d = a \text{ and } [a \wedge (b \vee c)] \wedge d \ll_f d/0$$

by Lemma 4.1.3. Now since

$$1 = a \vee b \vee c = [a \wedge (b \vee c)] \vee d \vee b \vee c = d \vee b \vee c$$

and since

$$(b \vee c) \wedge d = (b \vee c) \wedge d \wedge a = [(b \vee c) \wedge a] \wedge d \ll_f d/0,$$

d is an f-supplement of $b \vee c$ in L. Since $d \in a/0$, $d \leq a$ and therefore $b \vee d \leq a \vee b$. Also

$$(a \vee b) \vee c = (b \vee d) \vee c = 1.$$

Since c is an f-supplement of $a \vee b$ in L, c is an f-supplement of $b \vee d$ in L by Proposition 4.2.2. Then

$$(b \vee d) \vee c = 1 \text{ and } (b \vee d) \wedge c \ll_f c/0$$

by Lemma 4.1.3. So by Lemma 2.4.5 and by Lemma 3.1.9, we have

$$b \wedge (c \vee d) \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)] \ll_f (c \vee d)/0.$$

That is $c \vee d$ is an f-supplement of b in L. □

The following result generalizes [1, Proposition 7].

Proposition 4.2.15. *Let L be a compactly generated lattice. If $f \leq a_1$, $f \leq a_2$, $a_1 \vee a_2 = 1$, $a_1/0$ and $a_2/0$ are f-supplemented sublattices of L, then L is also f-supplemented.*

Proof. For an element b of L, there is an f-supplement of $a_1 \vee (a_2 \vee b) = 1$ in L. Since $a_1/0$ is f-supplemented, $a_2 \vee b$ has an f-supplement in L by Lemma 4.2.14. Since $a_2/0$ is f-supplemented, again by Lemma 4.2.14, b has an f-supplement in L. □

Definition 4.2.16. If $a \vee b = 1$ and $a \wedge b = 0$ for elements a and b of L, then we use the notation $a \oplus b = 1$ and call this a *direct sum*. In this case a and b are called *direct summands* of 1. Also a is said to be *complement* of b and b is said to be *complement* of a. If every element of a lattice L has a complement in L, then L is called a *complemented* lattice (see [6]).

Definition 4.2.17. Let L be a lattice and $a < b$ be element of L. If $a \leq c < b$ implies $c = a$, then it is said that a is *covered by* b. If 0 is covered by an element a of L, then a is called an *atom*. A lattice L is said to be *semiatomic*, if 1 is a join of atoms in L (see [2]).

The following result generalizes [1, Proposition 8].

Proposition 4.2.18. *Let L be a compactly generated compact lattice. If L is f-supplemented, then the quotient sublattice $1/\operatorname{rad}_f(L)$ of L is semiatomic.*

Proof. Since L is compactly generated, $1/\operatorname{rad}_f(L)$ is also compactly generated. Since L is f-supplemented, $1/\operatorname{rad}_f(L)$ is $(f \vee \operatorname{rad}_f(L))$-supplemented by Proposition 4.2.12. Also $1/\operatorname{rad}_f(L)$ does not contain any f-small element of L by Remark 4.2.6. Let $x \in 1/\operatorname{rad}_f(L)$. Since $1/\operatorname{rad}_f(L)$ is $(f \vee \operatorname{rad}_f(L))$-supplemented, there is an $(f \vee \operatorname{rad}_f(L))$-supplement y of x in $1/\operatorname{rad}_f(L)$. That is,

$$f \vee \operatorname{rad}_f(L) \leq y, x \vee y = 1 \text{ and } x \wedge y \ll_{(f \vee \operatorname{rad}_f(L))} y/\operatorname{rad}_f(L)$$

by Lemma 4.1.3. Therefore $x \wedge y \ll_{(f \vee \operatorname{rad}_f(L))} 1/\operatorname{rad}_f(L)$ by Lemma 3.1.5. Since L is compact, $\operatorname{rad}_f(L) \ll_f L$ by Proposition 4.2.10. Therefore $x \wedge y \ll_f L$ by Lemma 3.1.7. Again by Remark 4.2.6, $x \wedge y \leq \operatorname{rad}_f(L)$ and so $x \wedge y = \operatorname{rad}_f(L)$. This means that $1/\operatorname{rad}_f(L)$ is completed. Thus $1/\operatorname{rad}_f(L)$ is semiatomic by [2, Theorem 6.7]. □

Definition 4.2.19. [13, Definition 2.7] A lattice L is said to be *f-local* if the set $\{x \in L \mid f \leq x \neq 1\}$ of elements of L has the greatest element. Also an element l of L is called an *f-local element* if $f \leq l$ and the sublattice $l/0$ is an f-local lattice.

Definition 4.2.20. A lattice L is said to be *f-hollow* if every element a of with $f \leq a \neq 1$ is f-small.

Remark 4.2.21. Clearly a hollow lattice is f-hollow and the converse is true when $f = 0$. In general f-hollow lattices need not be hollow (see [1, Example 2]).

Lemma 4.2.22. *[2, Corollary 7.2] Let L be a compact lattice. Then for every element a of L there is a maximal element m of L such that $a \leq m$.*

Proof. $\Omega = \{x \in L \mid a \leq x, x \neq 1\}$. Let $\Gamma = \{x_\lambda \mid \lambda \in \Lambda\}$ be a chain in Ω. Take $x = \bigvee_{\lambda \in \Lambda} x_\lambda$. Since $a \leq x_\lambda$ for all $\lambda \in \Lambda$, $a \leq x$. If $1 = x = \bigvee_{\lambda \in \Lambda} x_\lambda$, then since L is compact, $1 = x = \bigvee_{\lambda \in F} x_\lambda = x_{\lambda_0}$ for some finite subset F of Λ and $x_{\lambda_0} \in F$. This is a contradiction. Therefore $x \in \Omega$. Also x is an upper bound for Γ. So there is a maximal element m in Ω with $a \leq m$ by Zorn's Lemma. □

Lemma 4.2.23. *If in a lattice L there exists a largest element m such that $f \leq m \neq 1$, then $m \ll_f L$.*

Lemma 4.2.24. *Let L be a compactly generated lattice. If L is f-local with $f \leq m$ the largest element, then $\mathrm{rad}_f(L) = m \ll_f L$.*

Proof. Clearly $\mathrm{rad}_f(L) = m$ is the only maximal element with $f \leq m$ and therefore $\mathrm{rad}_f(L) = m \ll_f L$ by Lemma 4.2.23. □

The following result generalizes [1, Proposition 10].

Proposition 4.2.25. *A compactly generated lattice L is f-local if and only if it is f-hollow and $\mathrm{rad}_f(L) \neq 1$.*

Proof. ($\Rightarrow$) Let a be an element of L with $f \leq a \neq 1$. Since L is f-local, $\mathrm{rad}_f(L)$ is the largest element with $f \leq \mathrm{rad}_f(L)$. Therefore $a \leq \mathrm{rad}_f(L)$. Since $\mathrm{rad}_f(L) = m \ll_f L$ by Lemma 4.2, $a \ll_f L$. Also $\mathrm{rad}_f(L) \neq 1$.
($\Leftarrow$) Suppose L is f-hollow and $\mathrm{rad}_f(L) \neq 1$. Since L is compactly generated, $\mathrm{rad}_f(L) = \bigvee_{i \in I}\{c_i \in L \mid c_i \ll_f L\}$. Since L is f-hollow, every element a with $f \leq a \neq 1$ is f-small in L. Hence there is a largest (f-small) element which is equal to $\mathrm{rad}_f(L)$ by Lemma 4.2. Therefore 1 is trivially compact i.e., $1 = \bigvee X$ implies $1 \in X$. Then $\mathrm{rad}_f(L) \ll_f L$. Thus L is f-local. □

Lemma 4.2.26. [13, Lemma 2.8] *Let* $\{l_i/0\}_{i\in I}$ *with* $f \leq l_i$ *for all* $i \in I$ *and* $I = \{1, \ldots, n\}$ *be a finite collection of* f*-local sublattices of a lattice* L *and* a *be an element of* L *such that* $a \vee (\bigvee_{i\in I} l_i)$ *has an* f*-supplement* b *in* L*. Then there exists a subset* J *of* I *such that* $b \vee (\bigvee_{i\in J} l_i)$ *is an* f*-supplement of* a *in* L*.*

Proof. Induction on n.
For $n = 1$, b is an f-supplement of $a \vee l_1$, i.e.

$$f \leq b, (a \vee l_1) \vee b = 1 \text{ and } (a \vee l_1) \wedge b \ll_f b/0.$$

Put $c = (a \vee b) \wedge l_1$. If $c = l_1$, then $l_1 \leq a \vee b$. So

$$1 = b \vee (a \vee l_1) = a \vee b$$

and

$$a \wedge b \leq (a \vee l_1) \wedge b \ll_f b/0.$$

Thus b is an f-supplement of a in L by Lemma 4.1.3. If $c \neq l_1$, then $(a \vee b) \wedge l_1 = c \ll_f l_1/0$. Therefore l_1 is an f-supplement of c in $l_1/0$. Now the following holds by Lemma 2.4.5 and Lemma 3.1.9:

$$a \wedge (b \vee l_1) \leq [b \wedge (a \vee l_1)] \vee [l_1 \wedge (a \vee b)] \ll_f (b \vee l_1)/0.$$

So $b \vee l_1$ is an f-supplement of a in L. Suppose that $n > 1$ and b is an f-supplement of $a' \vee (\bigvee_{i=2}^{n} l_i)$ in L where $a' = a \vee l_1$. By induction hypothesis there is a subset I' of $\{2, \ldots, n\}$ such that $b' = b \vee (\bigvee_{i\in I'} l_i)$ is an f-supplement of $a' = a \vee l_1$. Therefore either b' or $b' \vee l_1$ is an f-supplement of a in L. □

Lemma 4.2.27. [13, Lemma 2.9] *Let* m *be a maximal element of* L *such that* $f \leq m$*. If* l *is an* f*-supplement of* m *in* L*, then* $l/0$ *is* f*-local. Moreover* $l \wedge m$ *is the largest element of* $l/0$ *with* $f \leq l \wedge m$ *which is different from* l*.*

Proof. l is an f-supplement of m if and only if $f \leq l$, $m \vee l = 1$ and $m \wedge l \ll_f l/0$. Let $x \in l/0$ with $f \leq x$ and $x \neq l$. If $x \leq m$, then $x \leq l \wedge m$. If $x \nleq m$ ($x \nleq l \wedge m$), then since m is maximal $x \vee m = 1$. Now

$$l = l \wedge 1 = l \wedge (x \vee m) = x \vee (l \wedge m).$$

Since $l \wedge m \ll_f l/0$, $x = l$. This is a contradiction. Thus $l \wedge m$ is the largest element ($\neq l$) of $l/0$ such that $f \leq l \wedge m$. □

Let the join of f-local elements of L be denoted by $\mathrm{loc}_f(L)$. The following result is a generalization of [1, Corollary 7].

Theorem 4.2.28. *Let L be a compact lattice. Then L is f-supplemented if and only if every maximal element m of L with $f \leq m$ has an f-supplement in L.*

Proof. ($\Rightarrow$) Since L is f-supplemented by assumption, this part is clear.
($\Leftarrow$) Let $a \in L$. There is a maximal element m of L such that $a \leq m$ by Lemma 4.2.22. Suppose m is a maximal element of the quotient sublattice $1/\mathrm{loc}_f(L)$. m has an f-supplement b in L by assumption, i.e.

$$f \leq b, m \vee b = 1 \text{ and } m \wedge b \ll_f b/0$$

by Lemma 4.1.3. Therefore $b/0$ is an f-local sublattice by Lemma 4.2.27. That is b is an f-local element of L. Then $b \leq \mathrm{loc}_f(L) \leq m$ and so $1 = m \vee b = m$, which is contradicting with the maximality of m. So there is no maximal element in $1/\mathrm{loc}_f(L)$ and therefore $1/(a \vee \mathrm{loc}_f(L))$ has no maximal element. Since $1/(a \vee \mathrm{loc}_f(L))$ has at least one maximal element ($\neq 1$) whenever $a \vee \mathrm{loc}_f(L) \neq 1$ by [2, Lemma 2.4], $a \vee \mathrm{loc}_f(L) = 1$. Also since L is compact, $a \vee (l_1 \vee \ldots \vee l_n) = 1$ for some f-local elements $l_1, \ldots l_n$ of L. Now since f is an f-supplement of $a \vee (l_1 \vee \ldots \vee l_n) = 1$ in L, a has an f-supplement in L by Lemma 4.2.26. □

Using Theorem 4.2.28 we prove that if a lattice L is an arbitrary join of f-supplemented sublattices containing f, then L is also f-supplemented.

Theorem 4.2.29. *Let L be a compact lattice and $\{a_i/0\}_{i\in I}$ be a collection of f-supplemented sublattices of L with $1 = \bigvee_{i\in I} a_i$ and $f \leq a_i$ for each $i \in I$. Then L is f-supplemented.*

Proof. Let $f \leq m$ and m be a maximal element of L. If $a_i \leq m$ for all $i \in I$, then $1 = \bigvee_{i\in I} a_i \leq m$ which is a contradiction. So $a_j \nleq m$ for some $j \in I$. Therefore $1 = a_j \vee m$. Since

$$a_j/(a_j \wedge m) \cong (a_j \vee m)/m = 1/m,$$

the element $a_j \wedge m$ is maximal in $a_j/0$ and $f \leq a_j \wedge m$. There is an f-supplement b of $a_j \wedge m$ in $a_j/0$ by Theorem 4.2.28. That is,

$$f \leq b, (a_j \wedge m) \vee b = a_j \text{ and } (a_j \wedge m) \wedge b \ll_f b/0$$

by Lemma 4.1.3. If $b \leq m$, then $a_j = (a_j \wedge m) \vee b \leq m$, which is a contradiction. So $b \nleq m$. Therefore $1 = m \vee b$ and $m \wedge b = a_j \wedge m \wedge b \ll_f b/0$. Thus b is an f-supplement of m in L. Hence L is f-supplemented by Theorem 4.2.28. □

5. AMPLY f-SUPPLEMENTED LATTICES

5.1 Amply f-Supplemented Lattices

Definition 5.1.1. An element a of a lattice L has *ample f-supplements* in L if for every element b of L with $a \vee b = 1$, the sublattice $b/0$ contains an f-supplement of a in L. A lattice L is said to be *amply f-supplemented* if every element of L has ample f-supplements in L (see [13]).

Recall that a homomorphic image of an f-small element under a lattice homomorphism need not be f-small. Nevertheless, we will show that the quotient sublattice $1/a$ of an amply f-supplemented lattice L is amply $(f \vee a)$-supplemented by using properties of f-small elements given in Chapter 3.

Proposition 5.1.2. *If a lattice L is amply f-supplemented, then the quotient sublattice $1/a$ is amply $(f \vee a)$-supplemented for every element a of L.*

Proof. Let x be an element of $1/a$. If $x \vee y = 1$ for some $y \in 1/a$, then x has a supplement $y' \leq y$ in L since L is amply supplemented, i.e.

$$f \leq y', x \vee y' = 1 \text{ and } x \wedge y' \ll_f y'/0$$

by Lemma 4.1.3. Then

$$1 = x \vee y' = x \vee (y' \vee a).$$

By modular law,

$$x \wedge (y' \vee a) = (x \wedge y') \vee a).$$

Since $x \wedge y' \ll_f y'/0$,

$$(x \wedge y') \vee a) \ll_{(f \vee a)} (y' \vee a)/a$$

by Lemma 3.1.6 and since $f \leq y'$,

$$f \vee a \leq y' \vee a.$$

Therefore $y' \vee a$ is an $(f \vee a)$-supplement of x in $1/a$. □

The following result generalizes [1, Proposition 14].

Proposition 5.1.3. *If L is an amply f-supplemented lattice, then for every f-supplement a in L, $a/0$ is amply f-supplemented.*

Proof. Let a be an f-supplement of b in L, i.e.

$$f \leq a, a \vee b = 1 \text{ and } a \wedge b \ll_f a/0$$

by Lemma 4.1.3. Let $a = x \vee y$. Then

$$1 = a \vee b = x \vee y \vee b.$$

There is an f-supplement y' of $b \vee x$ in L with $y' \leq y$. Now

$$f \leq y', 1 = (b \vee x) \vee y' \text{ and } (b \vee x) \wedge y' \ll_f y'/0$$

by Lemma 4.1.3. Since

$$x \wedge y' \leq (b \vee x) \wedge y' \ll_f y'/0,$$

by Lemma 3.1.7

$$x \wedge y' \ll_f y'/0.$$

By modular law we have the following equalities:

$$a = a \wedge 1 = a \wedge [(b \vee x) \vee y'] = y' \vee [a \wedge (b \vee x)] = y' \vee [x \vee (a \wedge b)].$$

Now since $f \leq x \vee y'$ and $a \wedge b \ll_f a/0$, $a = x \vee y'$. Thus y' is an f-supplement of x in $a/0$. $\square$

Corollary 5.1.4. *If L is an amply f-supplemented lattice, then for a direct summand a of L with $f \leq a$, the sublattice $a/0$ is also amply f-supplemented.*

The following results generalize [1, Proposition 15 and Proposition 16] respectively.

Proposition 5.1.5. *Let a, b be elements of a lattice L with $a \vee b = 1$. If a and b have ample f-supplements in L, then $a \wedge b$ has ample f-supplements in L.*

Proof. Let $(a \wedge b) \vee c = 1$ for some $c \in L$. Then

$$1 = a \vee (b \wedge c) = b \vee (a \wedge c).$$

Now there is an f-supplement x of a in L with $x \leq b \wedge c$ and there is an f-supplement y of b in L with $y \leq a \wedge c$ by assumption. So

$$x \vee y \leq c \text{ and } (a \wedge b) \vee (x \vee y) = 1.$$

Also since $f \leq x$ and $f \leq y$, $f \leq x \vee y$. Moreover

$$(a \wedge b) \wedge (x \vee y) = (x \wedge a) \vee (y \wedge b) \ll_f L$$

by Lemma 3.1.8. Thus $(x \vee y)$ is an f-supplement of $(a \wedge b)$ in L. $\square$

Proposition 5.1.6. *Let a, b be elements of a lattice L such that $b \ll_f L$. If $a \vee b$ has ample f-supplements in L, then a has also ample f-supplements in L.*

Proof. Let $a \vee c = 1$ for some $c \in L$. Then

$$1 = a \vee c = a \vee b \vee c.$$

So by assumption there is an f-supplement $b' \leq b$ of $a \vee b$ in L, that is

$$f \leq b', a \vee b \vee b' = 1 \text{ and } (a \vee b) \wedge b' \ll_f b'/0$$

by Lemma 4.1.3. Since $b \ll_f L$ and $f \leq a \vee b'$, $a \vee b' = 1$. Also

$$a \wedge b' \leq (a \vee b) \wedge b' \ll_f b'/0$$

implies that

$$a \wedge b' \ll_f b'/0$$

by Lemma 3.1.7. Therefore b' is an f-supplement of a in L. □

Definition 5.1.7. Given elements $a \leq b$ of L, the inequality $a \leq b$ is said to be f-*cosmall* in L if $b \ll_{(f \vee a)} 1/a$.

The following result generalizes [1, Theorem 4].

Theorem 5.1.8. *The following statements are equivalent for a lattice L.*

(1) L is amply f-supplemented.

(2) Every element a of L is of the form $a = x \vee y$ with $x/0$ is f-supplemented and $y \ll_f L$.

(3) For every element a of L, there is an element $x \leq a$ such that the sublattice $x/0$ is f-supplemented and the inequality $x \leq a$ is f-cosmall in L.

Proof. (1) ⇒ (2) L is f-supplemented. Let b be an f-supplement of a in L, i.e.

$$f \leq b, a \vee b = 1 \text{ and } a \wedge b \ll_f b/0$$

by Lemma 4.1.3. Since L is amply f-supplemented, there is an f-supplement $x \leq a$ of b in L. That is,

$$f \leq x, b \vee x = 1 \text{ and } b \wedge x \ll_f x/0$$

by Lemma 4.1.3. Now

$$a = a \wedge 1 = a \wedge (b \vee x) = x \vee (a \wedge b)$$

by modular law. Since $a \wedge b \ll_f b/0$, $a \wedge b \ll_f L$ by Lemma 3.1.5. $x/0$ is amply f-supplemented and therefore it is f-supplemented by Proposition 5.1.3.
$(2) \Rightarrow (3)$ Let $a = x \vee y$ for which $x/0$ is f-supplemented and $y \ll_f L$. Then

$$a = x \vee y \ll_{(f \vee x)} 1/x$$

by Lemma 3.1.7.
$(3) \Rightarrow (1)$ Let $a \in L$ with $a \vee b = 1$. Then there is an f-supplement $x \leq a$ of b with $x \leq a$ f-cosmall in L by assumption. Since $1 = (a \vee x) \vee b$, $1 = a \vee x$. Now $a \wedge x \leq x$ has an f-supplement b' in $x/0$. That is,

$$f \leq b', x = (a \wedge x) \vee b' \text{ and } (a \wedge x) \wedge b' \ll_f b'/0$$

by Lemma 4.1.3. Therefore

$$1 = a \vee (a \wedge x) \vee b' = a \vee b'$$

and

$$a \wedge b' = (a \wedge x) \wedge b' \ll_f b'/0.$$

Thus b' is an f-supplement of a in L with $b' \leq b$. Hence L is amply f-supplemented. □

Corollary 5.1.9. *If the sublattice $a/0$ is f-supplemented for every element a of a lattice L, then L is amply f-supplemented.*

The following result is a new result for modules.

Corollary 5.1.10. *If every submodule of a left R-module M is F-supplemented, then M is amply F-supplemented.*

6. CONCLUSION

In this thesis we generalize some known results about F-supplemented and amply F-supplemented modules to complete modular lattices. In this work one of the most important motivation is to show that not every generalization is possible. Therefore we give an example showing that a homomorphic image of an f-small element under a lattice homomorphism need not be f-small unlike the module case (see Example 3.1.3). Another important motivation is to obtain different proofs of the results from those in modules such as the proof of Theorem 4.2.28 and Theorem 4.2.29. Also some proven results for lattices give new results for modules (see Corollary 5.1.10). Some of the important results given in Chapter 4 and Chapter 5 are as follows:

6.1 f-Supplemented Lattices

Theorem 6.1.1. *(Theorem 4.1.4) Let $a \leq b$ be elements of a compactly generated lattice L. Then the following properties hold:*

(1) If a is an f-supplement in L, then a is an f-supplement in $b/0$.

(2) If b is an f-supplement in L, then

(i) a is an f-supplement in L if and only if a is an f-supplement in $b/0$.

(ii) $a \ll_f L$ if and only if $a \ll_f b/0$.

Proposition 6.1.2. *(Proposition 4.2.3) Let L be a compactly generated compact lattice and c be an f-supplement of b in L. Then c is compact.*

Theorem 6.1.3. *(Theorem 4.2.9) Let L be a compactly generated lattice. Then* $\mathrm{rad}_f(L) = \bigvee_{i\in I}\{c_i \in L \mid c_i \ll_f L\}$.

Proposition 6.1.4. *(Proposition 4.2.10) If L is a compact lattice, then* $\mathrm{rad}_f(L) \ll_f L$.

Proposition 6.1.5. *(Proposition 4.2.11) Let L be a compactly generated compact lattice and c be an f-supplement of b in L. If $a \ll_f L$, then $a \wedge c \ll_f c/0$ and* $\mathrm{rad}_f(c/0) = c \wedge \mathrm{rad}_f(L)$.

Proposition 6.1.6. *(Proposition 4.2.12) Let L be a compactly generated f-supplemented lattice. Then $1/a$ is $(f \vee a)$-supplemented for every element a of L.*

Proposition 6.1.7. *(Proposition 4.2.15) Let L be a compactly generated lattice. If $f \leq a_1$, $f \leq a_1$, $a_1 \vee a_2 = 1$, $a_1/0$ and $a_2/0$ are f-supplemented sublattices of L, then L is also f-supplemented.*

Proposition 6.1.8. *(Proposition 4.2.18) Let L be a compactly generated compact lattice. If L is f-supplemented, then the quotient sublattice $1/\operatorname{rad}_f(L)$ of L is semiatomic.*

Theorem 6.1.9. *(Theorem 4.2.28) Let L be a compact lattice. Then L is f-supplemented if and only if every maximal element m of L with $f \leq m$ has an f-supplement in L.*

Theorem 6.1.10. *(Theorem 4.2.29) Let L be a compact lattice and $\{a_i/0\}_{i\in I}$ be a collection of f-supplemented sublattices of L with $1 = \bigvee_{i\in I} a_i$ and $f \leq a_i$ for each $i \in I$. Then L is f-supplemented.*

6.2 Amply f-Supplemented Lattices

Proposition 6.2.1. *(Proposition 5.1.2) If a lattice L is amply f-supplemented, then the quotient sublattice $1/a$ is amply $(f \vee a)$-supplemented for every element a of L.*

Proposition 6.2.2. *(Proposition 5.1.3) If L is an amply f-supplemented lattice, then for every f-supplement a in L, $a/0$ is amply f-supplemented.*

Corollary 6.2.3. *(Corollary 5.1.4) If L is an amply f-supplemented lattice, then for a direct summand a of L with $f \leq a$, the sublattice $a/0$ is also amply f-supplemented.*

Proposition 6.2.4. *(Proposition 5.1.5) Let a, b be elements of a lattice L with $a \vee b = 1$. If a and b have ample f-supplements in L, then $a \wedge b$ has ample f-supplements in L.*

Proposition 6.2.5. *(Proposition 5.1.6) Let a, b be elements of a lattice L such that $b \ll_f L$. If $a \vee b$ has ample f-supplements in L, then a has also ample f-supplements in L.*

Theorem 6.2.6. *(Theorem 5.1.8) The following statements are equivalent for a lattice L.*

(1) L is amply f-supplemented.

(2) Every element a of L is of the form $a = x \vee y$ with $x/0$ is f-supplemented and $y \ll_f L$.

(3) For every element a of L, there is an element $x \leq a$ such that the sublattice $x/0$ is f-supplemented and the inequality $x \leq a$ is f-cosmall in L.

Corollary 6.2.7. *(Corollary 5.1.9 If the sublattice $a/0$ is f-supplemented for every element a of a lattice L, then L is amply f-supplemented.*

Corollary 6.2.8. *(Corollary 5.1.10) If every submodule of a left R-module M is F-supplemented, then M is amply F-supplemented.*

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