CONDITIONAL DIRECT SUMMAND PROPERTIES VIA RELATIVE INJECTIVITY

GÖRECELİ İNJEKTİFLİK YARDIMIYLA KOŞULLU DİK TOPLANAN ÖZELLİKLERİ

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ABSTRACT

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In this study, CS-modules and some of their generalizations, conditional direct summands feature modules will be handled with the help of relative injectivity and the results in this direction will be compiled and the findings that may contribute to the literature will be given at the end of the thesis as an original section.

Keywords: CS-modules, relative injective modules, complement submodules, conditional direct summand modules, fully invariant submodules.

ÖZET

GÖRECELİ İNJEKTİFLİK YARDIMIYLA KOŞULLU DİK TOPLANAN ÖZELLİKLERİ

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Bu çalışmada CS-modüller ve belirlenmiş bazı genelleştirmeleri, koşullu dik toplanan özellikli modüller, göreceli injektiflik yardımıyla ele alınıp bu yöndeki sonuçlar derlenecek ve bu çerçevede literatüre katkısı olabilecek bulgular özgün bölüm olarak tezin sonunda verilecektir.

Anahtar Kelimeler: CS-modüller, göreceli injektif modüller, complement alt modüller, koşullu dik toplanan alt modüller, tam değişmez alt modüller.

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CONTENTS

ABBREVIATIONS

$\mathbb{N}$	The set of positive integers
$\mathbb{Z}$	The ring of integers
$\mathbb{N}$	The set of positive integers
$\mathbb{Z}$	The ring of integers
$\mathbb{Z}_n$ or $\mathbb{Z}/\mathbb{Z}n$ $(n > 1)$	The ring of integers modulo n
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$M_n(R)$ The $n \times n$	Matrix ring over R
$T_n(R)$ The $n \times n$	Upper triangular matrix ring over R
$Z(M_R)$ or $Z(M)$	The singular submodule of M_R
$Z_2(M_R)$ or $Z_2(M)$	The second singular submodule of M_R
soc(M) or soc(M_R)	The socle of M_R
rad(M) or rad(M_R)	The Jacobson radical of M_R
M^n or $M^{(n)}$	The direct sum of n copies of M
$J(R)$	The Jacobson radical of R
u-dim M	Uniform dimension
$E(M)$	The injective hull of M_R
$\widetilde{E}(M)$	The rational hull of M_R
acc (dcc)	The ascending (descending) chain condition
$N \leq M$	N is a submodule of M
$N \trianglelefteq M$	N is a fully invariant submodule of M
$N \leq_e M$	N is an essential (large) submodule of M
$N \leq_c M$	N is a complement submodule of M
$N \leq_d M$	N is a direct summand of M
$\|I\|$	The cardinal of I

1 Introduction

In this study, R is a ring with unit but not necessarily commutative, and M is a right R-module. In this chapter, some results and necessary definitions that will be used in other parts of the thesis will be given. In this context, the proofs of some of the main results are clearly shown for completeness. Basic notations and definitions as well as basic results we refer to [2], [4], [13].

1.1 Complement Submodules

In this section, we introduce the concept of complement submodules and establish their fundamental properties. Additionally, we delve into the topic of chain conditions concerning complement submodules.

Definition 1.1.1. A submodule N of a right R-module M is called *essential (or large)* provided $N \cap K \neq 0$ for each nonzero $K \leq M_R$, and in this case we write $N \leq_e M_R$. In particular $M \leq_e M_R$. On the other hand, $0 \leq_e M_R$ if and only if $M = 0$.

Example 1.1.2. Every non-zero submodule of $\mathbb{Z}_\mathbb{Z}$ are essential in $\mathbb{Z}_\mathbb{Z}$.

Propositon 1.1.3. *Let M be a module. Then*

(i) $N \leq_e M \iff N \cap mR \neq 0$ for all $0 \neq m \in M$.

(ii) Given $K \leq N \leq M$, $K \leq_e M \iff K \leq_e N$ and $N \leq_e M$.

(iii) For any integer $t \geq 1$, $N_i \leq_e K_i$ $(1 \leq i \leq t) \implies (N_1 \cap ... \cap N_t) \leq_e (K_1 \cap ... \cap K_t)$

(iv) For any nonempty index set Λ, $N_\lambda \leq_e K_\lambda$ $(\lambda \in \Lambda) \implies \bigoplus N_\lambda \leq_e \bigoplus K_\lambda$.

Definition 1.1.4. Let M_R be a module and $N \leq M_R$. If there exists a submodule $N' \leq M_R$ such that $N \cap N' = 0$ and $M = N + N'$, then N is said to be a *direct summand* of M and denoted by $N \leq_d M$. On the other hand N' is called a direct complement of N.

Example 1.1.5. Let $M_R = V_F$ be a vector space and $N \leq V_F$. Then $N \leq_d V_F$.

Definition 1.1.6. Given $L \leq M$, a *complement (submodule)* of L in M refers to a submodule K of M that is maximal with respect to the property $K \cap L = 0$. In other words, K is a complement of L in M if and only if it satisfies the following conditions:
(i) $K \cap L = 0$, and
(ii) For any submodule $K \subset N \leq M$, we have $N \cap L \neq 0$.

Example 1.1.7. Let F be any field and let $M_F = F \oplus F$. Now $L = F \oplus 0 \leq M_F$. For any $x \in F$, the subspace $(x, 1)F = \{xf, f : f \in F\} \leq M_F$ is a complement submodule of L in M_F.

Propositon 1.1.8. *Let L, $N \leq M$ with $N \cap L = 0$. Then there exists a complement K of L in M such that $N \subseteq K$.*

Proof. Clear on using Zorn's Lemma. □

Propositon 1.1.9. *Let $L \leq M$ and let K be any complement of L in M. Then $K \oplus L \leq_e M$.*

Proof. Assume $N \leq M$ and $(K \oplus L) \cap N = 0$. Let's suppose that $K \subset K + N$. Since $(K + N) \cap L \neq 0$, there exists $k \in K$, $n \in N$, and $0 \neq x \in L$ such that $x = k + n$. This implies that $n \in (K \oplus L) \cap N$, which means that $n = 0$. Consequently, we have $x = 0$, which leads to a contradiction. Therefore, we conclude that $K = K + N$, which implies that $N \subseteq K$. Consequently, we have $N = 0$. Thus, we have shown that if $(K \oplus L) \cap N = 0$, then $N = 0$. Therefore, $K \oplus L$ is an essential submodule of M. Hence, we can conclude that $K \oplus L \leq_e M$. □

Let M_R be a module. Then the sum of all minimal (or the direct sum of all simple) submodules of M is called the *socle* of M, and denoted by $\mathrm{Soc}(M)$.

Corollary 1.1.10. *For any module M, $Soc(M) = \bigcap \{N : N \leq_e M\}$.*

Proof. Let $N \leq_e M$ and U be a simple submodule of M. If $N \cap U \neq 0$, then it must be the case that $N \cap U = U$, implying that U is a subset of N. Thus, we have $soc(M) \subseteq \bigcap\{N : N \leq_e M\}$.

Conversely, let $m \in \bigcap\{N : N \leq_e M\}$. Suppose $soc(mR) \neq mR$. Then there exists a maximal submodule L of mR such that $soc(mR) \subseteq L$. Suppose $L \leq_e mR$. Let K be a complement of L in M. Then $mR \cap K = 0$ and $L \oplus K \leq_e mR \oplus K \leq_e M$ by Proposition 1.1.9. Thus, we have $L \oplus K \leq_e M$ (Proposition 1.1.3.) and $m \in L \oplus K$, which implies $m \in L$, leading to a contradiction. Therefore, $L \leq_e mR$ is not true, and there exists a non-zero submodule $V \leq mR$ such that $L \cap V = 0$. It follows that V is a simple submodule, and we have $V \subseteq soc(mR) \subseteq L$, which is a contradiction. Thus, we conclude that $mR = soc(mR)$, and $m \in soc(M)$. The desired result follows. □

A submodule K of a module M is called a complement submodule (in M), denoted as $K \leq_c M$, if there exists a submodule $L \leq M$ such that K is a complement of L in M. It is clear that $0 \leq_c M$ and $M \leq_c M$. Furthermore, for any direct summand K of M, we have $K \leq_c M$.

Propositon 1.1.11. *Let $N \leq M$. Then there exists $K \leq M$, containing N, such that $N \leq_e K \leq_c M$.*

Proof. Let N' be a complement of N in M. According to Proposition 1.1.8, there exists a complement K of N' in M such that $N \subseteq K$. Consider a non-zero submodule $L \leq K$. It follows that $N' \subsetneq L + N'$, and therefore $(L + N') \cap N \neq 0$. There exist $x \in L$, $n' \in N'$, and $0 \neq n \in N$ such that $n = x + n'$. This implies that $n' \in K \cap N'$, and therefore $n' = 0$, resulting in $n \in L \cap N$. Thus, we have shown that $N \leq_e K$. □

Propositon 1.1.12. *Let $K \leq_c M$ and $K \leq N \leq M$. Then $N \leq_e M \iff N/K \leq_e M/K$.*

Proof. ($\Longleftarrow$) By [13, Exercise 1.40.(iv)].

($\implies$) Assume that $N \leq_e M$. Let $M' = M/K$ and $N' = N/K$, where K is a complement of N in M. Consider a submodule $L' \leq M'$ with $N' \cap L' = 0$. There exists a submodule $L \leq M$ such that $K \subseteq L$, $L' = L/K$, and $N \cap L = K$. Let K' be a complement of K in M. Then we have $N \cap L \cap K' = 0$, which implies $L \cap K' = 0$. Since $K \subseteq L$, we conclude that $K = L$, and thus $L' = 0$. Therefore, we have shown that $N' \leq_e M'$. □

Propositon 1.1.13. *Assume $K \leq M$. Then $K \leq_c M$ if and only if whenever $K \leq_e L \leq M$, then $K = L$.*

Proof. Clear using Proposition 1.1.11. □

Propositon 1.1.14. *Let K, $L \leq_c M$. Then K is a complement of L in M if and only if L is a complement of K in M.*

Proof. Let K be a complement of L in M. Suppose $L \subseteq L' \leq M$ and $L' \cap K = 0$. According to Proposition 1.1.9., we have $K \oplus L \leq_e M$. Let $0 \neq y \in L'$. There exists $r \in R$ such that $0 \neq yr = k+x$ for some $k \in K$ and $x \in L$. Then we have $k = yr - x \in K \cap L' = 0$, which implies $yr \in L$. Therefore, we have $L \leq_e L'$. By Proposition 1.1.13., we conclude that $L = L'$. Hence, L is a complement of K in M. □

Propositon 1.1.15. *Assume $N \leq K \leq M$. Then*

(i) $K \leq_c M \implies K/N \leq_c M/N$.

(ii) $K/N \leq_c M/N$, $N \leq_c M \implies K \leq_c M$.

Proof. (i) Let L be a submodule of M such that $K \subseteq L$ and $K/N \leq_e L/N$. According to [13, Exercise 1.40.(iv)], $K \leq_e L$ and, by Proposition 1.1.13., it follows that $K = L$. Hence, $K/N = L/N$. Furthermore, by Proposition 1.1.13., we have $K/N \leq_c M/N$.

(ii) There exist submodules K' and N' of M such that $N \subseteq K'$, K/N is a complement of K'/N in M/N, and N is a complement of N' in M. Consequently, we have $K \cap K' = N$ and $N \cap N' = 0$, which implies $K \cap (K' \cap N') = 0$. Suppose $K \leq L \leq M$ and $L \cap (K' \cap N') = 0$. Since $N \subseteq L \cap K'$ and $(L \cap K') \cap N' = 0$, it follows that $L \cap K' = N$. Thus, $(L/N) \cap (K'/N) = 0$, and therefore $L/N = K/N$. We conclude that $L = K$, which implies that K is a complement of $K' \cap N'$ in M. □

Propositon 1.1.16. *Let $K \leq_c N$ and $N \leq_c M$. Then $K \leq_c M$.*

Proof. There exists a submodule K' of N such that K is a complement of K' in N, and there exists a submodule N' of M such that N is a complement of N' in M. It is evident that $K\cap(K'+N')=0$. Suppose $K\leq_e L\leq M$. Then $L\cap(K'+N')=0$, and consequently, $(N\cap(L+N'))\cap K'=K'\cap(L+N')=0$. However, $K\subseteq N\cap(L+N')$, which implies that $K=N\cap(L+N')$. Thus, $(N+L)\cap N'=0$. It follows that $L\subseteq N$, and by Proposition 1.1.13., we have $K=L$. Hence, by Proposition 1.1.13., we conclude that $K\leq_c M$. □

A module M satisfies the ascending chain condition on complements, denoted by acc-c, if for any chain

$$K_1\subseteq K_2\subseteq K_3\subseteq\dots$$

of complements, there exists a positive integer n such that $K_n=K_{n+1}=K_{n+2}=\dots$. Similarly, a module M satisfies the descending chain condition on complements, denoted by dcc-c, if for any chain

$$K_1\supseteq K_2\supseteq K_3\supseteq\dots$$

of complements, there exists a positive integer n such that $K_n=K_{n+1}=K_{n+2}=\dots$.

Propositon 1.1.17. *The following statements are equivalent for a module M.*

(i) M satisfies acc-c.

(ii) For any ascending chain of submodules $N_1\subseteq N_2\subseteq N_3\subseteq\cdots$ of M there exists $k\geq 1$ such that $N_i\leq_e N_{i+1}$ for all $i\geq k$.

(iii) M satisfies dcc-c.

(iv) For any descending chain of submodules $N_1\supseteq N_2\supseteq N_3\supseteq\cdots$ of M there exists $k\geq 1$ such that $N_{i+1}\leq_e N_i$ for all $i\geq k$.

(v) M does not contain an infinite direct sum of non-zero submodules.

(vi) There exists $N\leq_e M$ such that N does not contain an infinite direct sum of non-zero submodules.

(vii) For each $N\leq M$ there exists a finitely generated $K\leq_e N$.

(viii) For each $N\leq_e M$ there exists a finitely generated $K\leq_e N$.

Proof. (i) $\Longrightarrow$ (v) Assume that M satisfies the ascending chain condition on complements (acc-c). Suppose $N_1\oplus N_2\oplus N_3\oplus\dots$ is an infinite direct sum of non-zero submodules of M. By Proposition 1.1.11., there exists $K_1\leq_c M$ with $N_1\leq_e K_1$. Note that $K_1\cap(N_2\oplus N_3\oplus$

$\dots) = 0$. Again, by Proposition 1.1.11., there exists $K_2 \leq_c M$ such that $(K_1 \oplus N_2) \leq_e K_2$. Note that

$$K_2 \cap (N_3 \oplus N_4 \oplus \cdots) = 0.$$

Continuing this process, we obtain a chain of complements $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$. Thus, M does not satisfy the ascending chain condition on complements.

(v) $\implies$ (ii) Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ be an ascending chain of submodules of M such that N_i is not an essential submodule of N_{i+1} for all $i \geq 1$. For each $i \geq 1$, there exists $0 \neq K_i \leq N_{i+1}$ such that $N_i \cap K_i = 0$. It is easy to check that $K_1 + K_2 + K_3 + \dots$ is an infinite direct sum of non-zero submodules.

(ii) $\implies$ (i) This is implied by Proposition 1.1.13.

(iii) $\iff$ (iv) $\iff$ (v) Similar to (i) $\iff$ (ii) $\iff$ (v)

(v) $\iff$ (vi) Obvious.

(ii) $\iff$ (vii) Let $0 \neq N \leq M$. Let $0 \neq n_1 \in N$. Then either $n_1R \leq_e N$, or there exists $0 \neq n_2 \in N$ such that $n_1R \cap n_2R = 0$. Next, either $n_1R \oplus n_2R \leq_e N$ or there exists $0 \neq n_3 \in N$ such that $(n_1R \oplus n_2R) \cap n_3R = 0$. Repeat this process and note that, by (ii), it must stop after a finite number of steps. Thus there exists $k \geq 1$ such that $n_1R \oplus \cdots \oplus n_kR \leq_e N$.

(vii) $\iff$ (viii) Clear.

(viii) $\iff$ (v) Suppose (viii) holds. Let $N = N_1 \oplus N_2 \oplus N_3 \oplus \dots$ be a direct sum of submodules of M. Let N' be a complement of N in M. By Proposition 1.1.9., $N \oplus N' \leq_e M$, and hence there exists $K \leq_e N \oplus N'$ with K finitely generated. Since K is finitely generated, there exists $t \geq 1$ such that $K \subseteq N_1 \oplus N_2 \oplus \cdots \oplus N_t \oplus N'$. Then, for all $i \geq t+1$, $K \cap N_i = 0$, and hence $N_i = 0$. It follows that M satisfies property (v). □

Corollary 1.1.18. *Suppose that M satisfies acc-c and $N \leq M$. Then*

(i) N satisfies acc-c.

(ii) M/N satisfies acc-c provided $N \leq_c M$.

Proof. (i) is evident, and (ii) is a consequence of Proposition 1.1.15. □

We shall say that a module M satisfies acc-e (respectively, dcc-e) if every ascending (descending) chain of essential submodules terminates. The following statement is a direct implication of Proposition 1.1.17.

Propositon 1.1.19. *The following statements are equivalent for a module M.*

(i) M satisfies acc-e.

(ii) For all $K \leq_e N \leq M$, the module N/K is finitely generated.

(iii) M/ Soc (M) is Noetherian.

Proof. (i) $\implies$ (ii) Suppose that M satisfies acc-e. Let K be an essential submodule of N which is a submodule of M. Let L be a complement of K in N. Then $N \cap L = 0$, and by Proposition 1.1.9., $K \oplus L$ is an essential submodule of M. Hence $M/(K \oplus L)$ is Noetherian, and $N/K \cong (N \oplus L)/(K \oplus L)$ is finitely generated.

(ii) $\implies$ (iii) Suppose that (ii) holds. Let $S = \text{soc}(M)$. Let $S \subseteq N \subseteq M$. Let K be a complement of S in N. We first prove that K satisfies acc-c. Suppose not, and let $K' = K_1 \oplus K_2 \oplus K_3 \oplus \dots$ be a direct sum of non-zero submodules of K. For each $i \geq 1$, $S \cap K_i = 0$, and hence by Corollary 1.1.10., there exists $L_i \leq_e K_i$ with $L_i \neq K_i$. Let

$$L = L_1 + L_2 + L_3 + ... = L_1 \oplus L_2 \oplus L_3 \oplus \cdots.$$

By Proposition 1.1.3. (iv), $L \leq_e K'$ and hence K'/L is finitely generated. But

$$K'/L \cong (K_1/L_1) \oplus (K_2/L_2) \oplus (K_3/L_3) \oplus \cdots ,$$

which is an infinite direct sum of non-zero submodules, a contradiction. Thus K satisfies acc-c.

By Proposition 1.1.17., there exists $P \leq_e K$ with P finitely generated. Moreover, $P \oplus S \leq_e N$. By Propositions 1.1.3. and 1.1.9., $N/(P \oplus S)$ is finitely generated, and hence N/S is finitely generated. It follows that M/S is Noetherian.

(iii) $\implies$ (i)This is clear by Corollary 1.1.10. □

A similar argument yields the following result.

Propositon 1.1.20. *The following statements are equivalent for a module M.*

(i) M satisfies dcc-e.

(ii) For all $K \leq_e N \leq M$, the module N/K is finitely generated.

(iii) $M/$ soc (M) is Artinian.

Corollary 1.1.21.

(i) A module M is Noetherian if and only if M satisfies acc-c and acc-e.

(ii) A module M is Artinian if and only if M satisfies dcc-c and dcc-e.

Proof. By Propositions 1.1.17., 1.1.19., and 1.1.20. □

Another consequence of Proposition 1.1.17 is the following result.

Propositon 1.1.22. *Let $N \leq M$ be such that both N and M/N satisfy acc-c. Then M satisfies acc-c.*

Proof. Take K as a submodule of M. Consider a complement L of $N \cap K$ in K. Using Proposition 1.1.9., we have that $(N \cap K) \oplus L \leq_e K$. Applying Proposition 1.1.17., we find a submodule K' of $(N \cap K)$ such that K' is finitely generated. Additionally, since $N \cap L = 0$, we can see that L is isomorphic to a submodule of M/N. Thus, L satisfies acc-c. Using Proposition 1.1.17. once again, we can find a submodule L' of L that is finitely generated. It is clear that $K' \oplus L'$ is also finitely generated, and by Proposition 1.1.3., we have $K' \oplus L' \leq_e K$. Finally, Proposition 1.1.17. implies that M satisfies acc-c. □

Definition 1.1.23. A submodule U of M is called *uniform*, written $U \leq_u M$, if $U \neq 0$ and $X \cap Y \neq 0$ for all $0 \neq X, Y \leq U$. In other words, $U \leq_u M \iff X \leq_e U$ for all $0 \neq X \leq U$.

Example 1.1.24. $M_R = \mathbb{Z}_{\mathbb{Z}}$ is a uniform module.

The following result provides additional details regarding modules satisfying acc-c.

Propositon 1.1.25. *Suppose that M is a non-zero module satisfying acc-c. Then*

(i) M contains a uniform submodule.

(ii) There exist a positive integer n and uniform submodules U_i ($1 \leq i \leq n$) of M such

that $U_1 \oplus \cdots \oplus U_n \leq_e M$*.*

(iii) Given $N \leq M$*,* $N \leq_e M \iff N \cap U_i \neq 0$ *(*$1 \leq i \leq n$*).*

(iv) For any direct sum $N_1 \oplus \cdots \oplus N_k$ *of non-zero submodules of M,* $k \leq n$*.*

(v) If $V_1 \oplus \cdots \oplus V_k \leq_e M$*, with* $V_i \leq_u M$ *(*$1 \leq i \leq k$*), then* $k = n$*.*

Proof. (i)If M does not satisfy the uniform property, then there exist non-zero submodules L_1 and L_1' of M such that $L_1 \cap L_1' = 0$. If L_1' is not uniform, then there exist non-zero submodules L_2 and L_2' of L_1' such that $L_2 \cap L_2' = 0$. This process can be continued, generating a direct sum $L_1 \oplus L_2 \oplus L_3 \oplus \ldots$ of non-zero submodules. By Proposition 1.1.17, either M is uniform or there exists a positive integer t such that L_t' is uniform.

(ii) According to (i), there exists a uniform submodule U_1 of M. We consider two cases: either U_1 is an essential submodule of M, or there exists a non-zero submodule K_1 of M such that $U_1 \cap K_1 = 0$. In the latter case, we can apply (i) again to find a uniform submodule U_2 contained in K_1. Notably, U_1 and U_2 have zero intersection. We continue this process, either obtaining a direct sum $U_1 \oplus U_2 \oplus U_3 \oplus \ldots$ that is essential in M, or finding a non-zero submodule K_2 such that $(U_1 \oplus U_2) \cap K_2 = 0$. In the latter scenario, we repeat the process and find a uniform submodule U_3 contained in K_2. This pattern continues, allowing us to construct a direct sum of uniform submodules. By Proposition 1.1.17., there exists a positive integer n such that $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ is essential in M.

(iii) If $N \leq_e M$ then clearly $N \cap U_i \neq 0$ for each $1 \leq i \leq n$. Conversely, suppose that $N \cap U_i \neq 0$ ($1 \leq i \leq n$). Let $0 \neq K \leq M$. Let k be a non-zero element of K. Then $kR \cap (U_1 \oplus \cdots \oplus U_n) \neq 0$. Thus there exist $r \in R$, $u_i \in U_i$ ($1 \leq i \leq n$) such that

$$0 \neq kr = u_1 + \cdots + u_n.$$

Clearly, $u_i \neq 0$ for some $1 \leq i \leq n$. Note that, because U_i is uniform, we have $u_iR \cap (U \cap N) \neq 0$. Thus, there exists $s \in R$ such that $0 \neq u_is \in N$. Then

$$0 \neq krs = u_1\text{s} + \cdots + u_i\text{s} + \cdots + u_n\text{s}.$$

Let $x = krs$ - u_is. Let $V = \bigoplus_{j \neq i} U_j$. By induction on n, $N \cap V \leq_e V$. Either $x = 0$, or there exists $t \in R$ such that $0 \neq xt \in N \cap V$. Thus, either krs is a non-zero element of N, or $krst$ is a non-zero element of N. In any case, $N \cap K \neq 0$. Thus $N \leq_e M$.

(iv) Clearly, the direct sum $N_2 \oplus \cdots \oplus N_k$ is not an essential submodule of M. Without loss of generality, we can assume that $U_1 \cap (N_2 \oplus \cdots \oplus N_k) = 0$, using property (iii). We form the direct sum $U_1 \oplus N_2 \oplus \cdots \oplus N_k$. By applying the same argument, we can further assume, without loss of generality, that $U_1 + U_2 + N_3 + \cdots + N_k$ is a direct sum. If $k > n$, then repeating this process would yield the direct sum $U_1 \oplus \cdots \oplus U_n \oplus N_{n+1} \oplus \cdots \oplus N_k$. However, this would imply that

$$(U_1 \oplus \cdots \oplus U_n) \cap N_k = 0,$$

which contradicts the previous assumption. Therefore, we conclude that $k \leq n$.

(v) By (iv). □

Definition 1.1.26. Assume that M be a module with acc-c. There exists a positive integer n such that n is the number of non-zero direct summands in any essential direct sum of uniform submodules. n is called the *uniform dimension* or *Goldie dimension* of M, denoted as u-dim M. Clearly

$$u\text{−dim } M = 0 \iff M = 0.$$

Propositon 1.1.27. *Let M be a module which satisfies acc-c and let $N \leq M$. Then*

(i) N satisfies acc-c and u−dim $N \leq$ u−dim M. Moreover, u−dim $N =$ u−dim M if and only if $N \leq_e M$.

(ii) If $N \leq_c M$ then M/N satisfies acc-c and in this case u−dim $M =$ u−dim N + u−dim (M/N).

Proof. (i) Clear.

(ii) Suppose $N \leq_c M$. Let N' be a complement of N in M. By Proposition 1.1.9., $N \oplus N' \leq_e M$. Moreover, by Proposition 1.1.12., $N' \cong (N \oplus N')/N \leq_e M/N$. Now N' has acc-c, by (i), and hence so does M/N by Corollary 1.1.18 Moreover, it is clear that

$$u\text{−dim } M = u\text{−dim } N + u\text{−dim } N'$$

$$= u\text{−dim } N + u\text{−dim } (M/N).$$

□

Definition 1.1.28. Let M be a right R-module and $x \in M$. Let $r(x) = \{r \in R : xr = 0\}$ (resp., $l(x) = \{r \in R : rx = 0\}$) be the right (resp., left) *annihilator* of x. The singular submodule of M is defined by $Z(M) = \{x \in M : r(x) \leq_e R_R\}$. Then the module M is called *singular* if $M = Z(M)$, and *nonsingular* if $Z(M) = 0$.

Example 1.1.29. $\mathbb{Z}_{\mathbb{Z}}$ is a nonsingular module.

The following definition and corollary are very basic concepts in our work. Note that we will discuss fully invariant notion in details at Chapter 5. For more information we refer to [12], [2].

Definition 1.1.30. Let M_R be a module and $N \leq M_R$. If $f(N) \subseteq N$ for every $f \in \text{End}(M_R)$, N is called a *fully invariant* submodule of M.

Corollary 1.1.31. Let M be a right R-module and A, B, $C \leq M$. Then

$$(A \cap B) + (A \cap C) \leq A \cap (B \cap C).$$

If $B \leq A$ then

$$A \cap (B \cap C) = B \cap (A \cap C).$$

The latter equality is known as modular law in module theory.

Proof. Take any $x \in (A \cap C)$. Hence x $\in A$ and $x \in B + C$ and there exists $b \in B$ and $c \in C$ such that $x = b + c$. Since $B \leq A$ then $c = x$ - $b \in A \cap C$. From here $x = b + c \in B + (A \cap C)$. So

$$A \cap (B + C) \subseteq B + (A \cap C).$$

Conversely, since $B + (A \cap C) \leq B + C$ and $B + (A \cap C) \leq B + A \leq A$ then

$$B + (A \cap C) \subseteq A \cap (B + C).$$

Thus we obtain

$$A \cap (B + C) = B + (A \cap C).$$

□

1.2 Injective Modules

In this section, we focus on injective modules, which are a fundamental class of modules in our work. We introduce injective modules and provide an overview of their basic properties.

Definition 1.2.1. A right R-module M is *injective* provided that for any right R-module B and any submodule $C \leq B$, all homomorphisms $f\colon C \to M$ extend to homomorphism $g\colon B \to M$.

Equivalently M_R is injective if and only if for every monomorphism $f\colon A \to B$ and homomorphism $g\colon A \to M$, $\exists\, h\colon B \to M$ such that $h \text{ o } f = g$. In other words

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \quad \textit{exact} \\ & & \downarrow g & \swarrow h & \\ & & M & & \end{array}$$

which makes the diagram commutative. In this context, we define the lifting of g to h. Injective modules can be seen as the *duals* of projective modules, where the direction of arrows is reversed and epimorphisms are replaced by monomorphisms. The first result provides a valuable criterion for injectivity, often referred to as the *Injective Test Lemma.*

Theorem 1.2.2. (Baer's Lemma) *A right R-module M is injective if and only if for each right ideal I of R and each R-homomorphism $f : I \to M$ there exists $m \in M$ such that $f(r) = mr$ ($r \in I$).*

Proof. If M is injective then given $I \leq R_R$ any $f\colon I \to M$ extends to some $f_1 : R \to M$ and $f(r) = f_1(r) = f_1(1)r$ for all $r \in R$.

Conversely; assume that M satisfies given condition and consider right R-modules $C \leq B$ with $f : C \to M$. Let $X = \{(C_1,f_1) : C \leq C_1 \leq B;\ f_1 : C_1 \to M \text{ and } f_1|_C = f\}$

$\neq \emptyset$. Define a relation "$\leq$" on X by $(C_1,f_1) \leq (C_2,f_2) \iff C_1 \subseteq C_2$ and f_2 extends f_1. Thus $(X,\leq)$ is a poset and every chain in X has an upper bound. By Zorn's Lemma X has a maximal member (C^*,f^*) in X and if $C^* = B$ we are done. If not choose $b \in B/C$ and let $I = \{r \in R : br \in C^*\}$. The rule $r \mapsto f^*(br)$ defines a homomorphism $I \to M$, and hence by assumption $\exists\ m \in M \ni f^*(br) = mr$ for all $r \in I$. Now define $f_1 : C^* + bR \to M$, $c + br \mapsto f^*(c) + mr$ (for all $c \in C^*$, $r \in R$) is a well defined homomorphism. However this contradicts with the maximality of (C^*,f^*). M is injective. □

The above theorem has the following immediate consequence.

Corollary 1.2.3. *Let K be a field. Then every K-module is injective.*

Theorem 1.2.2 can be utilized to determine which Abelian groups are injective $\mathbb{Z}$-modules. Remarkably, the injective $\mathbb{Z}$-modules correspond precisely to the divisible groups. An Abelian group D is referred to as *divisible* if, for every $d \in D$ and nonzero integer k, there exists $d' \in D$ such that $d'k = d$. In this sense, every element of D is divisible by each nonzero integer. For any $k \in \mathbb{Z}$, let $Dk = dk : d \in D$. Then, D is divisible if and only if $D = Dk$ for every nonzero $k \in \mathbb{Z}$. It should be noted that the zero group is considered divisible. Furthermore, it is evident that the rational numbers $\mathbb{Q}$ form a divisible group.

Lemma 1.2.4.

(i) Every factor group of a divisible group is divisible.

(ii) Every direct sum and direct product of divisible groups is divisible.

(iii) Every Abelian group can be embedded in a divisible group.

Proof. (i) Suppose D is divisible and $C \leq D_{\mathbb{Z}}$. Then we have the following equality for any non-zero integer k:

$$(D/C)k = (Dk + C)/C = (D + C)/C = D/C,$$

(ii) Let D_λ $(\lambda \in \Lambda)$ be a non-empty collection of divisible groups, and let D be the direct product $\Pi_{\lambda\in\Lambda}\ D_\lambda$. Then, for any non-zero integer k, we have the following equality:

$$Dk = \Pi_{\lambda\in\Lambda} D_\lambda \mathrm{k} = \Pi_{\lambda\in\Lambda}(D_\lambda \mathrm{k}) = \Pi_{\lambda\in\Lambda} D_\lambda = D,$$

The argument for direct sums follows a similar pattern.

(iii) Consider an arbitrary Abelian group M. According to [13, Proposition 1.8.] there exists a set Λ and a surjective map $\theta : F \to M$, where F is the free Abelian group $F = \mathbb{Z}^{(\Lambda)}$. The kernel K of θ is a subgroup of F, and we have the isomorphism $M \cong F/K$. Assume $D = \mathbb{Q}^{(\Lambda)}$. By (ii) D is a divisible group and clearly F/K can be embedded in the group D/K which is divisible by (i). □

To utilize Lemma 1.2.4, we initially establish a proposition concerning the ring $\mathbb{Z}$.

Proposition 1.2.5. *An Abelian group M is an injective $\mathbb{Z}$-module if and only if M is a divisible group.*

Proof. Assuming M is an injective module over $\mathbb{Z}$, take an element $m \in M$ and a non-zero integer $n \in \mathbb{Z}$. Define a mapping $\phi : n\mathbb{Z} \to M$ by $\phi(nk) = mk$ for $k \in \mathbb{Z}$. We can observe that ϕ is well-defined because if $nk = nk'$ for some $k, k' \in \mathbb{Z}$, then $k = k'$ and consequently $mk = mk'$. It is clear that ϕ is a homomorphism, and by Theorem 1.2.2., there exists an element $b \in M$ such that $\phi(r) = br$ for $r \in n\mathbb{Z}$. In particular, $m = \phi(n) = bn$, indicating that M is divisible.

Conversely, assume that M is a divisible module. Consider a non-zero ideal I of $\mathbb{Z}$ and a homomorphism $\theta : I \to M$. Since $\mathbb{Z}$ is a principal ideal domain, there exists a non-zero element $a \in I$ such that $I = a\mathbb{Z}$. By assumption, there exists an element $b \in M$ such that $\theta(a) = ba$. For any $s \in I$, there exists $t \in \mathbb{Z}$ such that $s = at$, and we have

$$\theta(s) = \theta(at) = \theta(a)t = bat = bs.$$

Applying Theorem 1.2.2. once again, we conclude that M is an injective $\mathbb{Z}$-module. □

By combining Lemma 1.2.4. (iii) and Proposition 1.2.5., it becomes evident that any module over $\mathbb{Z}$ can be embedded in an injective module over $\mathbb{Z}$. In fact, this statement holds not only for the ring $\mathbb{Z}$ but also for any ring. This follows as a consequence of the next result.

Lemma 1.2.6. *Let D be a divisible group. Then Hom(R, D) is an injective right R-module.*

Proof. Assume $X = Hom(R, D)$ and consider the [13, Diagram 1.10.]. Associated with [13, Diagram 1.10.] is the following diagram of $\mathbb{Z}$-modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\theta} & B \quad \textit{exact} \\ & & \Big\downarrow{\scriptstyle \alpha} & & \\ & & D & & \end{array}$$

where $\alpha : A \to D$ is defined as $\alpha(a) = \phi(a)(1)$ for $a \in A$. By Proposition 1.2.5., there exists a $\mathbb{Z}$-homomorphism $\beta : B \to D$ such that $\alpha = \beta\theta$. Define $\psi : B \to X = \mathrm{Hom}(R, D)$ as follows: for each $b \in B$, $\psi(b)$ is the mapping from R to D defined by $\psi(b)(r) = \beta(rb)$ for $r \in R$. Note that $\psi(b)$ is a $\mathbb{Z}$-homomorphism since β is a $\mathbb{Z}$-homomorphism, thus $\psi(b) \in X$. It can be checked that ψ is an R-homomorphism. To show that $\psi\theta = \phi$, we need to verify that for any $a \in A$, the mappings $\psi\theta(a)$ and $\phi(a)$ are equal. Specifically, we want to prove:

$$\psi\theta(a)(r) = \phi(a)(r) \ (r \in R)$$

Let $r \in R$. Then

$$\begin{aligned} \psi\theta(a)(r) &= \psi[\theta(a)(r)] = \beta(r\theta(a)) = \beta\theta(ra) = \alpha(ra) \\ &= \phi(ra)(1) = (r\phi(a))(1) = \phi(a)(1r) = \phi(a)(r). \end{aligned}$$

This proves [13, Diagram 1.10.] and hence that $\psi\theta = \phi$. It follows that X is injective. □

Corollary 1.2.7. *For any ring R, any right R-module can be embedded in an injective right R-module.*

Lemma 1.2.8. *Let* $\theta : M_R \to B_R$ *be a monomorphism. Then there exists an extension C of M and an R-isomorphism* $\phi : C \to B$ *such that* $\phi|_M = \theta$.

Proof. Let $D = (0, b) : b \in B, b \notin \mathrm{im}\ \theta \leq (R \oplus B)_R$. Define the set $C = M \cup D$. We define a mapping $\phi : C \to B$ as follows:

$$\phi(c) = \begin{cases} \theta(c), & \text{if } c \in M \\ \pi(c), & \text{if } c \in D \end{cases} \tag{1}$$

where $\pi : R \oplus B \to B$ is the canonical projection. The mapping ϕ is a bijection, and thus has an inverse $\phi^{-1} : B \to C$. We can equip C with a right R-module structure by defining addition and scalar multiplication as follows:

$$c_1 + c_2 = \phi^{-1} \left[\phi(c_1) + \phi(c_2)\right], \text{ and}$$

$$rc = \phi^{-1}[\phi(rc)],$$

for all $r \in R$, $c, c_1, c_2 \in C$. It can be verified that with these definitions, C becomes a right R-module. Moreover, we have $M \leq C_R$, $\phi : C \to B$ is an R-isomorphism, and $\phi|_M = \theta$. □

Corollary 1.2.9. *Assume that $\theta : A_R \to B_R$ is an isomorphism and A_R does not have any nontrivial essential extension. B_R also does not have any nontrivial essential extension.*

Proof. Assume that B is an essential submodule of C_R. By Lemma 1.2.8., there exists an extension D of A and an R-isomorphism $\phi : D \to C$ such that $\phi|_A = \theta$. Let $0 \neq E \leq D_R$. Then $0 \neq \phi(E) \leq C_R$, implying that $B \cap \phi(E) \neq 0$. However, we have $B \cap \phi(E) \subseteq \phi(A \cap E)$. Thus, $A \cap E \neq 0$. This implies that $A \leq_e D_R$. By the hypothesis, we have $A = D$, and therefore,

$$C = \phi(D) = \phi(A) = \phi(A) = B.$$

Consequently, B does not have any proper essential extension. □

The combination of Corollary 1.2.7. and Lemma 1.2.8. immediately yields:

Proposition 1.2.10. *Any right R-module has an injective extension.*

The following result provides two characterizations of injective modules:

Proposition 1.2.11. *The following statements are equivalent for a right R-module E:*

(i) E is injective.

(ii) E is a direct summand of each of its extensions.

(iii) E has no proper essential extension.

Proof. (i) $\Longrightarrow$ (ii) Assume that $E \leq M_R$. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & E & \xrightarrow{\iota} & A \quad \textit{exact} \\ & & \big\downarrow{\scriptstyle i_E} & & \\ & & E & & \end{array}$$

By hypothesis, there exists $\theta : M \to E$ such that $\theta_\iota = i_E$. By [13, Exercise 1.30.] and [13, Exercise 1.31.], it follows that E is a direct summand of M.

(ii) $\Longrightarrow$ (iii)Assume that E is an essential submodule of B_R. By the hypothesis, there exists a submodule C of B_R such that $B = E \oplus C$ and $E \cap C = 0$. Consequently, $C = 0$, implying $E = B$.

(iii) $\Longrightarrow$ (i) According to Proposition 1.2.10., there exists an injective right R-module X such that $E \leq X_R$. By [13, Proposition 1.17.], there exists $D \leq X_R$ maximal with respect to the property $E \cap D = 0$ and in this case $E \oplus D \leq_e X_R$. Our goal is to prove

$$X = E \oplus D.$$

Suppose, for contradiction, $X \neq E \oplus D$. Note that

$$E \cong E/(E \cap D) \cong (E \oplus D)/D \leq (\mathrm{X/D})_R$$

and $(E \oplus D)/D \neq X/D$ (see [13, Proposition 1.6.]). By Corollary 1.2.9. it follows that $(E \oplus D)/D \nleq_e (X/D)_R$. Hence, there exists a submodule $D \subset H \leq X_R$ such that

$$(E \oplus D)/D \cap (H/D) = 0$$

This implies $(E \oplus D) \cap H = D$, and therefore $E \cap H \subseteq E \cap D = 0$. Thus, $E \cap H = 0$, contradicting the choice of D. This proves that E is a direct summand of the injective module X, and it can be shown that E is injective using a simple exercise. $\square$

Theorem 1.2.12. *Any right R-module M has an essential extension E which is an injective right R-module. Moreover, if E' is any essential extension of M such that E' is injective, then there exists an R-isomorphism $\theta : E \to E'$ such that $\theta|_M = i_M$.*

Proof. By Proposition 1.2.10., we know that M has an injective extension X. Applying [13, Exercise 1.63.] to obtain a submodule E of X maximal with respect to the property that $M \leq_e E_R$. Suppose that F is an essential extension of E. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & E & \xrightarrow{\iota} & F \quad \textit{exact} \\ & & \downarrow{\scriptstyle \iota} & & \\ & & X & & \end{array}$$

where ι denotes the inclusion mapping. Since X is injective, there exists an R-homomorphism $\theta : F \to X$ such that $\theta \circ \iota = \iota$. If $\ker, \theta \neq 0$, then $M \cap \ker, \theta \neq 0$ (see [13, Exercise 1.64.]). Thus, $\ker, \theta = 0$, which implies that θ is a monomorphism. Therefore, $E \leq_e R(\text{im}, \theta)$ because $E \leq_e F_R$. Since $E \subseteq \text{im}, \theta$, the maximality of E implies $E = \text{im}, \theta$. For any $f \in F$, we have $\theta(f) \in E$, so

$$\theta(f) = \iota(\theta(f)) = \theta(\theta(f)$$

which implies $f = \theta(f) \in E$. Thus, $F = E$, and E has no proper essential extension. By Proposition 1.2.11., we conclude that E is an injective module, and it is clear from the definition that $M \leq_e E_R$.

Now suppose that E' is an essential extension of M such that E' is injective. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{\iota} & E \quad \textit{exact} \\ & & \downarrow{\scriptstyle \iota} & & \\ & & E' & & \end{array}$$

where ι denotes the inclusion mapping. Since E' is injective, there exists a homomorphism $\theta : E \to E'$ such that $\theta \circ \iota = \iota$. We observe that $M \cap \ker, \theta = 0$ and $M \leq_e E_R$. Hence, $\ker, \theta = 0$. This implies that $\text{im}, \theta \cong E$ is an injective submodule of E' and, therefore, a direct summand of E' (Proposition 1.2.11.). Since $M \leq_e E'$ and $M \subseteq \text{im}, \theta$, we have $\text{im}, \theta \leq_e E'_R$. It follows that $\text{im}, \theta = E'$, which means $\theta : E \to E'$ is an isomorphism. Clearly, $\theta|_M = \iota$. □

Definiton 1.2.13. Assume that M be any right R-module. Theorem 1.2.12. states that there exists an injective module E such that $M \leq_e E_R$. This injective module E is known as the *injective envelope* or *injective hull* of M, and it will be denoted as $E(M_R)$ or simply $E(M)$. It is important to note that Theorem 1.2.12. also demonstrates the uniqueness of $E(M)$ up to isomorphism. Specifically, if E'_R is another injective module and $M \leq_e E'_R$, then E' is isomorphic to $E(M)$.

1.3 Quasi-Injective Modules

Recall that injective modules are based on lifting homomorphisms. In this section we deal with a weaker form of injective modules by restricting the main idea of injective modules so-called quasi-injective modules. (See [6]).

Definiton 1.3.1. A module M is said to be *quasi-injective* provided any homomorphism from a submodule of M into M extends to an endomorphism of M i.e if $N \leq M$ and $f\colon N \to M$ then $\exists$ $g\colon M \to M$ such that $g|_N = f$.

Proposition 1.3.2. *A right module M is quasi-injective if and only if M is a fully invariant submodule of E(M).*

Proof. Let $T = End_R\ (E(M))$. Assume that $TM \subseteq M$. Given $A \leq M$, any $f\colon A \to M$ must extend to some $g \in T$, where $g|_M$ is an endomorphism of M which extends to f. Thus M is quasi-injective.

Conversely, assume that M is quasi-injective and $g \in T$. Now, restricting g, we get a map from $M \cap g^{-1}(M)$ into M, which by quasi-injectivity, extends to an endomorphism h of M. Then h extends to a map $\alpha \in T$ such that $\alpha(M) \subseteq M$ and $(\alpha$ - $g)$ $(M \cap g^{-1}) =$ 0. Since $\alpha(M) \subseteq M$ we get

$$M \cap ((\alpha \text{ - } g)^{-1}\ (M)) \leq M \cap \alpha^{-1}\ (M) \leq \text{Ker}\ (\alpha \text{ - } g)$$

where $(\alpha$ - $g)$ $(M) \cap M = 0$. Then $(\alpha$ - $g)$ $(M) = 0$. (because $M \leq_e E(M)$). Hence $g(M)$ $= \alpha(M) \subseteq M$. Thus $TM \subseteq M$. $\square$

Corollary 1.3.3. *If M is any quasi-injective module then any decomposition E(M) = $\bigoplus_\alpha$ E_α induces a corresponding decomposition M = $\bigoplus_\alpha$ (M ∩ E_α)*

Proof. For each α, let $\Pi_\alpha : E(M) \to E(M)$ be the projection onto the direct summand E_α. Since Π_α $(M) \subseteq M$, we have E_α-component of any element of M also belongs to M. Thus $M = \bigoplus_\alpha (M \cap E_\alpha)$. □

Corollary 1.3.4. *Let M a quasi-injective module. Then all complement submodules of M are direct summand of M (i.e M is CS) and all direct summands of M are quasi-injective.*

Proof. Let $N \leq_c M$ choose injective hulls. $E(N) \leq E(M)$, $N \leq_c M \cap E(N) \leq M$, we obtain $M \cap E(N) =$ Now, $E(M) \oplus B$ for some B. Hence by Corollary $M = [M \cap E(N)]$ $\oplus$ $[M \cap B]$ i.e $N \leq_c M$.

Now assume $M = A \oplus B$. Now; $E(M) = E(A) \oplus E(B)$, and let $T = End_R$ $(E(M))$ if π the projection onto $E(A)$, then π T π $= End_R$ $(E(A))$ and so π T $\pi(A) \leq A$ by the Proposition again, A is quasi-injective. □

Corollary 1.3.5. *Let M_R be a quasi-injective module, and S = End M. Then J(S) = {f ∈ S : Kerf $\leq_e$ M}, and S/J(S) is a regular ring.*

Proof. Set $K = \{f \in S :$ Ker $f \leq_e M\}$ and consider f, $g \in K$. Since (Ker f) ∩ (Ker g) $\leq$ Ker (f - g), we have Ker (f - g) $\leq_e M$. So (f - g) $\in K$. Given any $h \in S$ we have Ker $(fh) = h^{-1}$(Ker f) $\leq_e M$ i.e $fh \in K$. Also, since Ker $f \leq_e$ Ker (hf), we have $hf \in K$. Thus K is an ideal of S. Given any $f \in K$, we have ker $f \leq_e M$ and (ker (1 - f)) ∩ Ker $f = 0$. Hence ker (1 - f) = 0. □

2 Relative injective module classes

In this chapter, we delve into the concepts of relative injectivity and ejectivity of modules. By examining lifting homomorphisms, we construct classes of lifting submodules. It is important to note that the majority of the results presented in this chapter can be found in [1], [12], [13].

2.1 Relative injective modules

Definition 2.1.1. Assume M and X be right R-modules. We say that X is *M-injective* if, for every submodule N of M and every R-homomorphism $\varphi : N \to X$, there exists an R-homomorphism $\theta : M \to X$ such that $\theta(n) = \varphi(n)$ for all $n \in N$. Recall that a module X is called *quasi-injective* (or *QI* or *self-injective*) if it is X-injective, as defined in Definition 1.3.1.

First, note that any injective module is M-injective for any module M. This means that if N is an injective module and M is any module, then N is also M-injective. On the other hand, any R_R-injective module is injective. This means that if N is an R_R-injective module, then N is also injective. Recall that for any module M, $E(M)$ stands for the injective hull of M. The injective hull of a module M is the smallest injective module containing M as a submodule.

Proposition 2.1.2. *The R-module X is M-injective if and only if the following conditions hold for any submodule K of an R-module M:*

(i) X is K-injective

(ii) X is (M/K)-injective.

(iii) Any homomorphism $\varphi : K \to X$ can be lifted to a homomorphism $\theta : M \to X$.

Proof. Suppose that X is M-injective. Then conditions (i) and (iii) are clearly satisfied. Now, suppose that $K \subseteq N \leq M$ and $\varphi \in \mathrm{Hom}_R(N/K, X)$. Define $\varphi' : N \to X$ by

$$\varphi'(n) = \varphi(n + K) \text{ for } n \in N.$$

Note that $\varphi' \in \mathrm{Hom}_R(N, X)$ and can be lifted to $\theta' \in \mathrm{Hom}_R(M, X)$. Furthermore, $\theta'(n) = \varphi'(n)$ for $n \in N$ and specifically,

$$\theta'(k) = \varphi'(k) = \varphi(k + K) = 0 \text{ for } k \in K.$$

Define $\theta : M/K \to X$ as $\theta(m + K) = \theta'(m)$ for $m \in M$. Since $\theta'(K) = 0$, it follows that θ is well-defined. Moreover,

$$\theta(n + K) = \theta(n + K) = \theta'(n) = \varphi'(n) = \varphi(n + K) \text{ for } n \in N.$$

Therefore, X is (M/K)-injective.

Conversely, assume that X satisfies conditions (i), (ii), and (iii). Let $N \leq M$ and $\varphi \in \mathrm{Hom}_R(N, X)$. Denote the restriction of φ to $N \cap K$ as φ'. Since X is K-injective, there exists $\alpha \in \mathrm{Hom}_R(K, X)$ that lifts φ'. By (iii), there exists $\beta \in \mathrm{Hom}_R(M, X)$ that lifts α. Hence,

$$\beta(k) = \varphi(k) \text{ for } k \in K \cap N.$$

Let $\gamma = \varphi - \beta$. Then $\gamma \in \mathrm{Hom}_R(N, X)$ and $\gamma(K \cap N) = 0$. Define $\varphi'' : (N + K)/K \to X$ as $\varphi''(n+K) = \gamma(n)$ for $n \in N$. Note that φ'' is well-defined since $\gamma(K \cap N) = 0$. Clearly, $\varphi'' \in \mathrm{Hom}_R((N+K)/K, X)$. By (ii), there exists $\theta' \in \mathrm{Hom}_R(M/K, X)$ that lifts φ''. Define $\theta'' \in \mathrm{Hom}_R(M, X)$ as $\theta''(m) = \theta'(m + K)$ for $m \in M$. Let $\theta = \beta + \theta'' \in \mathrm{Hom}_R(M, X)$. For $n \in N$, we have $\theta(n) = \beta(n) + \theta''(n) = \varphi(n) - \gamma(n) + \theta'(n + K) = \varphi(n)$. Thus, θ lifts φ. Therefore, X is M-injective. □

Proposition 2.1.3. *Consider R be a ring and M an R-module expressed as the the sum $\Sigma_{\lambda \in \Lambda} M_\lambda$ of its submodules M_λ ($\lambda \in \Lambda$). Then an R-module X is M-injective if and only if X is M_λ-injective for every $\lambda \in \Lambda$.*

Proof. Suppose that X is M-injective. Then X is also M_λ-injective for every λ in Λ, by Proposition 2.1.2.

Conversely, assume that X is M_λ-injective for all λ in Λ. Let N be a submodule of M and φ be an element of $Hom_R(N, X)$. Let $\underline{S}$ denote the collection of pairs (L, α), where $N \subseteq L \leq M$, $\alpha \in Hom_R(L, X)$, and $\alpha|N = \varphi$. If $(L, \alpha), (L', \alpha') \in \underline{S}$, then we define $(L, \alpha) \leq (L', \alpha')$ if $L \subseteq L'$ and $\alpha'|L = \alpha$. A non-empty collection of elements $(L\omega, \alpha\omega)$ in $\underline{S}$, where ω belongs to some index set Ω, is called a chain if for all ω, ω' in Ω, either $(L_\omega, \alpha_\omega) \leq (L_{\omega'}, \alpha_{\omega'})$, or $(L_{\omega'}, \alpha_{\omega'}) \leq (L_\omega, \alpha_\omega)$. Let $\{(L_\omega, \alpha_\omega) : \omega \in \Omega\}$ be a chain in $\underline{S}$. Let $L = \bigcup_{\omega \in \Omega} L_\omega$. Then $L \leq M$ and clearly $N \subseteq L$. Define $\alpha : L \to X$ by $\alpha(a) = \alpha_\omega(a)$, where $a \in L_\omega$. It can be easily verified that $(L, \alpha) \in \underline{S}$. By Zorn's Lemma, $\underline{S}$ contains a maximal member (K, θ).

Next we prove that $K = M$. Assume $\lambda \in \Lambda$. Let $P = M_\lambda \cap K$ and $\beta = \theta|_P$. Note that $\beta \in Hom_R(P, X)$ and, since X is $M\lambda$-injective, β can be lifted to a homomorphism $\gamma : M_\lambda \to X$. Define $\theta' : M_\lambda + K \to X$ by

$$\theta'(m+k) = \gamma(m) + \theta(k), \text{ where } m \in M_\lambda,\ k \in K.$$

Suppose $m \in M_\lambda$, $k \in K$, and $m + k = 0$. Then $m = -k \in M_\lambda \cap K = P$, so

$$\gamma(m) = \beta(m) = \theta(m) = \theta(-k) = -\theta(k),$$

and hence $\theta'(m+k) = 0$. Therefore, θ' is well-defined, and it can be verified that $\theta' \in Hom_R(M_\lambda + K, X)$ and $\theta'|_K = \theta$. Since $M\lambda + K = K$, we have $M_\lambda \subseteq K$. Thus $M \subseteq K$ and hence $M = K$. It follows that X is M-injective. □

Corollary 2.1.4. *An R-module X is said to be M-injective if and only if X is (mR)-injective for every element m in the module M.*

Proof. Evident from Proposition 2.1.3. □

Proposition 2.1.5. *Let R be a ring and M an R-module. An R-module X is M-injective if and only if $\varphi(M) \subseteq X$ for every $\varphi \in Hom_R(E(M), E(X))$.*

Proof. Assume first that for every φ in $Hom_R(E(M), E(X))$, we have $\varphi(M) \subseteq X$. Let N be a submodule of M and α be an element of $Hom_R(N, X)$. Consider the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{\iota} & E(M) \\
 & & \downarrow{\scriptstyle \alpha} & & & & \\
 & & X & & & & \\
 & & \downarrow{\scriptstyle \iota} & & & & \\
 & & E(X) & & & &
\end{array}$$

where each ι is an inclusion mapping. Since $E(X)$ is an injective module, there exists β in $Hom_R(E(M), E(X))$ such that β lifts α. By the assumption, $\beta(M) \subseteq X$, and thus the restriction γ of β to M is a homomorphism from M to X that lifts α. Hence, X is M-injective. Conversely, suppose that X is M-injective. Let φ be an element of $Hom_R(E(M), E(X))$. Define N as the set $\{m \in M : \varphi(m) \in X\}$. It is clear that N is a submodule of M. Let φ' denote the restriction of φ to N. By the M-injectivity of X, there exists θ in $Hom_R(M, X)$ that lifts φ'. Thus,

$$\theta(n) = \varphi(n) \text{ for all } n \text{ in } N.$$

Let φ'' denote the restriction of φ to M. Then $\lambda = \theta - \varphi''$ is an element of $Hom_R(M, E(X))$ and $\lambda(N) = 0$. Assume that $\lambda(M) \neq 0$. Then $X \cap \lambda(M) \neq 0$. Let m be a nonzero element of M such that $\lambda(m) \in X$. Then $\varphi(m) = \varphi''(m) = \theta(m) - \lambda(m) \in X$, which implies $m \in N$. However, in this case, $\lambda(m) = 0$, which is a contradiction. Hence, we have $\lambda(M) = 0$, and therefore $\varphi(m) = \theta(m) \in X$ for all m in M. Thus, $\varphi(M) \subseteq X$. This completes the proof. □

Proposition 2.1.6. *Assume R be a ring and M is an R-module. Let Y be a complement submodule of an M-injective R-module X. Then Y is M-injective.*

Proof. Without loss of generality, let's assume that the injective envelope $E(X)$ contains submodules X, Y, and $E(Y)$. Since $X \cap E(Y)$ is an essential extension of Y, we have $Y = X \cap E(Y)$. Consider $\varphi \in Hom_R(E(M), E(Y))$. Since $E(Y) \leq E(X)$, we can conclude that $\varphi \in Hom_R(E(M), E(X))$. According to Proposition 2.1.5., we have $\varphi(M) \subseteq X$. Hence, $\varphi(M) \subseteq X \cap E(Y) = Y$. Consequently, $\varphi(M) \subseteq Y$ holds for any $\varphi \in Hom_R(E(M), E(Y))$. Based on Proposition 2.1.5., we can assert that Y is M-injective. □

Lemma 2.1.7. *Assume $K \subseteq N$ be submodules of an R-module M such that N/K is M-injective. Then N/K is a direct summand of M/K.*

Proof. Based on Proposition 2.1.2., we conclude that N/K is injective with respect to the module M/K. The identity mapping $\iota : N/K \to N/K$ can be extend to a homomorphism $\theta : M/K \to N/K$. It can be verified that $M/K = (N/K) \oplus (ker\theta)$. □

Corollary 2.1.8. *Assume R be a ring and M a quasi-injective R-module. Then any complement in M is a direct summand of M.*

Proof. Using Proposition 2.1.6 and Lemma 2.1.7.

□

Proposition 2.1.9. *The following statements are equivalent for an R-module M.*

(i) M is semisimple. (i.e. M = Soc M)

(ii) Every R-module is M-injective.

(iii) Every submodule of M is M-injective.

(iv) Every submodule of a M-injective R-module is M-injective.

Proof. (i) $\implies$ (ii) Consider an arbitrary R-module X. Given a submodule N of M and a homomorphism φ from N to X, we can find a submodule N' of M such that M is the direct sum of N and N'. We define a mapping θ from M to X as follows:

$$\theta(n + n') = \varphi(n) \quad (n \in N, n' \in N').$$

It can be shown that θ is a homomorphism from M to X, and it lifts φ. Therefore, X is M-injective.

(ii) $\implies$ (iii), (iv) Clear.

(iv) $\implies$ (iii) Assuming condition (iv) is satisfied, we consider $E(M)$ as an injective envelope of M. According to (iv), every submodule of $E(M)$, and consequently every submodule of M, possesses the property of being M-injective.

(iii) $\implies$ (i) From Lemma 2.1.7. □

Lemma 2.1.10. *Let R be a ring and M an R-module.*

(i) Any direct summand of an M-injective R-module is M injective.

(ii) Let X_λ $(\lambda \in \Lambda)$ be any non-empty collection of M-injective R-modules. Then $X = \Pi_\lambda X_\lambda$ is M-injective.

Proof. Obvious. □

Let P be a module property, such as "Noetherian," "Artinian," etc. A right R-module M is referred to as being locally P if every submodule of M that is generated by a finite set of elements exhibits the property P.

Theorem 2.1.11. *An R-module M is locally Noetherian if and only if the direct sum of any family of M-injective modules is M-injective.*

Proof. Suppose M satisfies the local Noetherian property. Consider any non-empty collection X_λ $(\lambda \in \Lambda)$ of M-injective R-modules, and let $X = \bigoplus_\lambda X_\lambda$. Take any finitely generated submodule N of M, and let K be a submodule of N. Suppose $\varphi \in \mathrm{Hom}R(K, X)$. Since K is finitely generated, the image $\mathrm{im}(\varphi)$ is also finitely generated and is contained in a submodule X' of X that is the direct sum of a finite number of the submodules $X\lambda$. By Lemma 2.1.10. (ii), X' is M-injective, and thus φ can be lifted to a homomorphism

$\theta : M \to X'$. It follows that $\theta \in \text{Hom}_R(N, X)$, so X is N-injective for every finitely generated submodule N of M. Moreover, for every element $m \in M$, X is mR-injective. By Corollary 2.1.4., X is M-injective.

Conversely, assume that the direct sum of any family of M-injective modules is M-injective. Let L be a finitely generated submodule of M, and consider an ascending chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots$ of submodules of L. Define N as the submodule $\bigcup_i N_i$ of M, and let

$$X = E(M/N_1) \oplus E(M/N_2) \oplus E(M/N_3) \oplus \cdots.$$

Define the mapping $\varphi : N \to X$

$$\varphi(n) = (n + N_1,\, n + N_2,\, n + N_3, \ldots).$$

This mapping is well-defined because for any $n \in N$, there exists $k \geq 1$ such that $n \in N_k$. Since X is M-injective, φ can be lifted to a homomorphism $\theta \in \text{Hom}R(M, X)$. As L is finitely generated, $\theta(L)$ is finitely generated and is contained in $E(M/N_1) \oplus \cdots \oplus E(M/N_t)$ for some positive integer t. For any $n \in N$, we have

$$(n + N_1,\, n + N_2,\, n + N_3, \ldots) = \varphi(n) = \theta(n) = (e_1, \ldots, e_t,\, 0,\, 0, \ldots),$$

where $e_i \in E(M/N_i)$ for $1 \leq i \leq t$. Thus, $n \in N_{t+1}$. It follows that $N = N_{t+1}$, and hence $N_{t+1} = N_{t+2} = N_{t+3} = \ldots$. Therefore, L is Noetherian, implying that M is locally Noetherian. □

2.2 Lifting Submodules

On using lifting homomorphisms from submodules we build up class of lifting submodules. To this end this section is devoted to lifting submodules and their basic properties.

Definition 2.2.1. Let M and X be right R-modules. We are interested in the class of submodules of M for which X is relative injective with respect to each member of that class. A submodule N of M is called a *lifting submodule* for X in M if, for any $\varphi \in \text{Hom}_R(N, X)$, there exists $\theta \in \text{Hom}_R(M, X)$ such that $\varphi = \theta|_N$. In other words, any R-homomorphism φ from N to X can be extended or lifted to an R-homomorphism θ from M to X that restricts to φ on N. So we set

$$Lift_X(M) = \{N : N \leq M \text{ and } N \text{ is a lifting submodule for } X \text{ in } M\}.$$

Let's examine properties of this new class of submodules. First, observe that $0 \in \mathrm{Lift}_X(M)$, meaning that the zero submodule is always in the lifting submodule class for X in M. Additionally, $M \in \mathrm{Lift}_X(M)$, indicating that the entire module M itself is also in the lifting submodule class for X in M. More generally, we have

Lemma 2.2.2. *Assume N be a direct summand of the module M. Then $N \in Lift_X(M)$.*

Proof. Let $M = N \oplus N'$ be a decomposition of M into submodules, where N' is a submodule of M. Given $\varphi \in \mathrm{Hom}_R(N, X)$, we define $\theta : M \to X$ as $\theta(n + n') = \varphi(n)$ for $n \in N$ and $n' \in N'$. It is straightforward to verify that $\theta \in \mathrm{Hom}_R(M, X)$ and $\varphi = \theta|_N$. □

Lemma 2.2.3. *The following statements are equivalent.*

(i) X is M-injective.

(ii) Every submodule of M is a lifting submodule for X in M.

(iii) Every essential submodule of M is a lifting submodule for X in M.

Proof. The implications (i) $\implies$ (ii) and (ii) $\implies$ (iii) are obvious.

(iii) $\implies$ (i) Consider a submodule N of M. Let N' be a complement of N in M (as stated in Proposition 1.1.8.). We have $N \oplus N' \leq_e M$. For any $\varphi \in \mathrm{Hom}_R(N, X)$, Lemma 2.2.2. guarantees the existence of $\theta \in \mathrm{Hom}_R(N \oplus N', X)$ such that $\theta|_N = \varphi$. By property (iii), there exists $\chi \in \mathrm{Hom}R(M, X)$ such that $\chi|N \oplus N' = \theta$. Consequently, $\chi|_N = \varphi$. Therefore, X is M-injective. □

Lemma 2.2.4. *Let K, N be submodules of M such that $K \leq N$. Then*

(i) $K \in Lift_X(N)$, $N \in Lift_X(M)$ implies that $K \in Lift_X(M)$.

(ii) $K \in Lift_X(M)$ implies that $K \in Lift_X(N)$.

(iii) $N \in Lift_X(M)$ implies that $N/K \in Lift_X(M/K)$.

(iv) $K \in Lift_X(M)$, $N/K \in Lift_X(M/K)$ implies that $N \in Lift_X(M)$.

Proof. (i) and (ii) are obvious.

(iii) Consider $\varphi \in \mathrm{Hom}_R(N/K, X)$. Let $\pi : N \to N/K$ denote the canonical projection. Then $\varphi\pi : N \to X$ is a homomorphism. Since $N \in \mathrm{Lift}_X(M)$, there exists

$\theta \in \mathrm{Hom}_R(M, X)$ such that $\theta(n) = \varphi\pi(n) = \varphi(n+K)$ for all $n \in N$. Define $\overline{\theta} : M/K \to X$ by $\overline{\theta}(m + K) = \theta(m)$ for $m \in M$.

Suppose $m + K = m' + K$, where $m, m' \in M$. Then $m - m' \in K$, and hence $\varphi\pi(m - m') = 0$. Thus, $\theta(m - m') = 0$, implying $\theta(m) = \theta(m')$. Consequently, $\overline{\theta}$ is well-defined. It is clear that $\overline{\theta} \in \mathrm{Hom}_R(M/K, X)$. For any $n \in N$, $\overline{\theta}(n+K) = \theta(n) = \varphi(n+K)$. Therefore, $N/K \in \mathrm{Lift}_X(M/K)$.

(iv) Consider $\varphi \in \mathrm{Hom}_R(N, X)$. Then $\varphi|_K \in \mathrm{Hom}_R(K, X)$. There exists $\theta \in \mathrm{Hom}_R(M, X)$ such that $\varphi|_K = \theta|_K$. Define $\chi : N/K \to X$ by

$$\chi(n + K) = \varphi(n) - \theta(n) \text{ for } n \in N.$$

It can be verified that χ is well-defined and a homomorphism. There exists $\psi \in \mathrm{Hom}R(M/K, X)$ such that $\chi|N/K = \psi$. Let $\pi : M \to M/K$ denote the canonical projection. Let $\alpha = \psi\pi + \theta \in \mathrm{Hom}_R(M, X)$. For any $n \in N$, we have

$$\alpha(n) = \psi\pi(n) + \theta(n) = \psi(n + K) + \theta(n) = \chi(n + K) + \theta(n) = \varphi(n).$$

Therefore, $\alpha|_N = \varphi$, which implies that $N \in \mathrm{Lift}_X(M)$. □

Corollary 2.2.5. *For any $N \in Lift_X(M)$, $Lift_X(N) = \{K \leq N : K \in Lift_X(M).\}$*

Proof. If $K \in \mathrm{Lift}_X(N)$, then $K \leq N$ and, according to Lemma 2.2.4. (i), $K \in \mathrm{Lift}_X(M)$. Therefore, $\mathrm{Lift}_X(N) \subseteq K \leq N : K \in \mathrm{Lift}_X(M)$. Conversely, suppose $K \leq N$ and $K \in \mathrm{Lift}_X(M)$. Using Lemma 2.2.4. (ii), we conclude that $K \in \mathrm{Lift}_X(N)$. □

Let K and N be submodules of the module M such that $K \leq N$. However, it is important to note that the inclusion of K in $Lift_X(M)$ does not necessarily imply the inclusion of N in $Lift_X(M)$. To illustrate this point, consider the following example.

Example 2.2.6. Consider a non-injective right R-module X. There exists $E \leq_e R_R$ such that $E \notin Lift_X(R_R)$. Let $M = R \oplus R$, $K = R \oplus 0$, $N = R \oplus E$. According to Lemma 2.2.2., K is in $Lift_X(M)$ since it can be lifted to M. However, by Lemma 2.2.4., N is not in $Lift_X(M)$ since it cannot be lifted to M.

Proposition 2.2.7. *Assume $N, K \leq M$ be submodules such that $N + K$ and $N \cap K$ are both in $Lift_X(M)$. Then both N and K belong to $Lift_X(M)$.*

Proof. Let $\varphi \in Hom_R(N, X)$. The restriction $\varphi|N \cap K \in Hom_R(N \cap K, X)$. According to the given condition, there exists $\theta_1 \in Hom_R(M, X)$ such that $\theta_1|N \cap K = \varphi|_{N \cap K}$. Define $\chi : N + K \to X$ as

$$\chi(n + k) = \varphi(n) + \theta_1(k) \text{ for } n \in N \text{ and } k \in K.$$

Suppose $n, n' \in N$, $k, k' \in K$, and $n + k = n' + k'$. Then $n - n' = k' - k$, implying $k' - k \in N \cap K$. Consequently, $\theta_1(k') - \theta_1(k) = \theta_1(k' - k) = \varphi(k' - k) = \varphi(n - n') = \varphi(n) - \varphi(n')$. This implies $\varphi(n) + \theta_1(k) = \varphi(n') + \theta_1(k')$. Hence, χ is well-defined. Clearly, $\chi \in Hom_R(N + K, X)$. By the given hypothesis, there exists $\theta \in Hom_R(M, X)$ such that $\theta|_{N+K} = \chi$. For any $n \in N$, we have

$$\theta(n) = \chi(n) = \varphi(n).$$

Thus, $\theta|_N = \varphi$. It follows that $N \in Lift_X(M)$. Similarly, we can show that $K \in Lift_X(M)$. □

Corollary 2.2.8. *Let K, N be submodules of M.*

(i) If $N \cap K = 0$ and $N \oplus K \in Lift_X(M)$, then $N, K \in Lift_X(M)$.

(ii) If $N + K = M$ and $N \cap K \in Lift_X(M)$, then $N, K \in Lift_X(M)$.

Proof. This follows directly from Proposition 2.2.7. □

Lemma 2.2.9. *Consider $K \in Lift_X(M)$, $N \leq M$. Suppose $N \cap K \in Lift_X(K)$ and $(N + K)/K \in Lift_X(M/K)$. Then $N \in Lift_X(M)$.*

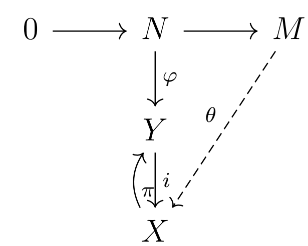
Proof. Using Lemma 2.2.4. (i) and (iv), we can conclude that both the intersection $N \cap K$ and the sum $N + K$ are elements of $Lift_X(M)$. Applying Proposition 2.2.7. yields the desired result. □

Corollary 2.2.10. *Consider $K \leq M$. Then X is M-injective if and only if (i) X is K-injective, (ii) X is (M/K)-injective, and (iii) $K \in Lift_X(M)$.*

Proof. By utilizing Lemma 2.2.4. and Lemma 2.2.9., we can establish the given result. □

Theorem 2.2.11. *Let $X = \Pi_{\lambda \in \Lambda} X_\lambda$. Then $Lift_X(M) = \bigcap_{\lambda \in \Lambda} Lift_{X_\lambda}(M)$, for any module M.*

Proof. Consider an arbitrary index λ in the set Λ and let $Y = X_\lambda$. Take N as an element of $Lift_X(M)$. Let φ belong to $Hom_R(N, Y)$. We define the inclusion mapping $i : Y \to X$ and the canonical projection $\pi : X \to Y$. It follows that $i\varphi$ is an element of $Hom_R(N, X)$. Based on the given hypothesis, we can find θ in $Hom_R(M, X)$ such that $\theta|_N = i\varphi$



We observe that $\pi\theta$ is an element of $Hom_R(M, Y)$. Furthermore, for any n in N, we have $\pi\theta(n) = \pi i\varphi(n) = \varphi(n)$. This implies that $\varphi = \pi\theta|_N$. Consequently, we conclude that N belongs to $Lift_Y(M)$. Therefore, we have shown that $Lift_X(M) \subseteq Lift_Y(M)$. As a result, we obtain $Lift_X(M) \subseteq \bigcap_{\lambda \in \Lambda} Lift_{X_\lambda}(M)$.

Conversely K be an element of $\bigcap_{\lambda \in \Lambda} Lift_{X_\lambda}(M)$. Consider α as an element of $Hom_R(K, X)$. For every λ in Λ, we have the canonical projection $\pi_\lambda : X \to X_\lambda$. It follows that $\pi_\lambda\alpha$ is an element of $Hom_R(K, X_\lambda)$ for each λ in Λ. By the assumption, for each λ in Λ, there exists β_λ in $Hom_R(M, X_\lambda)$ such that $\beta_\lambda(k) = \pi_\lambda\alpha(k)$ for all k in K. Now we define $\beta : M \to X$ as

$$\beta(m) = \{\beta_\lambda(m)\}_{\lambda \in \Lambda} \text{ for all } m \text{ in } M.$$

For any k in K, we have $\beta(k) = \alpha(k)$. Consequently, we conclude that K belongs to $Lift_X(M)$. □

Corollary 2.2.12. *Assume $X = \Pi_{\lambda \in \Lambda} X_\lambda$. Then X is M-injective if and only if X_λ is M-injective for all $\lambda \in \Lambda$.*

Proof. Using Lemma 2.2.3. and Theorem 2.2.11., we can conclude. □

Proposition 2.2.13. *Consider the following conditions for a any submodule N of a module M:*

(i) $\theta(M) \leq X$ for any $\theta \in Hom_R(M,\ E(X))$ with $\theta(N) \leq X$.

(ii) $N \in Lift_X(M)$.

(iii) $\theta(M) \leq X$ for any $\theta \in Hom_R(M,\ E(X))$ with $\theta(N) \leq X$ and $\theta^{-1}(X) \in Lift_X(M)$.

Then (i) $\implies$ (ii) $\implies$ (iii).

Proof. (i) $\implies$ (ii) For any φ in $Hom_R(N, X)$, there exists θ in $Hom_R(M, E(X))$ such that $\theta|_N = i\varphi$, where $i : X \to E(X)$ is the inclusion map. This implies that $\theta(N) \leq X$. By the hypothesis, we have $\theta(M) \leq X$, and thus $\theta \in Hom_R(M, X)$. Consequently, N belongs to $Lift_X(M)$.

(ii) $\implies$ (iii) Suppose (ii) holds.Let θ be an element of $Hom_R(M, E(X))$ such that $N \leq \theta^{-1}(X) \in Lift_X(M)$. There exists θ' in $Hom_R(M, X)$ such that $\theta'(k) = \theta(k)$ for k in $\theta^{-1}(X)$. Consider the function $\theta - \theta' : M \to E(X)$. If $(\theta - \theta')(M) \neq 0$, then $(\theta - \theta')(M) \cap X \neq 0$, which implies the existence of a non-zero element x in X and an element m in M such that $x = (\theta - \theta')(m) = \theta(m) - \theta'(m)$. Therefore, $\theta(m) = x + \theta'(m) \in X$, and hence $m \in \theta^{-1}(X)$. In this case, $\theta'(m) = \theta(m)$, leading to a contradiction since $x = 0$. We conclude that $(\theta - \theta')(M) = 0$, which implies $\theta(M) = \theta'(M) \leq X$. □

We provide two examples to demonstrate that the implication from (ii) to (i) in Proposition 2.2.13 does not hold.

Example 2.2.14. Consider R be the ring $\mathbb{Z}$ of integers. Take $X = M = \mathbb{Z}$ and $N = 0$ in Proposition 2.2.13. Then $N \in Lift_X(M)$. Let $0 \neq m \in \mathbb{Z}$ and define $\theta : \mathbb{Z} \to \mathbb{Q}$ by $\theta(n) = n/m$ ($n \in \mathbb{Z}$). Then $\theta(N) \leq X$, but $\theta(M) \nleq X$. Hence (ii) $\nRightarrow$ (i).

Example 2.2.15. Consider $R = \mathbb{Z}$, $X = \mathbb{Z}/p\mathbb{Z}$, where p be any prime integer, $M = \mathbb{Q}$ and $N = \mathbb{Z}$ in Proposition 2.2.13.

Assume $N \in Lift_X(M)$. Let $\pi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ denote the canonical epimorphism, defined by

$$\pi(n) = n + \mathbb{Z}/p\mathbb{Z}\ (n \in \mathbb{Z}).$$

Then there exists a homomorphism $\alpha : \mathbb{Q} \to \mathbb{Z}/p\mathbb{Z}$ such that $\alpha|_{\mathbb{Z}} = \pi$:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} \quad \textit{exact} \\ & & \downarrow \pi & \swarrow \alpha & \\ & & \mathbb{Z}/p\mathbb{Z} & & \end{array}$$

Now $\alpha(1/p) = x + \mathbb{Z}/p\mathbb{Z}$ for some $x \in \mathbb{Z}$. Thus

$$p\alpha(1/p) = \alpha(1) = \pi(1) = 1 + \mathbb{Z}/p\mathbb{Z}$$

It follows that $px + \mathbb{Z}/p\mathbb{Z} = 1 + \mathbb{Z}/p\mathbb{Z}$, and hence $1 \equiv 0 \pmod p$, a contradiction. Thus $N \notin Lift_X(M)$.

Next, we investigate a set of submodules within a module to derive general conclusions regarding specific types of modules based on the category of lifting submodules.

Theorem 2.2.16. *The following assertions are equivalent for a non-empty collection* $\underline{M}$ *consisting of submodules of M.*

(i) If $N \in \underline{M}$*, then* $N \leq_d M$.

(ii) $\underline{M} \subseteq Lift_X(M)$ *for all right R-modules X.*

(iii) $\underline{M} \subseteq Lift_X(M)$ *for all* $X \in M$.

Proof. (i) $\Longrightarrow$ (ii) According to Lemma 2.2.2.

(ii) $\Longrightarrow$ (iii) Clear.

(iii) $\Longrightarrow$ (i)Let B be an element of the collection $\underline{\text{M}}$. Considering the identity mapping $i_B : B \to B$, and using the fact that B belongs to $Lift_B(M)$ according to (iii), we conclude the existence of θ in $Hom_R(M, B)$ such that $\theta(m) = m$ for all m in M. It can be readily verified that M can be expressed as the direct sum of B and the kernel of θ. Therefore, we have $B \leq_d M$. □

Corollary 2.2.17. *The following statements are equivalent for a module M.*

(i) M is semisimple.

(ii) Every right R-module X is M-injective.

(iii) Every submodule of M is M-injective.

Proof. Applying Theorem 2.2.16. to the collection $\underline{\mathrm{M}} = \{N : N \leq M\}$, and utilizing Lemma 2.2.3. □

Corollary 2.2.18. *The following statements are equivalent for a module M*

(i) If $N \leq_c M$, then $N \leq_d M$.

(ii) If $N \leq_c M$, then $N \in Lift_X(M)$ for all right R-modules X

(iii) If $N \leq_c M$, then $N \in Lift_X(M)$ for all $X \leq_c M$.

Proof. Applying Theorem 2.2.16. to the collection $\underline{\mathrm{M}} = \{N : N \leq_c M\}$. □

The conditions presented in Corollary 2.2.18. establish that if N is a complement submodule in M, then N belongs to $\mathrm{Lift}_M(M)$. However, it is important to note that the converse statement, namely, N belonging to $\mathrm{Lift}_M(M)$ implies that N is a direct summand of M, is not always true, as exemplified by the following example.

Example 2.2.19. Consider p be any prime integer and let R be the local ring $\mathbb{Z}_{(p)}$. Let M denote the $\mathbb{Z}$-module $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$. Then

(i) M is an R-module.

(ii) $K \leq_c M$ if and only if $K \leq_d M$ or $K = R(1 + p\mathbb{Z}, q)$ for some non-zero element q in $\mathbb{Q}$.

(iii) There exists a complement submodule in M which is not a direct summand of M.

(iv) If $K \leq_c M$ then $K \in Lift_M(M)$.

Proof. (i) Consider the modules $M_1 = (\mathbb{Z}/p\mathbb{Z}) \oplus 0$ and $M_2 = 0 \oplus \mathbb{Q}$, which form a direct sum $M = M_1 \oplus M_2$. The ring R is defined as the subring of $\mathbb{Q}$ that consists of all rational numbers s/t, where $s, t \in \mathbb{Z}$, $t \neq 0$, and t is coprime to p. It is worth noting that for any element $m \in M$ and any $s, t \in \mathbb{Z}$ such that p does not divide t, there exists a unique element m' in M such that $m't = ms$. We can represent this element as $m(s/t)$. Thus, M can be viewed as an R-module.

(ii) Let q be a rational number and $K = (1 + p\mathbb{Z}, q)R$. We first show that K is a complement submodule of $M_{\mathbb{Z}}$. It can be observed that K is a uniform submodule of M. Suppose there exists a submodule N of M such that K is a complement in N. Let x be an element of N. Then the module $U = x\mathbb{Z} + (1 + p\mathbb{Z}, q)\mathbb{Z}$ is a finitely generated uniform $\mathbb{Z}$-module and is therefore cyclic. Assuming $U = (m + p\mathbb{Z}, b)\mathbb{Z}$, where $m \in \mathbb{Z}$ and $b \in \mathbb{Q}$, there exists an integer n such that $(1 + p\mathbb{Z}, q) = (m + p\mathbb{Z}, b)n$. Noting that $1 - mn \in p\mathbb{Z}$, we conclude that n is coprime to p, and thus $(m + p\mathbb{Z}, b) \in (1 + p\mathbb{Z}, q)R = K$. Therefore, $x \in K$, implying $N = K$. Thus, K is a complement submodule in M.

Now let L be a complement submodule in M and assume $L \neq M$. As M has uniform dimension 2, L is also uniform according to Proposition 1.1.27. Our aim is to show that L is an R-submodule of M. Let $L' = m \in M : mt \in L$ for some $t \in \mathbb{Z}, t$ coprime to p. It can be observed that L' is a submodule of M containing L, specifically $L' = LR$. If $0 \neq m \in L'$, then $mt \in L$ for some $t \in \mathbb{Z}$ coprime to p, implying $mt \neq 0$. Consequently, we have $L \leq_e L'$. Thus, $L = L'$, and hence L is an R-submodule of M.

Next, we prove that L can only be one of the following submodules: 0, M, M_1, M_2, or $(1+p\mathbb{Z}, q)R$ for some $q \in \mathbb{Q}$. Suppose L is not equal to 0, M, M_1, or M_2. Since M_1 and M_2 are both uniform submodules, L cannot be contained in either M_1 or M_2. Thus, there exists $(c+p\mathbb{Z}, d) \in L$ for some $c \in \mathbb{Z}$ coprime to p and $0 \neq d \in \mathbb{Q}$. Without loss of generality, we can assume $c = 1$. Since L is an R-submodule of M, we have $(1 + p\mathbb{Z}, d)R \subseteq L$. However, $(1 + p\mathbb{Z}, d)R \leq_c M$, implying $L = (1 + p\mathbb{Z}, d)R$. This completes the proof of (ii).

(iii) Note that $K = (1 + p\mathbb{Z}, 1)R$ is a complement submodule in M. Suppose K is a direct summand of M. Then $M = K \oplus L$ for some submodule L of M. Let $(m+p\mathbb{Z}, b) \in L$, where $m \in \mathbb{Z}$ and $b = \frac{m}{n} \in \mathbb{Q}$. It follows that $(m + p\mathbb{Z}, b)p = (0 + p\mathbb{Z}, pm/n) \in L$. Therefore, $(0 + p\mathbb{Z}, pm/n)n = (0 + p\mathbb{Z}, pm) = (0 + p\mathbb{Z}, 0)$, since $K \cap L = 0$. Consequently, $npb = pm = 0$, which implies $b = 0$. Hence, for $x \in L$, we have $x = (y + p\mathbb{Z}, 0)$ where $y \in \mathbb{Z}$. Thus, $L \leq M_1$, which is a simple submodule, and therefore $L = M_1$. Thus, $M = K \oplus M_1$. Hence, we have

$$K \cong M/M_1 \cong M_1 \cong \mathbb{Q} \cong \mathbb{Q}p.$$

However, this leads to the existence of an element $(c + p\mathbb{Z}, d) \in K$ such that $(1 + p\mathbb{Z}, 1) =$

$(c+p\mathbb{Z},d)p=(0+p\mathbb{Z},pd)$, where $c\in\mathbb{Z}$ and $d\in\mathbb{Q}$. Consequently, 1 belongs to $p\mathbb{Z}$, which is a contradiction. Therefore, K is not a direct summand of M.

(iv) To establish that if K is a complement submodule of M then K belongs to $Lift_M(M)$, it suffices to demonstrate that for any non-zero $q\in\mathbb{Q}$ and any homomorphism $\varphi:(1+p\mathbb{Z},q)R\to M$, there exists a lift θ of φ as an endomorphism of M. Let $K=(1+p\mathbb{Z},q)R$. Assume that $\varphi(1+p\mathbb{Z},q)=(m+p\mathbb{Z},b)$ for some $m\in\mathbb{Z}$ and $b\in\mathbb{Q}$. Define the mapping $\theta:M\to M$ by

$$\theta(c+p\mathbb{Z},d)=(ca+p\mathbb{Z},\tfrac{db}{q})\quad \text{for } c\in\mathbb{Z} \text{ and } d\in\mathbb{Q}.$$

It can be readily verified that θ is well-defined. Moreover, $\theta:M\to M$ is a homomorphism and φ is the restriction of θ to K. Thus, K belongs to $Lift_M(M)$. □

2.3 Ejectivity

On using essentiality of the submodule in the definition of M-injectivity, recently M-ejectivity was defined as a generalization of relative injectivity concept and studied in details in [1], [13].

Definition 2.3.1. Assume M and X be right R-modules. We define X to be *M-ejective* if, for every submodule $K\leq M$ and every homomorphism $\varphi:K\to X$, there exist a homomorphism $\theta:M\to X$ and an essential submodule $E\leq_e K$ such that $\theta(x)=\varphi(x)$ for all $x\in E$. In other words, the restriction of θ to E is equal to the restriction of φ to E. It is clear that if X is M-injective, then X is also M-ejective. If X is M-ejective for all right R-modules M, then we say that X is ejective.

Proposition 2.3.2. *Let M and X be R-modules. Then X is M-ejective if and only if there exists $E\leq_e M$ such that X is E-ejective and for any R-homomorphism $\varphi:E\to X$ there exists $K\leq_e E$ and an R-homomorphism $\theta:M\to X$ such that $\theta|_K=\varphi|_K$.*

Proof. ($\Longrightarrow$) For this direction, consider E to be equal to M.

($\Longleftarrow$) Consider $B\leq M$ and $\varphi:B\to X$ as an R-homomorphism. Let $B_1=B\cap E$. As X is E-ejective, there exist $B_2\leq_e B_1$ and $\varphi_1:E\to X$ such that $\varphi|B_2=\varphi_1|B_2$. Moreover, there exists $K \leq_e E$ and an R-homomorphism $\varphi_2 : M \to X$ such that $\varphi_2|K = \varphi_1|K$. Let $B_3 = B_2 \cap K$. Then $B_3 \leq_e B$ and $\varphi|B_3 = \varphi_2|B_3$. Hence, we can conclude that X is M-ejective. □

Theorem 2.3.3. *Let M_1 and M_2 be modules such that $M = M_1 \oplus M_2$. Then M_1 is M_2-ejective if and only if for every $K \leq M$ such that $K \cap M_1 = 0$, there exists $M_3 \leq M$ such that $M = M_1 \oplus M_3$ and $K \cap M_3 \leq_e K$.*

Proof. ($\Longrightarrow$) Assuming that M_1 is M_2-ejective, let $\pi_i : M \to M_i$ for $i = 1, 2$ denote the canonical projections. Consider $K \leq M$ such that $K \cap M_1 = 0$. Observe that $\pi_2 : K \to M_2$ is an injection. Let $\overline{K} = \pi_2(K)$. Since M_1 is M_2-ejective, there exists $E \leq_e \overline{K}$ and a homomorphism $\theta : M_2 \to M_1$ such that $\theta|_E = \pi_1\pi_2^{-1}|_E$. Define $M_3 = \theta(y) + y : y \in M_2$. For $m \in M$, there exist $m_i \in M_i$ such that $m = m_1 + m_2 = (m_1 - \theta(m_2)) + (\theta(m_2) + m_2) \in M_1 + M_3$. Suppose $y \in M_1 \cap M_3$. Then there exist $y_i \in M_i$ such that $y = y_1 = \theta(y_2) + y_2$. Hence, $y_1 - \theta(y_2) = y_2 \in M_1 \cap M_2 = 0$. Thus, $M_1 \cap M_3 = 0$. Therefore, $M = M_1 \oplus M_3$. Now, let $0 \neq k \in K$. Then $k = \pi_1(k) + \pi_2(k)$. Recall that $\pi_2(k) \neq 0$ because $K \cap M_1 = 0$. So, there exists $r \in R$ such that $0 \neq \pi_2(k)r = \pi_2(kr) \in E$. Hence, $0 \neq kr = \pi_1(kr) + \pi_2(kr)$. However, $\pi_1(kr) = \theta(\pi_2(kr))$, so $0 \neq kr = \theta(\pi_2(kr)) + \pi_2(kr) \in K \cap M_3$. Therefore, $K \cap M_3 \leq_e K$.

($\Longleftarrow$) Assuming that for every $K \leq M$ such that $K \cap M_1 = 0$, there exists $M_3 \leq M$ such that $M = M_1 \oplus M_3$ and $K \cap M_3 \leq K$, let $L \leq M_2$ and $\varphi : L \to M_1$ be a homomorphism. Define $H = -\varphi(x) + x : x \in L$. Then $H \leq M$ and $H \cap M_1 = 0$. By the assumption, there exists $H' \leq M$ such that $M = M_1 \oplus H'$ and $H' \cap H \leq_e H$. Let $K = H' \cap H \cap L$. There exists $C \leq L$ such that $K \cap C = 0$ and $K \oplus C \leq_e L$. Let $B = b \in C : -\varphi(b) + b \in H'$. Note that $B \leq C$. We claim that $B \leq_e C$. Consider $0 \neq c \in C$. Then $-\varphi(c) + c \in H$. If $-\varphi(c) + c = 0$, then $c \in M_1 \cap L = 0$, which is a contradiction. Hence, $-\varphi(c) + c \neq 0$. There exists $r \in R$ such that $0 \neq (-\varphi(c) + c)r = -\varphi(cr) + cr \in H' \cap H$. Thus, $0 \neq cr \in B$. Therefore, $B \leq_e C$.

Observe that $K \oplus B \leq_e L$. Now let $k + b \in K \oplus B$, where $k \in K$ and $b \in B$. Let $\pi : M \to M_1$ be the projection onto M_1 along H', i.e., ker$\pi = H'$. Then $\pi(k + b) = \pi(b) = \pi(\varphi(b) - \varphi(b) + b) = \pi(\varphi(b)) + \pi(-\varphi(b) + b) = \pi(\varphi(b)) = \varphi(b)$. Recall that $k \in H \cap L$.

Then there exists $y \in L$ such that $k = -\varphi(y) + y$. Hence, $\varphi(y) = y - k \in L \cap M_1 = 0$. Thus, $y = k$ and $0 = \varphi(y) = \varphi(k)$. We conclude that $\pi(k + b) = \varphi(b) = \varphi(k + b)$, and therefore M_1 is M_2-ejective. □

Corollary 2.3.4. *Let M_1 and M_2 be modules with $Z(M_1)=0$ and $M = M_1 \oplus M_2$. Then M_1 is M_2-injective if and only if M_1 is M_2-ejective.*

Proof. ($\Longrightarrow$) Obvious.

($\Longleftarrow$) Assuming M_1 is M_2-ejective, let $K \leq M$ such that $K \cap M_1 = 0$. By Theorem 2.3.3, there exists $M_3 \leq M$ such that $M = M_1 \oplus M_3$ and $K \cap M_3 \leq_e K$. Consider $0 \neq k \in K$. Then $0 \neq k = \pi_1(k) + \pi_3(k)$, where π_1 and π_3 are the canonical projections onto M_1 and M_3, respectively. There exists $L \leq_e R$ such that $kL \subseteq K \cap M_3$. Hence, $\pi_1(k)L = 0$. Since M_1 is nonsingular, $\pi_1(k) = 0$. Thus, $K = K \cap M_3 \subseteq M_3$. Therefore, M_1 is M_2-injective.[4]. □

Proposition 2.3.5. *Let X and M be right R-modules. Then*

(i) X_R is M_R-ejective for all $M_R \in R$-Mod if and only if X_R is injective.

(ii) Assume $Z(M_R)=0$. Then M_R is R_R-ejective if and only if M_R is injective.

(iii) If M_R is R_R-ejective and $M_R = D_R \oplus Y_R$, then D_R is R_R-ejective.

(iv) If $M/Z_2(M_R)$ is R_R-ejective, then $M/Z_2(M_R)$ is injective both as an R-module and $R/Z_2(R_R)$-module. In particular, if M_R is R_R-ejective and $M = Z_2(M_R) \oplus B$, then $M/Z_2(M_R)$ is injective both as an R-module and $R/Z_2(R_R)$-module.

(v) Assume that $M_R = Z_2(M_R)$ and $Z(R_R)) = 0$. Then M_R is R_R-ejective.

(vi) Assume that $soc(M_R) \leq_e M_R$ and $soc(R_R)=0$. Then M_R is R_R-ejective.

Proof. (i) Assuming that X_R is M_R-ejective for all $M_R \in R$-Mod, let $\varphi \in \mathrm{End}(X_R)$. There exists $Y_R \leq_e X_R$ and a homomorphism $\theta : E(X) \to X$ such that $\theta(x) = \varphi(x)$ for all $x \in Y$. Since $Y_R \leq_e E(X)$, θ is a monomorphism. Hence, $\theta(E(X))$ is a direct summand of X_R. But $Y \subseteq \theta(E(X))$, we have $\theta(E(X)) = X_R$. Therefore, X_R is injective. The converse is straightforward.

(ii) By part (i), if M_R is injective, then M_R is R_R-ejective. So assume M_R is R_R-ejective. Let $I_R \leq R_R$ and $\varphi : I \to M$ be an R-homomorphism. There exists $J_R \leq_e I_R$

and $\theta : R \to M$ such that $\theta(x) = \varphi(x)$ for all $x \in J$. Let $k \in I$. There exists $L_R \leq_e R_R$ such that $Lk \subseteq J$. Then $L(\varphi(k) - \theta(k)) = 0$. Since $Z(M_R) = 0$, we have $\varphi(k) = \theta(k)$. By Theorem 1.2.2., M_R is injective.

(iii) Assume $I_R \leq R_R$, $\varphi : I \to D$ be an R-homomorphism, $i : D \to M$ the inclusion homomorphism, and $\pi : M \to D$ the projection. Since M_R is R_R-ejective, there exists $J_R \leq_e I_R$ and a homomorphism $\theta : R \to M$ such that $\theta(x) = i(\varphi(x))$ for all $x \in J$. Then $\pi\theta : R \to D$ is a homomorphism and $\varphi(x) = \pi(\theta(x))$ for all $x \in J$. Therefore, D_R is R_R-ejective.

(iv) Note that $Z_2(R) \subseteq lR(M/Z_2(M_R))$. Thus, $M/Z_2(M_R)$ is an $R/Z_2(R_R)$-module where multiplication by scalars is defined by $(r + Z_2(R_R))(m + Z_2(M_R)) = mr + Z_2(M_R)$ for all $m \in M$ and $r \in R$. Let $\overline{M}$ and $\overline{R}$ denote $M/Z_2(M_R)$ and $R/Z_2(R_R)$, respectively. Thus, $\overline{M}$ and $\overline{R}$ are both R and $\overline{R}$-modules. Now, let $\overline{KR} \leq \overline{RR}$ and $\varphi : \overline{K} \to \overline{M}$ be an $\overline{R}$-homomorphism. By (ii), $\overline{M}$ is an injective R-module. Hence, there exists an R-homomorphism $\theta : \overline{R} \to \overline{M}$. But θ is also an $\overline{R}$-homomorphism. Thus, $\overline{M}$ is $\overline{R}$-injective. The particular case when M_R is R_R-ejective and $M = Z_2(M_R) \oplus Y$ follows from (iii) and the above argument. For (v) and (vi), let $I_R \leq R_R$ and $\varphi : I \to M$ be an R-homomorphism. Suppose that there is a $0 \neq J_R \leq I_R$ such that $J \cap \ker\varphi = 0$. Let $\ker\varphi = K$.

(v) There exists a $y \in J$ such that $\varphi(y) \neq 0$. Since $Z(M_R) \leq_e M_R$, there exist $r \in R$ and $L_R \leq_e R_R$ such that $0 \neq \varphi(y)r$ and $\varphi(yrL) = 0$. Then $0 \neq yr$, but $yrL = 0$. This is contrary to $Z(R_R) = 0$. Hence, $K_R \leq_e I_R$. Let $\theta : R_R \to M_R$ be the zero homomorphism. Then $\varphi(k) = \theta(k)$ for all $k \in K$, and so M_R is R_R-ejective.

(vi) Observe that $\varphi|_J : J_R \to M_R$ is a monomorphism. Hence, $\text{soc}(J_R) \neq 0$, which contradicts $\text{soc}(R_R) = 0$. Therefore, we must have $K_R \leq_e I_R$. By the argument above, we conclude that M_R is R_R-ejective. □

Proposition 2.3.6. *Let M and X be modules. Then X is M-ejective if and only if for each $\varphi : M \to E(X)$ there exist $E \leq_e m$ and $\theta : M \to X$ such that $\theta|_E = \varphi|_E$.*

Proof. ($\implies$) Let X is M-ejective and $\varphi : M \to E(X)$ is a homomorphism. Let $K = \varphi^{-1}(N)$. Then K is a submodule of M. Since X is M-ejective, there exists an essential submodule $E \leq_e K$ and a homomorphism $\theta : M \to X$ such that $\theta|_E = \varphi|_E$. Thus, we have found the essential submodule E and homomorphism θ satisfying the desired conditions.

($\Longleftarrow$) Assuming that for every homomorphism $\varphi : M \to E(X)$ there exists a submodule $E \leq_e M$ and a homomorphism $\theta : M \to X$ such that $\theta|_E = \varphi|_E$, let $K \leq M$ and $\varphi : K \to X$ be a homomorphism. We can find a homomorphism $\overline{\varphi} : M \to E(X)$ such that $\overline{\varphi}|_K = \varphi$. Let $Y = \overline{\varphi}^{-1}(X)$. It follows that $K \leq Y \leq_e M$. Therefore, there exists a submodule $E \leq_e Y$ and a homomorphism $\theta : M \to X$ such that $\theta|_E = \overline{\varphi}|_E$. Let $\overline{E} = K \cap E \leq_e K$. Then we have $\theta|_E = \overline{\varphi}|_E = \varphi|_E$. This shows that X is M-ejective. $\square$

3 Extending property and some generalizations

This chapter consists of basic properties of extending (CS) modules as well as their important generalizations which have already appeared in literature as CS, Continuous, Quasi-Continuous and C_{11} modules [13].

3.1 CS-modules

In this section, we introduce the concept of CS-modules and provide a new characterization based on idempotent endomorphisms of the injective hulls. One of the intriguing questions regarding CS-modules is whether a direct sum of CS-modules, whether finite or infinite, is also a CS-module. We explore several results in this direction. Additionally, we investigate the inheritance of the CS property by submodules.

Let R be any ring. A right R-module M is defined as a CS-module (or extending) if every submodule of M is essential in a direct summand of M. From Proposition 1.1.11, it becomes evident that a module M is a CS-module if and only if every closed (or complement) submodule of M is a direct summand. This explains the rationale behind the name "CS-module." Proposition 1.1.16 immediately follows, stating that any direct summand of a CS-module is also a CS-module. These concepts have two main origins:

(i) The work of von Neumann in the 1930s focused on continuous geometries and their realization as lattices of principal left ideals of (von Neumann) regular rings. Utumi and others further developed this work in the context of rings and modules. [14]

(ii) The theory of injective modules.

To understand the relationship between CS-modules and injective modules, we recall some definitions.

A module M is considered quasi-injective (or self-injective) if it is M-injective. It can be shown that M is quasi-injective if and only if $\theta(M) \subseteq M$ for every endomorphism θ of $E(M)$, where $E(M)$ represents the injective hull of M. For any ring R, a right R-module M is referred to as quasi-continuous if $\varphi(M) \subseteq M$ holds for every idempotent endomorphism φ of $E(M)$. The term "quasi-continuous" is derived from von Neumann's work. It can be demonstrated that M is quasi-continuous if and only if for every finite collection N_i $(1 \leq i \leq k)$ of submodules of M such that $\Sigma_i N_i$ is direct, there exist submodules L_i $(1 \leq i \leq k+1)$ of M such that $M = L_1 \oplus L_2 \oplus \cdots \oplus L_{k+1}$ and $N_i \leq_e L_i$ $(1 \leq i \leq k)$. Thus, for any ring R and right R-module M, we have:

$$M \text{ is injective} \implies M \text{ is quasi-injective} \implies M \text{ is quasi-continuous} \implies M \text{ is CS.}$$

As an example, consider the $\mathbb{Z}$-module $\mathbb{Z}$. It is a quasi-continuous module but not quasi-injective. Similarly, the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$ is CS but not quasi-continuous.

Our initial result provides various characterizations of quasi-continuous modules.

Proposition 3.1.1. *The following statements are equivalent for a module M with injective hull E.*

(i) M is quasi-continuous.
(ii) If $E = E_1 \oplus \cdots \oplus E_n$ is a finite direct sum of submodules E_i $(1 \leq i \leq n)$, then $M = (E_1 \cap M) \oplus \cdots \oplus (E_n \cap M)$.
(iii) If $E = E_1 \oplus E_2$ is a direct sum of submodules E_1 and E_2, then $M = (E_1 \cap M) \oplus (E_2 \cap M)$.
(iv) (a) M is CS.
(b)For any K, $L \leq_d M$ with $K \cap L = 0$, the submodule $K \oplus L$ is also a direct summand of M (i.e., M satisfies C3).
(v) If $L_i \leq M$ $(1 \leq i \leq n)$ with $L_1 \oplus \cdots \oplus L_n \leq M$, where n is a positive integer, then there

exist $M_i \leq M$ ($1 \leq i \leq n + 1$) such that $M = M_1 \oplus \cdots \oplus M_{n+1}$ and $L_i \leq_e M_i$ ($1 \leq i \leq n$).
(vi) If L_1, $L_2 \leq M$ with $L_1 \cap L_2 = 0$, then there exist $M_i \leq M$ ($1 \leq i \leq 3$) such that $M = M_1 \oplus M_2 \oplus M_3$ and $L_i \leq_e M_i$ ($i = 1, 2$).
(vii) If L_1, $L_2 \leq M$ with $L_1 \cap L_2 = 0$, then there exist M_1,$M_2 \leq M$ such that $M = M_1 \oplus M_2$ and $L_i \leq M_i$ ($i = 1, 2$).

Proof. (i) $\implies$ (ii) Let $\pi_i : E \to E_i$ ($1 \leq i \leq n$) be the canonical projections. Each π_i is an idempotent endomorphism of E. Therefore, $\pi_i(M) \leq M$ ($1 \leq i \leq n$), and consequently, $M \leq \pi_1(M) \oplus \ldots \oplus \pi_n(M) \leq (E_1 \cap M) \oplus \ldots \oplus (E_n \cap M) \leq M$. Thus, $M = (E_1 \cap M) \oplus \ldots \oplus (E_n \cap M)$.

(ii) $\implies$ (iii) This implication is straightforward.

(iii) $\implies$ (i)Let φ be an idempotent endomorphism of E. We have $E = \varphi(E) \oplus (1 - \varphi)(E)$, and thus $M = [\varphi(E) \cap M] \oplus [(1 - \varphi)(E) \cap M]$. It follows that $\varphi(M) = \varphi[\varphi(E) \cap M] \oplus \varphi[(1 - \varphi)(E) \cap M] \leq \varphi(E) \cap M \leq M$. Therefore, M is quasi-continuous.

(i) $\implies$ (iv) Let $N \leq M$. Then $E = E(N) \oplus F$ for some $F \leq E$. By (iii), $M = [E(N) \cap M] \oplus [F \cap M]$. Clearly, $N \leq_e E(N) \cap M$. Thus M is CS. Let K, $L \leq_d M$ with $K \cap L = 0$. Then $E = E(K) \oplus E(L) \oplus G$ for some $G \leq E$. By (ii), $M = [E(K) \cap M] \oplus [E(L) \cap M] \oplus [G \cap M] = K \oplus L \oplus (G \cap M)$, i.e., $K \oplus L \leq_d M$.

(iv) $\implies$ (v) Let n be a positive integer and let $L_i \leq M$ ($1 \leq i \leq n$) such that $L_1 + \cdots + L_n$ is direct. By (iv) (a), for each $1 \leq i \leq\leq n$, there exists $M_i \leq_d M$ such that $L_i \leq_e M_i$. Then $M_1 + \cdots + M_n$ is direct and, by (iv)(b), $M_1 \oplus \cdots \oplus M_n \leq_d M$.

(v) $\implies$ (vi) $\implies$ (vii) This implications holds trivially..

(vii) $\implies$ (iii) Suppose there exist $E_1, E_2 \leq E$ such that $E = E_1 \oplus E_2$. Let $L_i = E_i \cap M$ for $i = 1, 2$. Then $M = M_1 \oplus M_2$ for some $M_i \leq M$ such that $L_i \leq M_i$. Since $L_i \leq_e E_i$, it follows that $L_i \leq_e M_i$ for $i = 1, 2$. Let $x \in M_1$. There exist $y \in E_1$ and $z \in E_2$ such that $x = y + z$. Suppose $z \neq 0$. There exists $r \in R$ such that $0 \neq zr \in M$. Then $zr = xr - yr \in M_1 \cap M_2 = 0$, which is a contradiction. Hence, $z = 0$, and consequently, $x = y \in E_1 \cap M = L_1$. Therefore, $L_1 = M_1$. Similarly, $L_2 = M_2$. Thus, $M = (E_1 \cap M) \oplus (E_2 \cap M)$. □

Recall that R-modules M_i ($i \in I$) are called relatively injective if M_i is M_j-injective for all distinct i and j in I. We provide an alternative characterization of quasi-continuous modules.

Corollary 3.1.2. *A module M is quasi-continuous if and only if*

(i) M is CS, and

(ii) Whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1, M_2, then M_1 and M_2 are relatively injective.

Proof. If M is quasi-continuous, then the implications (i) and (ii) follow from Proposition 3.1.1. Conversely, assuming (i) and (ii) hold, let L_1 and L_2 be submodules of M with $L_1 \cap L_2 = 0$. By (i), we can find M_1 and M_2 such that $M = M_1 \oplus M_2$ and L_1 is essential in M_1. It is evident that $M_1 \cap L_2 = 0$. Using (ii), there exists a submodule M' of M satisfying $M = M \oplus M'$ and $L_2 \subseteq M'$. By applying Proposition 3.1.1., we conclude that M is quasi-continuous. □

For a module M, consider the following relations on the set of submodules of M:

(i) X is α-related to Y if there exists a submodule N of M such that X is essential in N and Y is essential in N.

(ii) X is β-related to Y if $X \cap Y$ is essential in both X and Y. (Alternatively, X is β-related to Y if and only if whenever $X \cap N = 0$, it implies $Y \cap N = 0$, and whenever $Y \cap K = 0$, it implies $X \cap K = 0$, for all submodules N and K of M.)

We can observe that if X and Y are submodules of M such that X is α-related to Y, then X is β-related to Y.

A module M is referred to as a UC-module if every submodule has a unique closure. (refer to Proposition 1.1.11.).

Instances of UC-modules include semisimple modules, uniform modules, and nonsingular modules. However, the $\mathbb{Z}$-module $(\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$ does not possess the property of being a UC-module [13].

Lemma 3.1.3. *Let M be a module. Then*

(i) α is reflexive and symmetric.

(ii) α is transitive if and only if M is a UC-module.

(iii) β is an equivalence relation.

Proof. Obvious. □

Our next goal is to provide a characterization of CS-modules using the β relation. To begin, we present an example that illustrates the distinction between the notions of isomorphism and the β equivalence relation when applied to submodules of a module.

Example 3.1.4. (i) Let F be any field. Let $R_R = \left[\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right]$. Take the right ideals $X = \left[\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right]$ and $Y = \left[\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}\right]$ of R. Then X and Y are not β-releated. However $X \cong Y$ clearly.
(ii)Consider a noncommutative nonprincipal ideal domain R. Let X be any right ideal of R. In this case, we observe that X is β-related to R, but X is not isomorphic to the right R-module R_R.

Proposition 3.1.5. *Assume M be a module. The following statements are equivalent.*

(i) M is CS.

(ii) For each $X \leq M$, there exists a direct summand D of M such that $X\alpha D$.

(iii) For each $c = c^2 \in$ End $(E(M))$ there is an $e = e^2 \in$ End $(E(M))$ such that $eM \leq M$, $eE(M)\beta cE(M)$, and there exists a homomorphism $h : cE(M) \to eE(M)$ such that $h|_{M\cap\, cE(M)}$ is the inclusion homomorphism.

Proof. (i) $\iff$ (ii) Obvious.

(i) $\implies$ (iii) Let $c = c^2 \in \text{End }(E(M))$ and $X = M \cap cE(A)$. Then there is $f = f^2 \in \text{End}(M)$ such that $X \leq_e fM$. Let $e \in \text{End}(E(M))$ be the projection $e : E(M) \to E(fM)$ (i.e., for $x \in E(M)$, $x = x_1 + x_2$, where $x_1 \in E(fM)$ and $x_2 \in E(M(1 - f))$ and $e(x) = x_1$). So $X \leq_e cE(M)$ and $X \leq_e eE(M)$. Hence $eE(M)\beta cE(M)$. Now $eM = (fM \oplus eM(1 - f)) = e(fM) \oplus eM(1 - f) = fM$ gives that $eM \leq M$. Since $eE(M)$ is injective, there is a monomorphism $h : cE(M) \to eE(M)$ that extends the inclusion $i : X \to eE(M)$.

(iii) $\implies$ (i) Let $Y \leq M$. Then $Y \leq_e E(Y) = cE(M)$ for some $c = c^2 \in \text{End}(E(M))$. So there is $e = e^2 \in \text{End}(E(M))$ such that $eM \leq M$, $E(eM)\beta cE(M)$, and there exists

$h : cE(M) \to eE(M)$ such that $h|_{M\cap\, cE(M)}$ is the inclusion homomorphism. Hence, for $y \in Y \subseteq M \cap cE(M)$, $h(y) = y \in M \cap eE(M) = eM$. Then $Y \leq eM \leq_d M$ and since $eE(M)\beta cE(M)$, $Y \leq_e eM$. Consequently, M is CS. $\square$

For any set I, the notation $|I|$ represents the cardinality of the set I.

Theorem 3.1.6. *Let R be a ring and let $M = \bigoplus_{i\in\, I} M_i$ be the direct sum of R-modules M_i ($i \in I$), for some index set I with $|I| \geq 2$. Then the following statements are equivalent.*

(i) M is CS-module.

(ii) There exist distinct elements i, j in the index set I such that for every closed submodule K of M, if K has trivial intersection with either M_i or M_j, then K is a direct summand.

(iii) There exist distinct elements i, j in the index set I such that for every complement of M_i or M_j in M, it is both a CS-module and a direct summand of M.

Proof. (i) $\Longrightarrow$ (ii) Obvious.

(ii) $\Longrightarrow$ (iii) Consider a complement K of M_i in M. According to property (ii), we can conclude that K is a direct summand of M. Now, let L be a closed submodule of K. Utilizing Proposition 1.1.16., we can establish that L is a closed submodule of M, and it is evident that L has trivial intersection with M_i. By virtue of property (ii), we can deduce that L is a direct summand of M and, consequently, a direct summand of K. Therefore, K is a CS-module.

(iii) $\Longrightarrow$ (i)Consider a closed submodule N of M. There exists a closed submodule H of N such that the intersection of N with M_i is an essential submodule of H. It is evident that H has trivial intersection with M_j. By utilizing Zorn's Lemma, we can establish the existence of a complement P of M_j in M such that H is a submodule of P. Applying Proposition 1.1.16 demonstrates that H is a closed submodule of M and, consequently, a closed submodule of P. By employing property (iii), we can deduce that H is a direct summand of the CS-module P, and P is a direct summand of M. Thus, H is a direct summand of M.

Furthermore, there exists a submodule H' of M such that M can be expressed as the direct sum $H \oplus H'$. By the modular law, we have $N = H \oplus (N \cap H')$. According to Proposition 1.1.16, $N \cap H'$ is a closed submodule of M, and it is clear that the intersection of $N \cap H'$ with M_i is trivial. Applying the argument mentioned above, property (iii) implies that $N \cap H'$ is a direct summand of M and, consequently, a direct summand of H'. Therefore, N is a direct summand of M. Consequently, we can conclude that M is a CS-module. □

Lemma 3.1.7. *Consider a module $M = M_1 \oplus M_2$, and let K be a submodule of M. We can say that K serves as a complement of M_2 in M if and only if there exists a homomorphism $\varphi : M_1 \to E(M_2)$ such that $K = \{x + \varphi(x) : x \in \varphi^{-1}(M_2)\}$.*

Proof. Assume that K serves as a complement of M_2 in M. Let $\pi_i : M \to M_i$ $(i = 1, 2)$ denote the canonical projections. It can be observed that $\pi_1|_K : K \to M_1$ is an injective homomorphism. By considering the inclusion mapping $i : M_2 \to E(M_2)$, we can establish the existence of a homomorphism $\varphi : M_1 \to E(M_2)$ such that $\varphi(\pi_1|_K) = i(\pi_2|_K)$. For any $x \in K$, it follows that $\varphi\pi_1(x) = \pi_2(x) \in M_2$. Thus, $\pi_1(x) \in \varphi^{-1}(M_2)$, and we have $x = \pi_1(x) \oplus \pi_2(x) = \pi_1(x) \oplus \varphi(\pi_1(x))$. Hence, we obtain $K \subseteq y + \varphi(y) : y \in \varphi^{-1}(M_2) = K_1$ (denoted as K_1 for clarity). Since K_1 is a submodule of M and $K_1 \cap M_2 = 0$, it follows that $K = K_1$, as desired.

Conversely, assume that $\theta : M_1 \to E(M_2)$ is a homomorphism, and let $K = \{x + \theta(x) : x \in \theta^{-1}(M_2)\}$. It is clear that K is a submodule of M and $K \cap M_2 = 0$. Now, suppose that L is a submodule of M such that $K \subseteq L$ and $L \cap M_2 = 0$. Let $u \in L$ be such that $\pi_2(u) \neq \theta\pi_1(u)$. Since $0 \neq \pi_2(u) - \theta\pi_1(u) \in E(M_2)$, there exists $r \in R$ such that $0 \neq (\pi_2(u) - \theta\pi_1(u))r \in M_2$. However, in this case, we have $\theta\pi_1(u)r \in M_2$ and $(\pi_2(u) - \theta\pi_1(u))r = \pi_2(ru) - \theta\pi_1(ur) = ur - (\pi_1(ur) + \theta\pi_1(ur)) \in (L + K) \cap M_2 = L \cap M_2 = 0$, leading to a contradiction.

Let $v \in L$. Then $\theta\pi_1(v) = \pi_2(v) \in M_2$, implying that $\pi_1(v) \in \theta^{-1}(M_2)$. Thus, we have $v = \pi_1(v) + \pi_2(v) = \pi_1(v) + \theta(\pi_1(v)) \in K$. Consequently, we obtain $L = K$. Therefore, K is a complement of M_2 in M. □

Theorem 3.1.8. *Assume R be a ring and let $M = \bigoplus_{i \in I} M_i$ be the direct sum of R-modules M_i $(i \in I)$, for some index set I with $|I| \geq 2$. Then the following statements* *are equivalent.*

(i) M is a CS-module.

(ii) For each $i \in I$ and each homomorphism $\varphi : M_{-i} \to E(M_i)$, the submodule $\{x + \varphi(x) : x \in \varphi^{-1}(M_i)\}$ is a CS-module and a direct summand of M.

(iii) There exist $i \neq j$ in I such that for each $k \in \{i, j\}$ and each homomorphism $\varphi : M_{-k} \to E(M_k)$, the submodule $\{x + \varphi(x) : x \in \varphi^{-1}(M_k)\}$ is a CS-module and a direct summand of M.

Proof. By [13, Theorem 3.9.]. □

Consider a module $M = \bigoplus_{i \in I} M_i$, where the modules M_i $(i \in I)$ are relatively injective. Let $i \in I$ and let $\varphi : M_{-i} \to E(M_i)$ be a homomorphism. According to Proposition 2.1.3., M_i is M_{-i}-injective, and thus $\varphi(M_{-i}) \subseteq M_i$ (by Proposition 2.1.5.). We define $K = \{x + \varphi(x) : x \in \varphi^{-1}(M_i)\} = \{x + \varphi(x) : x \in M_{-i}\}$. It follows that $M = K \oplus M_i$. Consequently, in Theorem 3.1.8., if the modules M_i $(i \in I)$ are relatively injective, condition (ii) can be equivalently expressed as (i') Each M_{-i} is CS, and condition (iii) can be equivalently expressed as (ii') There exist distinct $i, j \in I$ such that M_{-i} and M_{-j} are CS.

We also observe that if $M = \bigoplus_{i \in I} M_i$ is a CS-module, where M_i $(i \in I)$ is a family of modules, and if $i \in I$, then $\varphi^{-1}(M_i)$ is a CS-module for any homomorphism $\varphi : M_{-i} \to E(M_i)$. This follows from the fact that, in Theorem 3.1.8., $\varphi^{-1}(M_i)$ is isomorphic to $\{x + \varphi(x) : x \in \varphi^{-1}(M_i)\}$.

Proposition 3.1.9. *Assume R be a ring, M_1 be an R-module with zero socle, and M_2 be a semisimple R-module. Then the R-module $M = M_1 \oplus M_2$ is CS if and only if M_1 is CS and M_2 is M_1-injective.*

Proof. Assume that M is a CS-module and M_1 is also a CS-module. It is evident that M_2 is the socle of M. Now, consider any submodule N of M_1 and a homomorphism $\varphi : N \to M_2$. Let $L = \{x - \varphi(x) : x \in N\}$. Then L is a submodule of M and $L \cap M_2 = 0$. There exist submodules K and K' of M such that $M = K \oplus K'$, and L is an essential submodule of K. Notably, the socle of K is $K \cap M_2 = 0$, implying that $M_2 = \operatorname{soc}(M) \subseteq K'$. Therefore, we have $K' = M_2 \oplus (K' \cap M_1)$, and $M = K \oplus M_2 \oplus (K' \cap M_1)$. Let $\pi : M \to M_2$

denote the projection with kernel $K \oplus (K' \cap M_1)$. Consider $\theta = \pi|_{M_1}$. We can observe that $\theta : M_1 \to M_2$, and for all $x \in N$, we have $\theta(x) = \varphi(x)$. Therefore, M_2 is M_1-injective.

Conversely, assume that M_1 is a CS-module and M_2 is M_1-injective. It is clear that M_1 is also M_2-injective. Thus, M is a CS-module. □

Lemma 3.1.10. *Consider a ring R and R-modules M_1 and M_2, where M_2 is semisimple. The R-module $M_1 \oplus M_2$ is a CS-module if and only if every complement K of M_2 in M is both a CS-module and a direct summand of M.*

Proof. By [13, Lemma 3.14.]. □

Theorem 3.1.11. *Consider a ring R, where M_1 is a CS R-module and M_2 is a semisimple R-module such that M_2 is (M_1/N)-injective for every non-zero submodule N of M_1. In this case, the R-module $M = M_1 \oplus M_2$ is a CS-module.*

Proof. Let K be a complement of M_2 in M. According to Lemma 3.1.7., there exists a homomorphism $\varphi : M_1 \to E(M_2)$ such that $K = \{x + \varphi(x) : x \in \varphi^{-1}(M_2)\}$. Let $Q = \varphi^{-1}(M_2)$ and $P = \ker(\varphi)$. Both P and Q are submodules of M_1.

Suppose $P = 0$. In this case, $K \cap M_1 = 0$, and therefore $M_1 \oplus K = M_1 \oplus \varphi(Q)$, which is a direct summand of M since $\varphi(Q)$ is a direct summand of M_2. Thus, K is a direct summand of M, and since K embeds in $M/M_1 \cong M_2$, K is semisimple and thus CS.

Now suppose $P \neq 0$. By the hypothesis, M_2 is (M_1/P)-injective. We have $Q/P \cong \varphi(Q)$, which is a direct summand of M_2. Hence, Q/P is also (M_1/P)-injective. There exists a submodule Q' of M_1 such that $P \subseteq Q'$ and $(M_1/P) = (Q/P) \oplus (Q'/P)$. Define $\theta : M_1 \to E(M_2)$ by $\theta(q + q') = \varphi(q)$ for $q \in Q$ and $q' \in Q'$. It can be easily verified that θ is a well-defined homomorphism. Moreover, $\theta|_Q = \varphi$. Let $K' = \{x + \theta(x) : x \in \theta^{-1}(M_2)\} = \{x + \theta(x) : x \in M_1\}$, noting that $\theta(M_1) = \varphi(Q) \leq M_2$. By Lemma 3.1.7., K' is a complement of M_2 in M. However, we have $K \subseteq K'$, which implies $K = K'$. It is clear that $M = K \oplus M_2$. Thus, K is a CS-module and a direct summand of M. By Lemma 3.1.10., M is a CS-module. □

Theorem 3.1.12.

(i) Suppose M is a CS-module and X is a submodule of M. If the intersection of X

with any direct summand of M is a direct summand of X, then X is also a CS-module.

(ii) Consider a module M, where X is a CS submodule of M, and D is a direct summand of M. If $D+X$ is nonsingular, then $D \cap X$ is a direct summand of X.

(iii) If M is nonsingular and X is a CS submodule, then the intersection of X with any direct summand of M is a direct summand of X.

Proof. (i) Consider a submodule N of X. There exists a direct summand D of M such that N is essential in D. It follows that N is also essential in $D \cap X$, and $D \cap X$ is a direct summand of X. Therefore, X is a CS-module.

(ii) Let D be a direct summand of M, and let $Y = D \cap X$. There exists a submodule C of X such that C is a direct summand of X, and Y is essential in C. Assume $Y \neq C$. Then $D \neq D + C$. Take $d + c \in D + C$ such that $d + c \notin D$, where $d \in D$ and $c \in C$. Since $c \neq 0$, there exists an essential right ideal L of R such that $0 \neq cL \subseteq Y$. Since D is nonsingular, we have $0 \neq (d + c)L \subseteq D$. Thus, D is essential in $D + C$, which is a contradiction. We conclude that $Y = C$.

(iii) This part follows immediately using the same proof as in part (ii).

□

Let M be a module and $\mathcal{L}$ be the collection of all submodules of M. It is well known that $\mathcal{L}$ is a lattice with respect to inclusion, intersection and sum operations.A module is named as a *distributive* module if its lattice of submodules forms a distributive lattice.

Corollary 3.1.13. *Let M be a CS-module.*

(i) If M is a distributive module, then every submodule is CS.

(ii) If X is a submodule of M such that $e(X) \subseteq X$ for all $e = e^2 \in End\ M$, then X is a CS-module. In particular every fully invariant submodule of M is CS.

Proof. (i) follows directly from (i) of Theorem 3.1.12.

(ii) Consider D be a direct summand of M and π: $M \to D$ the projection map. Then $\pi(X) = X \cap D$. According to Theorem 3.1.12 (i), X is a CS-module. □

The following result is the most useful characterization of CS-modules in terms of decomposition as well as relative injectivity of component direct summand.

Theorem 3.1.14. *M_R satisfies CS if and only if $M = Z_2(M) \oplus N$ and $Z_2(M)$ is N-injective.*

Proof. Assume first that M is a CS-module. Since $Z_2(M)$ is a complement in M, we can express M as the direct sum $M = Z_2(M) \oplus N$, where N is a nonsingular module. Thus, both $Z_2(M)$ and N are CS-modules. Let $\varphi : X \to Z_2(M)$ be a homomorphism, where X is a submodule of N. Consider $X' = \{x - \varphi(x) : x \in X\}$. By hypothesis, there exists a direct summand L of M such that X' is an essential submodule of L. We can write $M = L \oplus Y$, where Y is another submodule. Since $X' \cap Z_2(M) = 0$ and X' is essential in L, it follows that L is nonsingular and $Z_2(M) = Z_2(Y)$. Consequently, $Z_2(M)$ is a direct summand of Y, denoted as $Y = Y' \oplus Z_2(M)$. Let $\pi : L \oplus Y' \oplus Z_2(M) \to Z_2(M)$ be the canonical projection. It can be easily verified that $\pi|_X = \varphi$.

Conversely, assume that $M = Z_2(M) \oplus N$, where $Z_2(M)$ and N are CS-modules and $Z_2(M)$ is N-injective. Let A be a complement of M. Since $Z_2(A)$ is a complement of A, it is also a complement of M. However, $Z_2(A) \subseteq Z_2(M)$, implying that $Z_2(A)$ is a complement of $Z_2(M)$. Therefore, $Z_2(A)$ is a direct summand of $Z_2(M)$ and, consequently, a direct summand of A. We can express A as $A = Z_2(A) \oplus B$, where B is a nonsingular submodule of A. Since $B \cap Z_2(M) = 0$ and $Z_2(M)$ is N-injective, there exists a homomorphism $\theta : N \to Z_2(M)$ such that $\theta\pi_2|_B = \pi_1|_B$, where π_1 and π_2 are the projections of M onto $Z_2(M)$ and N, respectively. Consider $N' = \{n + \theta(n) : n \in N\}$. It follows that B is contained in N'. Since $N' \cong N$ is a CS-module, B is a direct summand of N'. It is evident that $M = Z_2(M) \oplus N'$. Therefore, A is a direct summand of M. □

3.2 Continuous and quasi-continuous modules

This section focuses on various concepts related to the CS property. Specifically, we examine modules that possess the CS property along with the conditional direct summand conditions C_2 and C_3. These modules are of particular interest and are commonly referred to as continuous (quasi-continuous) modules in the literature. Let us provide the definitions for the C_2 and C_3 properties.

(i) property C_2: if $X \leq M$ is isomorphic to a direct summand of M, then X is a direct summand of M; in other words, for each direct summand N of M and each monomorphism $\varphi : N \to M$, the submodule $\varphi(N)$ is also a direct summand of M;

(ii) property C_3: if M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M;

Let R be a ring and M a right R-module. The module M is referred to as *continuous* if it satisfies the CS condition, as well as the condition C_2. It is important to note that any module satisfying C_2 also satisfies C_3. On the other hand, the module M is called *quasi-continuous* if it is a CS-module and satisfies the condition C_3. Therefore, continuous modules are quasi-continuous modules. We demonstrate that continuous and quasi-continuous modules can be characterized by the lifting of homomorphisms from certain submodules of M to M itself. It should be noted that there exists a distinct lifting condition to characterize continuous modules. Perhaps it is preferable to begin with this fact, which was presented in [8].

Theorem 3.2.1. *The following are equivalent for a module M.*

(i) M is continuous.

(ii) If $B \oplus C \leq M$ and $f : B \oplus C \to M$ is a homomorphism with im f closed in M and ker f = C, then there exists $g \in$ End (M) extending f.

(iii) If f is a partial endomorphism of M with both $kerf$ and imf closed in M, then f can be extended to an endomorphism of M.

Proof. i) $\implies$ (iii) Let N is a submodule of M and $f \in \mathrm{Hom}(N, M)$ such that $\ker(f)$ and $\mathrm{im}(f)$ are closed in M. By the given hypothesis, both $\ker(f)$ and $\mathrm{im}(f)$ are direct summands of M. Therefore, there exists a submodule B of M such that $M = \ker(f) \oplus B$ and $N = \ker(f) \oplus (N \cap B)$. Using the C_2 property, we have $N \cap B \cong \mathrm{im}(f)$, which is also a direct summand of M. Since continuous modules are quasi-continuous, we can conclude that $N = \ker(f) \oplus (N \cap B)$ is a direct summand of M. Moreover, this direct summand can be extended to an endomorphism of M.

(iii) $\implies$ (ii) Consider f be given as in (ii). Assume D be the closure of C in M.

Then f can be extended to $\overline{f} : B \oplus D \to M$ with $\overline{f}|_D = 0$. By applying (iii), we conclude that $\overline{f}$, and hence f, can be further extended to an endomorphism of M.

(ii) $\implies$ (i) M is quasi-continuous [9]. To show that M satisfies property C_2, suppose $B \leq_d M$ and $\varphi : N \to B$ is an isomorphism. By the assumption that M is quasi-continuous, there exists a submodule $D \leq_d M$ such that $N \leq_e D$. Let $M = D \oplus C$ for some submodule $C \leq M$. Define $f : N \oplus C \to B$ by $f(n+c) = \varphi(n)$, where $n \in N$ and $c \in C$. It can be verified that ker $f = C$ and im $f = B$ are closed submodules of M. Therefore, there exists an endomorphism $g \in \text{End}(M)$ extending f. Since $g|_N = \varphi$ is an isomorphism and $N \leq_e D$, it follows that $g|_D$ is also an isomorphism. Hence, $B = \varphi(N) = g(N) \leq_e g(D)$, which implies $g(N) = g(D)$ since B is closed in M. Consequently, $N = D \leq_d M$, demonstrating that M satisfies property C_2. Therefore, M is a continuous module. □

Lemma 3.2.2. *Let K be a complement in M. Then K is a direct summand of M if and only if there exists a complement L of K in M such that $K \oplus L \in Lift_M(M)$.*

Proof. If K is a direct summand of M, then we can write $M = K \oplus K'$, where K' is a submodule of M. It is evident that if we set $L = K'$, then K ⊕ L ∈ $Lift_M$(M).

Conversely assuming the existence of a complement L of K in M with the specified property, we define a homomorphism $\varphi : K \oplus L \to M$ as follows:

$$\varphi(x + y) = x \quad \text{for } x \in K,\ y \in L.$$

Given the hypothesis, there exists a homomorphism $\theta : M \to M$ such that $\theta(x + y) = x$ for $x \in K$, $y \in L$. It can be observed that K is contained in the image of θ, denoted as im θ, and L is contained in the kernel of θ, denoted as ker θ.

Let $0 \neq v \in$ im θ. Then there exists $u \in M$ such that $v = \theta(u)$. It is important to note that $u \notin L$. Consequently, $K \cap (L + uR) \neq 0$, where R denotes the underlying ring. There exist $x \in K$, $y \in L$, and $r \in R$ such that $0 \neq x = y + ur$. Consequently, $x = \theta(x) = \theta(y + ur) = vr$. It follows that $vR \cap K \neq 0$ for all non-zero $v \in$ im θ. Thus, K is an essential submodule of im θ. However, K is also a complement in M. Therefore, $K =$ im θ.

From this point, it can be easily verified that $M = K \oplus$ ker θ. Thus, K is a direct summand of M. □

Corollary 3.2.3. *A module M is CS if and only if for every complement K in M there exists a complement L of K in M such that $K \oplus L \in Lift_M(M)$.*

Proof. Immediate by using Lemma 3.2.2. $\square$

Let M be a module and n a positive integer. We define the following classes in conjunction with respect to the conditions C_2 and C_3:
$\underline{M'} = \{N \leq M$: there exists $K \leq_d M$ such that $K \cong N\}$,
$\underline{M}^{(n)} = \{L_1 + L_2 + \cdots + L_n$: $L_i \leq_d M$ for $1 \leq i \leq n$ and $L_1 + L_2 + \cdots + L_n$ is a direct sum$\}$, and
$\underline{C}^{(n)} = \{C_1 + C_2 + \cdots + C_n$: $C_i \leq_c M$ for $1 \leq i \leq n$ and $C_1 + C_2 + \cdots + C_n$ is a direct sum.

For a positive integer n, we examine the following criterion imposed on a module M:

$$P_n : \underline{C}^{(n)} \subseteq Lift_M(M).$$

Obviously if M satisfies P_n, then M satisfies P_{n-1}, for all $n \geq 2$.

Theorem 3.2.4. *For a module M, the following statements are equivalent.*

(i) M is quasi-continuous.

(ii) M satisfies P_n for every positive integer n.

(iii) M satisfies P_n for some integer $n \geq 2$.

(iv) M satisfies P_2.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) Clear.
(iv) $\Longrightarrow$ (i) This can be deduced from Proposition 4.1.16 and Corollary 3.2.3. $\square$

Consider the condition imposed on a module M for a given positive integer n:

Q_n : For every K *in* $\underline{M}^{(n)}$ such that $K = K_1 \oplus \cdots \oplus K_n$ and $K_i \in M'$ $(1 \leq i \leq n)$, $K \in Lift_M(M)$.

It is evident that for any positive integer $n \geq 2$, if a module M fulfills the condition Q_n, then it also fulfills the condition Q_{n-1}. Additionally, for every positive integer $n \geq 1$, if a module M satisfies Q_n, it also satisfies P_n.

Theorem 3.2.5. *For a module M, the following statements are equivalent.*

(i) M is continuous.

(ii) M satisfies Q_n for every positive integer n.

(iii) M satisfies Q_n for some integer $n \geq 2$.

(iv) M satisfies Q_2.

(v) M is CS and satisfies Q_1.

Proof. (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) Obvious.

(iv) $\implies$ (i) From Corollary 3.2.3. and Proposition 4.1.15.

(i) $\implies$ (v) Obvious.

(v) $\implies$ (i) From Proposition 4.1.15. □

The validity of the following implications is demonstrated by Theorems 3.2.4 and 3.2.5.

$$\begin{array}{ccccccc}
quasi-injective & \Longrightarrow & continuous & \Longrightarrow & quasi-continuous & & \\
\Downarrow & & \Updownarrow & & \Updownarrow & & \\
Q_n & \Longleftrightarrow & Q_2 & \Longrightarrow & P_2 & \Longleftrightarrow & P_n \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & Q_1 & \Longrightarrow & P_1 & \Longleftarrow & CS
\end{array}$$

For any integer $n \geq 2$ no other implications can be added to this table in general.

Example 3.2.6. Consider the ring $\mathbb{Z}$, which represents the set of rational integers. In this context, the $\mathbb{Z}$-module $\mathbb{Z}$ fulfills the condition P_2. However, it does not satisfy Q_1.

Proof. Consider the module $M = \mathbb{Z}_{\mathbb{Z}}$. It is evident that M satisfies the condition of being a completely reducible module (CS) and also fulfills C_3, thereby satisfying the property denoted by P_2 according to Theorem 3.2.4. Assume N denote the submodule $2\mathbb{Z}$ of $\mathbb{Z}$. Remarkably, N is isomorphic to M. However, the homomorphism $\varphi : N \to M$ defined by $\varphi(2n) = n$ (for $n \in \mathbb{Z}$) cannot be extended to a homomorphism on M. Consider there exists a homomorphism $\theta : M \to M$ such that $\theta|_N = \varphi$. Then, for any $m \in M$, there exists $x \in M$ such that $\theta(m) = mx$. Consequently, $2mx = \theta(2m) = \varphi(2m) = m$. Thus, $2x = 1$, which leads to a contradiction. Hence, M does not satisfy Q_1. □

Example 3.2.6 illustrates that none of the implications

$$\text{quasi-continuous} \implies \text{continuous}, P_2 \implies Q_2, P_1 \implies Q_1,$$

hold universally. Specifically, there exists a commutative local ring R for which the R-module R satisfies Q_1 but is not CS [9].

3.3 C_{11}-modules

In this section, we present C_{11}-modules as a generalization of CS-modules. The primary emphasis of our discussion revolves around exploring the properties of C_{11}-modules. Throughout this section, we will highlight the similarities and differences between C_{11}-modules and CS-modules. The majority of the material covered in this section can be found in references[10], [13], and [11].

Definition 3.3.1. A module M satisfies C_{11} if every submodule N of M has a complement that is a direct summand of M. In other words, for each submodule N of M, there exists a direct summand K of M such that K is a complement of N in M..

To facilitate comparison, we start by proving the given proposition.

Proposition 3.3.2. *A module M is CS if and only if, for every pair of submodules N and L satisfying $N \cap L = 0$, there exists a direct summand K of M such that L is a submodule of K and N has no intersection with K. Furthermore, in such cases, it holds that $N \oplus K \leq_e M$.*

Proof. Let's first consider the case where M is CS. Let N and L be submodules of M such that $N \cap L = 0$. There exists a complement K of N in M such that $L \leq K$. By the assumption, we conclude that K is a direct summand of M.

Conversely, let M satisfies the given condition. Let L be a complement in M. There exists a submodule N of M such that L is a complement of N in M. By the given hypothesis, there exists a direct summand K of M such that $L \leq K$ and $N \cap K = 0$. Consequently, we have $L = K$. It follows that every complement in M is a direct summand.

Therefore, M is CS. The last part can be deduced from Proposition 1.1.9 (also referred to in the proof of Lemma 3.3.3 below). □

Lemma 3.3.3. *Consider a submodule N of a module M and a direct summand K of M. The following condition holds: K is a complement of N in M if and only if $K \cap N = 0$ and $K \oplus N \leq_e M$.*

Proof. Assume that K is a complement of N in M. Therefore, it follows that $K \cap N = 0$. Now, consider an element $x \neq 0$ in M. If $x \in K$, then we have $0 \neq xR = xR \cap K \subseteq xR \cap (K \oplus N)$. On the other hand, if $x \notin K$, then $N \cap (xR + K) \neq 0$, which implies $xR \cap (K \oplus N) \neq 0$. Consequently, we can conclude that $xR \cap (K \oplus N) \neq 0$ for all $x \neq 0$ in M. Therefore, $K \oplus N \leq_e M$.

Conversely, let us suppose that K and N possess the properties stated. We can find a submodule K' of M such that $M = K \oplus K'$. Suppose there exists a submodule K_1 of M such that $K \subseteq K_1$ and $K_1 \cap N = 0$. Then, we have $K_1 = K_1 \cap M = K_1 \cap (K \oplus K') = K \oplus (K_1 \cap K')$. Assume that $y \neq 0$ belongs to $K_1 \cap K'$. Consequently, we have $0 \neq yr = n + k$ for some $n \in N$, $k \in K$, and $r \in R$. This implies that $yr - k = n \in K_1 \cap N = 0$. Hence, $yr = k \in K \cap K' = 0$, which leads to a contradiction. Therefore, we conclude that $K_1 \cap K' = 0$ and hence $K = K_1$. In other words, K serves as a complement of N in M. □

Proposition 3.3.4. *For a module M the followings are equivalent.*

(i) M is a C_{11}-module.

(ii) Every complement submodule L in M has a corresponding direct summand K in M that acts as a complement for L in M.

(iii) For any submodule N of M, there exists a direct summand K of M where N and K have no intersection ($N \cap K = 0$), and their direct sum $N \oplus K$ is included in M with essential inclusion ($N \oplus K \leq_e M$).

(iv) For any complement submodule L in M, there exists a direct summand K of M where L and K have no intersection ($L \cap K = 0$), and their direct sum $L \oplus K$ is included in M with essential inclusion ($L \oplus K \leq_e M$).

Proof. (i) $\implies$ (ii), (iii) $\implies$ (iv) Clear.

(i) $\iff$ (iii), (ii) $\iff$ (iv) Obviously by Lemma 3.3.3.

(iv) $\implies$ (i) Consider any submodule B of M. We can find a complement submodule C in M such that B is essentially included in C ($B \leq_e C$). By the given hypothesis, there exists a direct summand K of M satisfying $C \cap K = 0$ and $C \oplus K \leq_e M$. According to Lemma 3.3.3, we can deduce that K is a complement of C in M. Furthermore, it is important to note that K and B have no intersection ($K \cap B = 0$). Suppose there exists a submodule K' of M that properly contains K. Consequently, we have $K' \cap C \neq 0$, which implies $K' \cap C \cap B \neq 0$. In other words, $K' \cap B \neq 0$. This implies that K serves as a complement of B in M. □

Theorem 3.3.5. *The condition C_{11} is preserved under direct sums of modules. In other words, any direct sum of modules, each satisfying C_{11}, also satisfies C_{11}*

Proof. Let M_λ ($\lambda \in \Lambda$) be a non-empty collection of modules, each satisfying the C_{11} condition. Consider the module $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$. Let N be an arbitrary submodule of M. For each $\lambda \in \Lambda$, observe that $N \cap M_\lambda$ is a submodule of M_λ, and since M_λ satisfies C_{11}, Proposition 3.3.4. guarantees the existence of a direct summand K_λ of M_λ such that $(N \cap M_\lambda) \cap K_\lambda = 0$ and $(N \cap M_\lambda) \oplus K_\lambda \leq_e M_\lambda$. Furthermore, note that $N \cap M_\lambda = 0$, $(N \oplus K_\lambda) \cap M_\lambda = (N \cap M_\lambda) \oplus K_\lambda$, and $(N \oplus K_\lambda) \cap M_\lambda \leq_e M_\lambda$. Let Λ' be a non-empty subset of Λ that contains λ and also satisfies the condition that there exists a direct summand K' of $M' = \bigoplus_{\lambda \in \Lambda'} M_\lambda$ with $N \cap K' = 0$ and $(N \oplus K') \cap M' \leq_e M'$. Suppose $\Lambda' \neq \Lambda$. Choose $\mu \in \Lambda$ such that $\mu \notin \Lambda'$. Now, consider $L = (N \oplus K') \cap M_\mu$, which is a submodule of M_μ. According to Proposition 3.3.4, there exists a direct summand K_μ of M_μ such that $L \cap K_\mu = 0$ and $L \oplus K_\mu \leq_e M_\mu$. Let $\Lambda'' = \Lambda' \cup \mu$ and $M'' = \bigoplus_{\lambda \in \Lambda''} M_\lambda = M' \oplus M_\mu$. Notice that $K' \cap K_\mu = 0$. Define $K'' = K' \oplus K_\mu$. Then K'' is a direct summand of M'' and moreover $N \cap K'' = 0$.

Consider the submodule $N \oplus K''$. It is worth noting that $(N \oplus K'') \cap M'$ contains $(N \oplus K') \cap M'$, implying that $(N \oplus K'') \cap M' \leq_e M'$. Additionally, $(N \oplus K'') \cap M_\mu = (N \oplus K' \oplus K_\mu) \cap M_\mu = [(N \oplus K') \cap M_\mu] \oplus K_\mu = L \oplus K_\mu$, which is an essential submodule of M_μ. Therefore, it follows that $(N \oplus K'') \cap M''$ is an essential submodule of M''. By repeating this process, we can find a direct summand K of M such that $N \cap K = 0$ and $N \oplus K \leq_e M$. According to Proposition 3.3.4., M satisfies the C_{11} condition. □

Corollary 3.3.6. *Any direct sum of CS-modules provides the property C_{11}.*

Proof. By applying Theorem 3.3.5., the result follows immediately. □

Corollary 3.3.7. *Any direct sum of uniform modules provides property C_{11}.*

Proof. By applying Corollary 3.3.6., the result follows immediately. □

Next results provides characterization of C_{11}-modules in terms of decomposition like CS-modules (See Theorem 3.1.14.) Observe that this characterization does not contain relative injectivity which is unlike to the situation of CS-property.

Theorem 3.3.8. *An M module satisfies C_{11} if and only if it can be expressed as the direct sum of $Z_2(M)$ and a nonsingular submodule K of M, where both $Z_2(M)$ and K individually satisfy C_{11}.*

Proof. The sufficiency is an immediate consequence of Theorem 3.3.5. For the converse, let us begin by assuming that M satisfies C_{11}. Firstly, we show that $Z_2(M)$ is a direct summand of M. Let $L = Z_2(M)$. By Proposition 3.3.4., there exist submodules K and K' of M such that $M = K \oplus K'$, $L \cap K = 0$, and $L \oplus K \leq_e M$. Now, we have $L = Z_2(M) = Z_2(K \oplus K') = Z_2(K) \oplus Z_2(K')$. Since $Z_2(M) = 0$, we obtain $L = Z_2(K') \subseteq K'$. As $L \oplus K$ is essential in M, we conclude that L is essential in K', implying that K'/L is singular. Hence, $L = K'$, and L is a direct summand of M.

We have established that $M = L \oplus K$. Next, we prove that L satisfies C_{11}. Let N be any submodule of L. Then, $N \oplus K$ is a submodule of M. Since M satisfies C_{11}, there exist submodules P and P' of M such that $M = P \oplus P'$, $(N \oplus K) \cap P = 0$, and $N \oplus K \oplus P \leq_e M$. Notably, $P \cap K = 0$, and thus P embeds in $M/K \cong L$. This implies $P = Z_2(P)$ and $P \leq L$. Consequently, P is a direct summand of L (specifically, $L = P \oplus (L \cap P')$), and $N \oplus P \leq_e L$. According to Proposition 3.3.4., L satisfies C_{11}.

Lastly, we demonstrate that K satisfies C_{11}. Let $\pi : M \to K$ denote the canonical projection. Consider any submodule H of K. We have $L \cap H = 0$, and there exist submodules Q and Q' of M such that $M = Q \oplus Q'$, $(L \oplus H) \cap Q = 0$, and $L \oplus H \oplus Q \leq_e M$. Note that $L = Z_2(M) = Z_2(Q) \oplus Z_2(Q') = Z_2(Q')$ since $Q \cap L = 0$. Consequently, $L \leq Q'$, and we can express $Q' = L \oplus (Q' \cap K)$. Now, $M = Q \oplus Q' = Q \oplus L \oplus (Q' \cap K)$. This shows that $L \oplus Q$ is a direct summand of M. Moreover, $L \oplus Q = L \oplus \pi(Q)$. Therefore, the submodule $\pi(Q)$ of K is a direct summand of M and, hence, a direct summand of K. As $H \oplus \pi(Q) \oplus L \leq_e M$, we conclude that $H \oplus \pi(Q) \leq_e K$. By Proposition 3.3.4., K satisfies C_{11}. □

Lemma 3.3.9. *Let M be a module which satisfies C_{11}. Then $M = M_1 \oplus M_2$ where M_1 is a submodule of M with essential socle and M_2 a submodule of M with zero socle.*

Proof. Let S represent the socle of module M. We can find submodules K and K' of M such that $M = K \oplus K'$, $S \cap K = 0$, and $S \oplus K \leq_e M$. Consequently, we have $S = \text{soc } M = (\text{soc } K) \oplus (\text{soc } K')$. It is evident that $\text{soc } K = 0$, implying that $S \leq K'$. Additionally, since $S \oplus K \leq_e M$, we can conclude that $S \leq_e K'$. Thus, we have proved the desired result. □

Theorem 3.3.10. *A module M is nonsingular and satisfies C_{11} if and only if it can be decomposed as the direct sum $M = M_1 \oplus M_2$, where M_1 is a module satisfying C_{11} and possesses an essential socle, and M_2 is a module satisfying C_{11} and has a socle that is zero.*

Proof. The sufficiency is clear from Theorem 3.3.5. Conversely, assume that M satisfies C_{11}. According to Lemma 3.3.9., we can write M as the direct sum $M = M_1 \oplus M_2$, where M_1 has an essential socle and M_2 has a zero socle. Let S represent the socle of M. It is evident that $M_1 = c(S)$. We now proceed to prove that M_1 satisfies C_{11}. Consider any submodule N of M_1. By Proposition 3.3.4., there exists a direct summand P of M such that $(N \oplus M_2) \cap P = 0$ and $N \oplus M_2 \oplus P$ is an essential submodule of M. Since P embeds in M_1, it follows that P has an essential socle, denoted by $S \cap P$. Consequently, we have $P = c(S \cap P) \leq c(S) = M_1$. Hence, P is a direct summand of M_1, and $N \oplus P$ is an essential submodule of M_1. By applying Proposition 3.3.4., we conclude that M_1 satisfies C_{11}.

Let's now consider the module M_2. We denote the canonical projection from M to

M_2 as $\pi : M \to M_2$. Let H be an arbitrary submodule of M_2. According to Proposition 3.3.4., there exist submodules Q and Q' of M such that $M = Q \oplus Q'$, $(M_1 \oplus H) \cap Q = 0$, and $M_1 \oplus H \oplus Q$ is an essential submodule of M. Since $S \cap Q = 0$, we have $S \subseteq Q'$. Consequently, we obtain $M_1 = c(S) \subseteq Q'$. This implies that M_1 is a direct summand of Q', and thus $M_1 \oplus Q$ is a direct summand of M. As a result, we can deduce that $M_1 \oplus \pi(Q)$ is a direct summand of M, $\pi(Q)$ is a direct summand of M_2, and $H \oplus \pi(Q)$ is an essential submodule of M_2. By utilizing Proposition 3.3.4., we conclude that M_2 satisfies C_{11}. $\square$

Lemma 3.3.11. *Consider a module M with a direct summand N and an injective submodule K such that $N \cap K = 0$. We claim that $N \oplus K$ is a direct summand of M.*

Proof. Let N' be a submodule of M such that $M = N \oplus N'$. Consider the canonical projection $\pi : M \to N'$. Since $N \cap K = 0$, we have $K \cong \pi(K)$, which implies that $\pi(K)$ is injective. Hence, $\pi(K)$ is a direct summand of N'. Now, we observe that $N \oplus K = N \oplus \pi(K)$ since $\pi(K) \subseteq N'$. Therefore, $N \oplus K$ is a direct summand of M. $\square$

Proposition 3.3.12. *Suppose M is a module that fulfills the property C_{11}, and let N be a direct summand of M such that the quotient module M/N is injective. Then, N also satisfies C_{11}.*

Proof. Consider any submodule L of N. Let N' be an injective submodule of M such that $M = N \oplus N'$. Now, consider the submodule $L \oplus N'$. There exists a direct summand K of M such that $(L \oplus N') \cap K = 0$ and $(L \oplus N') \oplus K$ is an essential submodule of M (according to Proposition 3.3.4.). By Lemma 3.3.11., $N' \oplus K$ is a direct summand of M. Notably, $N' \oplus K = N' \oplus \pi(K)$, where $\pi : M \to N$ is the canonical projection. Consequently, $\pi(K)$ is a direct summand of N. Moreover, $L \oplus \pi(K) \oplus N'$ is an essential submodule of M. Hence, $L \oplus \pi(K)$ is an essential submodule of N. According to Proposition 3.3.4., N satisfies C_{11}. $\square$

Lemma 3.3.13. *Assume $M = M_1 \oplus M_2$. It follows that M_1 satisfies C_{11} if and only if, for every submodule N of M_1, there exists a direct summand K of M such that M_2 is contained in K, N intersects K trivially ($K \cap N = 0$), and $K \oplus N$ is an essential submodule of M.*

Proof. Assume that M_1 satisfies C_{11}. Let N be any submodule of M_1. According to Proposition 3.3.4., there exists a direct summand L of M_1 such that N intersects L trivially ($N \cap L = 0$) and $N \oplus L$ is an essential submodule of M_1. It is evident that $(L \oplus M_2) \cap N = 0$, and $(L \oplus M_2) \oplus N$ is an essential submodule of M.

Conversely, assume that M_1 satisfies the given property. Let H be a submodule of M_1. By hypothesis, there exists a direct summand K of M such that M_2 is contained in K, K intersects H trivially ($K \cap H = 0$), and $K \oplus H$ is an essential submodule of M. Now, observe that $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2$. This implies that $K \cap M_1$ is a direct summand of M, and thus a direct summand of M_1. Additionally, $H \cap (K \cap M_1) = 0$, and $H \oplus (K \cap M_1) = M_1 \cap (H \oplus K)$, which is an essential submodule of M_1. According to Proposition 3.3.4., M_1 satisfies C_{11}. □

Theorem 3.3.14. *Suppose that $M = M_1 \oplus M_2$ is a C_{11}-module such that for every direct summand K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M. Then M_1 is also a C_{11}-module.*

Proof. Consider any submodule N of M_1. According to the given hypothesis, there exists a direct summand K of M such that $(N \oplus M_2) \cap K = 0$ and $(N \oplus M_2) \oplus K$ is an essential submodule of M, as stated in Proposition 3.3.4.. Additionally, $M_2 \oplus K$ is a direct summand of M. Now, by applying Lemma 3.3.13., we can conclude that N satisfies C_{11}. □

Corollary 3.3.15. *If M is a module satisfying C_{11} and K is a direct summand of M with the property that M/K is K-injective, then K also satisfies C_{11}.*

Proof. We can find a submodule K' of M such that $M = K \oplus K'$ and K' is K-injective, as assumed. Let L be a direct summand of M such that $L \cap K' = 0$. There exists a submodule H of M with $H \cap K' = 0$, $M = H \oplus K'$, and $L \subseteq H$. Since L is a direct summand of H, it follows that $L \oplus K'$ is a direct summand of $M = H \oplus K'$. By Theorem 3.3.14., we conclude that K satisfies C_{11}. □

Corollary 3.3.16. *Assume $M = M_1 \oplus M_2$ be a direct sum of a submodule M_1 and an injective submodule M_2. Then M satisfies C_{11} if and only if M_1 satisfies C_{11}.*

Proof. By Corollary 3.3.15., if M satisfies C_{11}, then M_1 satisfies C_{11}. Conversely, according to Theorem 3.3.5., if M_1 satisfies C_{11}, then M satisfies C_{11}. □

4 Module classes with conditional summand properties

In this chapter firstly we collect some basic results on conditional direct summand properties secondly we consider C_{11}-modules with a special conditional direct summand property. For a good source of references, please look at [12], [14].

4.1 Conditional direct summands

Direct summands of a module hold significant importance in Ring and Module Theory and play a crucial role in our work. In this section, we specifically concentrate on this type of submodules. Following a common algebraic approach, we begin with the direct summand(s) of a module and introduce a condition that utilizes these direct summands to generate a new direct summand.

Let M be a right R-module. Recall the following properties of M:

(i) property C_2: if $X \leq M$ is isomorphic to a direct summand of M, then X is a direct summand of M; in other words, for each direct summand N of M and each monomorphism $\varphi : N \to M$, the submodule $\varphi(N)$ is also a direct summand of M;

(ii) property C_3: if M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M;

Furthermore, we also recall the following another type of conditional direct summand property:

(iii) the summand intersection property, SIP: if M_1 and M_2 are direct summands of M, then $M_1 \cap M_2$ is a direct summand of M [5], [3], [7].

We can establish a correspondence between direct summands and idempotent endomorphisms of a module. Let $M_R = K \oplus K'$. The canonical projection $\pi : M \to K$ is an idempotent endomorphism of M with $\pi^2 = \pi \in \text{End}(M_R)$, and $K = \pi M$ (where π

is a left operator on M). Thus, each direct summand of M corresponds to the image of an idempotent endomorphism of M. Conversely, if $e^2 = e \in \mathrm{End}(M_R)$, then $1 - e$ is an idempotent in $\mathrm{End}(M_R)$, and $M_R = eM \oplus M(1 - e)$.

Before establishing the relationships between the C_2, C_3, and SIP properties, we shall now present a straightforward lemma that will be employed at various points throughout our subsequent analysis.

Lemma 4.1.1. *Suppose $M = N \oplus N'$. Let $K \leq M$ with $N \cap K = 0$. We have $N \oplus K = N \oplus \pi(K)$, where $\pi : M \to N'$ is the canonical projection.*

Proof. Consider $x \in N \oplus K$. Then $x = n + k$, where $n \in N$ and $k = y + y'$ for some $y \in N$ and $y' \in N'$ such that $y' = \pi(k)$. Hence, $x = n + y + \pi(k) \in N + \pi(K)$. This implies that $N \oplus K \leq N + \pi(K) = N \oplus \pi(K)$.

Now, let $m \in N \oplus \pi(K)$. Then $m = b + c$, where $b \in N$, $c \in \pi(K)$, and $c = \pi(d)$ for some $d \in K$ where $d = e + c$ for some $e \in N$. Hence, $m = b + (d - e) = (b - e) + d \in N \oplus K$. Thus, we have $N \oplus K = N \oplus \pi(K)$, as required. □

Lemma 4.1.2. *If a module M satisfies property C_2, then it also satisfies property C_3.*

Proof. Let K and L be direct summands of M with $K \cap L = 0$. We have $M = K \oplus K'$ for some $K' \leq M$. Let $\pi : M \to K'$ denote the canonical projection. Since $K \cap L = 0$, we have $\pi(L) \cong L$ and $\pi(L) \leq K'$. By C_2, $\pi(L)$ is a direct summand of M, i.e., $M = \pi(L) \oplus L'$ for some $L' \leq M$. Thus, $K' = \pi(L) \oplus (K' \cap L')$ and $M = K \oplus \pi(L) \oplus (K' \cap L')$. Hence, $K \oplus \pi(L)$ is a direct summand of M. Moreover, $K \oplus L = K \oplus \pi(L)$. Thus, M satisfies property C_3. □

The following example demonstrates that none of the implications $C_3 \implies C_2$, SIP $\implies C_3$, and $C_3 \implies$ SIP hold in general.

Example 4.1.3.

(i) Assume $\mathbb{Z}$ be the $\mathbb{Z}$-module. Then $\mathbb{Z}$ satisfies C_3 but $\mathbb{Z}$ does not satisfy C_2.

(ii) If M is a free $\mathbb{Z}$-module of non-zero finite rank k, then M satisfies C_3 if and only if $k = 1$. Consequently, if $M = \bigoplus_{i=1}^{k} \mathbb{Z}$ with $k \geq 2$, then M has SIP but does not satisfy

C_3.

(iii) Assume $\mathbb{Z}$-module $M = \mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})$, where p is a prime integer, satisfies C_3 but does not satisfy SIP.

Proof. (i) The module $\mathbb{Z}$ satisfies C_3 since it is indecomposable. However, the submodule $N = 2\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ but is not a direct summand of $\mathbb{Z}$, illustrating that $\mathbb{Z}$ does not satisfy C_2.

(ii) For $k \geq 2$, let $M = \bigoplus_{i=1}^{k} \mathbb{Z}$ and consider the submodules $K_1 = f_1\mathbb{Z}$ and $K_2 = (f_1 + 2f_2)\mathbb{Z}$. We have $M = K_1 \oplus L = K_2 \oplus L$, where $L = f_2\mathbb{Z} + \ldots + f_k\mathbb{Z}$, and $K_1 \cap K_2 = 0$. However, $K_1 \oplus K_2 = f_1\mathbb{Z} \oplus 2f_2\mathbb{Z}$ is not a direct summand of M, illustrating that M does not satisfy C_3. Thus, $\bigoplus_{i=1}^{k} \mathbb{Z}$ $(k \geq 2)$ does not satisfy C_3, but it has the strong exchange property (SIP) as shown in [13].

(iii) Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})$, where p is a prime integer. Let $B = \mathbb{Z}(1, 0 + p\mathbb{Z})$ and $C = \mathbb{Z}(1, 1 + p\mathbb{Z})$ be direct summand submodules of M. However, $B \cap C$ is not a direct summand of M.[13]. □

Lemma 4.1.4. *Assume M be a module and $N \leq_d M$. Then*

(i) If M satisfies C_2 then N satisfies C_2.

(ii) If M satisfies C_3 then N satisfies C_3.

(iii) If M satisfies SIP then N satisfies SIP.

Proof. (i) Assume X and K be submodules of N such that X is isomorphic to K and K is a direct summand of N. By C_2, we have X is a direct summand of M. Therefore, M can be decomposed as $M = X \oplus X'$ for some submodule X' of M. Using the modular law, we can show that N can also be decomposed as $N = X \oplus (N \cap X')$. This implies that X is a direct summand of N.

(ii) Assume K_1 and K_2 be direct summands of N such that $K_1 \cap K_2 = 0$. According to C_3, we have $K_1 \oplus K_2$ is a direct summand of M. Therefore, M can be decomposed as $M = (K_1 \oplus K_2) \oplus K_3$ for some submodule K_3 of M. By applying the modular law, we find that $N = (K_1 \oplus K_2) \oplus (N \cap K_3)$, which implies that $K_1 \oplus K_2$ is a direct summand of N.

(iii) Assume K, L be direct summands of N. So K, L modules are direct summands of M. By hypothesis, $K \cap L \leq_d M$. Hence $M = (K \cap L) \oplus X$ for some $X \leq M$. By applying the modular law, $N = (K \cap L) \oplus (N \cap X)$, i.e., $K \cap L \leq_d N$. □

Lemma 4.1.5. *Let M be a right R-module, where $R = ReR$ for some idempotent e in R and $S = eRe$. For submodules $K, K' \leq M_R$ and $N, N' \leq (Me)_S$, we have the following:*

(i) $K = KeR$ and $N = NeR$.

(ii) $K \cap K' = 0$ if and only if $Ke \cap K'e = 0$.

(iii) $N \cap N' = 0$ if and only if $NR \cap N'R = 0$.

Proof. (i) Due to the fact that K is a submodule of M, we can express K as $K = KR = KReR = KeR$. Similarly, we can express N as $N = NS = NeRe = NeR$.

(ii) If $K \cap K' = 0$, then the condition $Ke \cap K'\text{e} \leq K \cap K'$ implies that $Ke \cap K'\text{e} = 0$. Conversely, if $Ke \cap K'\text{e} = 0$, let $x \in K \cap K'$. Then $xRe \leq Ke \cap K'\text{e} = 0$, which implies $xReR = 0$. Consequently, $xR = 0$, and hence $x = 0$. Thus, we conclude that $K \cap K' = 0$.

(iii) From (i) and (ii). □

Lemma 4.1.6. *Let $R = ReR$ and $S = eRe$ for some idempotent e in R, and let M be a right R-module. Suppose L and N are submodules of $(Me)_S$. Then L is a complement of N in $(Me)_S$ if and only if LR is a complement of NR in M_R.*

Proof. Let L be a complement of N in Me. Then L and N have disjoint intersections, i.e., $L \cap N = 0$, and consequently, $LR \cap NR = 0$. Assume $LR \leq K \leq M_R$ and $K \cap NR = 0$. By Lemma 4.1.5. (i), we have $L = LRe \leq Ke \leq Me$ and $Ke \cap N \leq K \cap NR = 0$. Hence, we conclude that $L = Ke$ and $LR = KeR = K$, which implies that LR is a complement of NR in M.

For the converse, suppose that LR is the complement of NR in M. This implies that the intersection of L and N is the zero element. Suppose we have the following inclusion relations: $L \leq H \leq (Me)_S$, and $H \cap N = 0$. According to Lemma 4.1.5, we know that $LR \leq RH$, and $RH \cap NR = 0$. Consequently, we can conclude that $LR = RH$. By transitivity, we have $L = LRe = HRe = H$, once again utilizing Lemma 4.1.5. Thus, we can assert that L is the complement of N in Me. □

Proposition 4.1.7. *Assume M be a right R-module, and consider L as a submodule of M, where $R = ReR$ for some idempotent element e in R, and $S = eRe$. Then*

(i) $L \leq_e M_R$ if and only if $Le \leq_e (Me)_S$,

(ii) $L \leq_c M_R$ if and only if $Le \leq_c (Me)_S$,

(iii) $L \leq_d M_R$ if and only if $Le \leq_d (Me)_S$.

Proof. (i) Suppose $L \leq_e M_R$, where L is a submodule of M and $R = ReR$ for some idempotent element e in R. Let N be a nonzero submodule of $(Me)_S$, such that $0 \neq N \leq (Me)_S$. Using Lemma 4.1.5, we have $K = KRe$ for any nonzero submodule $K \leq M_R$. Applying this lemma, we find that $0 \neq Ke \leq (Me)_S$. Therefore, we have $Ke \cap Le \neq 0$. Conversely, if $Le \leq_e (Me)_S$, then for any nonzero submodule $K \leq M_R$, we have $K = KRe$ according to Lemma 4.1.5. Thus, $0 \neq Ke \leq (Me)_S$. Consequently, $Ke \cap Le \neq 0$. So $K \cap L \neq 0$. Then $L \leq_e M_R$.

(ii) From Lemma 4.1.6.

(iii) Let L is a submodule of M such that $L \leq_d M_R$. Then M can be expressed as the direct sum $M = L \oplus L'$ for some submodule L' satisfying $L' \leq M_R$. Consequently, we have $Me = Le + L'e$. However, it is true that $Le \cap L'e \leq L \cap L' = 0$. Hence, we can conclude that $Me = Le \oplus L'e$. Conversely, let's assume that $Me = Le \oplus K$ for some submodule K satisfying $K \leq (Me)_S$. According to Lemma 4.1.5, we know that $L \cap KR = 0$. Furthermore, we have $M = MRe = (Le + K)R = LeR + KR = L + KR$. Thus, we can express M_R as the direct sum $M_R = L \oplus KR$, which implies that $L \leq_d M_R$. □

Proposition 4.1.8. *Let M be a right R-module where $R = ReR$ for some idempotent e in R and $S = eRe$. Then*

(i) e soc(M_R) = soc($(Me)_S$). In particular, M_R is semisimple if and only if $(Me)_S$ is semisimple,

(ii) Ze(M_R) = Z($(Me)_S$). In particular, M_R is nonsingular if and only if $(Me)_S$ is nonsingular.

Proof. (i) By Lemma 4.1.5. and Proposition 4.1.7.

(ii) Assume me $\in Ze(M_R)$. Then $m \in Z(M_R)$. There exists an essential right ideal F of R such that $meF = 0$. By Proposition 4.1.7. (i), $Re \cap F$ is essential in Re and hence

$(Re \cap F)e$ is essential in $(eRe)_S = S$. But $me \in Me$ and $(Re \cap F)e \leq Fe \leq F$. Thus $me[(Re \cap F)e] = 0$, and so $me \in Z((Me)_S)$. Now, let $me \in Z((Me)_S)$. Then $meG = 0$ for some essential right ideal G of S. By Proposition 4.1.7., RG is essential in Re. Thus $RG \oplus R(1 - e)$ is essential in R_R. Since $me[RG \oplus R(1 - e)] = 0$, we have $me \in Z(M_R)$ and hence $me \in Ze(M_R)$. The second part is clear. □

Theorem 4.1.9.. *Assume M be a right R-module, where $R = ReR$ for some idempotent e in R and $S = eRe$. Then*

(i) the right R-module M satisfies C_2 if and only if the right S-module Me satisfies C_2,

(ii) the right R-module M satisfies C_3 if and only if the right S-module Me satisfies C_3,

(iii) the right R-module M has the SIP if and only if the right S-module Me has the SIP.

Proof. (i) Assume K and L be submodules of M such that $K \cong L$, where L is a direct summand of M_R. Consider an R-isomorphism $f : K \to L$. Then Le is also a direct summand of $(Me)S$. Let $\varphi = f|Ke : Ke \to Le$ be an isomorphism. Thus, Ke is a direct summand of Me, and consequently, K is a direct summand of M_R. Conversely, let B and C be submodules of Me such that $B \cong C$, and C is a direct summand of $(Me)S$. Hence, CR is a direct summand of M_R. Suppose $\varphi : B \to C$ is an isomorphism. Define $\theta : BR \to CR$ and $\theta' : CR \to BR$ by $\theta\left(\sum i = 1^n r_i b_i\right) = \sum_{i=1}^n r_i\varphi(b_i)$ and $\theta'\left(\sum_{i=1}^n r_i c_i\right) = \sum_{i=1}^n r_i\varphi^{-1}(c_i)$ for all $n \geq 1$, $b_i \in B$, $c_i \in C$, and $r_i \in R$ $(1 \leq i \leq n)$. Now, suppose $\sum_{i=1}^n r_i b_i = 0$. Then $\sum_{i=1}^n ser_i b_i = 0$ for all $s \in R$. Therefore, $\sum_{i=1}^n ser_i eb_i = 0$, and hence $\sum_{i=1}^n ser_i e\varphi(b_i) = 0$. Thus, $\sum_{i=1}^n ser_i\varphi(b_i) = 0$. It follows that $Re\left(\sum_{i=1}^n r_i\varphi(b_i)\right) = 0$, implying $ReR\left(\sum_{i=1}^n r_i\varphi(b_i)\right) = 0$. That is, $\sum_{i=1}^n r_i\varphi(b_i) = 0$. Therefore, we see that θ is a well-defined mapping. It can be easily verified that θ is an R-homomorphism. Similarly, θ' is an R-homomorphism. Clearly, $\theta'\theta = 1|_{RB}$ and $\theta\theta' = 1|_{RC}$. Hence, θ is an isomorphism. By the hypothesis, BR is a direct summand of M_R. Then, using Proposition 4.1.7, we can conclude that B is a Let B and C be direct summands of $(Me)_S$ with $B \cap C = 0$. Thus, we have $B = BRe$ and $C = CRe$. Consequently, BR and CR are direct summands of M_R. Since $BRe \cap CRe = B \cap C = 0$, it follows that $BR \cap CR = 0$ in M_R. Hence, $BR \oplus CR$ is a direct summand of M_R. By Proposition 4.1.7, we can deduce that $B \oplus C = (BR \oplus CR)e$

is a direct summand of $(Me)_S$. Conversely, let K and L be direct summands of M_R with $K \cap L = 0$. Thus, Ke and Le are direct summands of $(Me)_S$. Consequently, $Ke \cap Le \leq K \cap L = 0$, which implies that $(Ke \oplus Le)R = KeR \oplus LeR = K \oplus L$ is a direct summand of M_R.

(iii) Clear from Proposition 4.1.7. □

Corollary 4.1.10. *Consider a ring R such that $R = ReR$ for an idempotent element e in R. Then, the right R-module R_R satisfies property C_2 (resp., C_3 or SIP) if and only if the right eRe-module Re satisfies property C_2 (resp., C_3 or SIP).*

Proof. By Theorem 4.1.9, this result follows immediately. □

Proposition 4.1.11. *The following statements are equivalent for a module M.*

(i) M has C_2.

(ii) $\underline{M'} \subseteq Lift_X(M)$ for all right R-modules X.

(iii) $\underline{M'} \subseteq Lift_X(M)$ for all $X \in \underline{M'}$.

(iv) $\underline{M'} \subseteq Lift_M(M)$.

Proof. (i) $\iff$ (ii) $\iff$ (iii) based on Theorem 2.2.16.

(iii) $\implies$ (iv) is clear.

(iv) $\implies$ (i) Let $N' \in \underline{M'}$. Then there exists $N \leq_d M$ and an isomorphism $\varphi : N' \to N$. According to Theorem 2.2.11, $Lift_M(M) \subseteq Lift_N(M)$. Hence, by (iv), $N' \in Lift_N(M)$, and there exists $\theta \in Hom_R(M, N)$ such that $\theta|_{N'} = \varphi$. For any $m \in M$, $\theta(m) \in N$, so we have $\theta(m) = \varphi(n')$ for some $n' \in N'$. Consequently, $\theta(m) = \theta(n')$, and thus $m - n' \in \ker \theta$. It follows that $M = N' + (\ker \theta)$. However, $N' \cap (\ker \theta) = \ker \varphi = 0$. Therefore, $M = N' \oplus (\ker \theta)$. Hence, M satisfies C_2. □

Proposition 4.1.12. *The following statements are equivalent for a module M.*

(i) M has C_3.

(ii) $\underline{M}^{(2)} \subseteq Lift_X(M)$ for all right R-modules X.

(iii) $\underline{M}^{(2)} \subseteq Lift_X(M)$ for all $X \in \underline{M}^{(2)}$.

(iv) $\underline{M}^{(2)} \subseteq Lift_M(M)$.

Proof. (i) $\iff$ (ii) $\iff$ (iii) based on Theorem 2.2.16.

(iii) $\implies$ (iv) is clear.

(iv) $\implies$ (i) Assume $K, L \leq_d M$ with $K \cap L = 0$. Consider the canonical projection $\pi : K \oplus L \to K$. By (iv) and Theorem 2.2.11, we have $K \oplus L \in Lift_M(M) \subseteq Lift_K(M)$, which implies the existence of $\theta \in Hom_R(M, K)$ such that $\theta|_{K\oplus L} = \pi$. Consequently, we have $M = K \oplus \ker \theta$. Moreover, since $\theta(L) = \pi(L) = 0$, we have $L \subseteq \ker \theta$. Considering that $M = L \oplus L'$ for some submodule L' of M, we can express $\ker \theta$ as $\ker \theta = L \oplus (\ker \theta \cap L')$. Consequently, we obtain $M = K \oplus L \oplus (\ker \theta \cap L')$. Thus, M satisfies C_3. □

Corollary 4.1.13. *Assume M be a module. If M has C_3, then for any integer $n \geq 3$, every element of $\underline{M}^{(n)}$ is a direct summand of M.*

Proof. Consider $L \in \underline{M}^{(n)}$. Then $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$, where $L_i \leq_d M$ for $1 \leq i \leq n$. Using induction, we can establish that $L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1} \leq_d M$. Therefore, by applying property C_3, we conclude that $L_1 \oplus L_2 \oplus \cdots \oplus L_n \leq_d M$. □

Proposition 4.1.14. *Let X be any right R-module. Then the following statements are equivalent for a module M.*

(i) $\underline{C}^{(2)} \subseteq Lift_X(M)$.

(ii) $\underline{C}^{(n)} \subseteq Lift_X(M)$ for all $n \geq 2$.

Proof. (ii) $\implies$ (i) Clear.

(i) $\implies$ (ii) Assume that (i) holds. Let $k \geq 3$ and $N_i \leq_c M$ $(1 \leq i \leq k)$ be submodules such that $N_1 + N_2 + \cdots + N_k$ is a direct sum. Let $N = N_1 + N_2 + \cdots + N_k$ and $\varphi \in Hom_R(N, X)$. There exists a submodule $N' \leq_c M$ such that $N_2 + N_3 + \cdots + N_k \leq_e N'$. By induction, we have $N_2 + N_3 + \cdots + N_k \in Lift_X(M)$, and therefore there exists $\alpha \in Hom_R(M, X)$ such that $\alpha(m) = \varphi(m)$ for $m \in N_2 + N_3 + \cdots + N_k$. Now, $N_1 \cap N' = 0$ because $N_1 \cap (N_2 + N_3 + \cdots + N_k) = 0$. Hence, we can define $\beta \in Hom_R(N_1 \oplus N', X)$ by $\beta(n + n') = \varphi(n) + \alpha(n')$ for $n \in N_1$ and $n' \in N$. Then, according to (i), there exists $\delta \in Hom_R(M, X)$ such that $\delta|_{N_1 \oplus N'} = \beta$. For any $n_i \in N$ $(1 \leq i \leq k)$, we have $\delta(n_1 + n_2 + \cdots + n_k) = \beta(n_1 + n_2 + \cdots + n_k) = \varphi(n_1) + \alpha(n_2 + \cdots + n_k) = \varphi(n_1) + \varphi(n_2 + \cdots + n_k) = \varphi(n_1 + n_2 + \cdots + n_k)$. Thus, $\delta|_N = \varphi$. It follows that $N \in Lift_X(M)$. Hence, $\underline{C}^{(k)} \subseteq Lift_X(M)$. □

Corollary 4.1.15. *$\underline{C}^2 \subseteq Lift_M(M)$ if and only if M has SIP.*

Proof. By Proposition 4.1.14. □

4.2 C_{11}-modules with conditional direct summand properties

In this section, we discuss several results concerning C_{11} modules with a conditional direct summand property. It is worth noting that direct summands of modules with C_{11} may not necessarily satisfy C_{11} themselves. This is in contrast to the behavior observed in CS-modules. Let P be a property of modules. We define the notion of "P^+" for a module M, indicating that every direct summand of M satisfies property P. For instance, an indecomposable module satisfies P^+ if and only if it satisfies property P. In the case of C_1 (i.e., CS), a module M satisfies C_1 if and only if it satisfies C_1^+. This equivalence can be abbreviated as $C_1^+ = C_1$ [13].

Before proceeding further, let us consider the following example.

Example 4.2.1. Consider the $\mathbb{Z}$-module $M = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$, where p is a prime number. We observe that M satisfies C_{11}^+ and C_2, but it does not satisfy C_1.

Proof. According to Corollary 3.3.7, the module M satisfies C_{11}. Since M has uniform dimension 2, it also satisfies C_{11}^+. However, M does not satisfy C_1 as it contains a complement submodule $K = R(1 + p\mathbb{Z}, 1)$ that is not a direct summand. Here, R represents the local ring $\mathbb{Z}(p)$.

Now we will establish that M satisfies C_2. Let L be a non-zero direct summand of M. If $L \neq M$, then L is uniform due to M having a uniform dimension of 2. Specifically, L can be expressed as $(\mathbb{Z}/p\mathbb{Z}) \oplus 0$, $0 \oplus \mathbb{Q}$, or $R(1 + p\mathbb{Z}, q)$, where q is a non-zero element in $\mathbb{Q}$ [13].

We have $M = L \oplus L'$, where L' is a submodule of M. Suppose $L = R(1 + p\mathbb{Z}, q)$ for some nonzero $q \in \mathbb{Q}$. Then $pL \cap L' = 0$, which implies $R(0, pq) \cap L' = 0$. Consequently, $L' \cap (0 \oplus \mathbb{Q}) = 0$. This implies that L' embeds in $\mathbb{Z}/p\mathbb{Z}$, which is a simple module. Therefore, $L' = (\mathbb{Z}/p\mathbb{Z}) \oplus 0$. Thus, $M = L \oplus L' = (\mathbb{Z}/p\mathbb{Z}) \oplus qR$, which contradicts the fact that $\mathbb{Q} \neq qR$. Hence, we conclude that $L = (\mathbb{Z}/p\mathbb{Z}) \oplus 0$ or $L = 0 \oplus \mathbb{Q}$.

Assume $\varphi : L \to M$ be a monomorphism. If $L = (\mathbb{Z}/p\mathbb{Z}) \oplus 0$, then $\varphi(L)$ is a simple submodule, so $\varphi(L) = L$. If $L = 0 \oplus \mathbb{Q}$, then $\varphi(L)$ is torsion-free injective, and hence $\varphi(L) = L$. If $L = M$, then $\varphi(L) = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q} = L$. Thus, $\varphi(L) = L$ holds for every direct summand L of M and monomorphism $\varphi : L \to M$. Hence, M satisfies C_2. □

Proposition 4.2.2.. *Assume M is a C_{11}-module and X is a submodule. If the intersection of X with any direct summand of M is itself a direct summand of X, then X is also a C_{11}-module.*

Proof. Consider a submodule B of X. We can find a direct summand N of M such that $B \cap N = 0$ and $B \oplus N$ is an essential submodule of M. Since $M = N \oplus K$ for some submodule K of M, we have $X \cap (B \oplus N) = B \oplus (X \cap N)$, which is an essential submodule of X. Based on the given hypothesis that $X \cap N$ is a direct summand of X, we conclude that X satisfies the C_{11} condition. □

Corollary 4.2.3. *Let M be a C_{11}-module.*

(i) If M is a distributive module, then every submodule of M is a CS-module.

(ii) If X is a submodule of M such that $Xe \subseteq X$ for all idempotent endomorphisms $e^2 = e \in End(M_R)$, then X is a C_{11}-module. In particular, every fully invariant submodule of M is a C_{11}-module.

(iii) If M has the SIP property, then M satisfies the C_{11}^{+} condition.

Proof. (i) Let N be a complement submodule of M. We can find an idempotent endomorphism $e^2 = e \in \text{End}(M_R)$ such that Me is a complement of N. Then we have $N = N \cap M = N \cap (Me \oplus M(1-e)) = (N \cap Me) \oplus (N \cap M(1-e)) = N \cap M(1-e) \leq M(1-e)$. Since N is a complement submodule in M, it follows that $N = M(1-e)$. Hence, M is a CS-module. By Corollary 3.1.13, every submodule of M is also a CS-module.

(ii) Consider a direct summand D of M, and let $e : M \to D$ be the canonical projection. We have $Xe = D \cap X$. According to Proposition 4.2.2, since $Xe = D \cap X$, X is a C_{11}-module.

(iii) This is a direct implication of Proposition 4.2.2. □

Theorem 4.2.4. *If M is a module satisfying C_{11} and C_2, then the quotient ring S/Δ is a von Neumann regular ring, and Δ is equal to the Jacobson radical J.*

Proof. Let $\alpha \in S$ and $K = \ker(\alpha)$. By the C_{11} condition, there exists a direct summand L of M that is a complement of K in M. Since $\alpha|_L$ is a monomorphism, $\alpha(L)$ is a direct summand of M by the C_2 condition. Hence, there exists $\beta \in S$ such that $\beta\alpha = i|_L$. Then $(\alpha - \alpha\beta\alpha)(K \oplus L) = (\alpha - \alpha\beta\alpha)(L) = 0$, which implies $K \oplus L \leq \ker(\alpha - \alpha\beta\alpha)$. Since $K \oplus L$ is an essential submodule of M, it follows that $\alpha - \alpha\beta\alpha \in \triangle$. Therefore, $S/\triangle$ is a regular ring. This also proves that $J \leq \triangle$.

Consider $m \in \triangle$. Since $\ker(m) \cap \ker(1-m) = 0$ and $\ker(m)$ is essential in M, we have $\ker(1-m) = 0$. Thus, $M(1-m)$ is a direct summand of M by the C_2 condition. Moreover, $M(1-m)$ is also essential in M since $\ker(m) \leq M(1-m)$. Therefore, $M(1-m) = M$, implying that $1-m$ is a unit in S. Consequently, we conclude that $m \in J$, which implies $\triangle \leq J$. □

Lemma 4.2.5. *For a nonsingular right R-module M, we have $\triangle = 0$.*

Proof. Consider $f \in \triangle$ and let $N = \ker(f)$. For any $x \in M$, there exists an essential right ideal L of R such that $0 \neq xL \leq N$. Consequently, we have $f(x)L = 0$. Since M is nonsingular, we conclude that $f(x) = 0$. Since x was arbitrary, it follows that $f = 0$. □

Corollary 4.2.6. *For a nonsingular right R-module M satisfying C_{11} and C_2, the ring S is a von Neumann regular ring.*

Proof. Based on Lemma 4.2.16, we have $\triangle = 0$. Therefore, the result follows from Theorem 4.2.5. □

5 Conditional direct summand properties relative to fully invariant submodules

This last chapter consists of two sections. First section exhibits basic properties of the class of fully invariant submodules of a module. Section two provides approaches to build up new classes of modules on using fully invariant submodules.

5.1 Fully invariant submodules

In module and ring theory, using basic and elite submodules to learn about the module or starting from these special submodules and researching a new module class are the rooted problems. Research in this direction is still being carried out for different classes of invariant submodules. In this context we will introduce fully invariant submodules in this section. [6], [12].

Definition 5.1.1. Let M be a right R-module and S be the ring of $\mathrm{End}(M_R)$. A submodule X of M is called *fully invariant* written $X \trianglelefteq M$ if $f(X) \subseteq X$ for all $f \in S$. According to this the Jacobson radical $J(R)$ of a ring R, the socle submodule $\mathrm{soc}(M_R)$ of a module M, the singular submodule $Z(M)$, the second singular submodule $Z_2(M)$, the torsion submodule $T(M)$ are examples of fully invariant submodules [4], [6], [12].

Lemma 5.1.2. *Let R be a ring and M be a right R-module. Then,*
(i) A right ideal I is fully invariant in R if and only if I is an ideal in R.
(ii) Let M be a multiplicative module. Then every submodule of M are fully invariant.

Proof. (i) Since $R \cong \mathrm{End}(R_R)$ the proof is clear.

(ii) Let M be a multiplicative module and $N \leq M$. Hence there exists an ideal I of R such that $N = IM$. Since $f(N) = f(IM) \subseteq \mathrm{If}(M) \subseteq IM = N$ for all $f \in \mathrm{End}(M_R)$ we get $N \trianglelefteq M$. □

Proposition 5.1.3. *Let M be a right R-module and S = End(M_R). So the followings are provided.*
(i) If $\{N_i : i \in I\}$ family of fully invariant submodules of M, then $\bigcap_{i\in\ I} N_i \trianglelefteq M$ and $\sum_{i\in\ I} N_i \trianglelefteq M$.
(ii) Let $X \leq Y \leq M$ submodules given. If $X \trianglelefteq Y$ and $Y \trianglelefteq M$ then $X \trianglelefteq M$.
(iii) If $M = \bigoplus_{i\in\ I} M_i$ and $N \trianglelefteq M$ for $M_i \leq M (i \in I)$ then $N = \bigoplus_{i\in\ I} (N \cap M_i)$.
(iv) If $M = \bigoplus_{i=1}^{n} M_i$ and $N \leq_d M$ for $M_i \trianglelefteq M$ ($i = 1, \dots, n$) then $N = \bigoplus_{i=1}^{n} (M_i \cap N)$.

Proof. (i) Let $\{N_i : i \in I\}$ family of fully invariant submodules of M. Take $f \in S$. $f(\bigcap_{i\in\ I} N_i) \subseteq \bigcap_{i\in\ I} f(N_i) \subseteq \bigcap_{i\in\ I} N_i$ and hence we get $\bigcap_{i\in\ I} N_i \trianglelefteq M$. Similarly, since $f(\sum_{i\in\ I}$

$N_i) = \sum_{i\in\ I} f(N_i) \subseteq \sum_{i\in\ I} N_i$ we get $\sum_{i\in\ I} N_i \trianglelefteq M$.

(ii) Let $X \trianglelefteq M$ and $Y \trianglelefteq M$ for $X \leq Y \leq M$. Hence $\alpha(Y) \subseteq Y$ for all α in S. If $\alpha|_Y : Y \to \alpha(Y)$ since $\alpha|_Y \in \mathrm{End}(Y_R)$ then $(\alpha|_Y)(X) \subseteq X$. Thus $\alpha(X) \subseteq (\alpha|_Y)(X) \subseteq X$ and $X \trianglelefteq M$.

(iii) Let $M = \bigoplus_{i\in\ I}$ and $N \trianglelefteq M$. Hence it is clear that since $N \cap M_i \subseteq N$ then $\bigoplus_{i\in\ I} (N \cap M_i)$ for all $i \in I$. Now $\pi_i : M \to M_i$ being a projection mapping, then from the definiton $\pi_i(n) = m_i \in M_i$. Moreover since $N \trianglelefteq M$ then $\pi_i(N) \subseteq N$ and we get $\pi_i(n) = m_i \in N \cap M_i$. Thus $n = \sum_{i\in\ I} \pi_i(n)$ and this implies that $n \in \bigoplus_{i\in\ I} (N \cap M_i)$. Hence $N = \bigoplus_{i\in\ I} (N \cap M_i)$.

(iv) Let $M = \bigoplus_{i=1}^{n} M_i$ and $N \leq_d M$ for $M_i \trianglelefteq M$ $(i = 1,..., n)$. Hence there exists $V \leq M$ such that $M = N \oplus V$. Since $M_i \trianglelefteq M$ by (iii), we get $M_i = (M_i \cap N) \oplus (M_i \cap V)$ for all $i = 1,..., n$. So $M = \bigoplus_{i=1}^{n} M_i = [\bigoplus_{i=1}^{n} (M_i \cap N)] \oplus [\bigoplus_{i=1}^{n} (M_i \cap V)]$. By modular law, we get $N = [\bigoplus_{i=1}^{n} (M_i \cap N)] \oplus [\bigoplus_{i=1}^{n} (M_i \cap V) \cap N]$. Hence since $V \cap N = 0$ $N = \bigoplus_{i=1}^{n} (M_i \cap V)$. □

Example 5.1.4. (i) Let R be a ring and $M_R = R \oplus R$. Take the submodule $X = R \oplus 0$ of M. It is clear that X is a direct summand in M. Now define a map f: $M \to M$, $f(x, y) = (y, x)$. Then f is an R-homomorphism and we obtain $f \in \mathrm{End}(M_R)$. Since $f(X) = 0 \oplus R$ then $f(X) \nsubseteq X$. Thus X is a direct summand in M but X is not a fully invariant submodule.

(ii) Let F be a field, V be a F-vector space and $\dim(V) \geq 1$. Hence let say $R_R = \begin{bmatrix} F & & V \\ & \ddots & \\ 0 & & F \end{bmatrix} = \{\left[\begin{smallmatrix} 0 & v \\ 0 & a \end{smallmatrix}\right] : m \in F,\, v \in V\}$. We take $I_v = \left[\begin{smallmatrix} 0 & vF \\ 0 & 0 \end{smallmatrix}\right] \leq R_R$ with $v \in V$. Since R is a commutative ring, every submodules of R are fully invariant. So I_v is fully invariant submodule. On the other hand since R_R is indecomposable, direct summands of R are only 0 and R. So I_v is not a direct summand of R_R.

Theorem 5.1.5. *Let M be a right R-module and $M = B \oplus C$ and $B \trianglelefteq M$. Then $B \oplus F \trianglelefteq M$ for $F \trianglelefteq C$.*

Proof. Let $h \in \mathrm{End}(M_R)$ and $b + c \in B \oplus F$ for $b \in B$, $c \in C$. Then $h(b + c) = h(b) + h(c) = h(b) + b_1 + c_1$ for $h(c) = b_1 + c_1$ with $b_1 \in B$ and $c_1 \in C$. Now $\pi : M \to C$ denote the projection and since $c_1 = \pi(h(c))$ and $B \trianglelefteq M$ then $h(b) + b_1 \in B$. Hence $\pi \mathrm{h}|_C : C \to C$ is an endomorphism. Since $F \trianglelefteq C$ then $(\pi \mathrm{h})c = c_1 \in F$ for $c \in F$. Thus $h(b + c) \in B \oplus F$ and we get $B \oplus F \trianglelefteq M$. □

Example 5.1.6. Let $R = T_2(\mathbb{Z}) = \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right]$. Let's find the left and right semicentral idempotent elements of R. Directly set $P = \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & b \\ 0 & 0 \end{smallmatrix}\right] : b \in \mathbb{Z}\}$ is the set of idempotent elements of R. Now take $\left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right] \in R$ and $\left[\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right] \in P$. Hence $\left[\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & bz \\ 0 & z \end{smallmatrix}\right]$. On the other hand since $\left[\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & bz \\ 0 & z \end{smallmatrix}\right]$ we get $\left[\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right] \in S_r(R)$. If similar ways apply $\left[\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right] \in P$ then $\left[\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right] \notin S_r(R)$. Hence the set of right semicentral idempotent elements of R is $S_r(R) = \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right] : b \in \mathbb{Z}\}$. Similarly we get the set of left semicentral idempotent elements of R is $S_l(R) = \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & b \\ 0 & 0 \end{smallmatrix}\right] : b \in \mathbb{Z}\}$.

5.2 FC_2, FC_3 and FSIP-modules

In this section, we define new generalizations of C_2, C_3 and SIP conditional direct summand properties on using fully invariant submodules and obtain their most basic properties. The detailed examination of such new module classes and their implications in the literature will form the basis for future studies.

Let's continue by giving the new definitions we mentioned above.

Definition 5.2.1.
FC_2 property: if $X \trianglelefteq M$ is isomorphic to a direct summand of M, then X is a direct summand of M.
FC_3 property: if M_1 is any fully invariant submodule and M_2 is any direct summand of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.
FSIP property: if M_1 is any fully invariant submodule and M_2 is any direct summand of M, then $M_1 \cap M_2$ is a direct summand of M.

Obviously $C_2 \implies FC_2$. However the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}$ does not satisfy C_2. It can be checked that M_R has FC_2.

Lemma 5.2.2. *If M_R satisfies FC_2 then M_R satisfies FC_3.*

Proof. Let K, $L \leq_d M_R$ with $K \cap L = 0$, $L \trianglelefteq M_R$. Then $M_R = K \oplus K'$ for some submodule $K' \leq M_R$. Let $\lambda : M \to K'$ be the projection map. Since $K \cap L = 0$, $\lambda(L) \cong L$ and $\lambda(L) \leq K'$. By FC_2 assumption $M = \lambda(L) \oplus L'$ for some $L' \leq M_R$. Hence $K' = \lambda(L) \oplus (K' \cap L')$ and $M = K \oplus \lambda(L) \oplus (K' \cap L')$. Thus $K \oplus \lambda(L)$ is a direct summand of M_R. By Lemma, $K \oplus L = K \oplus \lambda(L)$. It follows that M_R has FC_3. □

The following example shows that the converse of Lemma 5.2.2. does not true in general.

Example 5.2.3. (i) Let $R = \left[\begin{smallmatrix} K & K \\ 0 & K \end{smallmatrix}\right]$ where K is a field. Then R_R has FC_3. But R_R does not have FC_2.
(ii)Let $M_R = \mathbb{Z}_{\mathbb{Z}}$. Then M_R has FC_3. But M_R does not have FC_2.

Proof. (i) Let N $= \left[\begin{smallmatrix} 0 & K \\ 0 & 0 \end{smallmatrix}\right]$. Since N is an ideal of R then N $\trianglelefteq R_R$. Define an isomorphism $\varphi : \left[\begin{smallmatrix} 0 & K \\ 0 & 0 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} 0 & 0 \\ 0 & K \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right]$. Then N$\cong \left[\begin{smallmatrix} 0 & 0 \\ 0 & K \end{smallmatrix}\right] \leq_d R_R$. But N $\nleq_d R_R$.

(ii) Let N $= 2\mathbb{Z}$. Since N is an ideal of $\mathbb{Z}$ then N $\trianglelefteq M_R$. Define an isomorphism φ : $\mathbb{N}$ $\to \mathbb{Z}$, 2x $\mapsto$ x. Assume $2\mathbb{Z} \oplus$ X $= \mathbb{Z}$. Then X = m.$\mathbb{Z}$. But 2m $\in 2\mathbb{Z} \cap$ m$\mathbb{Z} = 0$. So m = 0, a contradiction. $2\mathbb{Z} \nleq_d \mathbb{Z}$. □

Theorem 5.2.4. *Let M_R be a module and N is a fully invariant direct summand of M. Then*
(i) If M satisfies FC_2 then N satisfies FC_2.
(ii) If M satisfies FC_3 then N satisfies FC_3.
(iii) If M satisfies FSIP then N satisfies FSIP.

Proof. (i) Let X, $K \leq N$ such that X fully invariant in N, K is a direct summand of N and $X \cong K$. Now K is a direct summand of M. By FC_2 assumption, X is a direct summand of M. Hence $M = X \oplus X'$ such that $X' \leq M$. By the Corollary 1.1.31., $N = N \cap M = N \cap (X \oplus X') = X \oplus (N \cap X')$. Thus X is a direct summand of N. So N satisfies FC_2.

(ii) Let K_1, K_2 be direct summands of N such that K_1 is fully invariant in N and $K_1 \cap K_2 = 0$. Since K_1, K_2 are direct summands of M, by FC_3 assumption, $K_1 \oplus K_2$ is a direct summand of M. Hence $M = (K_1 \oplus K_2) \oplus K_3$ for some $K_3 \leq M$. By the Corollary 1.1.31., $N = N \cap M = N \cap [(K_1 \oplus K_2) \oplus K_3] = (K_1 \oplus K_2) \oplus (N \cap K_3)$. Thus $K_1 \oplus K_2$ is a direct summand of N. Hence N satisfies FC_3.
(iii) Let K, L be direct summands of N such that K is fully invariant in N. Therefore K, L be direct summands of M. By Proposition 5.1.3. (ii), K is a fully invariant submodule of M. By FSIP assumption, $K \cap L$ is a direct summand of M. Thus $M = (K \cap L) \oplus X$ for some $X \leq M$. By Corollary 1.1.31., $N = (K \cap L) \oplus (N \cap X)$. So $K \cap L$ is a direct summand of N. $\square$

It is natural to think of whether any direct summand of a module with FC_2 (FC_3, FSIP) satisfies FC_2 (FC_3, FSIP) or not. So far we could not obtain counter example and left this situation for future work.

To this end we complete this section with the following problem:

Open Problem: Is N being fully invariant in Theorem 5.2.4. superfluous or not?

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