CONTACT STRUCTURES AND LEGENDRIAN LINKS

KONTAKT YAPILAR VE LEGENDRE LİNKLER

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ABSTRACT

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A contact structure on an oriented three-manifold is a maximally non-integrable 2-plane field distributed all over the three-manifold. Legendrian knots and Legendrian links in contact three-manifolds give important information about the contact three-manifold. A Legendrian knot/link is a knot/link which is everywhere tangent to the contact planes. In this thesis, the properties of Legendrian knots and links as well as the classification results are discussed. Legendrian unknots classification and Legendrian Hopf links classification in the contact 3-sphere S^3 are studied in detail.

Keywords: contact structures, legendrian knots, legendrian links, Legendrian unknot, Legendrian Hopf link

ÖZET

KONTAKT YAPILAR VE LEGENDRE LİNKLER

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Yönlü bir üç manifold üzerine dağılmış 2-düzlem alanı eğer integrallenemiyorsa kontakt yapı olarak adlandırılır. Kontakt üç manifoldlar içerisinde bulunan Legendre düğümler ve Legendre linkler kontakt üç manifoldlar ile ilgili önemli bilgiler verir. Legendre düğümü/linki kontak düzlemlere her yerde teğet olan bir düğümdür/linktir. Bu tezde Legendre düğümler ve linklerin özellikleri ile sınıflandırma sonuçları çalışılmıştır. Kontakt 3-küre S^3 içerisindeki Legendre çözük düğümün sınıflandırma sonucu ile Legendre Hopf linklerin sınıflandırma sonucu detaylı çalışılmıştır.

Keywords: kontakt yapılar, Legendre düğümler, Legendre linkler, Legendre çözük düğüm, Legendre Hopf link

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CONTENTS

Page

ABSTRACT i
ÖZET ii
ACKNOWLEDGEMENTS iii
CONTENTS iv
FIGURES vi
1. INTRODUCTION 1
1.1 Scope Of The Thesis 3
1.2 Organization 3
2. BACKGROUND OVERVIEW 4
2.1 Tangent Vectors and Tangent Spaces of $\mathbb{R}^3$ 4
2.2 Local Coordinates and Topological Manifolds 5
2.3 Differentiable Manifolds 6
2.4 Examples of 3-Manifolds 8
2.5 Submanifolds 9
2.6 C^∞ Functions and C^∞ Mappings On Manifolds 9
2.7 Tangent Vectors On Manifolds 10
2.8 The Differential of Maps 11
2.9 Immersion and Embedding 12
2.10 Vector Fields 13
2.11 Boundary and Orientation of Manifolds 13
2.12 Differential Forms on $\mathbb{R}^3$ 14
2.13 Differential Forms on General 3-Manifolds 17
2.14 Various Operations on Differential Forms 18
2.14.1 Exterior Product 18
2.14.2 Exterior Differentiation 19
2.14.3 Pullback Map 19
2.15 Knots and Links 19

2.15.1 Knots and Links 20
2.15.2 Reidemeister Moves 20
2.15.3 Linking Number 22
2.15.4 The Seifert Surface of a Knot 24
2.16 Contact 3-Manifolds 24
2.16.1 Contact Structures on $\mathbb{R}^3$ 24
2.16.2 Classification of Tight Contact Structures on 3-Manifolds 29
2.16.3 Legendrian Knots and Links 33
2.16.4 The Classical Invariants of Legendrian Knots 36
2.16.5 Topological Knot Type 36
2.16.6 Thurston-Bennequin Invariant 38
2.16.7 Rotation Number 39
2.16.8 Stabilizations 41
2.16.9 Knots in Overtwisted Contact Structures 43
3. CLASSIFICATION OF LEGENDRIAN KNOTS AND LINKS 43
3.1 Legendrian Knots Classification 43
3.2 Classification Results 44
3.3 Classification Results in Detail 50
3.3.1 Legendrian Unknots in Contact 3-Sphere S^3 50
3.3.2 Legendrian Hopf Links in Contact 3-Sphere S^3 55
4. CONCLUSION 87

FIGURES

		Page
Figure 2.1	Local chart	5
Figure 2.2	The 2-sphere S^2 in $\mathbb{R}^3$	10
Figure 2.3	(a) Immersion, (b) Embedding	12
Figure 2.4	Manifold with boundary	14
Figure 2.5	(a) The torus and (b) the Möbius band	15
Figure 2.6	(a) unknot and (b) trefoil knot	20
Figure 2.7	(a) Hopf link and (b) Whitehead link	21
Figure 2.8	Reidemeister moves	21
Figure 2.9	Oriented Figure-8 knot	22
Figure 2.10	Assigning number for crossing	22
Figure 2.11	Calculating linking number of a link	23
Figure 2.12	A positive and a negative Hopf link	24
Figure 2.13	Standard contact structure on $\mathbb{R}^3$	25
Figure 2.14	Overtwisted contact structure on $\mathbb{R}^3$	28
Figure 2.15	Dividing curve pointed out as red line	30
Figure 2.16	Front projections of a Legendrian unknot	35
Figure 2.17	Front projections of (a) a Legendrian left-handed trefoil and (b) a Legendrian right-handed trefoil	35
Figure 2.18	Converting a topological knot to the Legendrian knot	35
Figure 2.19	(a) Legendrian unknot, (b) Legendrian right-handed trefoil, (c) Legendrian figure-8 knot	36
Figure 2.20	Legendrian Reidemeister moves	37
Figure 2.21	(a) Legendrian left-handed trefoil and (b) Legendrian figure-8 knot	37
Figure 2.22	Contact framing of Legendrian unknot	38
Figure 2.23	Seifert framing of a Legendrian unknot	38
Figure 2.24	Right-handed trefoil	39

Figure 2.25 (a) Legendrian unknot with $rot = 2$ and (b) Legendrian unknot with $rot = -2$ 41
Figure 2.26 Two non-isotopic Legendrian unknots 41
Figure 2.27 Positive and negative stabilization of a Legendrian knot 42
Figure 2.28 (a) Legendrian unknot and (b) a positive and (c) a negative stabilization of Legendrian unknot 42
Figure 3.1 Classification of Legendrian unknots 45
Figure 3.2 Legendrian unknot with $tb = -1$ and $rot = 0$ 52
Figure 3.3 (a) Legendrian unknot with $tb = -2$ and $rot = 1$, and (b) Legendrian unknot with $tb = -2$ and $rot = -1$ 52
Figure 3.4 Legendrian Hopf link $H = L_0 \cup L_1$ with $tb(L_0) = tb(L_1) = -1$, $rot(L_0) = rot(L_1) = 0$ 83
Figure 3.5 (a) Legendrian Hopf link with $tb_0 = -1$, $rot_0 = 0$ and $tb_1 = -2$, $rot_1 = 1$ and (b) Legendrian Hopf link with $tb_0 = -1$, $rot_0 = 0$ and $tb_1 = -2$, $rot_1 = -1$ 84
Figure 3.6 (a) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = 1$ and $tb_1 = -3$, $rot_1 = 2$, (b) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = 1$ and $tb_1 = -3$, $rot_1 = 0$, (c) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = 1$ and $tb_1 = -3$, $rot_1 = -2$. 84
Figure 3.7 (a) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = -1$ and $tb_1 = -3$, $rot_1 = 2$, (b) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = -1$ and $tb_1 = -3$, $rot_1 = 0$, (c) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = -1$ and $tb_1 = -3$, $rot_1 = -2$. 85

1. INTRODUCTION

Contact topology is a field of smooth manifolds which is interested in geometric structures. These geometric structures are called contact structures. The contact structure on a smooth 3-manifold can be thought of as maximally non-integrable 2-plane fields. A 3-manifold with a contact structure on it is called a contact 3-manifold. Submanifolds of contact 3-manifolds help us to understand contact manifolds. In particular, knots in contact structures can also be used to understand contact structures. There are two types of knots in contact 3-manifolds: knots that are everywhere tangent to the contact planes and knots that are everywhere transverse to the contact planes.

Martinet proved that, in [1], each 3-manifold has a contact structure on it. There are two types of contact structures on the 3-manifolds: tight contact structure and overtwisted contact structure. There exists an overtwisted contact structure for all 3-manifolds. But there may not be a tight contact structure on every 3-manifold. Etnyre and Honda construct a manifold where there is no tight contact structure on it, in [2].

A knot in a contact 3-manifold is called a Legendrian knot, which is everywhere tangent to the contact planes. The disjoint union of Legendrian knots is called a Legendrian link. In this thesis, Legendrian knots, Legendrian links, and their classification will be studied in detail. Specifically, Legendrian unknots and Legendrian Hopf links on the contact 3-sphere S^3 are considered.

The classification of Legendrian unknots in tight contact 3-manifolds in [3] is the first classification result of Legendrian knots. In this paper of Eliashberg and Fraser, Legendrian unknots in tight contact 3-manifolds are completely classified. Also, Eliashberg and Fraser classify loose Legendrian unknots in overtwisted contact 3-manifolds whose complement is overtwisted up to coarse equivalence.

After Eliashberg and Fraser, the problem of classification of Legendrian knots has attracted the attention of many people. Etnyre and Honda classified Legendrian torus knots and figure-8 knots in tight contact S^3 in [4] and also in [5] they classified cables of knots.

The classification problem of Legendrian knots remains popular today. Baker and Etnyre study Legendrian rational unknots in [6]. Later, Geiges and Onaran classified Legendrian rational unknots in tight contact lens spaces and gave a clear list of these unknots in [7]. In addition, Geiges and Onaran classified Legendrian rational unknots with tight complement in overtwisted contact lens spaces, in [7].

The classification problems in other 3-manifolds are also studied by Ding, Chen, and Li. They study Legendrian torus knot classification in tight contact $S^1 \times S^2$ in [8].

Little was known about the classification of Legendrian links. The [9] and [10] articles of Ding and Geiges are the articles that study the classification of Legendrian links in the literature. The first complete Legendrian link type classification is given by Geiges and Onaran in [11]. In this paper, Geiges and Onaran classify Legendrian Hopf links in S^3 with any contact structure, up to coarse equivalence.

Loose Legendrian knot classification studied in [12]. In [12], Dymara shows that two loose knots (these are knots with overtwisted complements) in an overtwisted S^3 having the same invariants are contactomorphic. In addition to that, if knots have a common overtwisted disc in their complement, then Dymara showed that they are Legendrian isotopic in this case. Moreover, in her paper [12], Dymara presents the first example of a non-loose knot (that is a knot with a tight complement) in an overtwisted contact 3-manifold. Before that, no example was known.

Chatterjee, in [13], studied links in overtwisted contact structures. In this paper, Chatterjee's results for the classification of links in overtwisted contact structures is this: two null-homologous Legendrian links with the same knot type, Thurston-Bennequin invariant, and rotation number are contactomorphic.

Computer programs are also used to classify Legendrian knots and links. Chongchitmate in [14], used a program written in Java.

1.1 Scope Of The Thesis

This thesis studies contact 3-manifolds. In particular, this thesis studies Legendrian knots and Legendrian links. Invariants of Legendrian knots and Legendrian links as well as the classification problems of Legendrian knots and Legendrian links are discussed. Moreover, the proof of Legendrian unknots classification and the proof of Legendrian Hopf links classification in contact 3-sphere S^3 are discussed in detail.

1.2 Organization

The thesis is organized as follows:

- Chapter 1 presents our motivation, contributions, and the scope of the thesis.
- In Chapter 2, the main definitions and examples that will be used in this thesis are given. First, tangent spaces and manifolds are introduced, and their examples are mentioned. Then, differential forms on 3-manifolds are handled, and their examples are given. Before handling contact 3-manifolds, knots and links are briefly introduced, and their definitions and examples are given. Finally, contact 3-manifolds are introduced and examples are given. In addition, Legendrian knots are defined, properties are studied and examples are given.
- Chapter 3 provides a summary of classification results existing in the literature. In addition, this chapter gives detailed proof for some classification results. In particular, a detailed Legendrian unknot classification, as well as Legendrian Hopf link classification, is given in this chapter.
- Chapter 4 states the summary of the thesis and the classification problems of Legendrian knots and links are listed.

2. BACKGROUND OVERVIEW

In this section, we will give some important definitions and theorems, which we will use in the next sections. You can look into [15] for further readings.

2.1 Tangent Vectors and Tangent Spaces of $\mathbb{R}^3$

Definition 2.1.1. Let $(x, y, z) \in \mathbb{R}^3$ be a point in space. The *tangent space* of $\mathbb{R}^3$ at (x, y, z) is the set of all vectors starting from (x, y, z) and denoted by $T_{(x,y,z)}\mathbb{R}^3$. Its elements are called the *tangent vectors* at (x, y, z). In $\mathbb{R}^3$, the tangent space at any point looks like $\mathbb{R}^3$.

There are two important roles played by tangent vectors to characterize them and define them on a general manifold. These are the *velocity vector* of a curve and *directional derivative*.

Definition 2.1.2. A smooth curve on $\mathbb{R}^3$ is stated by a C^∞ map $c : \mathbb{R} \to \mathbb{R}^3$. The *velocity vector* at a point $c(t)$ $(t \in \mathbb{R})$ on the c is

$$\frac{\mathrm{d}c}{\mathrm{d}t}(\mathrm{t}) = \left(\frac{\mathrm{d}c_1}{\mathrm{d}t}(\mathrm{t}), \frac{\mathrm{d}c_2}{\mathrm{d}t}(\mathrm{t}), \frac{\mathrm{d}c_3}{\mathrm{d}t}(\mathrm{t})\right),$$

where $c = (c_1, c_2, c_3)$. Indeed, the velocity vector at $c(t)$ is the tangent vector of it. We can observe that if we move the point over the curve, then we have a variety of tangent vectors over the $c(t)$.

Definition 2.1.3. A function $f(x, y, z)$ is given, and partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ are considered. For an arbitrary tangent vector

$$v = u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} + u_3\frac{\partial}{\partial z}$$

at the $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, the *partial derivative* $v_t(f)$ of the function f at the direction of v is defined as

$$v_t(f) = u_1\frac{\partial f}{\partial x}(t) + u_2\frac{\partial f}{\partial y}(t) + u_3\frac{\partial f}{\partial z}(t).$$

Definition 2.1.4. A vector field V on $\mathbb{R}^3$ is an assignment of a tangent vector $V_{(x,y,z,)} \in T_{(x,y,z)}\mathbb{R}^3$ to each point $(x, y, z) \in \mathbb{R}^3$.

2.2 Local Coordinates and Topological Manifolds

In this part, we will introduce topological 3-manifolds by listing conditions one by one, and we will give a few examples.

The first condition is that M is a Hausdorff space: let M be a topological space and $p, q \in M$ such that $p \neq q$. The subsets U and V are open neighborhoods of p and q respectively. M is called a *Hausdorff space* if the open neighborhoods U and V do not intersect. For example, $\mathbb{R}^3$ and all its subspaces are Hausdorff spaces.

The second condition is that *M* locally looks like $\mathbb{R}^3$. Let $(n_1, n_2, n_3) \in M$ and let U be its open neighborhood. The subset U is homeomorphic to an open subset $V \in \mathbb{R}^3$. Let $\rho : U \to V$ be such a homeomorphism. Then, the image of (n_1, n_2, n_3) is shown below:

$$\rho(n_1, n_2, n_3) = (f_1(n_1, n_2, n_3), f_2(n_1, n_2, n_3), f_3(n_1, n_2, n_3)).$$

This 3-tuple is called the *local coordinates* of (n_1, n_2, n_3) and the neighborhood *U* is called a *coordinate neighborhood*. Additionally, f_1, f_2, f_3 is called *coordinate functions* on *U*. The pair (U, ρ) is called a *local chart* or *local coordinate system*, see Figure 2.1.

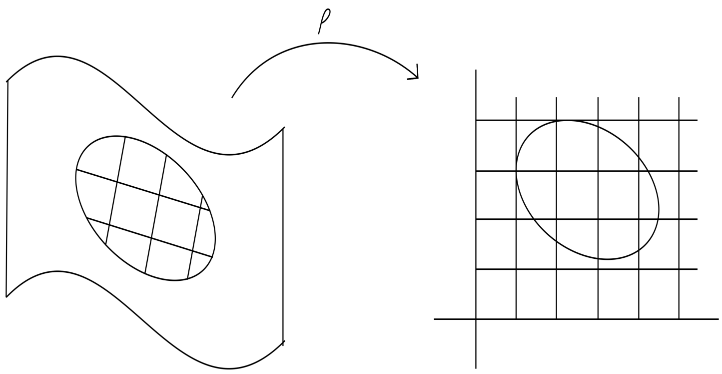


Figure 2.1 Local chart

The last condition is the *second countability axiom*. That is, *M* has a countable basis.

Definition 2.2.1. Let M be a second countable, Hausdorff space. If M locally looks like $\mathbb{R}^3$, then M is called a *topological* 3*-manifold*.

Example 2.2.1. $\mathbb{R}^3$, *the unit* 3*-sphere* S^3 *and the* 3*-torus* T^3 *are some examples of important* 3*-manifolds.*

A manifold can be connected or disconnected and compact or non-compact since it is a topological space. Usually, the term ”manifold” is used to mean manifold with boundaries. A non-compact manifold that has no boundary is called an *open manifold*.

Example 2.2.2. *A manifold is a topological space. If a manifold is compact as a topological space, then it is called a compact manifold. For example, the* 3*-dimensional sphere* S^3 *and the* 3*-torus* T^3 *are compact.*

Example 2.2.3. *Let* M *be a topological manifold. If* M *is a connected space, then* M *is called a connected manifold. For example, the* 3*-sphere* S^3 *is a connected* 3*-manifold.*

2.3 Differentiable Manifolds

Definition 2.3.1. Let *M* be a topological 3-manifold. An *atlas* for *M* is an indexed family $S = \{(U_\gamma, \rho_\gamma) | \gamma \in I\}$ of local charts on *M* which covers *M*, that is $\cup_{\gamma\in I} U_\gamma = M$.

Definition 2.3.2. An atlas S is called a maximal atlas such that there does not exist any atlas A such as $S \subset A$. The maximal atlas of a manifold is unique.

Example 2.3.1. *Let* $S^2 \subseteq \mathbb{R}^3$ *be the unit sphere, consisting of all* $(x, y, z) \in \mathbb{R}^3$ *satisfying the equation* $x^2 + y^2 + z^2 = 1$*. We will define an atlas with* (C_+, ρ_+) *and* (C_-, ρ_-)*. Let* $n = (0, 0, 1)$ *be the north pole and* $s = (0, 0, -1)$ *be the south pole, and put*

$$C_+ = S^2 \setminus \{s\}, \quad C_- = S^2 \setminus \{n\}.$$

Consider $\mathbb{R}^2$ *as a coordinate subspace of* $\mathbb{R}^3$ *with z=0. Let a stereographic projection from the south pole be*

$$\rho_+ : C_+ \to \mathbb{R}^2, \quad p \mapsto \rho_+(p).$$

Thus, $\rho_+(p)$ is the unique point where $\mathbb{R}^2$ and the affine line through p and s intersect. Similarly, a stereographic projection from the north pole is,

$$\rho_- : U_- \to \mathbb{R}^2, \quad p \mapsto \rho_-(p).$$

Thus, $\rho_-(p)$ is the unique point where $\mathbb{R}^2$ and the affine line through p and s intersect.

Definition 2.3.3. Let *M* be a 3-manifold and (U_γ, ρ_γ) and (V_β, ρ_β) be two local charts of *M* such that $U_\gamma \cap V_\beta \neq \emptyset$. The transition map $g_{\beta\gamma}$ is defined by,

$$g_{\beta\gamma} = \rho_\beta \circ \rho_\gamma^{-1} : \rho_\gamma(U_\gamma \cap V_\beta) \to \rho_\beta(U_\gamma \cap V_\beta).$$

This $g_{\beta\gamma}$ is a homeomorphism from an open subset of $\mathbb{R}^3$ to $\mathbb{R}^3$. So, it is described as $g_{\beta\gamma} = (g^1_{\beta\gamma}, g^2_{\beta\gamma}, g^3_{\beta\gamma})$ by continuous functions $g^1_{\beta\gamma}, g^2_{\beta\gamma}, g^3_{\beta\gamma}$. The local coordinates of any point *s* on $U_\gamma \cap V_\beta$, $(x_1(s), x_2(s), x_3(s))$ with respect to (U_γ, ρ_γ) and $(y_1(s), y_2(s), y_3(s))$ with respect to (V_β, ρ_β), then there exists a relation

$$\begin{aligned} y_1(s) &= g^1_{\beta\gamma}(x_1(s), x_2(s), x_3(s)) \\ y_2(s) &= g^2_{\beta\gamma}(x_1(s), x_2(s), x_3(s)) \\ y_3(s) &= g^3_{\beta\gamma}(x_1(s), x_2(s), x_3(s)) \end{aligned}$$

between them. The relationship between local coordinates (U_γ, ρ_γ) and (V_β, ρ_β) is defined by the homeomorphism $g_{\beta\gamma}$. The homeomorphism $g_{\beta\gamma}$ is called the *coordinate change*.

Definition 2.3.4. Take *M* as a topological manifold. An atlas $\mathcal{S} = \{(U_\gamma, \rho_\gamma) | \gamma \in I\}$ of *M* is called a C^∞ *atlas* if all coordinate changes $g_{\beta\gamma} = \rho_\beta \circ \rho_\gamma^{-1}$ are C^∞ maps. The atlas $\mathcal{S}$ defines a C^∞ structure on *M*. If M has a C^∞ structure on it, then M is called a C^∞ *differentiable manifold* or *smooth manifold*.

Example 2.3.2. *The unit 3-sphere S^3 and $\mathbb{R}^3$ are well-known smooth manifolds. Also, open subsets of a smooth manifold are still smooth manifolds. Let M, N be smooth manifolds, then their product $M \times N$ is also a smooth manifold.*

2.4 Examples of 3-Manifolds

Example 2.4.1. $\mathbb{R}^3$ *is a* 3*-dimensional smooth manifold. As an atlas, we can take only one local chart,* $(\mathbb{R}^3, id)$.

Example 2.4.2. *A* 3*-sphere with center* (m_0, m_1, m_2, m_3) *and radius r is the set of all points* (x_0, x_1, x_2, x_3) *in* $\mathbb{R}^4$ *is a* 3*-manifold, and it is expressed as*

$$\sum_{i=0}^{3}(x_i - m_i)^2 = (x_0 - m_0)^2 + (x_1 - m_1)^2 + (x_2 - m_2)^2 + (x_3 - m_3)^2 = r^2.$$

The unit 3*-sphere is the* 3*-sphere which is centered at the origin with radius* $r = 1$ *is a* 3*-manifold, and generally denoted as* S^3.

$$S^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$$

Example 2.4.3. *A topological space that is homeomorphic to the cartesian product of circles*

$$\mathbb{T}^3 = S^1 \times S^1 \times S^1$$

is called the 3*-torus or* 3*-dimensional torus.*

The usual 2*-torus is the cartesian product of two circles. It is a* 2*-manifold.*

Example 2.4.4. *Let two solid tori be given, and* M *be a manifold obtained by gluing these two solid tori along their boundaries with an orientation reversing homeomorphism* $g : T^2 \to T^2$. *The meridian-longitude bases of the two solid tori are* (μ_1, λ_1) *and* (μ_2, λ_2), *respectively. The homeomorphism* g *agree with a matrix*

$$B = \begin{pmatrix} -q & s \\ p & r \end{pmatrix}$$

with $qr + ps = 1$ *where* p *and* q *are relatively prime. In particular, the curve* $-q\mu_2 + p\lambda_2$ *is isotopic to the image of the meridian* μ_1 *of the first torus, i.e. winds* $-q$ *times in the* θ_2

direction and p times in the ψ_2 direction of the second torus. Here, θ_2 denotes the direction of the second meridian μ_2 and ψ_2 denotes the direction of the second longitude λ_2.

Hence, p and q determine the manifold M completely. Such a manifold is called a lens space and denoted by $L(p, q)$.

2.5 Submanifolds

Definition 2.5.1. Let *M* be a C^∞ 3-manifold and *N* be a subset of *M*. Then *N* is called a *submanifold* of M if it fulfills the following condition. For any point $s \in N$ there exists an open neighborhood U of s and coordinate functions x_1, x_2, x_3 that are defined on U such that

$$N \cap U = \{q \in U \mid x_{k+1}(q) = \ldots = x_3(q) = 0\ ,\ k \in \mathbb{Z}^+\}.$$

Example 2.5.1. *The* 2*-torus* $T^2 = f^{-1}(0) \subseteq \mathbb{R}^3$*, where*

$$f(x, y, z) = (\sqrt{x^2 + y^2} - R)^2 + z^2 - r^2$$

is a submanifold of $\mathbb{R}^3$.

Example 2.5.2. *The* 2*-sphere in Figure 2.2,*

$$S^2 = \{(x, y, z)|(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2\}$$

in $\mathbb{R}^3$ *with center* (x_0, y_0, z_0) *and radius* $r > 0$ *is a submanifold of* $\mathbb{R}^3$*. Also, the* 1*-sphere* S^1 *is a submanifold of* $\mathbb{R}^3$.

2.6 C^∞ Functions and C^∞ Mappings On Manifolds

Definition 2.6.1. Let M be a C^∞ manifold and $f : M \to \mathbb{R}^3$ be a function from M to $\mathbb{R}^3$. Then, f is called a C^∞ *function* if for all local charts (U, ρ) in an atlas that defines the C^∞ structure on M, the function

$$f \circ \rho^{-1} : \rho(U) \to \mathbb{R}^3$$

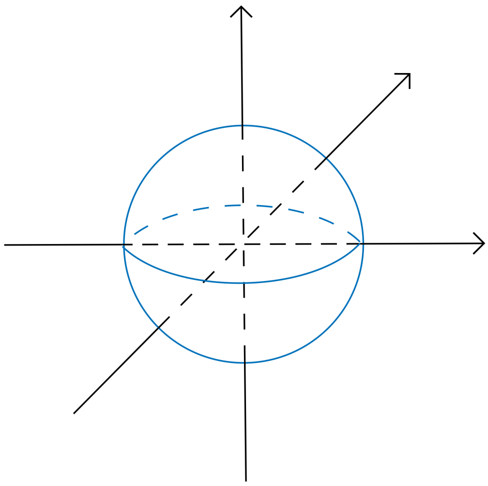


Figure 2.2 The 2-sphere S^2 in $\mathbb{R}^3$

is C^∞. The set of all C^∞ functions on M is denoted by $C^\infty(M)$.

Example 2.6.1. *The 'height function' $h : S^2 \to \mathbb{R}, \ \ (x, y, z) \mapsto z$ is a C^∞ function.*

Definition 2.6.2. Let M, N be C^∞ manifolds and $f : M \to N$ be a continuous function. Then, f is called a C^∞ $function$ at $p \in M$ such that the composition $\psi \circ f \circ \rho^{-1}$ is a C^∞ function where (U, ρ) is a local chart around p and (V, ψ) is a local chart around $f(p)$. If the function f is a C^∞ function at all $p \in M$, then f is called a C^∞ map.

Definition 2.6.3. Let M and N be C^∞ manifolds. A one to one and onto C^∞ function $f : M \to N$ is called a *diffeomorphism* if the f^{-1} is also a C^∞ function. The manifolds M and N are called *diffeomorphic* if there exists a diffeomorphism between them.

Example 2.6.2. *Let M be a manifold and (U, ρ) be a local chart on M. Then by definition, there is a diffeomorphism from (U, ρ) onto an open subset of* $\mathbb{R}^3$.

2.7 Tangent Vectors On Manifolds

Definition 2.7.1. Take a C^∞ manifold M and a point p on it. If a map $v : C^\infty(M) \to \mathbb{R}$ satisfies the following conditions for arbitrary functions h, g and $s \in \mathbb{R}$, v is called a *tangent vector* to M at p.

- $v(h+g) = v(h) + v(g), \quad v(sh) = sv(h),$
- $v(hg) = v(h)g(p) + h(p)v(g),$

The *tangent space* at p is defined as the set of all tangent vectors at p and denoted by T_pM.

Let (U, ρ) be a local chart around the point p on M and x_1, x_2, x_3 be its coordinate functions and $f \in C^\infty(M)$. Easily seen that the correspondence

$$f \mapsto \frac{\partial(f \circ \rho^{-1})}{\partial x_i}(\rho(p)) \quad (i = 1, 2, 3)$$

defines a tangent vector at p. It is denoted by $\left(\frac{\partial}{\partial x_i}\right)_p \in T_pM$. If the point p is known before, we write $\frac{\partial}{\partial x_i}$ for short.

Theorem 2.7.2. *Let M be a 3-dimensional C^∞ manifold. Therefore, the tangent space T_pM at a point p on M is a 3-dimensional vector space. Also, if $(U; x_1, x_2, x_3)$ is a local chart around p, then the tangent vectors*

$$\left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial x_2}\right)_p, \left(\frac{\partial}{\partial x_3}\right)_p$$

form a basis of T_pM. Here, x_1, x_2, x_3 denote the coordinate functions of the local chart.

2.8 The Differential of Maps

Definition 2.8.1. Let M, N be C^∞ manifolds and $f : M \to N$ be a C^∞ map. For any point p on M, a linear map

$$df_p = f_* = T_pM \to T_{f(p)}N$$

is called the *differential* of f at p and defined as follows. Let $v \in T_pM$ and $h \in C^\infty(N)$; then

$$f_*(v)(h) = v(h \circ f) \in \mathbb{R}$$

is a linear map.

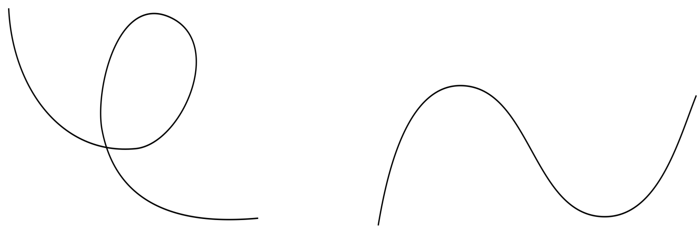
Example 2.8.1. *If $f : X_1 \to X_2$ is a C^∞ map that is the restriction of a linear map $r : \mathbb{R}^3 \to \mathbb{R}^3$, then for all $p \in X_1$, the differential df_p is equal to the restriction of dr_p which in turn is equal to itself. Thus, $df_p(a) = ra$ for every $a \in T_pX_1$ or equivalently $df_p = r|_{T_pX_1}$.*

2.9 Immersion and Embedding

Definition 2.9.1. Let M, N be C^∞ manifolds. Assume that $f : M \to N$ is a C^∞ map and $f_* : T_pM \to T_{f(p)}N$ is the differential of f.

- If for any $p \in M$ the differential f_* is an injection, then f is called an *immersion*.
- If f is an immersion and also $f : M \to f(M)$ is a homeomorphism, then f is called an *embedding*.
- If f is a surjection and at every point p the differential df_p is also a surjection, then f is called a *submersion*.

An embedding is an immersion where we no longer allow self-intersections, see Figure 2.3.



(a) *(b)*

Figure 2.3 (a) Immersion, (b) Embedding

Example 2.9.1. *The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by*

$$\alpha(t) = (t^3 - 4t, t^2 - 4)$$

is an immersion, since $\alpha'(t)$ is never zero (as $3t^2 - 4 = 2t = 0$ has no solution in t). However, it is not an injective map, as $\alpha(2) = \alpha(-2)$, so this is a curve with self-intersection at $\alpha(2) = \alpha(-2) = (0, 0)$.

Example 2.9.2. *The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by*

$$\alpha(t) = (t^3, t^2)$$

is not an immersion, since $d_t\alpha$ is the zero map for $t = 0$.

2.10 Vector Fields

Let M be a C^∞ 3-manifold. A *vector field* X on M assigns to a point $p \in M$ a tangent vector $X_p \in T_pM$. If we let $(U; x_1, x_2, x_3)$ be a local chart of M, for every point $p \in U$, the vector field X defined as follows:

$$X_p = \sum_{i=1}^{3} f_i(p)\frac{\partial}{\partial x_i} = f_1(p)\frac{\partial}{\partial x_1} + f_2(p)\frac{\partial}{\partial x_2} + f_3(p)\frac{\partial}{\partial x_3}$$

where f_1, f_2, f_3 are functions defined on U. If each f_i $(i = 1, 2, 3)$ is a C^∞ function, X_p is called class of C^∞ with respect to p.

The set of all vector fields on M is denoted by $\mathfrak{X}$. Let $X, Y \in \mathfrak{X}$. Then we can define their sum $X + Y \in \mathfrak{X}$ by putting $(X + Y)_p = X_p + Y_p$. Furthermore, for a real number $a \in \mathbb{R}$, $(aX)_p = a(X_p)$ defines multiplication $aX \in \mathfrak{X}$ of X by a. Thus, $\mathfrak{X}$ becomes a vector space over $\mathbb{R}$ with addition and multiplication on it.

2.11 Boundary and Orientation of Manifolds

Boundary and orientation are two fundamental facts concerning manifolds. Now, we will define a manifold with a boundary and the definition of an orientable/non-orientable surface. Then, we give a few examples.

Definition 2.11.1. Let X be a Hausdorff space which is second countable. X is called a *manifold with a boundary* such that for all $x \in X$ there is such a homeomorphism $\varphi : U \to \mathbb{R}^n_+$ where U is an open neighborhood of x. We denote ∂X as the *boundary* of X.

Example 2.11.1. *An example of a 2-dimensional manifold with a boundary is given in Figure 2.4. The point x is in the interior (it has a neighborhood homeomorphic to $\mathbb{R}^2$),*

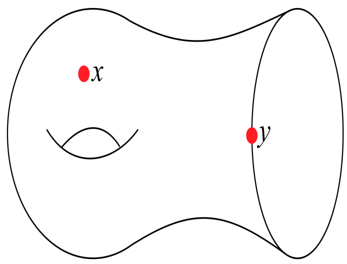


Figure 2.4 Manifold with boundary

and the point y is on the boundary (it has a neighborhood homeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}$).

Example 2.11.2. *The 3-dimensional disk $D^3 = \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 \leq 1\}$ is a manifold with a boundary, and $\partial D^3 = S^2$.*

If a compact manifold does not have a boundary, then it is called a *closed manifold*.

Definition 2.11.2. Let S be a surface in a 3-manifold. At each point of S, there are two choices for a unit normal n on S. Once a consistent choice of n is made at each point of S, we say that an *orientation* is assigned to S and S is *orientable*. A surface for which this can not be done is called a *non-orientable* surface.

Example 2.11.3. *The 2-torus given in Figure 2.5 (a) is a closed surface which is orientable. The surface in Figure 2.5 (b) is the Möbius band, it is a surface that has a boundary, so it is not a closed surface. Moreover, it is non-orientable.*

2.12 Differential Forms on $\mathbb{R}^3$

First, we will start with the definition of algebra. Let Λ be a vector space over the $\mathbb{R}$ with an associative product and for arbitrary $r \in \mathbb{R}$ and $\alpha, \beta \in \Lambda$. If the following condition is

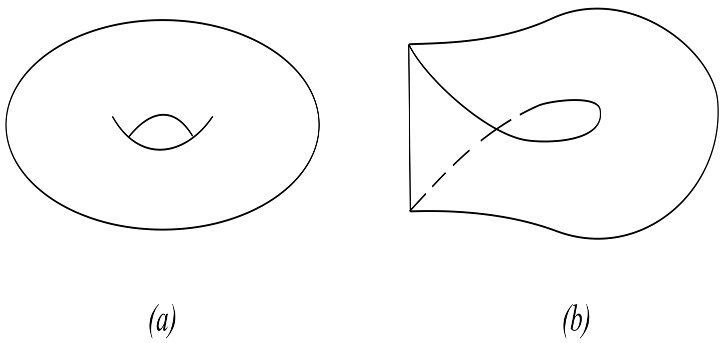


Figure 2.5 (a) The torus and (b) the Möbius band

satisfied, then Λ is called an *algebra* over $\mathbb{R}$.

$$r(\alpha\beta) = (r\alpha)\beta = \alpha(r\beta)$$

An algebra over $\mathbb{R}$ with unity 1 generated by $dx_1, dx_2, \ldots, dx_n$, that satisfy the equation

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

for arbitrary i, j is denoted by Λ_n^*. The operation $\wedge$ represents the product of this algebra. We can observe that $dx_i \wedge dx_i = 0$ for any i, from the above equation. The degree of each monomial in Λ_n^* is defined by taking the degree of dx_i to be 1. For example, the degree of $dx_1 \wedge dx_2 \wedge dx_3$ is 3.

The set of all linear decompositions of degree k is denoted by Λ_n^k. The direct sum decomposition

$$\Lambda_n^* = \bigoplus_{k=0}^{n} \Lambda_n^k = \Lambda_n^0 \oplus \Lambda_n^1 \oplus \cdots \oplus \Lambda_n^n$$

holds. Then, we can take the basis of Λ_n^k as

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq n.$$

Definition 2.12.1. A *k-form* (or *k-differential form*) on $\mathbb{R}^3$ is the linear combination of $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ with coefficients as C^∞ functions on $\mathbb{R}^3$. A k-form is shown as below:

$$\omega = \sum_{i_1 < \ldots < i_k} f_{i_1 \ldots i_k}(x_1, x_2, x_3) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

Sometimes we denote the above description simply as

$$\sum_I f_I(x_1, x_2, x_3) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

The set of all k-forms on $\mathbb{R}^3$ is denoted by $\mathcal{A}^k(\mathbb{R}^3)$.

Let ρ and ν be k and l forms respectively. They are expressed as

$$\rho = \sum_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \qquad \nu = \sum_J g_J dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

The product $\rho \wedge \nu \in \mathcal{A}^{k+l}(\mathbb{R}^3)$ is defined by,

$$\rho \wedge \nu = \sum_{I,J} f_I g_J \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

This is called the *exterior product* of ρ and ν.

The linear map $d : \mathcal{A}^k(\mathbb{R}^n) \to \mathcal{A}^{k+1}(\mathbb{R}^n)$ is called the *exterior differentiation*, and it is defined as follows. For $\rho = f(x_1, \ldots, x_n) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$,

$$d\rho = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Example 2.12.1. *Let ρ be a 2-form on $\mathbb{R}^3$ defined as $\rho = x^3 dx \wedge dy + 3xyz dy \wedge dz$. The exterior differentiation $d\rho$ is*

$$\begin{aligned} d\rho = & \frac{\partial}{\partial x}(x^3)\, dx \wedge dx \wedge dy + \frac{\partial}{\partial y}(x^3)\, dy \wedge dx \wedge dy + \frac{\partial}{\partial z}(x^3)\, dz \wedge dx \wedge dy \\ & + \frac{\partial}{\partial x}(3xyz)\, dx \wedge dy \wedge dz + \frac{\partial}{\partial y}(3xyz)\, dy \wedge dy \wedge dz + \frac{\partial}{\partial z}(3xyz)\, dz \wedge dy \wedge dz \end{aligned}$$

$$= 3yz\, dx \wedge dy \wedge dz,$$

$d\rho$ is a 3-form on $\mathbb{R}^3$. Also, $d(d\rho) = 0$, since there is no 4-form on $\mathbb{R}^3$.

Lemma 2.12.2. *If we operate the exterior differentiation twice, it is identically 0, that is, $d \circ d = 0$.*

Let ρ be a differential form. If $d\rho = 0$, then ρ is called a *closed form*. If for a differential form ν, there exists a differential form ρ such that $\nu = d\rho$, then ν is called an *exact form*.

Proposition 2.12.3. *For $\rho \in \mathcal{A}^k(\mathbb{R}^n)$ and $\nu \in \mathcal{A}^l(\mathbb{R}^n)$, we have*

(i) $\nu \wedge \rho = (-1)^{kl} \rho \wedge \nu$,

(ii) $d(\rho \wedge \nu) = d\rho \wedge \nu + (-1)^k \rho \wedge d\nu$.

2.13 Differential Forms on General 3-Manifolds

Let M be a 3-dimensional C^∞ manifold and $\{(U_\gamma, \rho_\gamma)\}$ be an atlas of it. A *k-differential form* on M is a family of k-forms ρ_γ on each coordinate neighborhood U_γ such that for an arbitrary γ, β with $U_\gamma \cap U_\beta \neq \emptyset$, ρ_γ and ρ_β are transformed to each other by the coordinate change. We denoted all k-forms on M by $\mathcal{A}^k(M)$, and

$$\mathcal{A}^*(M) = \bigoplus_{k=0}^{n} \mathcal{A}^k(M).$$

Let M be a C^∞ manifold and p be a point on M. The *cotangent space* at p is the dual space T_p^*M of the tangent space T_pM. Its exterior algebra is denoted as $\Lambda^* T_p^* M$.

Definition 2.13.1. Let M be a C^∞ 3-manifold. Then ρ is called a *k-form* on M, if it assigns $\rho_p \in \Lambda^* T_p^* M$ to each point $p \in M$ and ρ_p is of class C^∞ with respect to p.

2.14 Various Operations on Differential Forms

Let M be a 3-dimensional C^∞ manifold and $\mathcal{A}^k(M)$ be the set of all k-forms on M. The direct sum of these k-forms is defined as

$$\mathcal{A}^*(M) = \bigoplus_{k=0}^{n} \mathcal{A}^k(M)$$

according to k. $\mathcal{A}^*(M)$ denotes the set of all differential forms on M.

2.14.1 Exterior Product

Let $\rho \in \mathcal{A}^k(M)$ be a k-form and $\nu \in \mathcal{A}^l(M)$ be an l-form on M. We have $\rho_p \in \Lambda^k T_p^* M$, $\nu_p \in \Lambda^l T_p^* M$, since at each point $p \in M$. Then, their product $\rho_p \wedge \nu_p \in \Lambda^{k+l} T_p^* M$ is defined as

$$(\rho \wedge \nu)_p = \rho_p \wedge \nu_p.$$

This is called the *exterior product* of ρ and ν.

We can say that the exterior product is associative from the definition. That is, if $\omega \in \mathcal{A}^m(M)$, we have $(\rho\wedge\nu)\wedge\omega = \rho\wedge(\nu\wedge\omega)$. If $\rho = f dx_{i_1} \wedge\cdots\wedge dx_{i_k}$ and $\nu = g dx_{j_1} \wedge\cdots\wedge dx_{j_l}$, then

$$\rho \wedge \nu = fg dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

The exterior product induces a bilinear map

$$\mathcal{A}^k(M) \times \mathcal{A}^l(M) \ni (\rho, \nu) \mapsto \rho \wedge \nu \in \mathcal{A}^{k+l}(M),$$

and it has $\nu \wedge \rho = (-1)^{kl} \rho \wedge \nu$.

2.14.2 Exterior Differentiation

Let $\rho \in \mathcal{A}^k(M)$ be a k-form on M and locally it is expressed as $\rho = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Its exterior differentiation is defined by

$$d\rho = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and $d\rho \in \mathcal{A}^{k+1}(M)$. The exterior differentiation defines a degree 1 linear map $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$, and it has the following properties.

- $d \circ d = 0$
- For $\rho \in \mathcal{A}^k(M), d(\rho \wedge \nu) = d\rho \wedge \nu + (-1)^k \rho \wedge d\nu$.

2.14.3 Pullback Map

Let M and N be C^∞ 3-manifolds and $f : M \to N$ be a C^∞ map from M to N. Take the differential $f_* : T_pM \to T_{f(p)}N$ of f at every point $p \in M$. The dual map of $f^* : T^*_{f(p)}N \to T^*_pM$ is induced by f_*, i.e. the map $f^*(\alpha)(x) = \alpha(f_*(x))$ for $\alpha^*_{f(p)}$ and $x \in T_pM$. Moreover, the linear map $f^* : \Lambda^k T^*_{f(p)}N \to \Lambda^k T^*_pM$ is defined by for an arbitrary t

$$f^* : \mathcal{A}^*(N) \to \mathcal{A}^*(M),$$

is an algebra homeomorphism.

Let ω be a differential form in $\mathcal{A}^k(N)$. Then, $f^*\omega \in \mathcal{A}^k(M)$ is called the *pullback* by f.

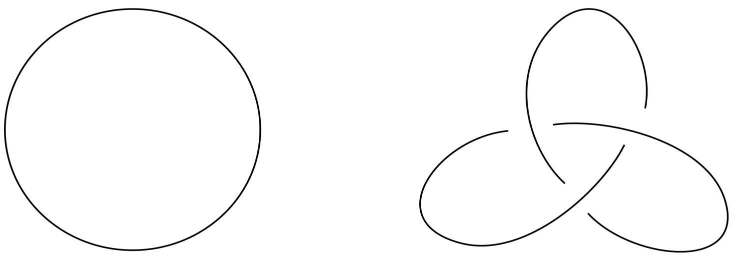
2.15 Knots and Links

Here, we will give some important definitions of knots and links, which we will use in the next chapters. Moreover, we will give some examples. See [16], [17] for more information.

2.15.1 Knots and Links

Definition 2.15.1. A knot is an embedded 1-dimensional subset of $\mathbb{R}^3(S^3, M^3)$ which is homeomorphic to S^1.

Example 2.15.1. *There are some knot examples shown below in Figure 2.6. These are called knot diagrams, that is, projections of knots onto the xz-plane in $\mathbb{R}^3$. The left one in Figure 2.6 (a) is called the unknot or the trivial knot. The right one in Figure 2.6 (b) is the trefoil knot, it has three crossings, and it is the only knot with this property.*



(a) *(b)*

Figure 2.6 (a) unknot and (b) trefoil knot

Definition 2.15.2. Let K_1 and K_2 be two knots in a 3-manifold M^3 and $f : M^3 \to M^3$ be an orientation preserving homeomorphism. Then, K_1 and K_2 are called *isotopic knots*, if $f(K_1) = K_2$.

Definition 2.15.3. A *link* is a disjoint union of knots.

Example 2.15.2. *There are different link examples shown below in Figure 2.7. The left one is called the Hopf link, given in Figure 2.7 (a), and the right one is called the Whitehead link, given in Figure 2.7 (b).*

2.15.2 Reidemeister Moves

Reidemeister moves are moves that change a projection of the knot by changing the relationship between the crossings.

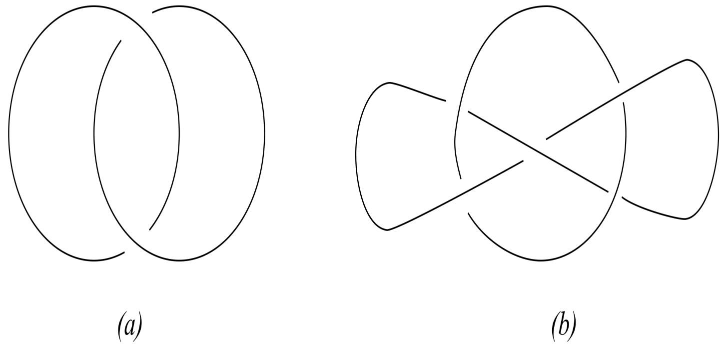


Figure 2.7 (a) Hopf link and (b) Whitehead link

Definition 2.15.4. [18] There are three types of Reidemeister moves. The Reidemeister moves R_1, R_2 and R_3 are shown in Figure 2.8.

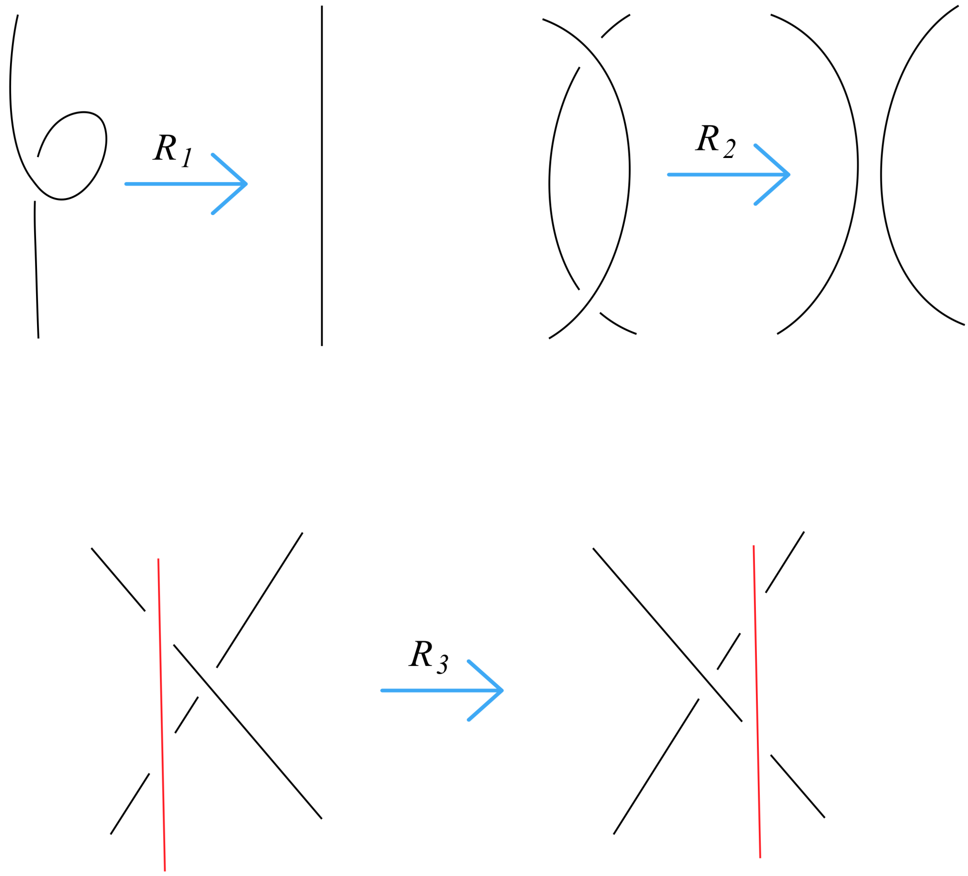


Figure 2.8 Reidemeister moves

Theorem 2.15.5. *[19][20] Two knot diagrams K_1 and K_2 are isotopic if and only if K_1 can be turned into K_2 by a finite sequence of Reidemeister moves R_1, R_2, R_3 or $R_1^{-1}, R_2^{-1}, R_3^{-1}$.*

2.15.3 Linking Number

Definition 2.15.6. An *oriented knot* is a knot where a clockwise orientation or counterclockwise orientation is given. An *oriented link* is a link where each component of the link is oriented.

Example 2.15.3. *There is an oriented Figure-8 knot shown in Figure 2.9.*

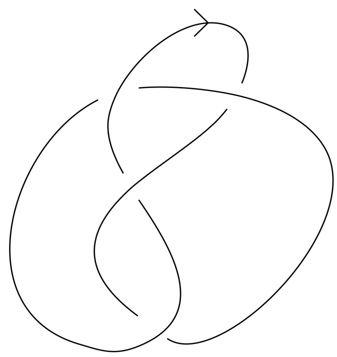


Figure 2.9 Oriented Figure-8 knot

Now, for a given knot or link, one can assign a ± 1 value to each crossing of a knot as in Figure 2.10.

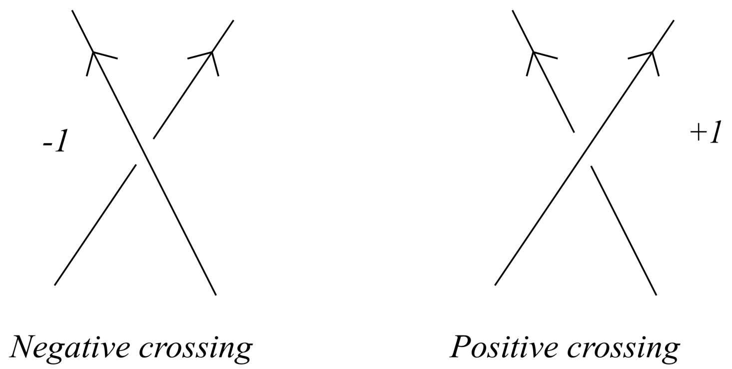


Figure 2.10 Assigning number for crossing

Definition 2.15.7. [16][17] Let K_1 and K_2 be two oriented knots. The *linking number* *lk(K_1, K_2)* is defined as the half sum of $+1$'s and -1's corresponding to the crossings of K_1 and K_2.

The linking number is a link invariant, i.e. any Reidemeister moves do not change the linking number of a diagram.

Example 2.15.4. *K_1, K_2 are the components of the link, which you can see in Figure 2.11. The linking number of this link is $lk(K_1, K_2) = \frac{1}{2}(-1-1) = -\frac{2}{2} = -1$. Note that we do not count the values at self-crossing of the individual knot components. The counted crossings must be between the components of the link.*

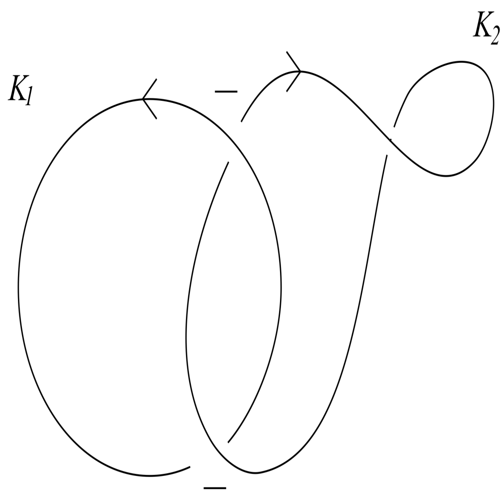


Figure 2.11 Calculating linking number of a link

Example 2.15.5. *Two Hopf links are given in Figure 2.12. Note that the Hopf link in Figure 2.12 (a) has a linking number equal to $+1$, it is called a positive Hopf link. The Hopf link in Figure 2.12 (b) has a linking number equal to -1, it is called a negative Hopf link.*

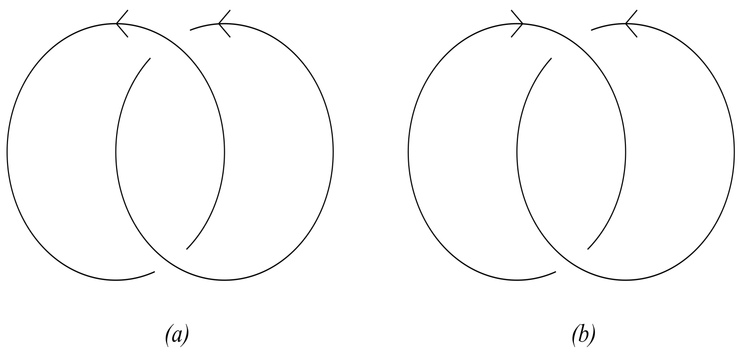


Figure 2.12 A positive and a negative Hopf link

2.15.4 The Seifert Surface of a Knot

Definition 2.15.8. [17] Let K be a knot in a 3-manifold M. If there exists a surface S such that $\partial S = K$, the surface S is called a *Seifert surface* of K. If a Seifert surface S exists for K, then K is called a *null-homologous* knot.

Theorem 2.15.9. *[21] Let K be a knot (link) in $\mathbb{R}^3(S^3)$. Then, there exists an orientable connected surface S such that the boundary of S is K. That is, all knots in $\mathbb{R}^3$ (S^3) are null-homologous.*

Example 2.15.6. *A disc is a Seifert surface for the unknot. Consider the Hopf link. An annulus is a Seifert surface of it.*

2.16 Contact 3-Manifolds

2.16.1 Contact Structures on $\mathbb{R}^3$

Suppose that M is a 3-manifold and T_pM is the tangent space of M at $p \in M$. Also, consider its tangent bundle $TM = \cup_{p\in M} T_pM$. Now, let us define contact structures. See [22], [23] for more information.

Definition 2.16.1. A *contact structure* on a 3-manifold is a 2-plane field such that there exists a 1-form $\rho : TM \to \mathbb{R}$ which is locally $\xi = ker\rho = \{v \in TM | \rho(v) = 0\}$ and $\rho \wedge d\rho \neq 0$. The 1-form ρ is called a *contact form.*

Definition 2.16.2. Consider a 3-manifold M and a contact structure ξ on M. Then, M is called a *contact* 3*-manifold*, and it is denoted by (M, ξ).

Example 2.16.1. *Let* $\rho = dx_3 - x_2 dx_1$ *be a* 1 *1-form and* $\xi_{std} = ker\rho = \left\langle \frac{\partial}{\partial x_2}, x_2\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_1} \right\rangle$ *in* $\mathbb{R}^3$*. The* 1*-form* ρ *is a contact form since*

$$\begin{aligned}
\rho \wedge d\rho &= (dx_3 - x_2 dx_1) \wedge d(dx_3 - x_2 dx_1) \\
&= (dx_3 - x_2 dx_1) \wedge d(dx_3) - d(x_2 dx_1) \text{ (d is a linear map)} \\
&= (dx_3 - x_2 dx_1) \wedge (-dx_2 \wedge dx_1) \text{ (}d(dx_3)\text{=0)} \\
&= (dx_3 - x_2 dx_1) \wedge (dx_1 \wedge dx_2) \text{ (}-dx_2 \wedge dx_1 = dx_1 \wedge dx_2\text{)} \\
&= (dx_3 \wedge dx_1 \wedge dx_2) - (x_2 dx_1 \wedge dx_1 \wedge dx_2) \text{ (}dx_1 \wedge dx_1\text{=0)} \\
&= dx_3 \wedge dx_1 \wedge dx_2 \\
&= dx_1 \wedge dx_2 \wedge dx_3 \neq 0.
\end{aligned}$$

Thus, ξ_{std} *is a contact structure on* $\mathbb{R}^3$*, see Figure 2.13.*

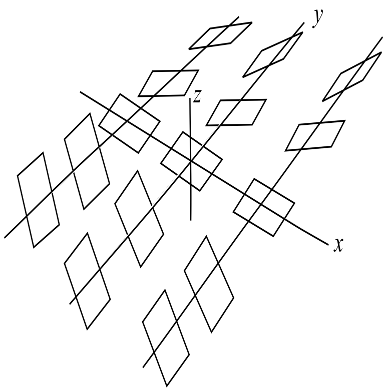


Figure 2.13 Standard contact structure on $\mathbb{R}^3$

Example 2.16.2. *Let* $\rho_1 = dx_3 + x_1 dx_2$ *be a* 1*-form and* $\xi_1 = ker\rho_1 = \left\langle \frac{\partial}{\partial x_1}, -x_1\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2} \right\rangle$ *in* $\mathbb{R}^3$*.* ξ_1 *is a contact structure on* $\mathbb{R}^3$*. Moreover,* 1*-form* ρ_1 *is a contact form:*

$$\begin{aligned}
\rho_1 \wedge d\rho_1 &= (dx_3 + x_1 dx_2) \wedge d(dx_3 + x_1 dx_2) \\
&= (dx_3 + x_1 dx_2) \wedge d(dx_3) + d(x_1 dx_2) \text{ (d is a linear map)}
\end{aligned}$$

$$= (dx_3 + x_1 dx_2) \wedge (dx_1 \wedge dx_2) \text{ (}d(dx_3) = 0\text{)}$$

$$= (dx_3 \wedge dx_1 \wedge dx_2) + (x_1 dx_2 \wedge dx_1 \wedge dx_2) \text{ (}dx_2 \wedge dx_2 = 0\text{)}$$

$$= dx_3 \wedge dx_1 \wedge dx_2$$

$$= dx_1 \wedge dx_2 \wedge dx_3 \neq 0.$$

Example 2.16.3. *Suppose* 1*-form* $\rho' = x_1dx_2 - x_2dx_1 + x_4dx_3 - x_3dx_4$ *is on the unit sphere* $S^3 \subset \mathbb{R}^4$. *Then,*

$$\begin{aligned}
\rho' \wedge d\rho' &= (x_1dx_2 - x_2dx_1 + x_4dx_3 - x_3dx_4) \wedge d(x_1dx_2 - x_2dx_1 + x_4dx_3 - x_3dx_4) \\
&= (x_1dx_2 - x_2dx_1 + x_4dx_3 - x_3dx_4) \wedge (2dx_1 \wedge dx_2 + 2dx_4 \wedge dx_3) \\
&= 2x_1dx_2 \wedge dx_4 \wedge dx_3 - 2x_2dx_1 \wedge dx_4 \wedge dx_3 + 2x_4dx_1 \wedge dx_2 \wedge dx_3 - 2dx_1 \wedge dx_2 \wedge dx_4.
\end{aligned}$$

We can generate the tangent space T_pS^3 *by the following set*

$$\left\langle \frac{\partial}{\partial x_1} - \frac{x_1}{y_1}\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} - \frac{x_2}{y_2}\frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_1} - \frac{x_1}{y_2}\frac{\partial}{\partial y_2} \right\rangle.$$

On this basis for the tangent space T_pS^3, $\rho' \wedge d\rho' \neq 0$. *Therefore,* $\xi_{std} = ker\rho'$ *is a contact structure on* S^3 *since* ρ' *is a contact form. This contact structure* ξ_{std} *is called the standard contact structure on* S^3, *and it is denoted by* (S^3, ξ_{std}).

Example 2.16.4. *There is another contact structure on* $\mathbb{R}^3$ *is the symmetric contact structure* $\xi_{sym} = ker\rho_2$ *where* 1*-form* $\rho_2 = dx_3 - x_2dx_1 + x_1dx_2$. *Indeed,* 1*-form* ρ_2 *is a contact form:*

$$\begin{aligned}
\rho_2 \wedge d\rho_2 &= (dx_3 - x_2dx_1 + x_1dx_2) \wedge d(dx_3 - x_2dx_1 + x_1dx_2) \\
&= (dx_3 - x_2dx_1 + x_1dx_2) \wedge d(dx_3) - d(x_2dx_1) + d(x_1dx_2) \text{ (}d\text{ is a linear map)} \\
&= (dx_3 - x_2dx_1 + x_1dx_2) \wedge (-dx_2 \wedge dx_1 + dx_1 \wedge dx_2) \text{ (}d(dx_3)\text{=0)} \\
&= (dx_3 - x_2dx_1 + x_1dx_2) \wedge (2dx_1 \wedge dx_2) \text{ (}-dx_2 \wedge dx_1 = dx_1 \wedge dx_2\text{)} \\
&= (dx_3 \wedge 2dx_1 \wedge dx_2) = 2dx_1 \wedge dx_2 \wedge dx_3 \neq 0.
\end{aligned}$$

Therefore, ξ_{sym} is a contact structure on $\mathbb{R}^3$. Contact structure ξ_{sym} on $\mathbb{R}^3$ is generated by following sets:

$$
\begin{aligned}
&\textit{if } y \neq 0 \quad ker\rho_2 = \left\langle x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}, x_2\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_1} \right\rangle \\
&\textit{if } x \neq 0 \quad ker\rho_2 = \left\langle x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}, x_1\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2} \right\rangle \\
&\textit{if } x = y = 0 \quad ker\rho_2 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle .
\end{aligned}
$$

Definition 2.16.3. [23] Let (M_1, ξ_1) and (M_2, ξ_2) be contact 3-manifolds. (M_1, ξ_1) and (M_2, ξ_2) are called *contactomorphic* if there exists a diffeomorphism $f : M_1 \to M_2$ such that $f_*(\xi_1)_p = (\xi_2)_{f(p)}$ for all $p \in M_1$ where $f_* : T_pM_1 \to T_{f(p)}M_2$ denotes the differential of f. Such an f is called a *contactomorphism.*

Theorem 2.16.4. *(Darboux's Theorem)[24] "Let M be a 3-manifold. Then, for any $p \in M$ there exists a neighborhood U of p and a neighborhood V of $p' = (0,0,0) \in (\mathbb{R}^3, \xi_{std})$, such that U is contactomorphic to V."*

Definition 2.16.5. [22][23] Let $\mathcal{D}_1$ be an embedded disc and (M, ξ) be a contact structure. If $\mathcal{D}_1$ is tangent to the contact planes along its boundary, which is denoted by $\partial\mathcal{D}_1$, then D^1 is called an *overtwisted disc.*

Example 2.16.5. *A 1-form $\rho_3 = \cos t dz + t \sin t d\beta$ in $\mathbb{R}^3$ with cylindrical coordinates, $\xi_{ot} = ker\rho_3 = \left\langle \frac{\partial}{\partial t}, -t\sin t\frac{\partial}{\partial z} + \cos t\frac{\partial}{\partial \beta} \right\rangle$ is a contact structure on $\mathbb{R}^3$. In fact, 1-form ρ_3 is a contact form. First calculate $d\rho_3$:*

$$
\begin{aligned}
d\rho_3 &= d(\cos t dz + t\sin t d\beta) \\
&= -\sin t dt \wedge dz + \cos t d(dz) + (\sin t + t\cos t) dt \wedge d\beta + t \sin t d(d\beta) \\
&= -\sin t dt \wedge dz + (\sin t + t\cos t) dr \wedge d\beta \; (d(dz) = d(d\beta) = 0).
\end{aligned}
$$

Now, let us compute $\rho_3 \wedge d\rho_3$.

$$\begin{aligned}
\rho_3 \wedge d\rho_3 &= (\cos t dz + t \sin t d\beta) \wedge (-\sin t dt \wedge dz + (\sin t + t \cos t) dt \wedge d\beta) \\
&= \cos t \sin t dz \wedge dt \wedge d\beta + t \cos^2 t dz \wedge dt \wedge d\beta - t \sin^2 t d\beta \wedge dt \wedge dz \\
&= \cos t \sin t dz \wedge dt \wedge d\beta + t \cos^2 t dz \wedge dt \wedge d\beta + t \sin^2 t dz \wedge dt \wedge d\beta \\
&= (\cos t \sin t + t) dz \wedge dt \wedge d\beta \\
&= \left(\frac{\cos t \sin t}{t} + 1 \right) dz \wedge dt \wedge d\beta \neq 0 \text{ (If } t > 0, \ \frac{\cos t \sin t}{t} + 1 > 0)
\end{aligned}$$

Therefore, ξ_{ot} is a contact structure on $\mathbb{R}^3$. Also, the boundary of the disc

$$\mathcal{D} = \{(t, \beta, z) \in \mathbb{R}^3 | z = 0, t \leq \pi\}$$

is tangent to the contact planes, and hence the disc $\mathcal{D}$ is an overtwisted disc.

Definition 2.16.6. [22][23] Let M be a contact 3-manifold. Then, M is called *an overtwisted contact 3-manifold* if there exists an overtwisted disc in M.

Example 2.16.6. $(\mathbb{R}^3, \xi_{ot})$ *has an overtwisted disc in it. Therefore, it is an overtwisted contact structure, shown in Figure 2.14. The red circle in Figure 2.14 is the boundary of an overtwisted disc.*

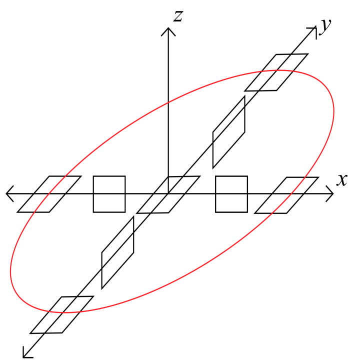


Figure 2.14 Overtwisted contact structure on $\mathbb{R}^3$

Definition 2.16.7. [22][23] Let M be a contact 3-manifold. If M does not contain an overtwisted disc, then M is called a *tight contact 3-manifold.*

Example 2.16.7. *The contact* 3*-manifolds* $(\mathbb{R}^3, \xi_{std})$ *and* $(\mathbb{R}^3, \xi_{sym})$ *are examples of tight contact manifolds.*

Theorem 2.16.8. *[25] Let* ξ *be a tight contact structure on* S^3*. Then,* ξ *is isotopic to the standard contact structure* ξ_{std} *on* S^3*, i.e.* ξ_{std} *is the unique tight contact structure on* S^3*.*

Definition 2.16.9. Two contact structures (M, ξ) and (M, ξ') are called *isotopic* if there exists a contactomorphism $f : (M, \xi) \to (M, \xi')$ such that $f_*(\xi)_p = (\xi')_{f(p)}$ for all $p \in M$ where $f_* : T_pM \to T_{f(p)}M$ denotes the differential of f and f is isotopic to the identity.

2.16.2 Classification of Tight Contact Structures on 3-Manifolds

Definition 2.16.10. [26] Let v be a vector field on a contact manifold (M, ξ). If v's flow ϕ_t preserves the contact planes, i.e. $(\phi_t)_*\xi = \xi$, then v is called a *contact vector field*.

Definition 2.16.11. [26] Let (M, ξ) be a contact 3-manifold and let S be a smooth surface in (M, ξ). The surface S is called *convex*, if a contact vector field v is transverse to S.

Definition 2.16.12. [26] Let $S \subset (M, \xi)$ be a convex surface and the contact vector field v on it. The *dividing set* of S is defined as follows:

$$\Gamma = \{x \in S | v(x) \in \xi_x\}.$$

Example 2.16.8. *You can see, in Figure 2.15, the dividing set of* S^2 *in* $(\mathbb{R}^3, \xi_{sym})$*.*

Definition 2.16.13. Let n be a natural number. A simple finite continued fraction expansion is an expression of the form

$$c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_n}}}}$$

where $c_0, c_1, \ldots, c_n \in \mathbb{R}$. It is denoted as $[c_0, c_1, \ldots, c_n]$.

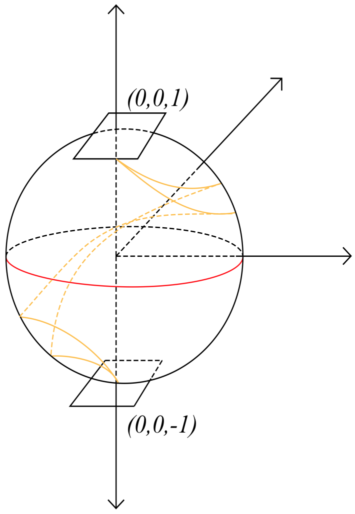


Figure 2.15 Dividing curve pointed out as red line

Example 2.16.9. *Let us calculate the continued fraction of* $\frac{187}{57}$.

$$\frac{187}{57} = 3 + \frac{16}{57} = 3 + \frac{1}{\frac{57}{16}} = 3 + \cfrac{1}{3 + \frac{1}{\frac{16}{9}}} = \ldots = 3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}}}$$

Also, it is expressed as $\frac{187}{57} = [3, 3, 1, 1, 3, 2]$.

Let p, q be integers where $p > q > 0$ and $(p, q) = 1$. For any negative rational number $-\frac{p}{q}$ and $r_i < -1$, there exists a unique continued fraction expansion. The continued fraction expansion of $-\dfrac{p}{q}$ is expressed by

$$r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_k}}}} = [r_0, r_1, \ldots, r_k]$$

with all $r_i < -1$, $i = 0, \ldots, k$.

Example 2.16.10. *Let us calculate the continued fraction expansion of* $-\dfrac{7}{5}$

$$-\frac{7}{5} = -2 - \frac{3}{-5} = -2 - \frac{1}{-\frac{5}{3}} = -2 - \cfrac{1}{-2 - \cfrac{1}{-3}}.$$

Also, it is expressed as $-\dfrac{7}{5} = [-2, -2, -3]$.

Definition 2.16.14. Let p, q be relatively prime integers. Consider (p, q)-curve on a torus, that is, (p, q)-curve wraps p times in the meridian direction, q times in the longitude direction. Then, the slope of the (p, q)-curve on the torus is defined as $\dfrac{q}{p}$.

Let us consider the tight contact structures on $S^1 \times D^2$.

Theorem 2.16.15. *[27][28] Consider $S^1 \times D^2$ with convex boundary with T^2, where T^2 has two dividing curves and the slope of dividing curves is $-\dfrac{p}{q}$ such that $p > q > 0$. Then, the number of the tight contact structures on $S^1 \times D^2$ is*

$$|(r_0 + 1)(r_1 + 1) \cdots (r_{k-1} + 1) r_k|.$$

Here, $r_0, \ldots, r_k$ denote the coefficients of the continued fraction expansion of $-\dfrac{p}{q}$ and $r_i < -1$, $i = 0, \ldots, k$.

Definition 2.16.16. [28] Suppose that ξ is a tight contact structure on $T^2 \times I$ with a convex boundary. The boundary slope on $T^2 \times \{0\}$ is sl_0 and the boundary slope on $T^2 \times \{1\}$ is sl_1. Then, ξ is called *minimally twisting*(in the I-direction), if each convex torus parallel to the boundary has slope sl between sl_0 and sl_1.

Theorem 2.16.17. *[27][28] Consider $T^2 \times I$ with Γ_0 dividing curve on $T^2 \times \{0\}$ and Γ_1 dividing curve on $T^2 \times \{1\}$. Let Γ_0 be a dividing curve with two components with slope $sl_0 = -1$. Let Γ_1 be a dividing curve with two components with slope $sl_1 = -\dfrac{p}{q}$ such that $p > q > 0$. Then, there exist exactly*

$$|(r_0 + 1)(r_1 + 1) \cdots (r_{k-1} + 1)r_k|$$

tight, minimally twisting contact structures on $T^2 \times I$ up to isotopy. Here, $r_0, r_1, \ldots, r_k$ denote the coefficients of the continued fraction expansion of $-\dfrac{p}{q}$ and $r_i < -1$, $i = 0, \ldots, k$.

Theorem 2.16.18. *[27][28] Consider $T^2 \times I$. Let Γ_i be a dividing curve on $T^2 \times I$ with two components with slopes $sl_i = -1$, $i = 0, 1$. Hence, there is a unique tight, minimally twisting contact structure on $T^2 \times I$.*

Now, let us consider the tight contact structures on lens spaces, see Example 2.4.4 for the definition of lens spaces.

Theorem 2.16.19. *[27][28] Let p, q be relatively prime integers and $L(p, q)$ be a lens space and $r_0, r_1, \ldots, r_k$ are the coefficients of the continued fraction expansion of $-\dfrac{p}{q}$ and $r_i < -1$, $i = 0, \ldots, k$. Then, there exist exactly*

$$|(r_0 + 1)(r_1 + 1) \cdots (r_k + 1)|$$

tight contact structures on the $L(p, q)$ up to isotopy.

2.16.3 Legendrian Knots and Links

Definition 2.16.20. [23][29] Suppose that, L is a knot in (M, ξ) which is an embedded S^1. L is called a *Legendrian knot*, if the tangent space at p is in ξ_p, i.e. $T_pL \in \xi_p$ for all $p \in L$. A disjoint union of Legendrian knots is called a *Legendrian link*.

Example 2.16.11. *The boundary of the overtwisted disc in (M^3, ξ_{ot}) is a Legendrian unknot since it is everywhere tangent to ξ_{ot}. The red circle in Figure 2.14 which is a boundary of an overtwisted disk is a Legendrian unknot.*

By Darboux's theorem, every contact structure locally looks like $(\mathbb{R}^3, \xi_{std})$. If we add a point at infinity to $\mathbb{R}^3$, then we obtain S^3. Hence, we can imagine that a Legendrian knot L in $\mathbb{R}^3$ is a Legendrian knot in S^3.

Front projection is a useful tool to picture the knots. It projects curves in $\mathbb{R}^3$ to xz-plane.

Definition 2.16.21. [23][29] Let L be Legendrian knot at $(\mathbb{R}^3, \xi_{std})$ and $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection map such that $f(x, y, z) = f(x, z)$. The image $f(L)$ is called a *front projection* of L.

Let L be a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ and a parametrization γ of L is defined as:

$$\gamma : S^1 \to \mathbb{R}^3; \;\; \alpha \mapsto (x(\alpha), y(\alpha), z(\alpha)).$$

Assume that γ is an immersion. Therefore, γ is differentiable, and its derivative

$$d_p\gamma : T_pS^1 \to T_{\gamma(p)}\mathbb{R}^3$$

is an injective map for all points $p \in S^1$. From the definition of a Legendrian knot, $\gamma'(\alpha) = (x'(\alpha), y'(\alpha), z'(\alpha)) \in \xi_{std}$ and since we take the ξ_{std} as $ker(dz - ydx)$, the following equation is obtained:

$$z'(\alpha) - y(\alpha)x'(\alpha) = 0. \tag{1}$$

We know that, γ is a parametrization of L. The *front projection* of L is defined as

$$\gamma_f : S^1 \to \mathbb{R}^2; \alpha \mapsto (x(\alpha), z(\alpha)).$$

The front projection γ_f is not an immersion as distinct from γ. If we take $x'(\alpha) = 0$, also $z'(\alpha) = 0$ from (1). Therefore, $x'(\alpha)$ could never be 0, if γ_f was an immersion. Therefore, the vertical tangencies are not contained in the front projection γ_f except can be seen at isolated points.

For a common C^1 Legendrian embedding in R^3, $x'(\alpha)$ only vanishes at isolated points. *Generalized cusps* are the isolated points in the front projection of Legendrian knot L where there exist well-defined tangent lines which are horizontal [29].

Also, we can express y-coordinate of γ from $z'(\alpha) - y(\alpha)x'(\alpha) = 0$:

$$y(\alpha) = \frac{z'(\alpha)}{x'(\alpha)}$$

In summary, a front projection of a Legendrian knot satisfies the below conditions:

- Does not contain vertical tangencies.
- No singular points except generalized cusps.
- At each crossing, the slope of the overcrossing is smaller than the undercrossing.

Example 2.16.12. *There are different front projections of Legendrian unknots given in Figure 2.16.*

Example 2.16.13. *Front projections of Legendrian left-handed trefoil and Legendrian right-handed trefoil can be seen below, in Figure 2.17.*

Theorem 2.16.22. *[29] Let K be a topological knot. Then, K can be converted to a Legendrian knot.*

Proof. We can convert any topological knot into a Legendrian knot by the following movements in Figure 2.18.

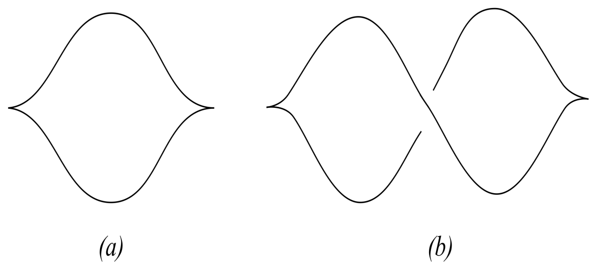


Figure 2.16 Front projections of a Legendrian unknot

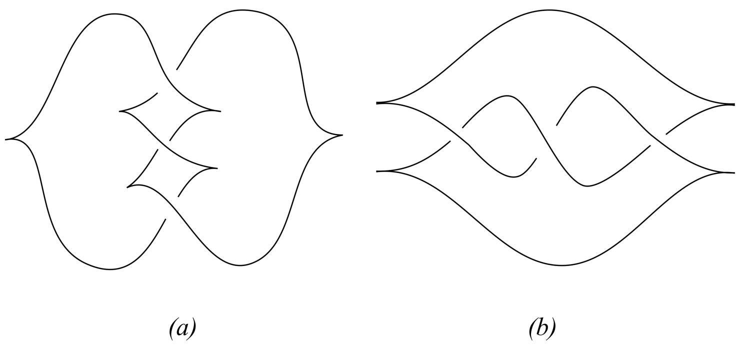


Figure 2.17 Front projections of (a) a Legendrian left-handed trefoil and (b) a Legendrian right-handed trefoil

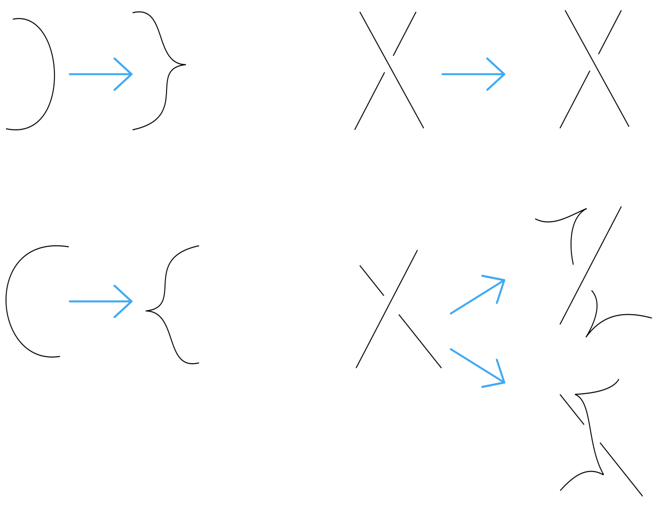
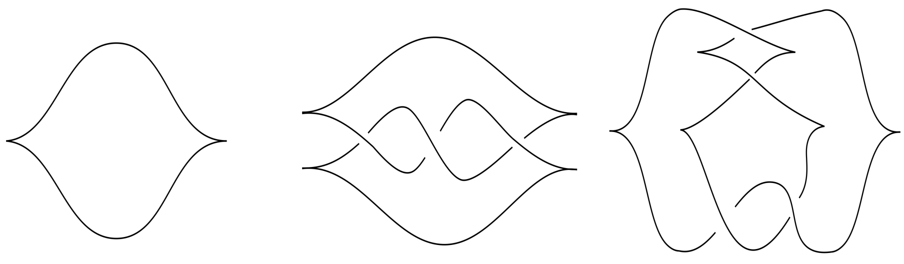


Figure 2.18 Converting a topological knot to the Legendrian knot

Example 2.16.14. *The type of a Legendrian knot is its topological knot type. There are examples of different types of Legendrian knots converted from topological knots given in Figure 2.19.*



(a) (b) (c)

Figure 2.19 (a) Legendrian unknot, (b) Legendrian right-handed trefoil, (c) Legendrian figure-8 knot

Definition 2.16.23. Let $L_1, L_2 \in (M^3, \xi)$. L_1 and L_2 are called *Legendrian isotopic* if there exists a contactomorphism $f : (M^3, \xi) \to (M^3, \xi)$ such that $f(L_1) = L_2$ and f is contact isotopic to the id.

Theorem 2.16.24. *[30] "The Legendrian isotopic Legendrian knots can be expressed by two front diagrams if and only if these diagrams are associated with regular homotopy and a series of Legendrian Reidemeister moves", given in Figure 2.20.*

Example 2.16.15. *In Figure 2.16, we can see the two front projections of a Legendrian unknot. They are Legendrian isotopic because we can convert one to the other by Legendrian Reidemeister moves.*

2.16.4 The Classical Invariants of Legendrian Knots

2.16.5 Topological Knot Type

The topological knot type of a Legendrian knot is invariant for the knot.

Example 2.16.16. *You can see two different Legendrian knots in Figure 2.21. The left one is a Legendrian left-handed trefoil, and the other one is a Legendrian Figure-8 knot. They have different knot types. Therefore, they are not Legendrian isotopic.*

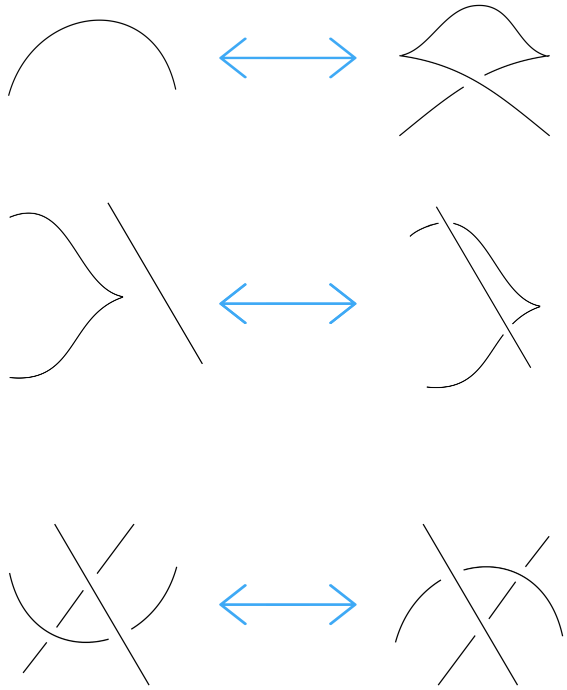
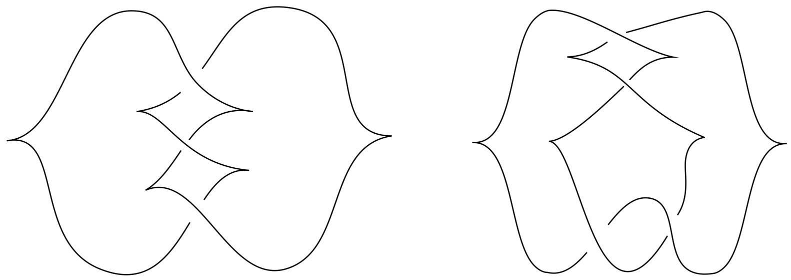


Figure 2.20 Legendrian Reidemeister moves



(a) *(b)*

Figure 2.21 (a) Legendrian left-handed trefoil and (b) Legendrian figure-8 knot

Definition 2.16.25. [23] Let L be a Legendrian knot in (M^3, ξ) and v be a non-zero vector field that is perpendicular to L. A *contact framing* of L is defined as the parallel push-off of L in the normal direction to ξ.

Example 2.16.17. *Let L be a Legendrian unknot in $(\mathbb{R}^3, \xi_{std})$. The vector field $v = \dfrac{\partial}{\partial z}$ is transverse to ξ_{std}. The contact framing of a Legendrian unknot is given in Figure 2.22.*

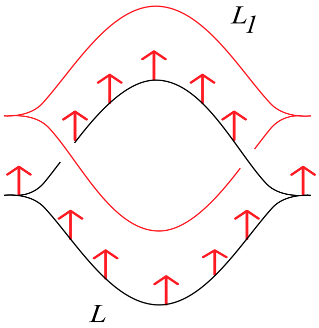


Figure 2.22 Contact framing of Legendrian unknot

Definition 2.16.26. [16] Consider a null-homologous knot L in (M^3, ξ), and consider a Seifert surface of L. Take a parallel push-off of L. The parallel push-off of L is called the *Seifert framing* of L which stays on the Seifert surface, and the parallel push-off and the linking number of L are zero.

Example 2.16.18. *Let U be the unknot. Then, the Seifert surface of the unknot U is a disc. You can see the Seifert framing of U in Figure 2.23.*

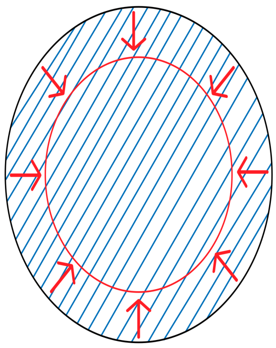


Figure 2.23 Seifert framing of a Legendrian unknot

2.16.6 Thurston-Bennequin Invariant

Definition 2.16.27. [29] Let L be a Legendrian knot in (M, ξ) and assume that L is null-homologous. The *Thurston-Bennequin invariant* of the Legendrian knot L is defined as the twisting of the contact framing relative to the Seifert framing of L and denoted by $tb(L)$.

Consider a Legendrian null-homologous knot $L \in (\mathbb{R}^3, \xi_{std})$. Let v be a non-zero vector field which is transverse to ξ_{std} along L. Take a parallel push off L' of L in the direction of v. Then the Thurston-Bennequin invariant tb(L) equals the linking number between L and L', so tb(L) = $lk(L, L')$. We can obtain crossings between L and L' at the crossings and the cusps of L. Therefore,

$$\text{tb}(L) = lk(L, L') = writhe(L) - \frac{1}{2}(\text{total number of cusps in } L).$$

Here, *writhe number* denotes the sum of ± 1s at each crossing in the front projection.

Example 2.16.19. *Let us calculate the Thurston-Bennequin invariant of the Legendrian right-handed trefoil, which is given below, in Figure 2.24. The writhe number of a Legendrian right-handed trefoil is* 3 *and it has* 4 *cusps.*

$$\begin{aligned} tb(L) &= writhe(L) - \frac{1}{2}(\#\ cusps\ of\ L) \\ &= 3 - \frac{1}{2} \cdot 4 = 1 \end{aligned}$$

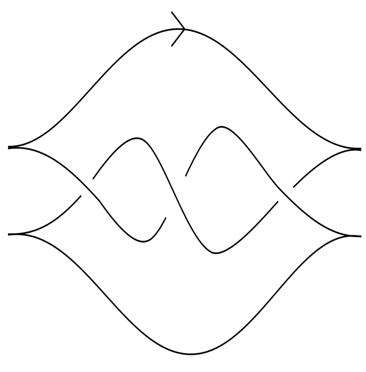


Figure 2.24 Right-handed trefoil

2.16.7 Rotation Number

Definition 2.16.28. [29] Let L be an oriented Legendrian knot which is null-homologous. Suppose that the Seifert surface of L is Σ_g^1. The restriction $\xi|_{\Sigma_g^1}$ is a trivial two-dimensional tangent bundle. A trivialization $\xi|_L = L \times \mathbb{R}^2$ is obtained from the trivialization of $\xi|_{\Sigma_g^1}$. Let v be a non-zero vector field which is tangent to L and which points in the direction of the orientation on L. Later, consider v in $\mathbb{R}^2$, a path of vector fields in $\mathbb{R}^2$, then the winding number of v is called *the rotation number* of L. The rotation number of L is denoted as $rot(L)$.

Assume that L is an oriented Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ and it is null-homologous. The trivialization of L is generated by using the vector field $u = \dfrac{\partial}{\partial y}$ which is a non-zero part of ξ_{std}. Let v be a vector field that is tangent to the L. So, the rotation number can be found by calculating the signed sum of how many times u and v point in the same direction. If v passes u counterclockwise, it is described as $(+1)$. If v passes u clockwise, it is described as (-1). The intersection of u and v will be positive at the down cusps, and the intersection of u and v will be negative at the up cusps. Throughout this process, the rotation number counts the number of v intersects both u and $-u$, so divide by two to get rot(L). Then, the rotation number of L is

$$rot(L) = \frac{1}{2}(D - U)$$

where D denotes the down cusps and U denotes the up cusps.

Example 2.16.20. *The Legendrian unknots with different orientations are given in Figure 2.25. Let left one be named by L_1 and the other one by L_2. The rotation number of L_1 is $rot(L_1) = \frac{1}{2}(5-1) = 2$ and the rotation number of L_2 is $rot(L_2) = \frac{1}{2}(1-5) = -2$. Observe that the rotation number depends on the orientation of the knot.*

Example 2.16.21. *Two Legendrian unknots are not Legendrian isotopic to each other, which are given in Figure 2.26. Let us show that.*

Both Legendrian unknots in Figure 2.26 have the same topological knot type. Such a knot type is the unknot. But their other classical invariants are different. The Thurston-Bennequin invariant of a is $tb(a) = writhe(a) - \frac{1}{2}$(# cusps of a) $= 0 - \frac{1}{2} \cdot 2 = -1$ and the Thurston-Bennequin invariant of b is $tb(b) = writhe(b) - \frac{1}{2}$(# cusps of b) $= 0 - \frac{1}{2} \cdot 4 = -2$. Their rotation numbers are also different. The rotation number of a is $rot(a) = 0$ since it

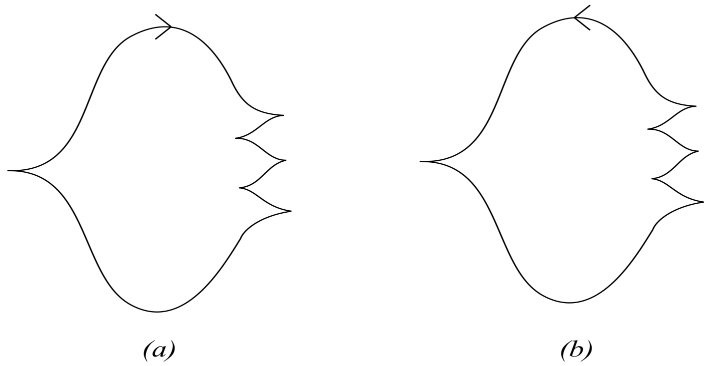


Figure 2.25 (a) Legendrian unknot with $rot = 2$ and (b) Legendrian unknot with $rot = -2$

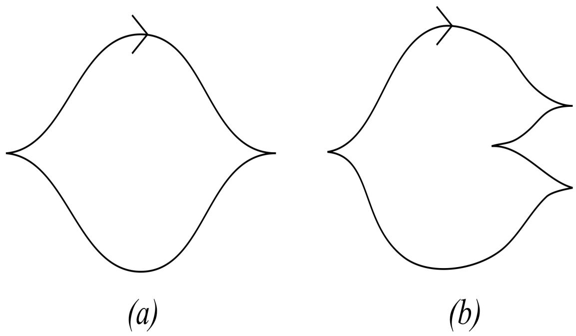


Figure 2.26 Two non-isotopic Legendrian unknots

has one up cusp and one down cusp. Similarly, $rot(b) = 1$ *since it has three down cusps and one up cusp. Thus, these two Legendrian unknots are not Legendrian isotopic.*

2.16.8 Stabilizations

Let a Legendrian knot L be given. We can get another Legendrian knot from L in the same topological knot type with stabilizations.

Definition 2.16.29. [29] Let L be an oriented Legendrian knot in $(\mathbb{R}^3, \xi_{std})$. If we change a strand of L by adding a down cusp by replacing a zigzag like in Figure 2.27, the resulting Legendrian knot is called the *positive stabilization* of L. It is denoted as $St_+(L)$. If we change a strand of L by adding an up cusp by replacing a zigzag like in Figure 2.27, the resulting Legendrian knot is called the *negative stabilization* of L. It is denoted as $St_-(L)$. In Figure 2.27, you can see positive and negative stabilizations.

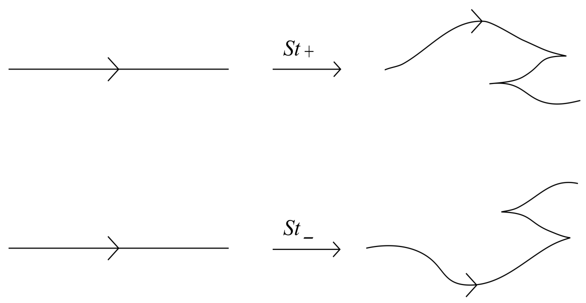


Figure 2.27 Positive and negative stabilization of a Legendrian knot

After stabilizations the classical invariants change to:

$$tb(St_{\pm}(L)) = tb(L) - 1 \quad \text{and} \quad rot(St_{\pm}(L)) = rot(L) \pm 1.$$

Example 2.16.22. *There are examples of stabilizations of Legendrian knots, given in Figure 2.28.*

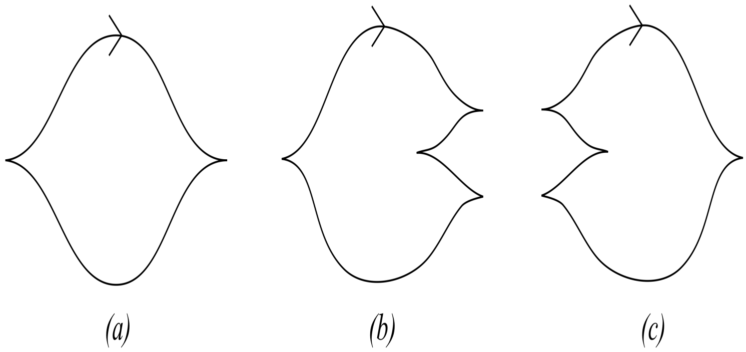


Figure 2.28 (a) Legendrian unknot and (b) a positive and (c) a negative stabilization of Legendrian unknot

2.16.9 Knots in Overtwisted Contact Structures

Legendrian knots in overtwisted contact structures are divided into two types. If a Legendrian knot L in an overtwisted (M, ξ_{ot}) has a tight complement, then L is called an *exceptional knot*. If a Legendrian knot L in an overtwisted (M, ξ_{ot}) has an overtwisted complement, then L is called a *loose knot*.

Example 2.16.23. *The boundary of an overtwisted disc is a loose Legendrian unknot in* (S^3, ξ_{ot}).

3. CLASSIFICATION OF LEGENDRIAN KNOTS AND LINKS

3.1 Legendrian Knots Classification

There are two ways to classify Legendrian knots. One way to classify Legendrian knots is by contactomorphism. The other way is by Legendrian isotopy. Here, we will give the definitions and in the next chapter, we will give some classification results of Legendrian knots and links in detail.

Definition 3.1.1. [29] Consider L_1 and L_2 two Legendrian knots in (M, ξ). We say that L_1 and L_2 are *coarsely equivalent* when there exists a contactomorphism $g : (M, \xi) \to (M, \xi)$ such that $g(L_1) = L_2$.

Definition 3.1.2. [29] Consider L_1 and L_2 two oriented Legendrian knots in (M, ξ). The knots L_1 and L_2 are called *Legendrian isotopic knots* if there exists a contactomorphism $g : (M, \xi) \to (M, \xi)$ so that $g(L_1) = L_2$ and g is contact isotopic to the identity function.

3.2 Classification Results

The classification of Legendrian knots is one of the most studied problems in 3-dimensional contact topology. In this section, we will mention classification results of Legendrian knots and links ([3], [4], [11]).

In [3], Eliashberg and Fraser focus on the topologically trivial Legendrian knots, bounding embedded 2-disc. When we say topologically trivial knot, we are mentioning the unknot. So, from now on, we will use Legendrian unknot instead of the topologically trivial Legendrian knot.

Classical invariants play an important role in the classification of Legendrian knots. These are the topological knot type, the Thurston-Bennequin invariant, and the rotation number. Theorem 3.2.1 gives us a complete classification for unknots in tight contact 3-manifolds.

Theorem 3.2.1. *[3] Let L and L' be two oriented Legendrian unknots in a tight contact 3-manifold (M, ξ). If the Thurston-Bennequin invariants of L and L' are the same that is $tb(L) = tb(L')$ and the rotation number of L and L' are the same that is $rot(L) = rot(L')$ then L and L' are Legendrian isotopic.*

With this theorem, the Thurston-Bennequin invariant and the rotation number determine Legendrian unknots in tight contact 3-manifolds. Any Legendrian unknot can be obtained from stabilizations of the unique Legendrian unknot with $tb = -1$ and $rot = 0$, see Figure 3.1.

Eliashberg and Fraser also give us the classification results for Legendrian unknots in overtwisted contact structures. There are two types of Legendrian unknots in overtwisted contact structures: loose Legendrian unknots and exceptional(non-loose) Legendrian unknots. The following theorem is about loose ones.

Theorem 3.2.2. *[3] Suppose L and L' be two loose Legendrian unknots in an overtwisted contact 3-manifold (M, ξ_{ot}). The Legendrian unknots L and L' are coarsely equivalent if and only if their classical invariants agree.*

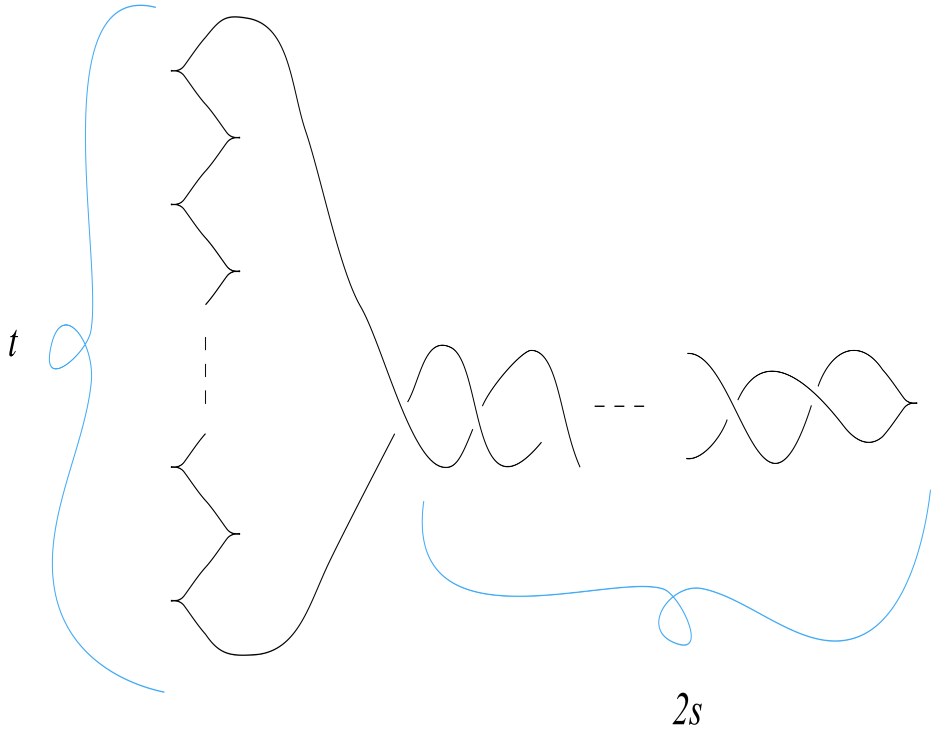


Figure 3.1 Classification of Legendrian unknots

In the first version of [3], there is no known example of an exceptional knot in an overtwisted contact structure. Eliashberg and Fraser say it could be that all Legendrian knots in overtwisted contact manifolds are loose. The first exceptional Legendrian knot examples are given in [12]. Also, there is a result for exceptional Legendrian unknots on overtwisted S^3 in [3].

Theorem 3.2.3. *[3] Let L and L' be two exceptional Legendrian unknots in (S^3, ξ_{ot}) are coarsely equivalent if and only if they have the same Thurston-Bennequin invariant and rotation number. A complete list of equivalence classes;*

$$(tb, rot) = (1, 0), (tb, rot) = (n, \pm(n-1)), (n \in \mathbb{Z}^+).$$

Eliashberg and Fraser completely classify Legendrian unknots in any tight contact structure on 3-manifolds up to Legendrian isotopy. They also classify loose Legendrian unknots for any overtwisted contact structure on 3-manifolds up to coarse equivalence.

In [4], the classification of Legendrian torus knots and the classification of Legendrian figure-eight knots in the tight contact structure on the 3-sphere S^3 is done by Etnyre and Honda, up to Legendrian isotopy.

Torus knots are a special class of knots that can be drawn on a torus without any self-intersection. A (p, q)-torus knot is a knot that wraps p times along the meridian and q times along the longitude where p and q are relatively prime.

Etnyre and Honda find the range of classical invariants for Legendrian torus knots. They gave us a complete classification for Legendrian torus knots in the tight contact structure on S^3.

Theorem 3.2.4. *[4] Let T and T' be oriented Legendrian torus knots in tight contact manifold (S^3, ξ). T and T' are Legendrian isotopic if and only if their classical invariants $tb(T) = tb(T')$ and $rot(T) = rot(T')$.*

For example, let T be a positive torus knot *i.e.* $p, q > 0$. Then, the Thurston-Bennequin invariant of T is $tb(T) \leq pq - p - q$. If $tb(T) = pq - p - q - n$ where n is a non-negative integer, then $rot(T) \in \{-n, -n + 2, \ldots, n\}$.

Now, we will mention the figure-eight knots. First, Etnyre and Honda prove that the maximal Thurston-Bennequin invariant for the figure-eight knots is -3. Here is the theorem by which they classify figure-eight knots with maximal Thurston-Bennequin invariant.

Theorem 3.2.5. *[4] Let F and F' be two oriented Legendrian figure eight knots. The maximal Thurston-Bennequin invariant of F and F' is -3 and the rotation number $rot(F) = rot(F') = 0$. Then, F and F' are Legendrian isotopic.*

Theorem 3.2.6. *[4] Let F and F' be two oriented Legendrian figure-eight knots. F and F' are called Legendrian isotopic if and only if their classical invariants agree.*

So far, we focus on Legendrian knots on contact 3-manifolds. Now, we will look out for Legendrian links in contact 3-manifolds. The classification of Legendrian links studied by Ding and Geiges in [9], [10].

Also, Geiges and Onaran give us classification results for Hopf links up to coarse equivalence, in [11]. Before this paper, there were few resources in the literature on the classification of Legendrian links. Previous studies gave us more information about the classification of Legendrian links in the standard tight contact structure ξ_{st} on S^3 or another tight contact 3-manifolds. The main theorem of their paper gives the first complete classification of a Legendrian link with Legendrian realizations in overtwisted contact structures.

An exceptional knot is called strongly exceptional if the complement has zero Giroux torsion [11]. Briefly, Giroux torsion is an invariant for contact 3-manifolds. It is defined as follows. $Tor(M, \xi)$ is the supremum of the integers $n \in \mathbb{Z}^+$ for which there is a contact embedding of

$$\mathbb{T}_n := \{T^2 \times [0, 1], ker(cos(2\pi nz)dx - sin(2\pi nz)dy)\}$$

into (M, ξ). If no such embedding exists, $Tor(M, \xi) = 0$.

The standard tight contact structure on S^3 is $ker(x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2)$ and denoted as ξ_{st}. It is the unique tight contact structure on S^3 up to isotopy. Also, there are countably many overtwisted contact structures up to d_3 invariant. Briefly, d_3 invariant of ξ is a homotopy invariant of ξ. The overtwisted contact structure on S^3 is denoted by ξ_d with a d_3 invariant $d_3 \in \mathbb{Z} + \frac{1}{2}$.

Like Legendrian knots, Legendrian realizations of the Legendrian Hopf link have classical invariants. Suppose $L_0 \cup L_1$ is a Legendrian realization of a Legendrian Hopf link. Here $tb(L_i) = tb_i$, $i = 0, 1$ is the Thurston-Bennequin invariant of L_0 and L_1 and $rot(L_i) = rot_i$, $i = 0, 1$ is the rotation number of L_0 and L_1.

Theorem 3.2.7. *[11] The Legendrian realizations of the Legendrian Hopf link are given as follows, up to coarse equivalence. Legendrian realizations are determined by the classical invariants in all cases.*

1. *Consider* (S^3, ξ_{st}). *There exists a unique Legendrian realization for all combinations of the Thurston-Bennequin invariant* tb *and the rotation number* rot, *in the range of* $tb_0, tb_1 < 0$ *and*

$$rot_i \in \{tb_i + 1, tb_i + 3, \ldots, -tb_i - 3, -tb_i - 1\}.$$

There are $tb_0 tb_1$ *realizations for fixed values of* $tb_0, tb_1 < 0$.

2. *Let* $tb_0 < 0$ *and* $tb_1 > 0$. *The strongly exceptional realizations of the Legendrian Hopf link are as follows.*

   (a) *In* $(S^3, \xi_{1/2})$ *there are realizations* $L_0 \cup L_1$ *consist of an exceptional Legendrian unknot* L_1 *with invariants* $(tb_1, rot_1) = (n+2, \pm(n+1))$, *where* $n \in \mathbb{N}_0$ *and a loose Legendrian unknot* L_0 *whose Thurston-Bennequin invariant* $tb_0 \in \mathbb{Z}^-$ *and* $rot_0 \in \{tb_0, tb_0 + 2, \ldots, tb_0 - 2, -tb_0\}$. *For a given* $tb_0 < 0$, *there are* $2|tb_0 - 1|$ *realizations.*

   (b) *In* $(S^3, \xi_{1/2})$ *there are realizations* $L_0 \cup L_1$ *with an exceptional Legendrian unknot* L_1 *and a loose Legendrian unknot* L_0. *The invariants of* L_1 *are* $(tb_1, rot_1) = (1, 0)$ *and the invariants of* L_0 *are* $tb_0 \in \mathbb{Z}^-$ *and* rot_0 *lies in the range*

   $$\{tb_0 - 1, tb_0 + 1, \ldots, tb_0 - 1, -tb_0 + 1\}.$$

   There are $|tb_0 - 2|$ *Legendrian realizations for a given* $tb_0 < 0$.

3. *For* $tb_0, tb_1 > 0$ *the strongly exceptional realizations are as follows.*

   (a) *There is a unique Legendrian realization* $L_0 \cup L_1$ *in* $(S^3, \xi_{1/2})$ *with invariants* $(tb_i, rot_i) = (1, 0)$, $i = 0, 1$. *The components* L_0 *and* L_1 *are both exceptional.*

   (b) *There is a pair of realizations with invariants* $(tb_0, rot_0) = (2, \pm 3)$ *and* $(tb_1, rot_1) = (1, \pm 2)$ *in* $(S^3, \xi_{-1/2})$.

There are two realizations in $(S^3, \xi_{-1/2})$ with invariants $(tb_0, rot_0) = (3, \pm 4)$ and $(tb_1, rot_1) = (1, \pm 2)$. Also, there is a realization live in $(S^3, \xi_{-1/2})$ with $(tb_0, rot_0) = (3, 0)$ and $(tb_1, rot_1) = (1, 0)$.

There are four realizations with $tb_0 = tb_1 = 2$: two with $rot_0 = rot_1 = \pm 3$ in $(S^3, \xi_{-1/2})$ and two with $rot_0 = rot_1 = \pm 1$ in $(S^3, \xi_{3/2})$.

The link components are loose in all cases.

(c) For any $tb_0 \geq 4$ and $tb_1 = 1$ there are four Legendrian realizations $L_0 \cup L_1$ with these Thurston-Bennequin invariants. For $tb_0 \geq 3$ and $tb_1 = 2$ there are six realizations. All link components are loose.

(d) Let $tb_0, tb_1 \geq 3$. Then, there are eight realizations and in all cases link components are loose.

4. *For $tb_0 = 0$, there exist two strongly exceptional realizations with the Thurston-Bennequin invariant $tb_1 = k$ for each $k \in \mathbb{Z}$ and the rotation numbers are $rot_0 = \pm 1$ and $rot_1 = \pm(k-1)$. The component L_0 is loose for all cases. The component L_1 is exceptional for the case $k \geq 1$, loose for the case $k \leq 0$.*

5. *If $tb_0 \neq \pm 1$ and $tb_1 \neq \pm 1$ and $n \in \mathbb{N}$, then there exist exactly a pair of exceptional Legendrian Hopf links $L_0 \cup L_1$. These pair of Legendrian Hopf links are recognized by the rotation numbers with $tb(L_i) = tb_i$ $(i = 0, 1)$ and n is equal to π-twisting in the link complement.*

6. *For any choice of $tb_0, rot_0, tb_1, rot_1 \in \mathbb{Z}$ with $tb_i + rot_i$ odd, and for any $d \in \mathbb{Z} + \frac{1}{2}$, there is a unique loose Hopf link $L_0 \cup L_1$ in (S^3, ξ_d) with invariants $tb(L_i) = tb_i$ and $rot(L_i) = rot_i$.*

This paper gives us a complete classification of the Legendrian realizations of the Legendrian Hopf link.

We mentioned three different papers to classify Legendrian knots and links, and the common point of these papers is that all of them use classical invariants to classify Legendrian knots and links. So, the classical invariants are useful tools to determine Legendrian knots and links.

3.3 Classification Results in Detail

Now, in this section, we will look at the classification of Legendrian unknots and Legendrian Hopf links in the contact 3-sphere S^3 in detail.

3.3.1 Legendrian Unknots in Contact 3-Sphere S^3

Theorem 3.3.1. *[3][7]*

1. *Let U be a Legendrian unknot in (S^3, ξ_{std}). Let, $tb(U) = n$ with $n \in \mathbb{Z}^-$, and $rot(U)$ lies in the range*
$$\{n+1, n+3, \ldots, -n-3, -n-1\}.$$
The invariants (tb, rot) determines U up to coarse equivalence and there exist $|n|$ distinct Legendrian unknots for all $n \leq -1$.

2. *Let U be an exceptional unknot in an overtwisted contact structure (S^3, ξ) which is determined by $d_3(\xi) = 1/2$ up to isotopy. Then the invariants of U lie in the range*
$$(tb(U), rot(U)) \in \{(n, \pm(n-1)) \mid n \in \mathbb{N}\}.$$
Legendrian unknot U with these pair of invariants exist and U is determined by these Thurston-Bennequin invariants and rotation numbers.

Proof. Consider a Legendrian unknot U in (S^3, ξ_{std}) or an exceptional unknot U in an overtwisted S^3. Now, the 3-sphere S^3 can be thought of as a decomposition of two solid tori, *i.e.* $S^3 = N_1 \cup N_2$, with N_1 is the standard neighborhood of U. Let μ_i, λ_i denote the meridian and longitude of the solid tori N_i, $i = 1, 2$. We assume that ∂N_1 is a convex torus with two dividing curves of slope $sl_1 = \frac{1}{tb}$, where $tb := tb(U)$ is the Thurston-Bennequin invariant of U. Gluing map of two solid tori is defined by the identifications $\mu_1 = \lambda_2$ and $\lambda_1 = \mu_2$.

By assumption, the contact structure on N_2 is tight for each case. By using the gluing map, we see that the boundary ∂N_2 is a convex torus with two dividing curves of slope tb. The Legendrian unknot U is determined by the total number of tight contact structures on N_2 up to coarse equivalence. To find the total number of tight contact structures on N_2, we need Theorem 2.16.15. Let N be a solid torus that has a convex boundary ∂N, and the boundary ∂N has two dividing curves of slope $-\frac{p}{q} < -1$. Then, N has

$$|(r_0 + 1) \cdots (r_{k-1} + 1) r_k|,$$

number of tight contact structures by Honda [28] and Giroux [27] by Theorem 2.16.15. Here for $i = 0, \ldots, k$, $r_i < -1$ and $r_0, r_1, \ldots, r_k$ are the coefficients of the continued fraction expansion of $-\frac{p}{q}$; for slope -1 there is a unique tight contact structure.

Now, let U be a Legendrian unknot with tight complement, and having the Thurston-Bennequin invariant $tb(U) = tb < 0$. Then, the complement of U is a tight solid torus N_2 with convex boundary ∂N_2 having two dividing curves of slope $sl_2 = tb < 0$. The continued fraction expansion of sl_2 is $sl_2 = tb = |r_0| = |tb|$. By Theorem 2.16.15, there exist $|tb|$ different tight contact structures on N_2. This means that, there are at most $|tb|$ different Legendrian unknots with $tb < 0$. We can explicitly find such Legendrian unknots. For example, there is at most one $|tb| = |-1| = 1$, Legendrian unknot U with $tb = -1$. Such an unknot is given in Figure 3.2.

There are at most two, $|tb| = |-2| = 2$, Legendrian unknot U with $tb = -2$. Such unknots are explicitly given in Figure 3.3.

Note that the Legendrian unknots in Figure 3.3 are stabilizations of the Legendrian unknot in Figure 3.2. Each tight contact structure on the complement of a Legendrian unknot U with

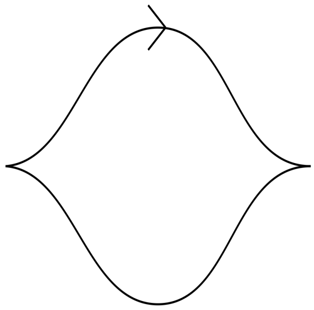
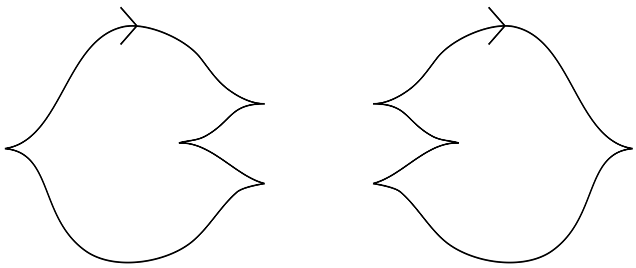


Figure 3.2 Legendrian unknot with $tb = -1$ and $rot = 0$



(a) *(b)*

Figure 3.3 (a) Legendrian unknot with $tb = -2$ and $rot = 1$, and (b) Legendrian unknot with $tb = -2$ and $rot = -1$

$tb < 0$ corresponds to the $|tb|$ realizations of a Legendrian unknot with Thurston-Bennequin invariant equals to tb in (S^3, ξ_{std}) where we obtain each realization as a stabilization of the Legendrian unknot with $tb = -1$ and $rot = 0$. There are at most $|tb|$ different Legendrian unknots with $tb < 0$, and we explicitly find $|tb|$ different such knots. Therefore, there are exactly $|tb|$ different Legendrian unknots with $tb < 0$.

Now, let U be a Legendrian unknot with the Thurston-Bennequin invariant $tb(U) = tb = 0$. Then the complement of U is a solid torus N_2 with convex boundary ∂N_2 having two dividing curves of slope $sl_2 = tb = 0$. Note that when the slope $sl_2 = tb = 0$ on ∂N_2, the contact structure on the solid torus N_2 would be overtwisted, so this case does not occur when the Legendrian unknot is in (S^3, ξ_{std}) or it is an exceptional unknot in an overtwisted S^3.

Now, let U be a Legendrian unknot with tight complement and having the

Thurston-Bennequin invariant $tb(U) = tb > 0$. Then the complement of U is a tight solid torus N_2. The solid torus N_2 has convex boundary ∂N_2. On the convex boundary ∂N_2, there are two dividing curves of slope $sl_2 = tb > 0$. When the slope $sl_2 = tb > 0$, note that we can not use the classification results of Honda [28] and Giroux [27], namely Theorem 2.16.15. Since, to use the theorem, the slope of the dividing curve on the convex boundary must be less than or equal to -1. Therefore, we have to modify N_2 so that the dividing curves have slope ≤ -1. To modify, we replace λ_2 by $\lambda_2' = \lambda_2 + m\mu_2$, then we have:

$$\begin{aligned}\mu_2 + tb \cdot \lambda_2 &= \mu_2 + tb \cdot (\lambda_2' - m \cdot \mu_2)\\ &= \mu_2 + tb \cdot \lambda_2' - m \cdot tb \cdot \mu_2\\ &= (1 - m \cdot tb)\mu_2 + tb \cdot \lambda_2'.\end{aligned}$$

That changes the slope $sl_2 = tb > 0$ to $sl_2' = \dfrac{tb}{1 - m \cdot tb}$.

For the slope $tb = 1$, we can take $m = 2$:

$$\begin{aligned}\mu_2 + \lambda_2 &= \mu_2 + (\lambda_2' - 2\mu_2)\\ &= -\mu_2 + \lambda_2'\end{aligned}$$

which changes the slope $sl_2 = 1$ to $sl_2' = -1$

For this slope $sl_2' = -1$ by Theorem 2.16.15 N_2 has a unique tight contact structure.

Now for slope $tb = 2$, take $\lambda_2' = \lambda_2 + \mu_2$:

$$\begin{aligned}\mu_2 + 2 \cdot \lambda_2 &= \mu_2 + 2 \cdot (\lambda_2' - \mu_2)\\ &= \mu_2 + 2 \cdot \lambda_2' - 2 \cdot \mu_2\\ &= (1 - 2) \cdot \mu_2 + 2 \cdot \lambda_2'\\ &= -1 \cdot \mu_2 + 2 \cdot \lambda_2'.\end{aligned}$$

Hence, this replacement changes the slope $sl_2 = 2$ to $sl_2' = -2$, then by the formula in Theorem 2.16.15, $sl_2' = -2 = [-2]$, N_2 has only two tight contact structure.

Similarly, for $sl_2 = tb = 3$, the new slope is $sl_2' = -\frac{3}{2} = -2 - \frac{1}{-2} = [-2, -2]$.

For $sl_2 = tb = 4$, the new slope is

$$sl_2' = -\frac{4}{3} = -2 - \frac{2}{-3} = -2 - \frac{1}{-\frac{3}{2}} = -2 - \frac{1}{-2 - \frac{1}{-2}} = [-2, -2, -2]$$

For $sl_2 = tb = 5$, the new slope is

$$sl_2' = -\frac{5}{4} = -2 - \frac{3}{-4} = -2 - \frac{1}{-2 - \frac{2}{-3}} = -2 - \frac{1}{-2 - \frac{1}{-\frac{3}{2}}}$$

$$= -2 - \frac{1}{-2 - \frac{1}{-2 - \frac{1}{-2}}} = [-2, -2, -2, -2]$$

$$\vdots$$

For $sl_2 = tb = n$, $sl_2' = -\frac{n}{n-1} = -2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{-2}}}} = [-2, -2, \ldots, -2]$

By Theorem 2.16.15, since $sl_2' = -\frac{n}{n-1} = [-2, -2, \ldots, -2]$, the total number of tight contact structures on solid torus N_2 is equal to $|(-2+1)(-2+1)\cdots(-2)| = 2$. This means that there are at most two Legendrian unknots with a given $tb > 0$. In Proposition 4.1 of [31] on page 174, two distinct exceptional Legendrian unknots with a given $tb > 0$ have been described explicitly. Therefore, there are exactly two Legendrian unknots that corresponds to the stated invariants in the theorem. $\square$

3.3.2 Legendrian Hopf Links in Contact 3-Sphere S^3

Now, consider a Legendrian Hopf link $L_0 \cup L_1$ in a contact 3-sphere S^3. We will examine the classification of Legendrian Hopf links having tight, minimally twisting complement in contact 3-sphere S^3 in detail.

Let the Thurston-Bennequin invariant of L_0 be $tb(L_0) = tb_0$ and the Thurston-Bennequin invariant of L_1 be $tb(L_1) = tb_1$. We decompose S^3 into two solid tori and a minimally twisting, thickened torus $T^2 \times [0, 1]$ where solid tori N_0, N_1 are tubular neighborhoods of L_0 and L_1 respectively. Hence,

$$S^3 = N_0 \cup T^2 \times I \cup N_1.$$

By standard Legendrian neighborhood theorem [28], we can assume that the boundary of a tubular neighborhood of L_0 is a convex torus with two dividing curves of slope $sl_0 = \dfrac{1}{tb(L_0)} = \dfrac{1}{tb_0}$. Similarly, the boundary of a tubular neighborhood of L_1 is a convex torus with two dividing curves of slope $sl_1 = \dfrac{1}{tb(L_1)} = \dfrac{1}{tb_1}$.

Now, let us determine the slopes of dividing curves on the boundary of $T^2 \times I$. $T^2 \times \{0\}$ has two dividing curves with slope $sl_0 = \dfrac{1}{tb_0}$. To find the slope of the dividing curves on $T^2 \times \{1\}$, we will use the gluing map

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The slope of the dividing curves on the boundary of N_1 is $sl_1 = \dfrac{1}{tb_1}$. The slope $sl_1 = \dfrac{1}{tb_1}$ curve is the $(tb_1, 1)$-curve, that is $tb_1\mu_1 + \lambda_1$-curve. After we apply the gluing map:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} tb_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ tb_1 \end{pmatrix},$$

we get $(1, tb_1)$-curve on $T^2 \times \{1\}$ which is a $1\mu_1 + tb_1\lambda_1$ curve, so that it has slope $\dfrac{tb_1}{1}$. Therefore, the slope of the dividing curve on $T^2 \times \{1\}$ is $sl_1 = tb_1$.

In conclusion, the complement of Hopf link $L_0 \cup L_1$, with the Thurston-Bennequin invariants $tb(L_i) = tb_i$, $i = 0, 1$ is a minimally twisting $T^2 \times I$ where the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1$.

Proposition 3.3.2. *[11] Let $L_0 \cup L_1$ be a Legendrian Hopf link in the contact 3-sphere S^3 with the Thurston-Bennequin invariants $tb(L_i) = tb_i$, $i = 0, 1$. Then the number of tight minimally twisting contact structures $Tight^{min}_{cs}$ on the complement $T^2 \times I$ where the slope of the dividing curve on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0}$ and the slope of the dividing curve on $T^2 \times \{1\}$ is $sl_1 = tb_1$ is given as:*

(1a) If $tb_0 = tb_1 = -1$, there is a unique tight contact structure up to diffeomorphism.

(1b) If $tb_0, tb_1 < 0$, then we have $Tight^{min}_{cs} = |tb_0 \cdot tb_1|$ (except the case $tb_0 = tb_1 = -1$).

(2a) If $tb_0 < 0$ and $tb_1 \geq 2$, we have $Tight^{min}_{cs} = 2 \cdot |tb_0 - 1|$.

(2b) If $tb_0 < 0$ and $tb_1 = 1$, then $Tight^{min}_{cs} = |tb_0 - 2|$.

(3a) If $tb_0 = tb_1 = 1$, there is a unique tight contact structure up to diffeomorphism.

(3b) If $tb_0 = 2$ and $tb_1 = 1$, then $Tight^{min}_{cs} = 2$.

(3c) If $tb_0 = 3$ and $tb_1 = 1$, then $Tight^{min}_{cs} = 3$.

(3d) If $tb_0 = 2$ and $tb_1 = 2$, then $Tight^{min}_{cs} = 4$.

(3e) If $tb_0 \geq 4$ and $tb_1 = 1$, then $Tight_{cs}^{min} = 4$

(3f) If $tb_0 \geq 3$ and $tb_1 = 2$, then $Tight_{cs}^{min} = 6$.

(3g) For all $tb_0 \geq tb_1 \geq 3$, we have $Tight_{cs}^{min} = 8$.

(4) For all $tb_1 \in \mathbb{Z}$ and $tb_0 = 0$, we have $Tight_{cs}^{min} = 2$.

Proof. By the Theorem 2.16.17 *(Giroux & Honda [27][28])*, we know how to compute the number of tight, minimally twisting contact structures on $T^2 \times I$ when the slope of the dividing curves of $T^2 \times \{0\}$ is $sl_0 = -1$ and when the slope of the dividing curves of $T^2 \times \{1\}$ is $sl_1 < -1$. By the Theorem 2.16.17, the number of tight, minimally twisting contact structure on $T^2 \times I$ can be calculated using the continued fraction expansion of the slope $sl_1 < -1$:

$$sl_1 = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_k}}}} = [r_0, r_1, \ldots, r_k],$$

where $r_i < -1$, $i = 0, \ldots, k$. By the formula in the Theorem 2.16.17, the number of tight, minimally twisting contact structure on $T^2 \times I$ when $s_0 = -1$ and $s_1 < -1$ is:

$$Tight_{cs}^{min} = |(r_0 + 1)(r_1 + 1)(r_{k-1} + 1)r_k|. \tag{2}$$

In our case, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1$. To use the formula in the Theorem 2.16.17, we need to arrange the slope of the dividing curves on $T^2 \times \{0\}$ to be -1 and the slope of the dividing curves on $T^2 \times \{1\}$ to be less than -1. In order to do that, we will replace the given $T^2 \times I$ by a diffeomorphic $T^2 \times I$ having proper slopes where such an operation corresponds to applying an element of $SL(2, \mathbb{Z})$.

For example, assume L_0 has $tb_0 = -2$ and L_1 has $tb_1 = -3$. Now, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} = -\dfrac{1}{2}$ and it is a $(-2, 1)$-curve, that is $-2\mu_0 + \lambda_0$-curve or $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$-curve and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = -3$ which is a $(1, -3)$ curve, that is $\mu_1 - 3\lambda_1$-curve or $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$-curve.

Now, we apply an element $\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \in SL(2, \mathbb{Z})$ to this $T^2 \times I$ to obtain a diffeomorphic $T^2 \times I$ so that we can use the formula in Theorem 2.16.17. After applying $\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix}$ on $T^2 \times \{0\}$, the $(-2, 1)$-curve becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve, that is a $\mu_0 - \lambda_0$-curve which has slope $sl_0' = -1$.

Now, on $T^2 \times \{1\}$, the $(1, -3)$-curve becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1+9 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$$

the $(-3, 8)$-curve, that is a $-3\mu_1 + 8\lambda_1$-curve which has slope $sl_1' = -\dfrac{8}{3}$.

Since, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0' = -1$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1' = -\dfrac{8}{3}$ after the transformation, now, we can use the formula in the Theorem 2.16.17. First, we find the continued fraction expansion of $sl_1' = -\dfrac{8}{3}$.

$$sl_1' = -\frac{8}{3} = -3 - \frac{1}{-3} = [-3, -3].$$

Then, by the formula

$$Tight_{cs}^{min} = |(-3+1)(-3)| = 6.$$

That is the number of tight, minimally twisting contact structures on the complement of $L_0 \cup L_1$ with Thurston-Bennequin invariants $tb(L_0) = tb_0 = -2$ and $tb(L_1) = tb_1 = -3$ is equal to 6.

Now, we will explicitly find the transformations for each case of the proof, and we will compute the number of tight contact structures in each case in detail.

Case (1a): Assume that $tb_0 = -1$ and $tb_1 = -1$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{-1} = -1$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = -1$. Then, by the Theorem 2.16.18 there is a unique tight, minimally twisting contact structure on this $T^2 \times I$.

Case (1b): Assume that $tb_0 < 0$ and $tb_1 < 0$, $tb_0 \neq -1$ and $tb_1 \neq -1$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} < 0$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 < 0$. In this case, we use the transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & tb_0 - 1 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0}$-curve is a $(tb_0, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & tb_0 - 1 \end{pmatrix} \begin{pmatrix} tb_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl_0' = -1$.

The slope $sl_1 = tb_1$-curve is a $(1, tb_1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & tb_0 - 1 \end{pmatrix} \begin{pmatrix} 1 \\ tb_1 \end{pmatrix} = \begin{pmatrix} tb_1 \\ tb_0 tb_1 - tb_1 - 1 \end{pmatrix}$$

the $(tb_1, tb_0 tb_1 - tb_1 - 1)$-curve that is $tb_1 \mu_1 + (tb_0 tb_1 - tb_1 - 1)\lambda_1$-curve which has slope

$$sl_1' = \frac{tb_0 tb_1 - tb_1 - 1}{tb_1} = tb_0 - 1 - \frac{1}{tb_1}.$$

Then, when $tb_1 \leq -2$, the continued fraction expansion of

$$sl_1' = tb_0 - 1 - \frac{1}{tb_1} = [tb_0 - 1, tb_1].$$

According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equals to $Tight_{cs}^{min} = |(tb_0 - 1 + 1)(tb_1)| = |tb_0 tb_1|$, by (2).

On the other hand, when $tb_1 = -1$ and $tb_0 \leq -2$, we have the slope

$$sl_1' = tb_0 - 1 - \frac{1}{tb_1} = tb_0 - 1 - \frac{1}{-1} = tb_0 - 1 + 1 = tb_0$$

which has the continued fraction expansion $sl_1' = tb_0 = [tb_0]$. In this case, according to the Theorem 2.16.17, we have $Tight_{cs}^{min} = |tb_0|$ tight, minimally twisting contact structures on $T^2 \times I$.

Case (2a): Assume that $tb_0 < 0$ and $tb_1 \geq 2$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} < 0$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 \geq 2$. In this case, we apply the same transformation as in $(1a)$:

$$\begin{pmatrix} 0 & 1 \\ -1 & tb_0 - 1 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0}$-curve is a $(tb_0, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & tb_0 - 1 \end{pmatrix} \begin{pmatrix} tb_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl_0' = -1$.

The slope $sl_1 = tb_1$-curve is a $(1, tb_1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & tb_0 - 1 \end{pmatrix} \begin{pmatrix} 1 \\ tb_1 \end{pmatrix} = \begin{pmatrix} tb_1 \\ tb_0 tb_1 - tb_1 - 1 \end{pmatrix}$$

the $(tb_1, tb_0 tb_1 - tb_1 - 1)$-curve that is $tb_1\mu_1 + (tb_0 tb_1 - tb_1 - 1)\lambda_1$-curve which has the slope

$$sl_1' = \frac{tb_0 tb_1 - tb_1 - 1}{tb_1} = tb_0 - \frac{tb_1 + 1}{tb_1}.$$

Since $tb_0 < 0$ and $tb_1 \geq 2$, then the slope $sl_1' = tb_0 - \dfrac{tb_1 + 1}{tb_1} < -1$. Now we need to find the continued fraction expansion of sl_1'. We know that $-\dfrac{tb_1 + 1}{tb_1}$ has the continued fraction expansion

$$-\frac{tb_1 + 1}{tb_1} = -2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{-2}}}} = [\underbrace{-2, -2, \ldots, -2}_{tb_1}].$$

Therefore, the continued fraction expansion of sl'_1 is

$$sl'_1 = tb_0 - \frac{tb_1+1}{tb_1} = tb_0 - 2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{-2}}}} = [tb_0 - 2, \underbrace{-2, -2, \ldots, -2}_{tb_1 - 1}].$$

According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equals to $Tight^{min}_{cs} = |(tb_0 - 2 + 1)tb_1| = 2|tb_0 - 1|$, by 2.

(2b) Assume that $tb_0 < 0$ and $tb_1 = 1$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} < 0$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = 1$. In this case, we apply the same transformation as in $(1a)$:

$$\begin{pmatrix} 0 & 1 \\ -1 & tb_0 - 1 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0}$-curve is a $(tb_0, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & tb_0 - 1 \end{pmatrix} \begin{pmatrix} tb_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl'_0 = -1$.

The slope $sl_1 = tb_1 = 1$-curve is a $(1, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & tb_0 - 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ tb_0 - 2 \end{pmatrix}$$

the $(1, tb_0 - 2)$-curve that is $\mu_1 + (tb_0 - 2)\lambda_1$-curve which has slope

$$sl_1' = tb_0 - 2.$$

Since $tb_0 < 0$, $sl_1' = tb_0 - 2 < -1$, now we can use the Theorem 2.16.17. The continued fraction expansion of $sl_1' = tb_0 - 2 = [tb_0 - 2]$. According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equals to $Tight_{cs}^{min} = |tb_0 - 2|$, by (2).

Case (3a): Assume that $tb_0 = 1, tb_1 = 1$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} = 1$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = 1$. In this case, we apply the transformation:

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0} = 1$-curve is a $(1, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl_0' = -1$.

The slope $sl_1 = tb_1 = 1$-curve is a $(1, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve that is $\mu_1 - \lambda_1$-curve which has slope $sl_1' = -1$.

Since $sl_0' = sl_1' = -1$, by the Theorem 2.16.18, there is a unique tight, minimally twisting contact structure on this $T^2 \times I$.

Case (3b): Assume that $tb_0 = 2$, $tb_1 = 1$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{2}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = 1$. In this case, we apply the transformation:

$$\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{2}$-curve is a $(2, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl'_0 = -1$.

The slope $sl_1 = tb_1 = 1$-curve is a $(1, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

the $(-1, 2)$-curve that is $-\mu_1 + 2\lambda_1$-curve which has slope $sl'_1 = -2$.

Since $sl'_1 = -2 < -1$, now we can use the Theorem 2.16.17. The continued fraction expansion of $sl'_1 = -2 = [-2]$. According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equals to $Tight^{min}_{cs} = |-2| = 2$, by (2).

Case (3c): Assume that $tb_0 = 3$, $tb_1 = 1$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{3}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = 1$. In this case, we apply the transformation:

$$\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{3}$-curve is a $(3,1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1,-1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl'_0 = -1$.

The slope $sl_1 = tb_1 = 1$-curve is a $(1,1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

the $(-1,3)$-curve that is $-\mu_1 + 3\lambda_1$-curve which has slope $sl'_1 = -3$.

Since $sl'_1 = -3 < -1$, now we can use the Theorem 2.16.17. The continued fraction expansion of $sl'_1 = -3 = [-3]$. According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equals to $Tight^{min}_{cs} = |-3| = 3$, by (2).

Case (3d): Assume that $tb_0 = 2$, $tb_1 = 2$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{2}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = 2$. In this case, we apply the transformation:

$$\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{2}$-curve is a $(2,1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1,-1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl'_0 = -1$.

The slope $sl_1 = tb_1 = 2$-curve is a $(1, 2)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

the $(-1, 4)$-curve that is $-\mu_1 + 4\lambda_1$-curve which has slope $sl'_1 = -4$.

Since $sl'_1 = -4 < -1$, now we can use the Theorem 2.16.17. The continued fraction expansion of $sl'_1 = -4 = [-4]$. According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equals to $Tight^{min}_{cs} = |-4| = 4$, by (2).

Case (3e) Assume that $tb_0 \geq 4$, $tb_1 = 1$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = 1$. In this case, we apply the transformation:

$$\begin{pmatrix} 1 & 1 - tb_0 \\ -2 & -1 + 2tb_0 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0}$-curve is a $(tb_0, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 1 - tb_0 \\ -2 & -1 + 2tb_0 \end{pmatrix} \begin{pmatrix} tb_0 \\ 1 \end{pmatrix} = \begin{pmatrix} tb_0 + 1 - tb_0 \\ -2tb_0 - 1 + 2tb_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl'_0 = -1$.

The slope $sl_1 = tb_1 = 1$-curve is a $(1, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 1 - tb_0 \\ -2 & -1 + 2tb_0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 1 - tb_0 \\ -2 - 1 + 2tb_0 \end{pmatrix} = \begin{pmatrix} 2 - tb_0 \\ -3 + 2tb_0 \end{pmatrix}$$

the $(2 - tb_0, -3 + 2tb_0)$-curve that is $(2 - tb_0)\mu_1 + (-3 + 2tb_0)\lambda_1$-curve which has slope $sl'_1 = \dfrac{-3 + 2tb_0}{2 - tb_0}$. Note that $2 - tb_0 \neq 0$ since $tb_0 \geq 4$.

The slope sl_1' can be written as $sl_1' = \dfrac{-3+2tb_0}{2-tb_0} = -3 + \dfrac{tb_0-3}{tb_0-2} = -3 - \cfrac{1}{-\cfrac{tb_0-2}{tb_0-3}}$.

Note also that, for $tb \geq 4$, the slope $sl_1' < -1$, now we can use the Theorem 2.16.17. To use the theorem, we need to find the continued fraction expansion of sl_1'.

For example, for $tb_0 = 4$, we have $-\dfrac{tb_0-2}{tb_0-3} = -\dfrac{2}{1} = [-2]$.

For $tb_0 = 5$, we have $-\dfrac{tb_0-2}{tb_0-3} = -\dfrac{3}{2} = -2 - \dfrac{1}{-2} = [-2,-2]$.

For $tb_0 = 6$, we have $-\dfrac{tb_0-2}{tb_0-3} = -\dfrac{4}{3} = -2 - \cfrac{1}{-2-\cfrac{1}{-2}} = [-2,-2,-2]$, and so on.

Note that we have $-\dfrac{tb_0-2}{tb_0-3} = -2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{-2}}}} = [\underbrace{-2,-2,\ldots,-2}_{tb_0-3}]$.

By using this, we can compute the continued fraction expansion of sl_1' as:

$$sl_1' = -3 - \cfrac{1}{-\cfrac{tb_0-2}{tb_0-3}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{-2}}}} = [-3, \underbrace{-2,-2,\ldots,-2}_{tb_0-3}].$$

The continued fraction expansion of $sl_1' = [-3, \underbrace{-2,-2,\ldots,-2}_{tb_0-3}]$. According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$

equals to

$$Tight_{cs}^{min} = |(-3+1)(-2+1)\cdots(-2+1)(-2)| = |(-2)(-2)| = 4.$$

Case (3f): Assume that $tb_0 \geq 3$, $tb_1 = 2$. Then, the slope of the dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = 2$. In this case, we apply the transformation:

$$\begin{pmatrix} 1 & 1 - tb_0 \\ -2 & -1 + 2tb_0 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0}$-curve is a $(tb_0, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 1 - tb_0 \\ -2 & -1 + 2tb_0 \end{pmatrix} \begin{pmatrix} tb_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl'_0 = -1$.

The slope $sl_1 = tb_1 = 2$-curve is a $(1, 2)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 1 - tb_0 \\ -2 & -1 + 2tb_0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 2 - 2tb_0 \\ -2 - 2 + 4tb_0 \end{pmatrix} = \begin{pmatrix} 3 - 2tb_0 \\ -4 + 4tb_0 \end{pmatrix}$$

the $(3 - 2tb_0, -4 + 4tb_0)$-curve that is $(3 - 2tb_0)\mu_1 + (-4 + 4tb_0)\lambda_1$-curve which has slope

$$sl'_1 = \frac{-4 + 4tb_0}{3 - 2tb_0} = -3 + \frac{5 - 2tb_0}{2tb_0 - 3} = -3 - \cfrac{1}{2tb_0 - 3 \over -\cfrac{}{5 - 2tb_0}}.$$

Note that for $tb_0 \geq 3$, the slope $sl'_1 < -1$, so now we can use the Theorem 2.16.17. To use the theorem, we need to find the continued fraction expansion of sl'_1.

For example, for $tb_0 = 3$, we have $-\dfrac{2tb_0 - 3}{5 - 2tb_0} = -3 = [-3]$.

For $tb_0 = 4$, we have $-\dfrac{2tb_0 - 3}{5 - 2tb_0} = -\dfrac{5}{3} = -2 - \dfrac{1}{-3} = [-2, -3]$.

For $tb_0 = 5$, we have $-\dfrac{2tb_0 - 3}{5 - 2tb_0} = -\dfrac{7}{5} = -2 - \dfrac{1}{-\dfrac{5}{3}} = -2 - \dfrac{1}{-2 - \dfrac{1}{-3}} = [-2, -2, -3]$.

For $tb_0 = 6$, we have $-\dfrac{2tb_0 - 3}{5 - 2tb_0} = -\dfrac{9}{7} = -2 - \dfrac{1}{-\dfrac{7}{5}} = -2 - \dfrac{1}{-2 - \dfrac{1}{-2 - \dfrac{1}{-3}}} =$ $[-2, -2, -2, -3]$, and so on.

Note that we have $-\dfrac{2tb_0 - 3}{5 - 2tb_0} = -2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{-3}}}} = [\underbrace{-2, -2, \ldots, -2}_{tb_0 - 3}, -3]$.

By using this, we can compute the continued fraction expansion of sl_1' as

$$sl_1' = -3 - \cfrac{1}{-\cfrac{2tb_0 - 3}{5 - 2tb_0}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{-3}}}} = [-3, \underbrace{-2, -2, \ldots, -2}_{tb_0 - 3}, -3].$$

The continued fraction expansion of $sl_1' = [-3, \underbrace{-2, -2, \ldots, -2}_{tb_0 - 3}, -3]$. According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equals to

$$Tight_{cs}^{min} = |(-3 + 1)(-2 + 1) \cdots (-2 + 1)(-3)| = |(-2)(-3)| = 6.$$

Case (3g): Assume that $tb_0 \geq tb_1 \geq 3$. Then, the slope of dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1$. In this case, we use the transformation

$$\begin{pmatrix} 1 & 1 - tb_0 \\ -2 & -1 + 2tb_0 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0}$-curve is a $(tb_0, 1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 1 - tb_0 \\ -2 & -1 + 2tb_0 \end{pmatrix} \begin{pmatrix} tb_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1, -1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl_0' = -1$.

The slope $sl_1 = tb_1$-curve is a $(1, tb_1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 1 - tb_0 \\ -2 & -1 + 2tb_0 \end{pmatrix} \begin{pmatrix} 1 \\ tb_1 \end{pmatrix} = \begin{pmatrix} 1 + tb_1 - tb_0 tb_1 \\ -2 - tb_1 + 2tb_0 tb_1 \end{pmatrix}$$

the $(1 + tb_1 - tb_0 tb_1, -2 - tb_1 + 2tb_0 tb_1)$-curve that is $(1 + tb_1 - tb_0 tb_1)\mu_1 + (-2 - tb_1 + 2tb_0 tb_1)\lambda_1$-curve which has slope

$$sl_1' = \frac{-2 - tb_1 + 2tb_0 tb_1}{1 + tb_1 - tb_0 tb_1}.$$

Note that for $tb_0 \geq 3$, $tb_1 \geq 3$, the slope $sl_1' < -1$, now we can use the Theorem 2.16.17. To use the theorem, we need to find the continued fraction expansion of sl_1'.

For example, for $tb_0 = tb_1 = 3$, we have

$$sl_1' = \frac{-2 - 3 + 18}{1 + 3 - 9} = -\frac{13}{5} = -3 - \frac{2}{5} = -3 - \cfrac{1}{-\cfrac{5}{2}} = -3 - \cfrac{1}{-3 - \cfrac{1}{-2}} = [-3, -3, -2].$$

For $tb_0 = 3$ and $tb_1 = 4$, we have

$$sl'_1 = -\frac{18}{7} = -3 - \frac{3}{7} = -3 - \cfrac{1}{-\cfrac{7}{-3}} = -3 - \cfrac{1}{-3 - \cfrac{2}{-3}} = -3 - \cfrac{1}{-3 - \cfrac{1}{-\cfrac{3}{-2}}}$$

$$= -3 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2}}} = [-3, -3, -2, -2].$$

For $tb_0 = 3$ and $tb_1 = 5$, we have

$$sl'_1 = -\frac{23}{9} = -3 - \frac{4}{-9} = -3 - \cfrac{1}{-\cfrac{9}{-4}} = -3 - \cfrac{1}{-3 - \cfrac{3}{-4}} = -3 - \cfrac{1}{-3 - \cfrac{1}{\cfrac{4}{-3}}}$$

$$= -3 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{2}{-3}}} = -3 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{3}{-2}}}} = -3 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2}}}}$$

$$= [-3, -3, -2, -2, -2].$$

For $tb_0 = 3$ and $tb_1 = 6$, we have

$$sl'_1 = -\frac{28}{11} = -3 - \frac{5}{-11} = -3 - \cfrac{1}{-\cfrac{11}{-5}} = -3 - \cfrac{1}{-3 - \cfrac{4}{-5}} = -3 - \cfrac{1}{-3 - \cfrac{1}{-\cfrac{5}{-4}}}$$

$$= -3 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{3}{-4}}} = -3 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{4}{-3}}}} = -3 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{2}{-3}}}}$$

$$= -3 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{3}{-2}}}}} = -3 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2}}}}}$$

$$= [-3, -3, -2, -2, -2, -2],$$

and so on.

For $tb_0 = 4$ and $tb_1 = 5$, we have

$$sl_1' = -\frac{33}{14} = -3 - \frac{9}{-14} = -3 - \cfrac{1}{-\cfrac{14}{-9}} = -3 - \cfrac{1}{-2 - \cfrac{4}{-9}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{9}{-4}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{3}{-4}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-\cfrac{4}{-3}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{2}{-3}}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{3}{-2}}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2}}}}}$$

$$= [-3, -2, -3, -2, -2, -2].$$

For $tb_0 = 4$ and $tb_1 = 6$, we have

$$sl_1' = -\frac{40}{17} = -3 - \frac{11}{-17} = -3 - \cfrac{1}{-\cfrac{17}{-11}} = -3 - \cfrac{1}{-2 - \cfrac{5}{-11}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{11}{-5}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{4}{-5}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-\cfrac{5}{-4}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{3}{-4}}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{4}{-3}}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{2}{-3}}}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-\frac{3}{-2}}}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \frac{1}{-2}}}}}}$$

$$= [-3, -2, -3, -2, -2, -2, -2].$$

For $tb_0 = 5$ and $tb_1 = 6$, we have

$$sl'_1 = -\frac{52}{23} = -3 - \frac{17}{-23} = -3 - \cfrac{1}{-\frac{23}{-17}} = -3 - \cfrac{1}{-2 - \frac{11}{-17}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-\frac{17}{-11}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \frac{5}{-11}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-\frac{11}{-5}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \frac{4}{-5}}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-\frac{5}{-4}}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \frac{3}{-4}}}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{4}{-3}}}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{2}{-3}}}}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{3}{-2}}}}}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2}}}}}}}$$

$$= [-3, -2, -2, -3, -2, -2, -2, -2].$$

For $tb_0 = 6$ and $tb_1 = 7$, we have

$$sl'_1 = -\frac{75}{34} = -3 - \frac{27}{-34} = -3 - \cfrac{1}{-\cfrac{34}{-27}} = -3 - \cfrac{1}{-2 - \cfrac{20}{-27}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{27}{-20}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{13}{-20}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{20}{-13}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{6}{-13}}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{13}{-6}}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{5}{-6}}}}}$$

$$\vdots$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2}}}}}}}}}$$

$$= [-3,-2,-2,-2,-3,-2,-2,-2,-2,-2]$$

For $tb_0 = 7$ and $tb_1 = 8$, we have

$$sl_1' = -\frac{102}{47} = -3 - \frac{39}{-47} = -3 - \cfrac{1}{-\cfrac{47}{-39}} = -3 - \cfrac{1}{-2 - \cfrac{31}{-39}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{39}{-31}}}$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{23}{-31}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-\cfrac{31}{-23}}}} = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{15}{-23}}}}$$

$$\vdots$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{6}{-7}}}}}} = \cdots$$

$$\cdots = -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{5}{-6}}}}}}}$$

$$\vdots$$

$$= -3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2}}}}}}}}}}}$$

$$= [-3, -2, -2, -2, -2, -3, -2, -2, -2, -2, -2, -2],$$

and so on. Note that we have seen the pattern

$$sl_1' = \frac{-2 - tb_1 + 2tb_0tb_1}{1 + tb_1 - tb_0tb_1} = -3 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 3 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{-2}}}}}}$$

$$= [-3, \underbrace{-2, -2, \ldots, -2}_{tb_0-3}, -3, \underbrace{-2, -2, \ldots, -2}_{tb_1-2}].$$

The continued fraction expansion of $sl_1' = [-3, \underbrace{-2, -2, \ldots, -2}_{tb_0-3}, -3, \underbrace{-2, -2, \ldots, -2}_{tb_1-2}]$. According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equals to

$$Tight^{min}_{cs} = |(-3+1)\underbrace{(-2+1)\cdots(-2+1)}_{tb_0-3}(-3+1)\underbrace{(-2+1)\cdots(-2+1)}_{tb_1-1}(-2)|$$
$$= |(-2)(-2)(-2)| = 8.$$

Case (4a): Assume that $tb_0 = 0$ and $tb_1 > 0$. Then, the slope of dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{0}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1$. In this case, we use the transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{0}$-curve is a $(0,1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1,-1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl'_0 = -1$.

The slope $sl_1 = tb_1$-curve is a $(1, tb_1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ tb_1 1 \end{pmatrix} = \begin{pmatrix} tb_1 \\ -1 - tb_1 \end{pmatrix}$$

the $(tb_1, -1-tb_1)$-curve that is $tb_1\mu_1 + (-1-tb_1)\lambda_1$-curve which has slope $sl'_1 = -\dfrac{tb_1+1}{tb_1}$.

Since $tb_1 > 0$, then the slope $sl'_1 = -\dfrac{tb_1+1}{tb_1} < -1$. Now we need to find the continued fraction expansion of sl'_1. We know that $-\dfrac{tb_1+1}{tb_1}$ has continued fraction expansion

$$sl'_1 = -\frac{tb_1+1}{tb_1} = -2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{-2}}}} = [\underbrace{-2, -2, \ldots, -2}_{tb_1}].$$

According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equal to

$$Tight^{min}_{cs} = |(-2+1)(-2+1)\cdots(-2+1)(-2)| = |-2| = 2.$$

Case (4b): Assume that $tb_0 = 0$ and $tb_1 < 0$. Then, the slope of dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{0}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1$. In this case, we use the transformation

$$\begin{pmatrix} -tb_1+1 & 1 \\ tb_1-2 & -1 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{0}$-curve is a $(0,1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} -tb_1+1 & 1 \\ tb_1-2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1,-1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl'_0 = -1$.

The slope $sl_1 = tb_1$-curve is a $(1, tb_1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} -tb_1+1 & 1 \\ tb_1-2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ tb_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

the $(1,-2)$-curve that is $\mu_1 - 2\lambda_1$-curve which has slope $sl'_1 = -2$.

Since $sl'_1 = -2 < -1$, now we can use the Theorem 2.16.17. The continued fraction expansion of $sl'_1 = -2 = [-2]$. According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equal to $Tight^{min}_{cs} = |-2| = 2$ by (2).

Case (4c): Assume that $tb_0 = 0$ and $tb_1 = 0$. Then, the slope of dividing curves on $T^2 \times \{0\}$ is $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{0}$ and the slope of the dividing curves on $T^2 \times \{1\}$ is $sl_1 = tb_1 = 0$. In this case, we use the transformation

$$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

The slope $sl_0 = \dfrac{1}{tb_0} = \dfrac{1}{0}$-curve is a $(0,1)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the $(1,-1)$-curve that is $\mu_0 - \lambda_0$-curve which has slope $sl'_0 = -1$.

The slope $sl_1 = tb_1 = 0$-curve is a $(1,0)$-curve, after applying the transformation it becomes

$$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

the $(1,-2)$-curve that is $\mu_1 - 2\lambda_1$-curve which has slope $sl'_1 = -2$.

Since $sl'_1 = -2 < -1$, now we can use the Theorem 2.16.17. The continued fraction expansion of $sl'_1 = -2 = [-2]$. According to the Theorem 2.16.17, the number of tight, minimally twisting contact structures on this $T^2 \times I$ equal to $Tight^{min}_{cs} = |-2| = 2$ by (2). $\square$

Now, with the help of Proposition 3.3.2, we are ready to discuss the classification of Hopf links in the contact 3-sphere S^3.

Proof of Theorem 3.2.7 in the case of Legendrian Hopf links with tight, minimally twisting complements.

Assume that $H = L_0 \cup L_1$ is a Legendrian Hopf link in a contact 3-sphere with the Thurston-Bennequin invariants $tb(L_0) = tb_0$ and $tb(L_1) = tb_1$. Also, assume that the complement of $H = L_0 \cup L_1$ in S^3 is a tight, minimally twisting $T^2 \times I$. From the discussion at the beginning of the Section 3.2.2., we know that $T^2 \times I$ has two convex boundary components. One of them is $T^2 \times \{0\}$ with two dividing curves of slope $sl_0 = \dfrac{1}{tb_0}$. The second boundary component is $T^2 \times \{1\}$ with two dividing curves of slope $sl_1 = tb_1$.

Case (1) Now, assume that $tb_0 = tb_1 = -1$, then by Proposition 3.3.2 $(1a)$ there is a unique tight contact structure on the complement $T^2 \times I$. This means that there is at most one Legendrian Hopf link $H = L_0 \cup L_1$ with tight, minimally twisting complement and with $tb_0 = tb_1 = -1$. We can explicitly find such a Legendrian Hopf link in standard tight (S^3, ξ_{std}) which is given in Figure 3.4.

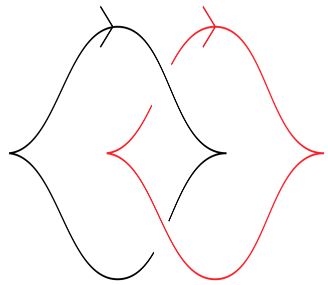


Figure 3.4 Legendrian Hopf link $H = L_0 \cup L_1$ with $tb(L_0) = tb(L_1) = -1$, $rot(L_0) = rot(L_1) = 0$

Since there is at most one such Legendrian Hopf link, and we find the Hopf link explicitly which is given in Figure 3.4, there is exactly one $H = L_0 \cup L_1$ with tight, minimally twisting complement and with $tb(L_0) = tb(L_1) = -1$.

Now, assume that $tb_0 < 0$ and $tb_1 < 0$ except $tb_0 = -1$ and $tb_1 = -1$. Then, by Proposition 3.3.2 $(1b)$, there are $|tb_0 tb_1|$ contact structures on the complement $T^2 \times I$. This means that there are at most $|tb_0 tb_1|$ many Legendrian Hopf link $H = L_0 \cup L_1$, with tight, minimally twisting complement having $tb_0 < 0$ and $tb_1 < 0$. We can find such Hopf links explicitly. For example, $tb_0 = -1$ and $tb_1 = -2$, there are at most two such Legendrian Hopf links. Two Legendrian Hopf links with corresponding invariants are given in Figure 3.5.

Therefore, there are exactly two Legendrian Hopf links with $tb_0 = -1$ and $tb_1 = -2$ having tight, minimally twisting complements.

Let us now look at the first example we examined in the proof of Proposition 3.3.2 $(1b)$ on the page 56. That is, let us look at Legendrian Hopf link $H = L_0 \cup L_1$ with $tb_0 = -2$ and $tb_1 = -3$. Then, in this case, by Proposition 3.3.2 $(1b)$, there are $|tb_0 tb_1| = |(-2)(-3)| = 6$ contact structures on the complement $T^2 \times I$. This means that there are at most 6 such

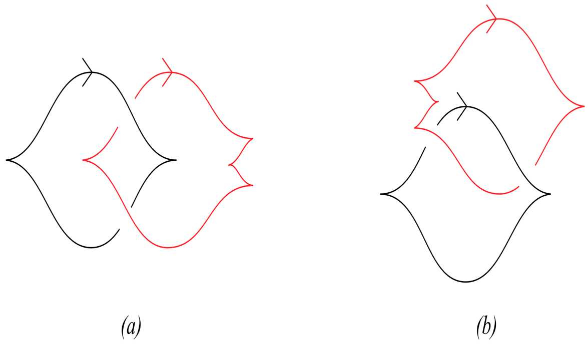


Figure 3.5 (a) Legendrian Hopf link with $tb_0 = -1$, $rot_0 = 0$ and $tb_1 = -2$, $rot_1 = 1$ and
(b) Legendrian Hopf link with $tb_0 = -1$, $rot_0 = 0$ and $tb_1 = -2$, $rot_1 = -1$

Legendrian Hopf links. Now, in Figure 3.6 and Figure 3.7, we present such Legendrian Hopf links explicitly.

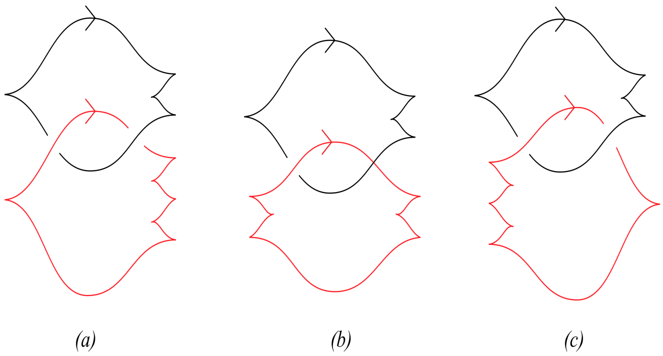


Figure 3.6 (a) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = 1$ and $tb_1 = -3$, $rot_1 = 2$,
(b) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = 1$ and $tb_1 = -3$, $rot_1 = 0$,
(c) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = 1$ and $tb_1 = -3$, $rot_1 = -2$.

There are at most 6 Legendrian Hopf links having $tb_0 = -2$ and $tb_1 = -3$ with tight, minimally twisting complement, and we present these 6 Legendrian Hopf links explicitly above in the Figure 3.6 and the Figure 3.7. Therefore, there are exactly $|tb_0 tb_1| = 6$ Legendrian Hopf links with given invariants.

Now, let us prove the general case. For $tb_0 < 0$ and $tb_1 < 0$ by Proposition 3.3.2 $(1b)$ we know that there are at most $|tb_0 tb_1|$ many Legendrian Hopf links with given invariants and with tight, minimally twisting complements. The $|tb_0 tb_1|$ explicit Legendrian Hopf links are in standard tight contact 3-sphere (S^3, ξ_{std}) which come from stabilization of Legendrian

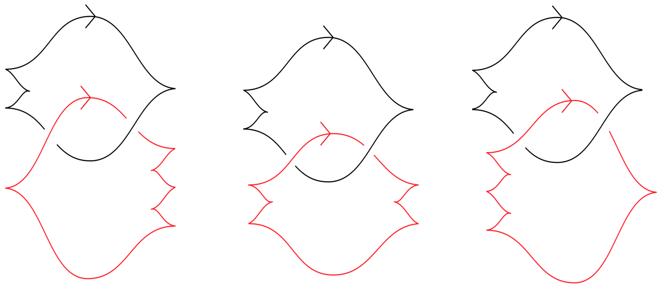


Figure 3.7 (a) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = -1$ and $tb_1 = -3$, $rot_1 = 2$,
(b) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = -1$ and $tb_1 = -3$, $rot_1 = 0$,
(c) Legendrian Hopf link with $tb_0 = -2$, $rot_0 = -1$ and $tb_1 = -3$, $rot_1 = -2$.

Hopf link $H = L_0 \cup L_1$ with $tb_0 = tb_1 = -1$ in Figure 3.4. Note that all Legendrian Hopf links with $tb_0 < 0$ and $tb_1 < 0$ are in tight contact 3-sphere S^3.

Since there are at most $|tb_0 tb_1|$ such Legendrian Hopf links with $tb_0 < 0$ and $tb_1 < 0$ and since we explicitly find $|tb_0 tb_1|$ such Legendrian Hopf links, there are exactly $|tb_0 tb_1|$ Legendrian Hopf links with the given invariants.

Case (2a) Now, assume that $tb_0 < 0$ and $tb_1 \geq 2$, then by Proposition 3.3.2 $(2a)$ there are $2|tb_0 - 1|$ tight contact structures on the complement $T^2 \times I$. This means that there are at most $2|tb_0 - 1|$ many Legendrian Hopf links having $tb_0 < 0$ and $tb_1 \geq 2$ with tight, minimally twisting complement. Such Legendrian Hopf link diagrams are explicitly given in Figure 7 of the paper [11] on page 1435. Therefore, there are exactly $2|t_0 - 1|$ many such Legendrian Hopf links.

Note that constructing explicit examples of Hopf links in this case as well as in the remaining cases of the proof requires the Dehn surgery theory and the contact surgery theory. Studying the surgery theory-related constructions is future work planned as the continuation of these studies.

Case (2b) Now, assume that $tb_0 < 0$ and $tb_1 = 1$, then by Proposition 3.3.2 $(2b)$ there are $|tb_0 - 2|$ tight contact structures on the complement $T^2 \times I$. This means that there are at most $|tb_0 - 2|$ many Legendrian Hopf links having $tb_0 < 0$ and $tb_1 = 1$ with tight, minimally twisting complement. Such Legendrian Hopf link diagrams are explicitly given in Figure 8 of the paper [11] on page 1437. Therefore, there are exactly $|tb_0 - 2|$ many such Legendrian Hopf links.

Case (3a) Now, assume that $tb_0 = tb_1 = 1$, then by Proposition 3.3.2 $(3a)$ there is a unique tight contact structure on the complement $T^2 \times I$. This means that there is at most one Legendrian Hopf link $H = L_0 \cup L_1$ with tight, minimally twisting complement and with $tb_0 = tb_1 = 1$. Explicit example is also given in Figure 8 of the paper [11] on page 1437 as in previous case $(2b)$ above. In this case, L_1 is just a parallel copy of L_0. Therefore, there is a unique exceptional Legendrian Hopf link with $tb_0 = tb_1 = 1$.

Case (3b) Now, assume that $tb_0 = 2$ and $tb_1 = 1$, then by Proposition 3.3.2 $(3b)$ there are two tight contact structures on the complement $T^2 \times I$. This means that there are at most two many Legendrian Hopf links having $tb_0 = 2$ and $tb_1 = 1$ with tight, minimally twisting complement. Such explicit two Legendrian Hopf links are described in Section 7.4 of the paper [11] on page 1451. Therefore, there are exactly two many such Legendrian Hopf links.

Now, assume that $tb_0 = 3$ and $tb_1 = 1$, then by Proposition 3.3.2 $(3c)$ there are three tight contact structures on the complement $T^2 \times I$. This means that there are at most three many Legendrian Hopf links having $tb_0 = 3$ and $tb_1 = 1$ with tight, minimally twisting complement. Such Legendrian Hopf link diagrams are explicitly given in Figure 9 of the paper [11] on page 1438. Therefore, there are exactly three many such Legendrian Hopf links.

Now, assume that $tb_0 = 2$ and $tb_1 = 2$, then by Proposition 3.3.2 $(3d)$ there are four tight contact structures on the complement $T^2 \times I$. This means that there are at most four many Legendrian Hopf links having $tb_0 = 2$ and $tb_1 = 2$ with tight, minimally twisting complement. Such Legendrian Hopf link diagrams are explicitly given in Figure 10 of the paper [11] on page 1438. Therefore, there are exactly four many such Legendrian Hopf links.

Case (3c) Now, assume that $tb_0 \geq 4$ and $tb_1 = 1$, then by Proposition 3.3.2 $(3e)$ there are four tight contact structures on the complement $T^2 \times I$. This means that there are at most four many Legendrian Hopf links having $tb_0 \geq 4$ and $tb_1 = 1$ with tight, minimally twisting complement. Such Legendrian Hopf link diagrams are explicitly given in Figure 11 of the paper [11] on page 1439. Therefore, there are exactly four many such Legendrian Hopf links.

Now, assume that $tb_0 \geq 3$ and $tb_1 = 2$, then by Proposition 3.3.2 $(3f)$ there are six tight contact structures on the complement $T^2 \times I$. This means that there are at most six many Legendrian Hopf links having $tb_0 \geq 3$ and $tb_1 = 2$ with tight, minimally twisting complement. Such Legendrian Hopf link diagrams are explicitly given in Figure 12 of the paper [11] on page 1440. Therefore, there are exactly six many such Legendrian Hopf links.

Case (3d) Now, assume that $tb_0 \geq 3$ and $tb_1 \geq 3$, then by Proposition 3.3.2 $(3g)$ there are eight tight contact structures on the complement $T^2 \times I$. This means that there are at most eight many Legendrian Hopf links having $tb_0 \geq 3$ and $tb_1 \geq 2$ with tight, minimally twisting complement. Such Legendrian Hopf link diagrams are explicitly given in Figure 13 of the paper [11] on page 1441. Therefore, there are exactly eight many such Legendrian Hopf links.

Case (4) Now, assume that $tb_0 = 0$ and $tb_1 \in \mathbb{Z}$, then by Proposition 3.3.2 (4) there are two tight contact structures on the complement $T^2 \times I$. This means that there are at most two many Legendrian Hopf links having $tb_0 = 0$ and $tb_1 \in \mathbb{Z}$ with tight, minimally twisting complement. Such Legendrian Hopf link diagrams are explicitly given in Figure 14 of the paper [11] on page 1442. Therefore, there are exactly two many such Legendrian Hopf links. □

4. CONCLUSION

In contact topology, distinguishing Legendrian knots and links by their classical invariants is an important problem. In this thesis, we studied the classification of Legendrian knots and links by their classical invariants. We look into the classification of Legendrian unknots and the classification of Legendrian Hopf links in detail in the standard tight contact 3-sphere S^3.

Moreover, the classification of exceptional Legendrian unknots and exceptional Legendrian Hopf links in the overtwisted contact 3-sphere S^3 are studied.

It has been observed that the classification of Legendrian unknots in tight contact structures or overtwisted contact structures is done with similar techniques. In both cases, it is necessary to know the classification of tight contact structures on the solid torus $S^1 \times D^2$, which is the complement of the neighborhood of the Legendrian unknot in S^3. As mentioned in Theorem 2.16.15, the classification of tight contact structures on solid torus was done by Giroux [27] and Honda [28]. Since the classification of tight contact structures on the complement of the Legendrian unknot is known, these knots can be classified. However, when the knot type changes its complement also changes, so it may be difficult to classify.

In the Hopf links case, classification is done similarly. The Hopf link is a union of two unknots. Since we are working in contact 3-sphere S^3, the complement of the two unknots that are components of the Hopf link is $T^2 \times I$. Therefore, it is necessary to know the classification of tight contact structures on $T^2 \times I$. The classification of tight contact structures on $T^2 \times I$ was made by Giroux [27] and Honda [28], as mentioned in Theorem 2.16.17. Since the classification of tight contact structures of the complement of the Legendrian Hopf links is known, these links can be classified. Note that, constructing explicit diagrams of Legendrian unknots and Legendrian Hopf links requires the Dehn surgery theory and the contact surgery theory. Studying the surgery theory-related constructions is future work planned as the continuation of this thesis.

If the number of tight contact structures on the 3-manifold in the complement of a given Legendrian knot or link is computed, an upper bound can be found for this Legendrian knot or link, with this technique. Surgery theory is needed to draw explicit diagrams of this Legendrian knot or link.

There are open problems in the classification of Legendrian knots and links. The following open problems will be studied in future works.

Open Problem 1: Find the explicit diagrams of Legendrian rational unknots in arbitrary lens spaces $L(p, q)$ by using Dehn surgery theory and contact surgery theory.

Open Problem 2: The classification of Legendrian Whitehead link in contact 3-sphere S^3.

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