MULTIPLAYER FUZZY GAME MODELS AND AN APPLICATION

ÇOK OYUNCULU BULANIK OYUN MODELLERİ VE BİR UYGULAMA

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I dedicated this thesis to my lovely son Can Onur

and my wonderful wife Hande.

For their endless love, patience and support.

ÖZET

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Günümüzün rekabet ortamının oluştuğu piyasalara ve ülkelerin stratejik hedeflerinin çatıştığı pozisyonlara bakıldığında 20. yüzyılın başında çok rastlanmayan ve giderek etkisini artıran yeni durumlar göze çarpmaktadır.

Öncelikle modern zamanın rekabet ortamında herhangi bir hedefin ikiden çok talibi bulunmaktadır. Bu durum tarafların amaca ulaşmak için birçok seçeneği gözden geçirmesini, çoklu işbirliğini ve farklı stratejilerde elde edeceği bireysel faydayı analiz etme zorunluluğunu ortaya çıkarmıştır. Ayrıca, modern zamanın rekabet ortamında belirsizlikler oldukça fazladır. Küresel rekabet, beraberinde çeşitli tanımlamalardaki farklılıkları ve belirsizlikleri getirmiştir. Böylece bulanık tanımlamalar gün geçtikçe önem kazanmaktadır.

Son olarak, değişken risk seviyeleri günümüz rekabet ortamındaki tarafların faydalarını dinamik bir şekilde incelemelerini ve sürekli takibi zorunlu kılmaktadır.

Yukarıda bahsedilen durumların tamamı göz önünde bulundurulduğunda rekabet ve çatışma ortamları çok taraflı, bulanık ve değişken riskler içermektedir. Son yıllarda yapılan çalışmalarda günümüzün çatışma ortamlarını modellemek amacıyla çok oyunculu oyun modelleri ile bulanık oyun modelleri akademisyenler tarafından sıkça incelenmektedir. Çok oyunculu bulanık oyun modelleri ise nispeten daha az ele alınmakta ve farklı çözüm yaklaşımları kullanılmaktadır. İlk türdeki yaklaşımda, oyuncuların olası koalisyonlara farklı oranlarda katılımları Tsurumi (2001) tarafından incelenmiş ve Choquet integrali yardımı ile bir çözüm yaklaşımı sunulmuştur. Mares (2001) ise tarafların ödemelerinin bulanık olacağını değerlendirdiği bir yaklaşımı öne sürmüştür. Bu amaçla Shapley vektörü bulanık şekilde işleme tabi tutulmuş ve bulanık Shapley değeri yardımıyla tarafların oyuna katkısı hesaplanmıştır. Bu tezin amacı ise Mares'in uyguladığı yöntemdeki bulanık Shapley değerini hesaplamak, işbirlikli oyunların çözümünde sunulan kurallardan bireysel rasyonalite kuralı ile oyuncuların koalisyona gireceği kritik α değerini tespit etmek, farklı belirsizlik ortamında oyuncuların bu belirsizliğe nasıl reaksiyon gösterdiğini incelemektir. Literatürde daha önce çalışılmamış bu yöntemle değişen belirsizlik durumlarında tarafların, oyuna dahil olup olmamaya karar verebileceği değerlendirilmektedir.

Önerilen yöntemin etkinliği gerçek hayat modellemesi üzerinde uygulanarak gösterilmiştir. Bu amaçla bir simülasyon problem hazırlanmıştır. Problemde yeraltı kaynaklarının kullanım hakkının elde edilmesi amacıyla 4 ülkenin taraf olduğu bir çatışma ortamı simüle edilmiştir. Ülkelerin etkinlikleri ve güçlerini hesaplamak için kullanılacak kategoriler ile yeraltı kaynakları üzerinde hak talep edebilmesini sağlayacak olan bu kategoriler bazında ülkelerin bireysel güçleri, uzman görüşleri alınarak hesaplanmıştır. Ülkelerin birbirleriyle kuracağı tüm olası koalisyonların değeri de aynı kategoriler baz alınarak uzman görüşü yardımıyla hesaplanmıştır. Daha sonra elde edilen veriler bulanıklaştırılmış ve Bulanık Shapley değerleri hesaplanmıştır. Her bir ülkenin bireysel olarak oyuna katkıları ile kendi güçleri bireysel rasyonalite açısından α değerleri yardımıyla karşılaştırılarak farklı risk seviyelerinde en uygun risk ortamı tespit edilmiştir. Bu yöntemle ülkelerin muhtemel koalisyonları içeren çatışma ortamına hangi risk seviyelerinde girmesinin anlamlı olacağı görülmüştür.

Anahtar Kelimeler: Bulanık Mantık, Oyun Teorisi, Çok Oyunculu Oyun Modelleri, Shapley Değeri, Bulanık Shapley Değeri

ABSTRACT

MULTIPLAYER FUZZY GAME MODELS AND AN APPLICATION

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Considering markets of today's competitive environment and conditions where strategic goals of countries conflict, new situations stand out. These situations were not common in the beginning of the 20th century and now the effects of them gradually increase.

First of all, in the competitive environment of modern times, any goal has more than two aspiring sides. This situation has revealed the necessity of the parties to review many options to achieve the goal, to analyze opportunities multiple coalitions and the individual benefit to be obtained in different strategies. Additionally, uncertainties are quite high in the competitive environment of modern times. Global competition has caused uncertainties and differences in various definitions. Thus, fuzzy definitions gain importance day by day.

Finally, variable risk levels require parties in competitive environment to dynamically examine their benefits continuously.

Considering all the situations mentioned above, competition and conflict environments contain multilateral, fuzzified and variable risks. In recent studies, multiplayer game models and fuzzy game models are frequently examined by academicians aiming to model today's conflict environments. Multiplayer fuzzy game models are relatively less discussed and different solution approaches are used. In the first approach, Tsurumi examines the players' participation in possible coalitions at different rates and a solution approach is presented with the help of Choquet integral. Mares, on the other hand, suggested an approach in which the parties have fuzzy payoffs. For this purpose, Shapley Vector was processed in a fuzzy way and the contribution of the parties to the game was calculated with the help of the fuzzy Shapley value by Mares. The purpose of this thesis is to calculate the fuzzy Shapley value with the method of Mares, to determine the critical α value that players will enter to coalition with the individual rationality rule which is offered in the solution of cooperative games, and to examine players’ reaction in different uncertainty environment. With this method, that has not been studied before in the literature, it is evaluated that the players can decide whether to participate in coalition or not in cases of changing uncertainty.

The effectiveness of the proposed method has been demonstrated by applying it on real life model. For this purpose a simulation is prepared. In the problem, a conflict environment is modeled so as to obtain the right to use underground resources where four countries are parties. The categories were determined by taking the expert opinions to be used in determining the effectiveness and strengths of the countries. The strengths of the countries have been calculated by using these categories again with experts. The value of all possible coalitions that the countries will establish with each other is calculated with the help of expert opinion on the basis of same categories. Then, the obtained data were fuzzified and fuzzy Shapley values were calculated. Each country's individual contribution to the game and their own strengths were compared at different risk levels with the help of α-values and the most reasonable risk level is calculated in terms of individual rationality. With this method, it has been investigated that what risk levels would be meaningful for countries to enter the conflict environment that includes possible coalitions.

Keywords: Fuzzy Logic, Game Theory, Multiplayer Game Models, Shapley Value, Fuzzy Shapley Value, Individual Rationality

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SYMBOLS AND ABBREVIATIONS

Symbols:

m:	Number of game strategies for player A
n:	Number of game strategies for player B
a_{ij}:	i^{th} row j^{th} column member of payoff matrix
A_i:	i^{th}strategy of player *A*
B_j:	j^{th}strategy of *B*
p_i:	Possibility that row *i* will be used by player *A* in a game
q_j:	Possibility that column *j* will be used by player *B* in a game
S_A:	Set of strategies for player *A*
P_A.	*Set* of probabilities of strategies will be played by player *A*
N:	Set of players
v(i):	Value for player *i* in a game
X:	Payoff vector that players receives as a result of a game
I(v):	Set of imputations
$\Phi_I(V)$:	Shapley value (Outcome of the game on behalf of player (*V*))
K:	Number of players in a coalition
V(K):	Total value of coalition *K*
V(K-i):	Value of coalition *K* without player *i*
A:	Crisp set
$\tilde{A}$:	Fuzzy set
$\tilde{X}_\alpha$:	α cut of fuzzy set *X*
$\mu_{\tilde{A}}(x)$:	Membership function for fuzzy Set $\tilde{A}$
w(i):	Fuzzy value of player *i*
T_i :	Fuzzy Shapley value of *i*
I:	Coalition of all players
α_i^-:	α value on the increasing side of triangular membership function

α_i^+: α value on the decreasing side of triangular membership function

Abbreviations:

TOPSIS: Technique for order preference by similarity to ideal solution

MODM: Multi-objective decision making

MCDM: Multi-criteria decision making

MOCDM: Multi objective and criteria decision making

1. INTRODUCTION

Since Neumann first published his famous article "On the theory of parlor games", in 1928, the game theory has become in the field of interest for academicians, companies and countries. For the purpose of solving conflicts between parties and during the strategy development phase or on the purpose of cooperation between sides, game theory has become quite helpful for researchers. In the fierce competition environment developing a strategy or eliminating the competitors is considerably vital. In modern world relations between countries, rivalry of parties brings about to competition environment. For instance a company should develop a strategy for the purpose of surviving in marketplace or a country should have a plan against other countries. In the sense of making decision, game theory has become rather useful for decision makers.

However, the real life is full of uncertainties. Defining the variables of the problem may change from person to person especially for linguistic notions. For instance, while clustering the trees according to their leaf colors, there are plenty of different hues for identifying the cluster. In other words, green is not just green. It also belongs to blue and yellow clusters in different memberships. In 1965, Zadeh first theorized Fuzzy Sets with the intention of suggesting a solution to these uncertainties. It was a milestone for analyzing the uncertain situations in real life. Thanks to this theory, linguistic terms could be considered while analyzing problems. Thus, the real situations could be integrated to analytic world.

On the other hand the real life is far greater chaotic than theory. Clearly saying, most rivalry in modern world does not have two sides. For example, in marketplace there are plenty of companies trying to earn maximum income, and in diplomatic area there are more than two countries in conflicts for most cases. Also, there may be coalitions between sides or relations based on self interest. All these situations must be taken into consideration for solving a real life problem.

All these considerations led us to implicate real life restrictions in a mathematical model. Therefore we used fuzzy logic in multiplayer cooperative games by fuzzifying Shapley value.

When the literature is studied, there are two types of fuzzification for cooperative game models. In first type, Tsurumi studied on cooperative games where players' participation rates are fuzzy. Shapley values of players are exact numbers. In second type, Mares suggested a different method for fuzzy Shapley value, considering that the payoffs would be fuzzy. Shapley value is defined with its membership function in Mares' method. Both studies focused on fuzzification of cooperative games. However, this fuzzification reveals a complex situation for players to join coalitions. Because, any solution offered for cooperative games must meet the two rules. The first is group rationality. According to this rule, the profits or costs of the game should be distributed fairly among the players at the end of the game. The second one is individual rationality rule which states that any player should get higher payment than its beginning value for joining a coalition. In fuzzy sense, the comparisons of the fuzzy values are complex. But in fuzzy cooperative game models, the individual rationality rule can be utilized for determining critical α value for joining the coalitions. Considering the literature, this method was not taken into consideration for offering a solution of cooperative games. It will be studied in this thesis.

Games in modern times are dynamic which means the decisions of the decision makers may change frequently. On the other hand, vagueness of the situation may change dynamically. So the reaction of the players may not stay same. Because of this, the reaction of the players should be investigated in case of dynamic uncertainties. Critical α values which verifies the individual rationality may change according to uncertainty of the situation.

To sum up, cooperative fuzzy games are newly studied by academicians but highly represent today’s complex situations. So this new method can be just a beginning but inspiring for future studies.

2. GAME THEORY

Game theory is an extraordinary subject. Nearly in every aspects of life people meet with games even without realizing. Chess, monopoly, computer games, card games and even head or tails games are examples of entertaining games. Stock markets, competitions between firms are economic games. Conflicts and alliances between countries are political games. Psychological games are played with mimics words and actions. Likewise, natural selection is a biological game which makes our species strong and able to cope with nature.

Interaction between number of players or decision makers in uncertainty brings about game. Players threaten each other or form coalition during the game. At the end of the game, players can get benefits or income. But they can also experience loss or punishment [1].

As the history of games is from far in the past, in scientific and modern sense, game theory is suggested by mathematician John Von Neumann and economist Oskar Morgennstern, in 1944. However, in 1921, it is first proposed by Emile Borel, who was a French mathematician [2].

Game theory generally used in the solution of cooperative or non-cooperative decision making problems. Game theory, which is a method of decision making process, is widely used in analyzing of military issues, politics, elections, decision making of leaders, international conflicts, the utilization of natural resources and etc. The first form of game theory was carried to an advanced level by Neumann and Morgenstern and cooperative games were introduced [3].

Three elements of games are players(decision makers), strategies and payoffs. The first element is players who play the game. Players are also decision makers. Strategies are set of possible moves and payoffs are returns of game as a consequence of strategies. In game theory, player’s aim is to maximize their own payoff function by establishing coalition or eliminating the other players, instead of beating the rivals [4]. Game theory assumes that players are intelligent and behave rationally in the game. The word

rational here means that the players act consistently. Intelligent means that players of the game know rules and the results of the game in any strategy pair.
[5].

2.1. Literature Review:

Dresher provided solution of three military game problems. The examples were about air battle The first one was strategic air war Secondly a tactical air war problem is solved, and third example was target prediction for aircrafts. Firstly strategic air war example is solved as finite and infinite game. It was a target selection problem for aircrafts about choosing attack and defense options. Then tactical air war example is solved. It has various moves or strikes which include various defense operations, attack operations air defense so as to complete a scenario. Finally, the third problem was about scheduling the launching of missiles. It was described as an example of target prediction. These problems are solved as a model of game theory[6].

Cruz et.al. proposed a model for air operations with discrete-time dynamic game. Blue and red forces were players of proposed model. The aim of blue forces was to destroy bridge and airport which are critical targets with combat air unit. However the red forces have air defense units and ground troops. Objective function for players is to define constraints on the control and state variables are modeled [7].

Staffin studied application of cooperative game theory to political science. Strengths of the politicians has been determined by Shapley and Banzhaf values. For voting process, feasibility of Shapley and Banzhaf values for determining the strength of politicians is illustrated with simple applications. Finally Shapley function and Banzhaf results are compared to each other [8].

S. Brams published his book Game Theory and Politics. A new perspective is submitted with this book. Various political situations and crisis are analyzed with payoff tables. Chapters of book include different political issues such as international relations or voting process. It provides computational view for social sciences, which is widely used in real life [9].

Sandler studied terrorism with the help of the game theory. Decision making process of government is improved and get easier with payoff tables. Interactions with terrorist groups are modeled and famous phenomenon of game theory such as Prisoner's Dilemma, which is a good example of Nash Equilibrium and Chicken Game are utilized to solve the problem. There is a relation between actions of government and terrorist groups. These relations are analyzed with most known basic games. Strategic decisions of government which are preemptive strategies and reactive strategies are compared [10].

Game theory has been popular topic since the mid-20th century. That's why there are plenty of books, articles and surveys about this subject. Therefore, we could have just mentioned only few of them. For more information about game theory a detailed literature review is mentioned in an article by Demirci, Palanci, An Expended Literature Review on Game Theory in National Literature [8]

2.2. Definitions:

Game theory has its own terminology. Firstly, some of this terminology will be introduced for being acquainted with the field. Then this terminology will be used to understand the proposed model of this thesis.

2.2.1. Payoff Matrix:

Payoff matrix is a table which shows strategies of players. For a two player games rows show one players strategies while columns show other players strategies. Then incomes of the player are indicated in the cells of the matrix. Considering the two player zero sum games, row players income is equal to columns player's loss. Table 2.1 illustrates an example of two player zero sum game payoff matrix. As illustrated in Table 2.1 row player can choose one of the m strategies while column player has n strategies. Cells of the table show incomes of row player. As mentioned before the cells are also loss of the column player. Incomes or loss of the player is called as payoff. For example if row player selects strategy 2 and column player chooses strategy 1 the payoff will be a_{12}. In other words a_{ij} will be the payoff of the game where row player of the game plays strategy i and column player of the game plays strategy j.

Table 2.1: Payoff matrix for Two Players Zero Sum Game

		Strategies of Column Player			
		Strategy (1)	**Strategy (2)**	……	**Strategy (*n*)**
Strategies of Row Player	**Strategy (1)**	a_{11}	a_{12}	……	a_{1n}
	Strategy (2)	a_{21}	a_{22}	……	a_{2n}
	……	…….	…….	……	……
	Strategy (*m*)	a_{m1}	a_{m2}	……	A_{mn}

2.2.2.Pure Strategy:

In a pure strategy, players choose just one strategy which has the best payoffs. In other words, in pure strategies players play with certainty. For example if a salesman decides to export his product, he chooses the country which pays the highest price. His decision will be independent from probability or rival company decision.

2.2.3. Mixed Strategy:

Mixed strategies contain a probability distribution, contrast to pure strategies. A probability is assigned to the player for each possible actions [12].

For example, in example of Matching Pennies game, two players can select two options Heads (H) or Tails (T). Players make a selection and if selection matches, A wins a dollar from player B. However, if the selection does not match player A loses one dollar to player B. This game is illustrated in Table 2.2 [12];

Table 2.2: Nash Equilibrium for Mixed Games

		Player B's choice	
		HEAD	TAIL
Player A's choice	HEAD	(1,-1)	(-1,1)
	TAIL	(-1,1)	(1,-1)

Table 2.2 shows that player A’s possible choices are illustrated in a row and player B’s possible choices are shown in a column, and the cells of the table represents the payoffs of player A and player B. It is possible to choose each action with one-half probability for players. Each player's strategy depends on a probability. Therefore the game is an example of a mixed strategy [12]. Finally the most characteristic of a mixed strategy is that profit from the game depends on rivalry’s strategy.

2.2.4. Saddle Point:

Considering two-person zero-sum game, if;

$\max_{All\ Rows}(row\ minimum) = \min_{All\ Rows}(column\ maximum)$ is satisfied by the payoff matrix, it can be said that the game has **saddle point**. Saddle point is a payoff which satisfies all players. In other words, the players are reluctant to change their strategies when the game has saddle point. In this situation row player choose maximum of the minimum row values as indicated on the left side of the formula. Similarly if a two player zero sum game has saddle point, column player chooses minimum of the maximum column values. This is also illustrated on left side of the formula [13].

2.3. Two Player Zero Sum Games

Actually two players zero sum games are mentioned but it will be elaborated in this section. Simply there are two players in two player zero sum games. Row of the matrix represent player1 and column of the matrix represents player 2. The row player has m strategies while column player has n strategies to choose. Both players make their selection simultaneously. If row player selects strategy i and column player selects strategy j, the payoff of the game will be a_{ij}. As explained before it means that row player gets the income of a_{ij} while the column player loses a_{ij}. In other words, the income of row player equals to loss of column player. This is the main characteristic of two player zero sum games and players of the game are fully opponents and try to get payoff of the rival. These types of games are very common for game theory but in real world most of the games are non-zero sum games [13].

2.3.1. Optimal Solution of Two Player Zero Sum Games:

Because the origin of the games is based on conflict of the players and assuming the players behaving rational during the game, the optimum solution chooses specific strategies for players so that selecting another possible strategy does not increase the returns for players [14].

For example, two drug companies named Company A and Company B, try to market their product. For this purpose Company A gives advertisement in radio (A_1), television channel (A_2) and also in newspapers (A_3). On the other hand Company B mails brochures to customers (B_4), in addition to using radio (B_1), television (B_2) and newspapers (B_3). The payoff matrix of the problem can be explained as any company gets market share of the opponent as a payoff according to efficiency of advertisement campaign. The Table 2.3 illustrates payoff matrix that summarizes the percentage of the market income or lost according to strategies [14].

Table 2.3: Two Player Zero Sum Game Example

	B_1	**B_2**	**B_3**	**B_4**	**Row Min.**
A_1	8	-2	9	-3	-3
A_2	6	5	6	8	5
A_3	-2	4	-9	5	-9
Column Max.	8	5	9	8	

Saddle point will be detected to find the solution of the problem. For two player zero sum games, best of the worst rule is the basic method for detecting saddle point. Considering the row player, company A has three strategies. These strategies are shown in rows. If Company A chooses strategy A_1, the worst possible thing that can happen is that A loses 3% of its market share to company B. Minimum value of *Row 1* which belongs to player A_1 is illustrated in the end of the first row. Similarly, if Company A chooses strategy A_2, the worst scenario for Company A is to get 5% market share of the Company B and it is clearly better than worst scenario of strategy A_1. Finally the worst thing for Company A when choosing Strategy A_3 is to lose 9% market share to Company B and this one is the worst of worst scenario. These results are illustrated in the "row min" column as (-3, 5,-9). Best choice of Company A among the worst scenarios is obviously selecting strategy A_2 [14].

Considering the Company B's strategy, best of the worst criteria will be different. Because the payoff matrix, which is illustrated in Table 2.3, belongs to Country A, B's best of the worst criterion will be determined by using minimax value. It can be explained as minimum of maximum loss. As a conclusion Company B will select strategy B_2.

So the saddle point (5) shows that Company A should select strategy A_2 while Company B chooses strategy B_2. According to this, both companies select using television for advertising. It can be said that company A will get some market share of the Company B. In this case, saddle point of A and B is strategy A_2 and B_2. Also value of the game is %5 for Company A and %(-5) for the Company. So that summation of the incomes is zero. That's why the game is zero sum game [14].

The saddle point solution prevents both companies from choosing a better strategy. If Company B shifts to another strategy (B_1, B_3 *or* B_4) and Company A stays with strategy A_2, market share of Company B decreases to (6% or 8%). Likewise, Company A doesn't want to use a different strategy either. Assume that the strategy of Company A shifts to A_1. Then, B can move to B_4 and get %3 market share of the Company A [14].

2.3.2.Dominance:

Dominance occurs when a strategy is definitely advantageous for a player in comparison with other strategies, no matter how other players may play. The dominated rows or columns can be taken away from the payoff matrix.

For example, if the situation is showed by a payoff matrix, the dominance can be illustrated as below. Supposing that rows of matrix are the strategies of the player A and columns of matrix are strategies of player B.

$$
A = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array}
\begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\
\begin{bmatrix} 3 & 4 & 2 & 2 & 0 \\ 6 & 5 & 2 & 3 & 2 \\ 3 & -2 & -3 & -1 & 0 \\ 0 & 0 & 1 & 7 & 6 \\ 7 & -2 & -1 & 5 & 5 \end{bmatrix} \end{array}
\qquad
A = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array}
\begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\
\begin{bmatrix} 3 & 4 & 2 & 2 & 0 \\ 6 & 5 & 2 & 3 & 2 \\ 3 & -2 & -3 & -1 & 0 \\ 0 & 0 & 1 & 7 & 6 \\ 7 & -2 & -1 & 5 & 5 \end{bmatrix} \end{array}
$$

Figure 2.1: Illustration of Dominance

In Figure 2.1, strategy A_1 and A_3 are dominated by A_2, then B_1, B_4,B_5 are dominated by B_3. Later, A_4 and A_5 are dominated by A_2, finally B_2, is dominated by B_3. As a conclusion player A selects strategy A_2 and player$B$ prefers strategy B_3.

2.3.3. Mixed Strategy in Two Player Zero Sum Games:

Mixed strategy for two players zero sum games are mentioned as a short summary before. But in this section detailed explanation and solution method for two player zero sum games will be explained. Especially complex games do not have a saddle point. If all games had a saddle point, there would be no need to play the game. Players would know the result already. But in real life there is always randomness in such situations. It is not easy to find solution without saddle point. Aiming to solve such problems, there should be probability distribution for strategies. In mixed games probability distributions are assigned to each strategies and the solution can be found with linear programming method as illustrated below[15].

Let p_i is the probability that row (strategy of player A) i will be used by player A and,
q_i is the probability that columns (strategy of player B) i will be used by player B

$p_i \geq 0, y_i \geq 0$

$$\sum_{i=1}^{m} p_i = p_1 + p_2 + p_3 + \ \dots\dots\dots + p_m = 1$$

$$\sum_{j=1}^{n} q_i = q_1 + q_2 + q_3 + \ \dots\dots\dots + q_n = 1$$

Table 2.4: Illustration of Mixed Strategy in Two Player Zero Sum Games

		B_1	B_2	...	B_n
		q_1	q_2	...	q_n
A_1	p_1	a_{11}	a_{12}	...	a_{1n}
A_2	p_2	a_{21}	a_{22}	...	a_{2n}
...	...	...	...	...	...
A_m	p_m	a_{m1}	a_{m2}	...	a_{mn}

Expected payoff to Player 1 of game is;

$v = p_1 q_1 a_{11} + p_1 q_2 a_{12} + \dots \dots \dots + p_1 q_n a_{1n}$

$+ \boldsymbol{p_2 q_1} a_{21} + \boldsymbol{p_2 q_2} a_{22} + \dots \dots \dots + \boldsymbol{p_2 q_n} a_{2n}$

$+ \dots \dots \dots \dots \dots \dots$

$+ \boldsymbol{p_m q_1} a_{m1} + \boldsymbol{p_m q_2} a_{m2} + \dots \dots \dots + \boldsymbol{p_m q_n} a_{mn}$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij}$$

$$\max_{i} \min_{j}{}_{a_{ij}} \leq v \leq \min\max{}_{a_{ij}}$$

In two player games we assume that, the set of strategies of player A is;

$S_A = \{A_1, A_2, A_3, \dots, A_m\}$ and the set of strategies of player B is;

$S_B = \{B_1, B_2, B_3, \dots, B_n\}$ and mxn payoff matrix is;

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

$P_A = \{p_1, p_2, \dots, p_i, \dots, p_m\}$ denotes to set of probabilities of strategies will be played by player A.

a_{ij} is the payoff of the players in case of player A chooses strategy A_i and player B chooses strategy B_j.

The payoff of the player A against the each strategy of player B is,

max v,

Subject to

$p_1 a_{11} + p_2 a_{21} + p_3 a_{31} + \dots \dots + p_m a_{m1} \geq v$

$p_1 a_{12} + p_2 a_{22} + p_3 a_{32} + \dots \dots + p_m a_{m2} \geq v$

... .

$p_1 a_{1n} + p_2 a_{2n} + p_3 a_{3n} + \dots \dots + p_m a_{mn} \geq v$

$p_1 + p_2 + p_3 + \dots \dots + p_m = 1$

If the linear programming problem of above is solved the value of game for player A can be found.

Similarly the value of the game for player B is

$P_B = \{ q_1, q_2, \ldots, q_i, \ldots, q_n \}$ denotes to set of probabilities of strategies will be played by player

min v,

Subject to

$q_1a_{11} + q_2a_{12} + q_3a_{13} + \ldots\ldots\ldots + q_na_{1n} \leq v$

$q_1a_{21} + q_2a_{22} + q_3a_{23} + \ldots\ldots\ldots + q_na_{2n} \leq v$

... .

$q_1a_{m1} + q_2a_{m2} + q_3a_{m3} + \ldots\ldots\ldots + q_na_{mn} \leq v$

$q_1 + q_2 + q_3 + \ldots\ldots\ldots + q_m = 1$

The solution of linear programming problem above gives the value of the game for player B [16].

2.4. Two Player Non-Zero Sum Games

Zero sum games assume that one player gets other players loss and the summation of players payoffs equals zero. But this is quite restricted for implementing the real life. In most issues of our world, both players may lose or gain. Non zero sum games can be indicated with discrete payoffs of the players. This can be illustrated with the most prominent example of the game theory which is called Prisoner's Dilemma [15].

2.4.1.Prisoner's Dilemma:

Prisoner's Dilemma, which is a famous model of game theory will be summarized briefly. Two prisoners are waiting for the trials. But the court doesn't have enough evidences. The attorney offers a proposal to the prisoners who are unaware of each other. If only one of the prisoners confess the crime, the prisoner who confesses will be free to go and other prisoner will be convicted to 20 years. If both of the prisoners confess the crime then punishment will be divided into two and each prisoners will be convicted to 5 years. Finally if both prisoner do not confess, each prisoners will stay one year in jail. This is modeled in Table 2.5[13].

Table:2.5: Prisoner's Dilemma Game

		PRISONER 2	
		CONFESS	DON'T CONFESS
PRISONER 1	CONFESS	(-5,-5)	(0,-20)
	DON'T CONFESS	(-20,0)	(-1,-1)

As illustrated in Table 2.5 each cells have two payoffs. Left one belongs to row player and right one belongs to column player.

Sum of the payoffs are not zero in prisoner's dilemma game that's why the game is not a zero sum game anymore. Dominated strategy is explained before and it will determine the solution of this problem. For both players "don't confess" strategy is dominated by "confess" strategy. As a result of this, it is expected from prisoners to confess their crimes and convicted to five years. Actually better choice for both players is exist. If both prisoners don't confess their crime they will be prison only for one year. This ironic situation makes the prisoner's dilemma game so famous. Because of the prisoners do not know which strategy will be used by other prisoner, both of them choose undominated strategy. Assume that both player decided not to confess. If one of the player shift his decision, other one will be convicted to 20 years. So this risk causes an equilibrium point. So the (-5,-5) is the equilibrium point for prisoner's dilemma game. [13].

In Prisoner's Dilemma game the decision makers did not choose the best strategy for game, on the contrary they chose the best strategy for themselves. In other words undominated strategies for each prisoner are selected, it is called Nash Equilibrium.

2.5. N-Player Games:

Considering the real world of today, in many situations, there are more than two competitors. For example, marketplace of today, international conflicts or allies, political issues such as elections and various strategic campaigns have more than two players. Basically any game with more than two players is called *n*-player games [13].

Firstly Non-cooperative games for n-player games are very restricted in that all the individual players, have their own strategy and the Nash Equilibrium is not different from two player games. Moreover non cooperative games are weak about representation of the real situation. Because interaction between opponents mostly brings coalition that's why, cooperative games are more meaningful for explaining to n-player games.

2.5.1. N-Player Cooperative Games:

If set of players allies for increasing their profits it is called cooperative games. Intending to analyze today's conflict environments, the cooperative game models may be quite helpful. For attracting players to the cooperation, the outcome of the game should be allocated fairly between the members of the coalition. Solution concept of cooperative games should explain the distribution of the outcome to the players. Various solution method is proposed about the cooperative games but the main investigation area for cooperative game models is about distribution of the profits [16]. Like any other mathematical methods, cooperative games have specific characteristic features and rules. These features and rules will help to describe the concept. These rules are illustrated below.

For an N-player cooperative game,
Let N: Set of player in a game and
$v(i)$:Payoff of a player in a coalition.

The cooperative game is:

a. Zero-Normalized if $v(\{i\}) = 0$ for all $i \in N$,

b. Superadditive if for all $S, T \subset N$, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$,

c. Additive if for all $S, T \subset N$, if $S \cap T = \emptyset$, then $v(S \cup T) = v(S) + v(T)$,

d. Monotonic if $S \subseteq T \subseteq N$ implies that $v(S) \leq v(T)$,

e. Convex if $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$ for all S, $T \subset N$

f. Constant-Sum if for all $S \subset N$, $v(S) + v(N \setminus S) = v(N)$,

Many solution concepts are exist for n-player games. Common rule of solution concepts is a solution concept provides payoff that each player will receive. A vector represents payoff of the players which they get at the end of the game. Such vector can be illustrated as $x = \{x_1, x_2, \ldots, x_n\}$. Such a vector is called payoff vector. A payoff vector $x = (x_1, x_2, \ldots, x_n)$ can only be a reasonable candidate for a solution if x satisfies,

$$x_i \geq v(\{i\}) \; (for\ each\ i \in N) \quad \text{(Individual Rationality)} \quad (2.1)$$

$$v(N) = \sum_{i=1}^{i=n} x_i \quad \text{(Group Rationality)}$$

If x is satisfied both these formula, we say that x is an imputation.

More formally;

a. **Individual rationality**: $x_i \geq v(i)$ for all i $\in$ N. Meaning of this rule is payoff of a player should be at least the amount which it can get by acting alone in the game.

b. **Group rationality**: $v(N) = \sum_{i=1}^{i=n} x_i$ for all i $\in$ N. Meaning of this rule is, coalition of all player should be equal to sum of the players' payoffs.

c. Imputation: Considering a cooperative game (N, v), if any x satisfies, individual rationality and group rationality, then x is an imputation [13].

2.5.2. Shapley Value:

The Shapley value of a player can be considered as power of that player in game. Alternatively, the Shapley value may be considered as fair allocation of outcome to the players. The Shapley value concept first proposed by Shapley in 1953 [17]. The formula of the Shapley Value is as given in Equation (2.1),

$$\boldsymbol{\Phi_i}(\boldsymbol{V}) = \sum_{\substack{\boldsymbol{K} \subset \boldsymbol{N} \\ \boldsymbol{i} \in \boldsymbol{K}}} \frac{(\boldsymbol{N}-\boldsymbol{K})!(\boldsymbol{K}-\mathbf{1})!}{\boldsymbol{N}!} [\boldsymbol{V}(\boldsymbol{K}) - \boldsymbol{V}(\boldsymbol{K}-\boldsymbol{i})] \quad (2.2)$$

where,

$\boldsymbol{\Phi}(\boldsymbol{V})$**:** Shapley Value (Outcome of the game on behalf of player i

$\boldsymbol{N}$**:** Number of players in game

$\boldsymbol{K}$**:** Number of players in coalition

$\boldsymbol{V}(\boldsymbol{K})$**:** Total payoff of coalition K

$\boldsymbol{V}(\boldsymbol{K}-\boldsymbol{i})$:: Payoff of the coalition K without player i

$[\boldsymbol{V}(\boldsymbol{K}) - \boldsymbol{V}(\boldsymbol{K}-\boldsymbol{i})]$**:** Player i's contribution to the coalition

[18]

3. FUZZY LOGIC

Complexity of the decision making problem mostly derives from uncertainty. Some of the uncertain processes may be expressed with probability theory. However probability theory is a tool for solving random processes. In other words probability theory is applicable for events which happen by chance. Nevertheless, uncertainty may result from insufficient information, unreliable information about the problem, changeable linguistic terms or multiple sources. In other words decision makers may have some clue about the problem. But they may not be enough for solution. It is better for decision-makers to use the available data than to treat the problem as purely random. Fuzzy logic is an excellent tool for decision makers to use the limited information they have. It was first proposed by Zadeh in 1965 so as to express linguistic terms. Later, it was effectively applied in many areas such as machine learning, decision-making processes and artificial intelligence [19].

3.1. Literature Review:

Fuzzy logic was first developed in early 1960's by Zadeh. He first devised an interaction between machines and human by encoding linguistic human knowledge into numerical forms [20].

Atanassov introduced the functions of terms in the form of belongingness and not belongingness degree to fuzzy set. It was proposed with intuitionistic fuzzy sets which was characterized by hesitation degree of the terms [21].

Young and Hwang solved linear programming problem in fuzzy terms. Different approaches are proposed for solution method. The flexibility of the problem was increased by benefiting from expert opinions. In a particular area of linear programming problem, all possible variations are taken into consideration with expert opinions and correctness of the problem is improved [22].

Aydin and Pakdil measured service quality of an airline company with the help of passenger questionnaire. Since the difference between the passengers will make a difference in linguistic terms, the answers were converted into trapezoidal fuzzy numbers. Passengers' ideas were collected with the help of a Likert scale. Different types of passengers are identified with α-cuts and decision makers are able to set their strategy according to passenger types [23].

Since first studies published about fuzzy sets by Zadeh in 1965, there have been plenty of academic studies about fuzzy game theory. We just mentioned the studies which are related to our study. For more literature reviews about the Fuzzy Sets, Kahraman, Oztaysi, Onar, A Comprehensive Literature Review of 50 Years of Fuzzy Set Theory, 2016 may be studied [24].

Military applications of fuzzy logic are also remarkable. Sánchez-Lozano et al. has combined multicriteria decision making process with fuzzy logic. They proposed a model for selecting aircraft model in the Spanish Air Force. Technical and other criteria are weighted with Analytic Hierarchy Process (AHP) and best aircraft model is selected with Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method [25].

3.2. Fuzzy Sets:

In real world it is not that easy to classify materials. Simply a Labrador in your backyard is certain member of land animals. It breaths with lungs and has four legs which is convenient for walking. On the other hand, a goldfish is definitely an aquatic animal because it dies if you take it from your aquarium. However, a salamander can live both in land and water. So if we try to put it in two groups there will not be certain group. It has an ambiguous status. Similarly some linguistic definitions have the same problem. For example, when you describe a man to your colleagues and if you use the word “tall man”, each of your friends will visualize different height. But if you take all the answers we may have set of heights. In other words, these heights will be belonging to a class which can be named as tall man.

In 1965 Zadeh, first introduced fuzzy sets, in order to deal with such problems. Thanks to his studies, ambiguous situations in absence of sharply definition could be denoted by fuzzy sets [26].

The members of the fuzzy sets have belongingness to set in various grades. So the memberships of the elements unrelated to any chance.

To illustrate with an example, consider a set called *A* in which its members are just even natural numbers. In a set of *A* = {*(1, 0), (2, 1), (3, 0), (4, 1), ...* } first values are natural numbers and second values are the grade of belongingness. The value of belongingness will be 0 or 1. Considering this set of *A*, 0 denotes "not belongingness" and 1 denotes "certain belongingness". In a set of *A* the belongingness of natural numbers are illustrated [22].

In first example the memberships of the elements were binary. However by using this concept, different membership grades can be identified. For example, in a fuzzy set of *Ã;*

Ã = {(Can, 0.2), (Onur, 1), (Hande, 0.8), (Beliz, 0,3), (Beril, 0.5) }

Consider a fuzzy set *A*, showing the tall students of a class. The word of "tall" can mean a different length for each person. If "Onur" is 190 cm tall and "Can" is 170 cm tall, membership of Onur to the set *A*, should be more than membership of Can. As a result the memberships of the elements to the fuzzy set of *A* should be between [0,1].

Explanation of definitions and terminology will be as referred below.

Again considering the first example, if an element belongs to set of $\tilde{A}$ the membership is 0. On the contrary if an element does not belong to set of $\tilde{A}$ the membership of this element to the fuzzy set is 1 and characteristic function of a fuzzy set of $\tilde{A}$ can be illustrated as $\mu_A(x)$. This refers to the degree of membership of the element *x* in the set of *A*. Characteristic function of $\mu_A(x)$ can be illustrated as Equation 3.1

$$\mu_A(x) = \begin{cases} 1 \text{ if and only if } x \in A \\ 0 \qquad\qquad \text{otherwise} \end{cases} \tag{3.1}$$

If the membership function of a set is binary as illustrated in Equation 3.1, it is called valuation set. However a valuation set can be in the range of [0,1]. In this case, the set of $\tilde{A}$is called a fuzzy set. If the membership of the element gets closer to 1, the element

x more belongs to set of A. A fuzzy set is illustrated as with a function of $\tilde{A}$= {(x; $\mu_{\tilde{A}}(x)$)| $x \in X$ } [22].

Graphical representations and tables are quite important for understanding fuzzy set theory. The memberships of the continuous functions can easily be seen on figures. On the other hand analyzing complex problems is easier. A graphical representation of fuzzy membership function and interpretation of figure is explained with an example below.

For example if we divide population into three groups as young, middle aged and aged the graph may be as given in Figure 3.1

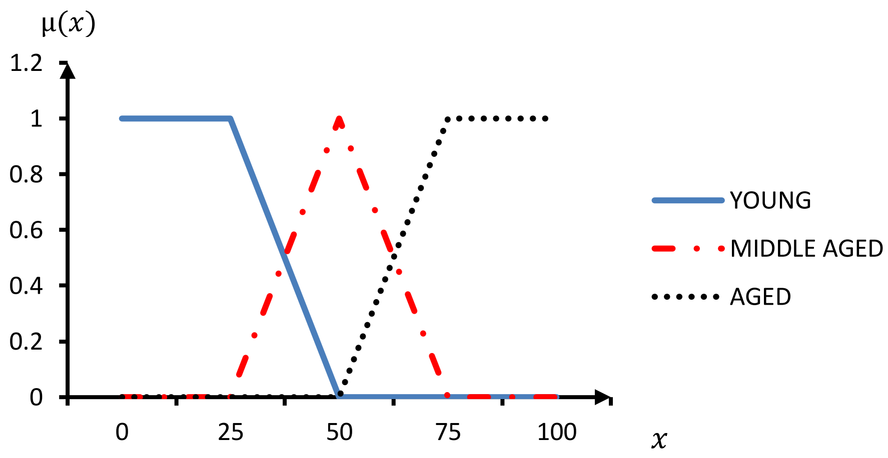

Figure 3.1: Age as a Fuzzy Variable

In Figure 3.1 x axis represents ages and y axis present their memberships. Graphical representation shows that there are three groups of fuzzy sets. Each of them extends between their own intervals. Three fuzzy sets can be named as $\tilde{Y}(Fuzzy\ Young\ Set)$, $\tilde{M}(Fuzzy\ Middle\ Aged\ Set)$ and finally $\tilde{A}(Fuzzy\ Aged\ Set)$.

Basically, a person in age of 20 is definitely young because $\mu_{\tilde{Y}}(20) = 1$, $\mu_{M}(20) = 0$, and $\mu_{\tilde{A}}(20) = 0$. On the other hand a 60 years old man is not in any crisp sets. In other words he is neither middle aged, nor aged. Because $\mu_{\tilde{M}}(60) = 0{,}60$ and $\mu_{\tilde{A}}(60) = 0{,}40$. So we could not assign it into a single (crisp) set. This is main logic of the fuzzy set concept.

3.3. Fuzzy Arithmetic

Fuzzy numbers have specific rules for arithmetical operations. Since the structure of fuzzy numbers has a membership function, unlike crisp numbers, arithmetic operation rules should be different. Addition, subtraction, multiplication and division for fuzzy sets are shown below.

3.3.1. Addition:

Let us consider, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are fuzzy sets;

If $\tilde{A} + \tilde{B} = \tilde{C}$ then, (3.2)

$\mu_{\tilde{C}}(Z) = max_{x+y=z}\{\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$

3.3.2. Substraction

There is no significant difference between addition and substraction.

Let us consider, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are fuzzy sets;

If $\tilde{A} - \tilde{B} = \tilde{C}$ then, (3.3)

$\mu_{\tilde{C}}(Z) = max_{x-y=z}\{\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$

3.3.3. Multiplication:

Let us consider, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are fuzzy sets;

If $\tilde{A}.\tilde{B} = \tilde{C}$ then, (3.4)

$\mu_{\tilde{C}}(Z) = max_{xy=z}\{\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$

3.3.4. Division

Let us consider, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are fuzzy sets;

If $\tilde{A}/\tilde{B} = \tilde{C}$ then, (3.5)

$\mu_{\tilde{C}}(Z) = max_{x/y=z}\{\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$

[27].

The formulations of fuzzy arithmetic operations are indicated above. Considering the continuous structure of fuzzy sets, the results should be continuous too. So analysis can be done by using the continuous structure of arithmetic operations in solving a problem. α-cuts of a fuzzy set is one of this kind of operations.

3.4.α-Cuts of Fuzzy Sets:

α-cut of a fuzzy set is the points which have similar or greater than α value membership. In other words α-cut is some kind of limitation in fuzzy set.

Let, $\tilde{A}$ be a fuzzy set and X is kind of a crisp set. α-cut of $\tilde{A}$ is illustrated as,

$\tilde{A}_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha \text{ and } x \in X$

If α =1, x should definitely be the member of the set and it is a crisp number.

If α = 0, there is no limitations for x to be in the set and x is absolute fuzzy number

3.5. Arithmetical Operation for Fuzzy α-Cut Fuzzy Sets:

α-cut method is very practical method for performing arithmetical operations for fuzzy sets. The results of the problem can easily be calculated with the help of α-cut fuzzy arithmetical operations. Addition, subtraction, multiplication and division methods with α-cut fuzzy arithmetical operations are shown below.

3.5.1. Addition:

Let $\tilde{X}$= [a, b, c] and $\tilde{Y}$= [p, q, r] are fuzzy numbers which may be considered as edges of a triangular membership and membership functions are,

$$\mu_x(x) = \begin{cases} \frac{x-a}{b-a}, & a < x < b \\ \frac{c-x}{c-b}, & b < x < c \\ 0, & otherwise \end{cases} \tag{3.6}$$

Triangular membership functions are shown in Equation 3.6 and Equation 3.7. The memberships of x linearly increases and linearly decreases.

$$\mu_Y(x) = \begin{cases} \frac{x-p}{p-q}, & p < x < q \\ \frac{r-x}{r-q}, & q < x < r \\ 0, & otherwise \end{cases} \tag{3.7}$$

Then;

$\tilde{X}_\alpha = [(b-a)\alpha + a, \quad c - (c-b)\alpha]$ and $\tilde{Y}_\alpha = [(q-p)\alpha + p, \; r - (r-q)\alpha]$

α cuts of fuzzy numbers $\tilde{X}$ and $\tilde{Y}$ are illustrated above. For any α values the new membership functions can be identified. The first value of new membership function shows beginning point of membership in α value and second one denotes the end of the membership. When first value equals to second value, membership in highest α value can be found.

Addition of fuzzy numbers with α cut is denoted in Equation 3.8

$$\tilde{X}_\alpha + \tilde{Y}_\alpha = [(\mathrm{a} + \mathrm{p} + (\mathrm{b} - \mathrm{a} + \mathrm{q} - \mathrm{p})\alpha, \quad \mathrm{c} + \mathrm{r} - (\mathrm{c} - \mathrm{b} + \mathrm{r} - \mathrm{q})\alpha] \tag{3.8}$$

3.5.2. Subtraction

Subtraction in α cuts can also be calculated with same method. Let $\tilde{X}$= [*a*, *b*, *c*] and $\tilde{Y}$= [*p*, *q*, *r*] be fuzzy numbers. The formula of subtraction in α cut values is illustrated below,

$$\tilde{X}_\alpha = [(\mathrm{b} - \mathrm{a})\alpha + \mathrm{a}, \quad \mathrm{c} - (\mathrm{c} - \mathrm{b})\alpha] \text{ and } \tilde{\mathrm{Y}}_\alpha = [(\mathrm{q} - \mathrm{p})\alpha + \mathrm{p}, \ \mathrm{r} - (\mathrm{r} - \mathrm{q})\alpha] \tag{3.9}$$

α- cuts of fuzzy numbers $\tilde{X}$ and $\tilde{Y}$ are shown in Equation 3.9.. Substraction of α-cuts of $\tilde{X}$ and $\tilde{Y}$ can be calculated with using interval arithmetic and the result is as Equation 3.10

$$\tilde{X}_\alpha - \tilde{Y}_\alpha = [(\mathrm{a} - \mathrm{r}) + (\mathrm{b} - \mathrm{a} + \mathrm{r} - \mathrm{q})\alpha, \quad (\mathrm{c} - \mathrm{p}) - (\mathrm{c} - \mathrm{b} + \mathrm{q} - \mathrm{p})\alpha] \tag{3.10}$$

3.5.3. Multiplication

Let $\tilde{X}$= [*a*, *b*, *c*] and $\tilde{Y}$= [*p*, *q*, *r*] are fuzzy numbers again.

$$\tilde{X}_\alpha = [(\mathrm{b} - \mathrm{a})\alpha + \mathrm{a}, \quad \mathrm{c} - (\mathrm{c} - \mathrm{b})\alpha] \text{ and } \tilde{\mathrm{Y}}_\alpha = [(\mathrm{q} - \mathrm{p})\alpha + \mathrm{p}, \ \mathrm{r} - (\mathrm{r} - \mathrm{q})\alpha] \tag{3.11}$$

α- cuts formulation of $\tilde{X}$ and $\tilde{Y}$ is illustrated in Equation (3.11). For the purpose of calculating α-cuts of $\tilde{X}$ and $\tilde{Y}$, following interval arithmetic formulation can be used.

$$\tilde{X}_{\alpha} * \tilde{Y}_{\alpha} = [\big((\mathrm{b}-\mathrm{a})\alpha + \mathrm{a}\big) * \big((\mathrm{q}-\mathrm{p})\alpha + \mathrm{p}), (\mathrm{c}-(\mathrm{c}-\mathrm{b})\ \alpha * \quad (\mathrm{r}-(\mathrm{r}-\mathrm{q})\alpha)\big]$$

(3.12)

3.5.4. Division

$\tilde{X}$= [*a*, *b*, *c*] and $\tilde{Y}$= [*p*, *q*, *r*] are fuzzy numbers.

$$\tilde{X}_{\alpha} = [(b-a)\alpha + a, \quad c-(c-b)\alpha] \text{ and } \tilde{Y}_{\alpha} = [(q-p)\alpha + p,\ r-(r-q)\alpha]$$

(3.13)

Equation 3.13 shows α.cuts of fuzzy numbers. Following formulation can be applied to calculate division of fuzzy numbers in α-cuts [28].

$$\frac{\tilde{\mathrm{X}}_{\alpha}}{\tilde{\mathrm{Y}}_{\alpha}} = \left[\frac{(\mathrm{b}-\mathrm{a})\alpha+\mathrm{a}}{(\mathrm{r}-(\mathrm{r}-\mathrm{q})\alpha)}, \frac{(\mathrm{c}-(\mathrm{c}-\mathrm{b})\ \alpha}{(\mathrm{q}-\mathrm{p})\alpha+\mathrm{p})}\right]$$

(3.14)

4. FUZZY GAMES

Most mathematical models assume that there is no uncertainty for problems. As other branches of analytic sciences game theory also ignores uncertainty of real life. However the real life is full of uncertainties. For example, in an international conflict environment, the expectations of countries may be uncertain. Similarly, the importance of the game can be expressed differently for each company in the market share competition of the two business companies. On the other hand, linguistic expressions of terms may be uncertain and payoff table of players need to fuzzification. There are many more examples of the need for fuzzification of the game theory. Therefore fuzzification is extremely useful for implementation of game models into real life. Due to this need, some applications are already done by academicians.

4.1.Literature Review

Butnariu and Aubin first adapted fuzzy sets into game models. Butnariu presented the heuristical description and a mathematical approach to fuzzy games [2].

Koca and Aydin solved a two players zero sum game in different risk levels. They implemented their model in real life problem that two online shopping sides are considered as players. They utilized different risk levels for their problem because the importance of the game should be different for each e-store for competitive environment [5].

Aplak, combined different decision making processes in one model. According to this “multi-objective decision making (MODM)” and “multi-criteria decision making (MCDM)” processes are combined. The new model is called as “multi objective and criteria decision making (MOCDM)” process. New model is investigated in two person non zero sum game theory perspective. In this process linguistic terms converted to fuzzy numerical values [30].

Azrieli and Lehrer discussed different types of cooperative fuzzy games such as convex games, exact games, and extendable games. Then the games are also investigated according to their cores as large core and stable core. The comparison of these types of games is studied. Also interrelation between classical cooperative games and fuzzy cooperative games are illustrated [31].

Tsurumi, proposed to use choquet integral for solving cooperative games. In his model, the rates of the participation of players in games were fuzzy [3].

Borkotokey, studied fuzzy Shapley value. His model proposed that both participation of players to the coalition and expectation of payoffs by the players are fuzzy [3].

Mares investigated the basic characteristics of cooperative games. Firstly he discussed interrelation of deterministic games and their fuzzy extensions. Fuzzy cooperative games with side payments and without side payments are studied. Fuzzy Shapley value is also redefined.The difference of his model from Butnariu's model was coalition degrees were still crisp but payoffs were fuzzy [33].

Zhang proposed a new method for calculating the fuzzy Shapley value of players. Participation of the players in coalition was fuzzy in proposed model. This new expression of fuzzy Shapley value is compared with Butnariu and Tsurumi's model [34].

Branzei, Dimitrov, studied different types of fuzzy cooperative games. Shapley value, cooperative crisp games and clan games are introduced in the first chapter of book. In the second chapter of the book, cooperative games with fuzzy coalitions are explained. Different solution concepts are studied. Convex fuzzy games and fuzzy clan games are also introduced [35].

Çevikel, offered a solution concept for two players zero sum games. He complicated the problem by considering the problem as multiobjective model. According to this the players have fuzzy payoffs and fuzzy goals. Linear interactive solution concept proposed and it is proved that if the players of a game cooperates in two player zero sum game (the game should be considered against nature) the joint payoff and individual payoff may increase. It also has been shown that a joint game can be constructed in which players combine their strategies and act as a single player with a fuzzy characteristic function. The fuzzy characteristic functions based on the optimal game values of zero-sum games are studied with fuzzy returns against nature. He also showed that the fuzzy characteristic function created in this way, satisfies the super additive condition. [36].

Oderanti F,O, studied decision making process in business by using fuzzy games. Each business organizations are considered as players and a competition of business organizations are modelled as games. Different situations of fuzzy game models are exemplified with experiments. These are fuzzy two player games, fuzzy n-player games, negotiation models in fuzzy environment [37].

4.2. Cooperative Fuzzy Games:

Cooperative game theory is a branch of game theory that studies coalitions and players who cooperate and act together for the purpose of increasing their payoffs [35].

Since the first publications of game theory, models have been mostly established on considering two players. However, the real life is more complex than theory. Considering conflictions between sides in modern world, we mostly face with more than two players in the game. For example in marketplace of today there are plenty of companies. Each of the companies may be sides of the game individually or some of them can establish coalitions against other companies. Additionally the modern world is full of uncertainties as referred before. Uncertainty can be expected in most situations for coalition. The following example illustrates only some of them. Consider two economic issues that players want to allocate their capital with maximum profit. They can make investment individually and get a profit of 20% of the stocks they invested. On the other hand they can accumulate their capital and make joint investment. However, the possible profit is subject to uncertainty. It may be at least 15%, but with a probability of 0.6 degrees (here 1.0 is certainty and 0.0 is impossible) it can even reach 30%. Will there be co-investment or players will make their investments individually?[33]. These modern time restrictions bring about fuzzy situations.

Literature reviews show that there are two types of coalition games. The first one is fuzzy coalition. According to this type of coalition, players take part in any coalition partially. Players get exact incomes as a result of games. Aubin and Butnariu (1978) introduced this type of games [38]. The level of being in coalition varies from acting individually to being completely in coalition [39]. This can be named as participation level of players. The solution of this type of games is proposed with Choquet integral.

Second type of coalition game is proposed by Mares and Vlach (2001). In this type of games, the payoffs of the players are fuzzy but their level of participation in coalition is crisp [33]. These types of games are solved with fuzzification of Shapley value. Because the payoffs of players are fuzzy, the fuzzy Shapley values of the players are calculated with fuzzy artihmetic operations. The rules of cooperative games are also applicable here. Individual rationality and group rationality of players are taken into consideration for analyzing the problem.

Sakawa and Nishizaki proposed a model for games with fuzzy payoffs and fuzzy goals. This kind of multiobjective fuzzy problem is solved with two phase linear programming. The solution was also fuzzy. However a defuzzification method is also introduced [39].

We will focus on fuzzy payoffs and crisp game coalitions that is to say income and correspondingly outcome of the coalition is vague. In other words uncertain definition of the power will be expressed as fuzzy values. Due to the fuzzy operations on the problem basis, the result will naturally be fuzzy too. On the other hand joining a coalition will not be partially for players.

Shapley value should be fuzzified to calculate fuzzy payoffs. As mentioned before, the best expression for outcome of a cooperative game is Shapley value. Owing to fuzzy calculations about the cooperative game, the outcome will be fuzzy too. So the Shapley value should be converted to fuzzy Shapley value.

4.2.1.Fuzzy Shapley Value:

It is clear that if income of a cooperative game is fuzzy then the outcome should be fuzzy too. So the Shapley value should converted to fuzzy Shapley value. It is possible to calculate fuzzy Shapley value with fuzzy quantities. The formulation of Shapley value is already illustrated in previous chapters. The Shapley value is calculated with the sum of the players' contribution to all possible coalitions. Considering the formula, calculation of Shapley value only needs arithmetic operations. In this sense, fuzzy Shapley value can be calculated with arithmetic operations of fuzzy quantities. By doing this, the result is also be a fuzzy quantity. The membership of fuzzy Shapley value will be in the range of, $\mu_{T_i}(x): R \rightarrow [0,1]$ and fuzzy numbers $w(K)$ will be used instead of $v(K)$. Terminology of fuzzy Shapley value is denoted below.

T_i: "Fuzzy Shapley value of player *i*"
N: "Number of players in the game"
K: "Set of players in the coalition"
w(K): "Fuzzy extension of coalition *K*'s payoff"
I: Set of players in a coalition

Finally fuzzy Shapley value for player *i* can be calculated as,

$$T_i = \sum_{K\subset I}^{\oplus} \frac{(N-K)!(N-1)!}{N!}\left(w(K) \oplus \left(-w(K-\{i\})\right)\right) \tag{4.1}$$

The summation symbol here $\oplus$ denotes the fuzzy addition operation. So the fuzzy Shapley value will be calculated with arithmetic operations of fuzzy quantities.

Fuzzy Shapley value of a player will be calculated briefly with an example. In a two player game, If (I, v) is the value of all players coalition and (I, w) is the fuzzy extension of (I, v). Assuming that $I = \{1,2\}$, $v(\{1\}) = 1$, $v(\{2\}) = 2$, $v(I) = 5$ and for any $K \subset I$, the membership function of any coalition *K is*;

$$\mu_K(x) = \begin{cases} x + 1 - v(K)\ for\ x \in [v(K) - 1, v(K)] \\ 1 + v(K) - x\ for\ x \in [v(K), v(K) + 1] \\ 0 \text{ otherwise} \end{cases} \qquad [30]$$

Then fuzzy membership functions of coalitions will be,

$$\mu_1(x) = \begin{cases} x, & x \in [0,1] \\ 2 - x, & x \in [1,2] \\ 0, & otherwise \end{cases} \tag{4.2}$$

$$\mu_2(x) = \begin{cases} x-1, & x \in [1,2] \\ 3-x, & x \in [2,3] \\ 0, & otherwise \end{cases} \tag{4.3}$$

$$\mu_I(x) = \begin{cases} x-4, & x \in [4,5] \\ 6-x, & x \in [5,6] \\ 0, & otherwise \end{cases} \tag{4.4}$$

So the fuzzy Shapley value of each player will be calculated by using Equation (4.1). we will divide fuzzy membership functions in two so as to calculate it easily . New membership functions and graphical representations are shown below,

$$\frac{1}{2}w(1) = \frac{1}{2}\mu_1(\mathrm{x}) = \begin{cases} 2\mathrm{x}, & \mathrm{x} \in [0,\frac{1}{2}] \\ 2-2\mathrm{x}, & \mathrm{x} \in [\frac{1}{2},1] \\ 0, & \text{otherwise} \end{cases} \tag{4.5}$$

The new membership function of $w(1)$ is represented in Equation (4.5). Fuzzy numbers can be divided by crisp numbers. Figure 4.1 shows the graphical representation of $\frac{1}{2}w(1)$

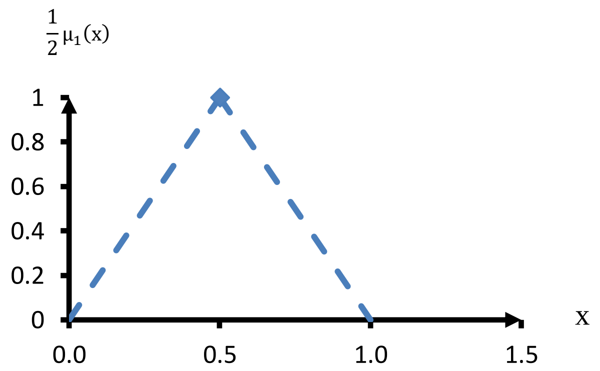

Figure 4.1 : Graphical Representation of $\frac{1}{2}w(1)$

Same calculations are done for *w(2)* and *w(I)*. The membership functions of $\frac{1}{2}w(2)$ and $\frac{1}{2}w(I)$ are shown in Equations (4.6) and (4.7). Also graphical representations are shown in Figure (4.2) and Figure (4.3).

$$\frac{1}{2}w(2) = \frac{1}{2}\mu_2(x) = \begin{cases} 2x - 1, & x \in [\frac{1}{2}, 1] \\ 3 - 2x, & x \in [1, \frac{3}{2}] \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

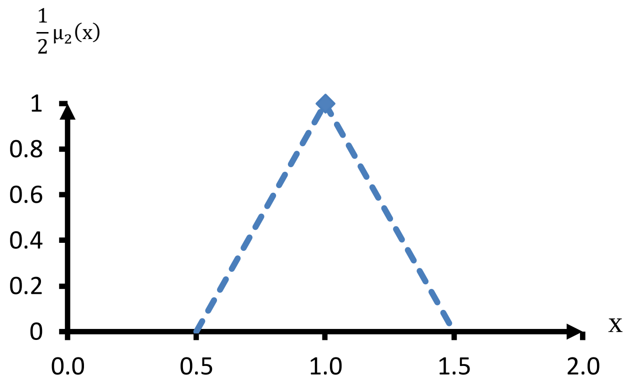

Figure 4.2 : Graphical Representation of $\frac{1}{2}w(2)$

$$\frac{1}{2}w(I) = \frac{1}{2}\mu_I(x) = \begin{cases} 2x - 4, & x \in [2, \frac{5}{2}] \\ 6 - 2x, & x \in [\frac{5}{2}, 3] \\ 0, & \text{otherwise} \end{cases} \tag{4.7}$$

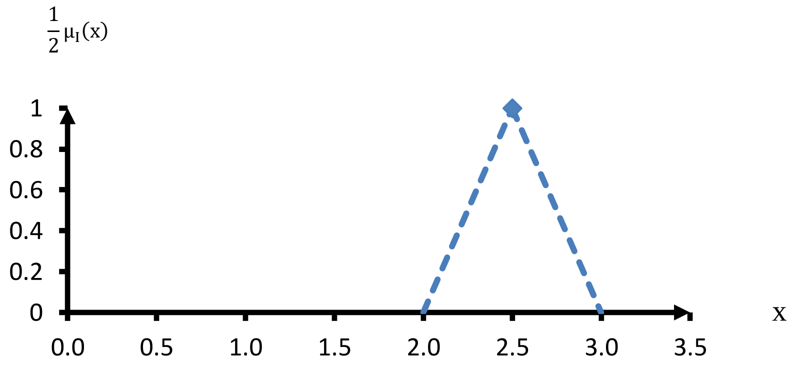

Figure 4.3 : Graphical Representation of $\frac{1}{2}w(I)$

After transforming fuzzy membership functions into new form, the a- cut fuzzy arithmetic operations are used below to find fuzzy Shapley value of each player. The formula of fuzzy Shapley value for player 1 is,

$$T_1 = \frac{1!\,0!}{2!}(w(\{1\}) \oplus \frac{0!\,1!}{2!}\big(w(I) - w(\{2\})\big))$$

$$= \frac{1}{2}w(\{1\}) \oplus \left(-\frac{1}{2}\right)w(\{2\}) \oplus \frac{1}{2}\,w(\{I\})$$

Elements of the formula are found in Equations (4.5), (4.6) and (4.7). As it is seen using Equation (3.8) and (3.10) is the easiest way to calculate fuzzy Shapley value for player 1, so the result is,

$$= \left(\frac{3}{2}\alpha + \frac{1}{2},\ \frac{7}{2} - \frac{3}{2}\alpha\right)$$

And membership function for fuzzy Shapley Value of player 1 is calculated with the help of Equations (3.9) and (3.11). The membership function is;

$$\mu_{T_1}(x) = \begin{cases} (2x-1)/3 \ \text{ for x } \in \ [0.5, 2] \\ (7-2x)/3 \ \text{ for x } \in \ [2, 7/2] \\ 0 \text{ otherwise} \end{cases}$$

Analogously, same Equations can be used to find fuzzy Shapley value of player 2. The formula of fuzzy Shapley value of player 2 is,

$$T_2 = \frac{1}{2}w(\{2\}) \oplus -\frac{1}{2}w(\{1\}) \oplus \frac{1}{2}w(I)$$

The result is calculated as;

$$= \left(\frac{3\alpha+3}{2},\ \frac{9-3\alpha}{2}\right)$$

Finally the membership function for fuzzy Shapley Value of player 1 is shown as,

$$\mu_{T_2}(x) = \begin{cases} (2x-3)/3 \ for \ \ x \in \ [1.5, 3] \\ (9-2x)/3 \ for \ \ x \in \ [3, 4.5] \\ 0 \ otherwise \end{cases}$$

By calculating fuzzy membership functions with the help of a values, we have the chance to determine the Shapley value at different risk levels. If the a value increases, the risk level decreases. Higher a values states lower risky situations. During all the calculations of this thesis, this case will be taken into consideration.

4.2.2. Critical Risk Level of Fuzzy Cooperative Game

The rules to be considered in the solutions for cooperative games were stated in Chapter 2. Any offered solution should satisfy the individual rationality and group rationality. Critical α value for players to join coalitions can be found by applying these two rules to fuzzy cooperative games. The individual rationality and group rationality for crisp games are illustrated below;

x_i: Profit or cost of player i at the end of the game.

$v(\{i\})$: Payoff (investment) of player i

N: Number of players in game

$v(N) = \sum_{i=1}^{i=n} x_i$ (Group Rationality)

$x_i \geq v(\{i\})$ (for each $i \in N$) (Individual Rationality)

Fuzzification of the cooperative game should not change features of games. According to individual rationality rule, players still want to enter the coalition if they will increase their investments at the end of the game [41]. On the other hand, group rationality states that the profit or cost of the fuzzy game should be allocated between players in a stable way. So the rules of the cooperative crisp game should stay the same. However, the comparison of fuzzy values has to be done. Therefore, equation and inequation must be restructured. So the fuzzy rules of individual rationality and group rationality is stated in Equation (2.1) but in order to see difference between fuzzy values it is rewritten below,

$w(\{i\})$: Fuzzy payoff (investment) of player i

T_i: Fuzzy Shapley value of player i

N: Number of players in game

$w(N) = \sum_{i=1}^{i=N} T_i$ (Group Rationality)

$T_i \geq w(\{i\})\ (for\ each\ i \in N)$ (Individual Rationality)

Group rationality rule should be controlled with fuzzy addition of players' fuzzy Shapley value. When checking the individual rationality rule, two fuzzy sets will be compared. This, unlike other studies in the literature, allows finding the critical α value that the player will want to join in coalition. The contribution of individual rationality rule to finding the critical α value in fuzzy games can be shown with the combined graphical representation of w_i and T_i on the Figure 4.4.

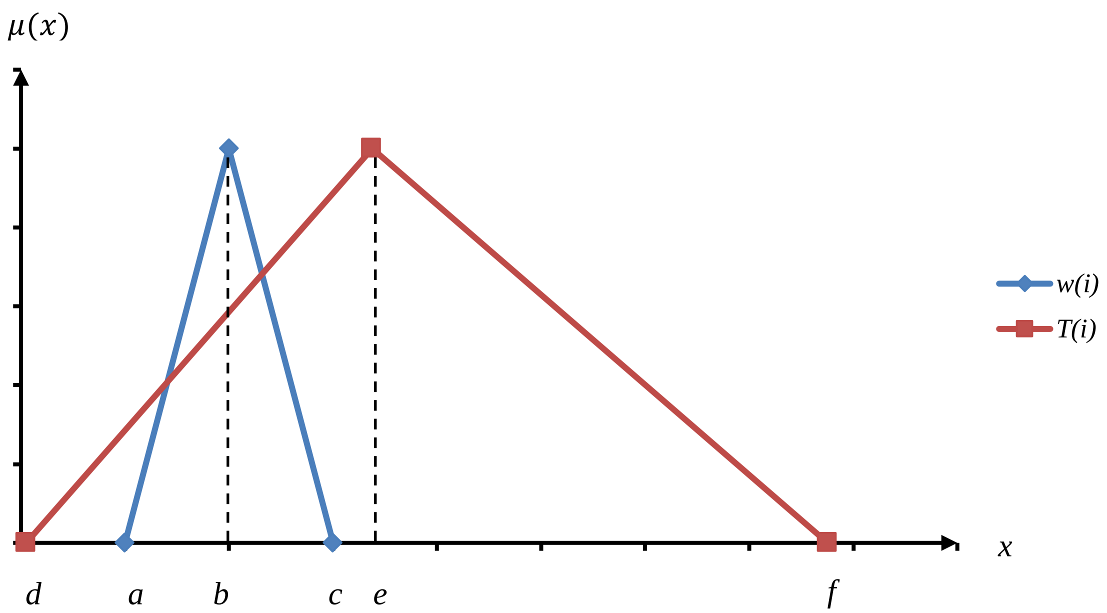

Figure 4.4: Combined Graphical Representation of w_i and T_i

Membership function of fuzzy Shapley value for w_i is illustrated below,

$$\mu_{w_i}(x) = \begin{cases} \dfrac{x-a}{b-a} for\ x \in [a,b] \\ \dfrac{c-x}{c-b}\ for\ x \in [b,c] \\ 0, \qquad otherwise \end{cases}$$

Then membership function of fuzzy Shapley value for T_i is stated below,

$$\mu_{T_i}(x) = \begin{cases} \dfrac{x-d}{e-d} for\ x \in [d,e] \\ \dfrac{f-x}{f-e}\ for\ x \in [e,f] \\ 0, \qquad otherwise \end{cases}$$

μ(x) axis of the first intersection point of T_i and w_i should be the critical α value for player *i* to be in coalition Because above this α level, the player *i* guarantees that the Shapley value at least belongs to the fuzzy set of the value she entered the game. Therefore, the intersection of T_i and w_i indicates the minimum risk level at which individual rationality is verified. To determine this point, we need to equalize the first lines of T_i and w_i membership functions.

$$\frac{x-a}{b-a} = \frac{x-d}{e-d}$$

The *x* axis of the critical α value can be calculated as $x = \frac{ae-db}{e-d-b+a}$

Finally the critical α value of player *i* for verifying the individual rationality is;

$$\mu_{w_i}\left(\frac{ae-db}{e-d-b+a}\right) = \mu_{T_i}\left(\frac{ae-db}{e-d-b+a}\right) = \alpha$$

This α value is the commensurable risk level for player *i*. Over this α value player *i* can prefer to be in coalition.

5. APPLICATION

In this section we will focus on a simulated real life application. In different risk levels we will discuss how logic is to be in coalition on behalf of the payoffs for players. For this purpose, an imaginary conflict between four different countries will be simulated. The power of countries from a conflict will be discussed with five expert's opinions. To do this, different categories which represent the power of countries will be identified. Finally fuzzified value of different coalitions will be determined by expert opinions and fuzzy Shapley value of these coalitions will be calculated. As mentioned before if Shapley value is lower than individual power of a country it is meaningless to be in coalition for this country. Actually we already know this from individual rationality rule. However, this rule will be used for determining player's motivation for entering the coalition. In different risk levels the motivation of a country may be different. That means; payoff of a country may vary in different α-cut levels. So the most reasonable risk level to be in coalition will be detected.

Today, solution of disagreements between countries has shifted. Coalitions between countries may easily enforce rivals to negotiations. Also various categories such as military power, diplomatic power, economic power and social power may determine the power of a country. So, multinational conflicts are so complicated to identify for sides. This situation brings about fuzzy logic to our concept. According to our simulation four different countries are in conflict. They will be named as country A, country B, country C and country D. The aim of the countries is to get underground sources. All these countries claim that they legally have rights to get a share from this source. Also all of them have some legal basis to deny other countries claims about their rights. Aiming to solve this conflict, countries will look for coalition to get power. So, the countries which are in coalition may support each other's claims about right to get a share from the sources.

Firstly, we consult five experts in order to identify the categories which encapsulate the power of a country. We asked them to write down five categories which epitomize the power of a country.

The answers of the experts are listed below.

M. O. K. (LtJG in Navy);

Economy, Military, Population, Justice, Technology

E. S. (Lt, Responsible for Operation Analysis in Southern Sea Area Command Headquarter)

Economy, Education, Military, Agriculture, Democracy

A.G.A. (Lieutenant Commander, Military Instructor in Turkish Naval Academy)

Population, Technology, Underground sources, Economy, Military

M.K.Ç (Psychological Operations Analysis Expert)

Economy, Military, Location, Technology, Education

M .T.(Deputy Commanding General of Task Unit Command in Syria)

Population, Technology, Military, Economy, Education

Table 5.1 illustrates the number of times each criteria is said by the experts.

Table 5.1: Descriptive Categories of Power According to Experts

Military	**Economy**	**Technology**	**Population**	**Education**
5	5	4	3	3
Justice	**Location**	**Underground Sources**	**Agriculture**	**Democracy**
1	1	1	1	1

It is clear that five different categories are common idea of majority. These are economy, military, technology, population and education. Now Country A, Country B, Country C and Country D can be described and same experts are going to grade these countries according to these categories. Four countries in an imaginary geography are defined as follows.

Country A: Country A has the largest military population among the rivals. It also has the growing technology especially in defense industry. However Country A has problems with keeping their qualified individuals in the country because of the living conditions. The economy of the country is in the process of burning up however it is not irrevocable. Population is the second crowded country among others. The education of the country is not stable because perpetually the education system changes and unemployment is in the realm of possibility due to the unplanned number of graduated people. Despite all, Country A has always been ascendant for the region.

Country B: After the military turmoil, Country B has relatively stable political landscape. Also this country is enthusiastic to establish good relations with countries which have strategic importance. During the turmoil period Country A supported opponents of the current government, therefore Country B is unwilling to rebuilt new relations with Country A. Due to the strong relations with great powers, there is not difficulty for Country B to reach weapons. Owing to the archaic history and geopolitical position, the country has experienced military forces. The education of the country is based on religion and not innovative. Also gross domestic product is one-third of Country A and same as Country C. Country B has the largest population among other countries however the technology of the country is underwhelming. Finally, Country B believes that coalition with the Country C may be more useful because Country C is the nemesis of Country A.

Country C: As indicated above, Country C has also other problems with Country A beside underground sources. That is not to say they can never solve their problems. Despite their disagreements, they both have a culture to find common way to their problems. Country C has the third largest population of four countries. Due to the economic crises, the technology of the country is underdeveloped. However the relationship with other countries and membership to the unions make the Country C advantageous for using technology. Besides, undergoing conflictions with Country A, keep military forces ready to move. Education system of Country B is well-established and stabile for years. Also culture of the Country C is ancient and inspiring to neighbor countries. Coalition with Country B may put Country A into trouble. However because of the geopolitical position of Country C and Country A the coalition of these countries may be advantageous for both of them

Country D: Country D is still in civil war. Current government was going to lose control if Country A would not help them through military assistance. Therefore the government in Country D seems Country A as assurance of existence. Nevertheless other coalitions may also guarantee them present conditions. The economic size of the Country D is the smallest one among others. Thus Country D may be more willing to be in coalition with other countries. Country D has an advantage; it is the closest country to the underground sources. The education in Country D is the worst one. Great part of the population has difficulties to reach the proper education. Also population of the country is the smallest one. Military forces of the country is divided into two groups. One of them supports current government while other is with opponents. Despite the military assistance of Country A, most part of the Country D is still under the control of opponents. Predictably, the technology of the country D depends on Country A. Country D has the lowest population among all these countries.

After presenting these information about the simulated countries we will evaluate the coalitions. Therefore we will ask to experts. They will assess to all possible coalitions and then we will fuzzify it.

A questionnaire is prepared for the experts. All possible coalitions were asked to be evaluated. A crisp preference scale from 0 to 100 is filled by experts. Considering five categories which define the power of the countries, each of the possible coalitions are evaluated. According to experts, worst coalition is graded with 0 and best coalition is graded with 100. Then they are converted to fuzzy numbers and median of the results are calculated as the value of the coalition. The questionnaire can be seen in the Appendix. Finally median of the experts' answers have been converted to fuzzy form with a scale which is given below. Each score given by experts is shown in Table 5.2.

Table 5.2: Scores of the Coalitions According to Experts

COALITIONS	A.G.A	E.S.	M.O.K	M.K.Ç.	M.T.
A	20	30	20	25	20
B	20	20	15	20	23
C	15	15	17	20	19
D	6	5	5	9	3
A,B	50	60	50	60	70
A,C	40	60	48	50	45
A,D	45	40	30	35	50
B,C	50	50	40	55	65
B,D	26	25	22	30	55
C,D	25	25	25	32	30
A,B,C	90	85	90	85	75
A,B,D	60	78	75	80	60
B,C,D	50	70	50	60	65
A,C,D	55	80	72	75	85
A,B,C,D	98	95	95	100	95

Then the median of the choices are calculated and the results are illustrated on Table 5.3 as;

Table 5.3: Crisp Scores of Coalitions

COALITIONS	SCORES
v(A)	20
v(B)	20
v(C)	17
v(D)	5
v(A,B)	60
v(A,C)	48
v(A,D)	40
v(B,C)	50
v(B,D)	26
v(C,D)	25
v(A,B,C)	85
v(A,B,D)	75
v(B,C,D)	60
v(A,C,D)	75
v(A,B,C,D)	95

Because of the linguistic differences between experts, there will be vagueness for determining power of the countries. Fuzzy extensions of these values are going to be used in order to overcome this problem. Table 5.2 shows that Country D has maximum scoring range. M.T. gives 3 points while M.K.Ç gives 9 points. Considering triangular membership function, Country D should have 50% extension. So other values will be extended 50% for each side on the purpose of fuzzification. Then the results will be investigated in different extensions. But firstly fuzzified values in 50% extension will be as given in Table.5.4.

Table 5.4: Fuzzified Values of Coalitions (50% Expanded for Both Sides)

FUZZY COALITIONS	**SCORES**
w(A)	10,20,30
w(B)	10,20,30
w(C)	8.5,17,25.5
w(D)	2.5,5,7.5
w(A,B)	30,60,90
w(A,C)	24,48,72
w(A,D)	20,40,60
w(B,C)	25,50,75
w(B,D)	13,26,39
w(C,D)	12.5,25,37.5
w(A,B,C)	42.5,85,127.5
w(A,B,D)	37.5,75,112.5
w(B,C,D)	30,60,90
w(A,C,D)	37.5,75,112.5
w(A,B,C,D)	47.5,95,142.5

Once fuzzified values are obtained, solution method will be implemented to problem. The formula will be used for country A. Equation (4.1) is implemented to problem and fuzzy Shapley value of country A is calculated with Equation (5.1).

$$T_A = \frac{1}{4}w(A) \oplus \frac{1}{12}(w(\{A,B\}) \oplus (-w(\{B\})) \oplus \frac{1}{12}(w(\{A,C\}) \oplus (-w(C)) \oplus \frac{1}{12}(w(\{A,D\}) \oplus (-w(\{D\})) \oplus \frac{1}{12}(w(\{A,B,C\}) \oplus (-w(\{B,C\})) \oplus \frac{1}{12}(w(\{A,B,D\}) \oplus (-w(\{B,D\})) \oplus \frac{1}{12}(w(\{A,C,D\}) \oplus (-w(\{C,D\})) \oplus \frac{1}{4}(w(\{A,B,C,D\}) \oplus (-w(\{B,C,D\})) \quad (5.1)$$

Addition of the values should be calculated with fuzzy arithmetic operations. fuzzy Shapley value for Country A is illustrated in Equation (5.2).

$$T_A = (-10.04, 33.75, 77.54) \quad (5.2)$$

(-10,04) denotes the beginning point of membership function for the fuzzy Shapley value. (33,75) is the point where membership equals 1 and it can be inferred from (77,54) that membership again decreases to 0. Furthermore the membership function should be triangular. So the membership function for the fuzzy Shapley value for the Country A can be shown with a graphical representation.

Graphical representation for fuzzy Shapley value of Country A is illustrated on Figure (5.1).

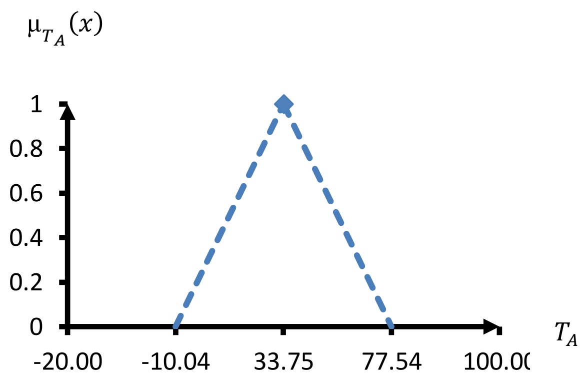

Figure 5.1: Fuzzy Shapley Value of Country A

The membership function of fuzzy Shapley value of Country A can be found with Equation (3.9) or simple geometric operations on graphical representation. The membership function of fuzzy Shapley value is illustrated in Equation (5.3);

$$\mu_{T_A}(x) = \begin{cases} \frac{x+10,04}{43,79} \; for\ x \in [-10.04, 33.75] \\ \frac{77.54-x}{43.79} \; for\ x \in [33.75, 77.54] \\ 0, \qquad otherwise \end{cases} \quad (5.3)$$

For the purpose of calculating fuzzy Shapley values of country A in different risk levels, we need α-cuts formulation for fuzzy Shapley values of country A. Equation (3.8) will be used to find the result. The α-cuts formulation for fuzzy Shapley value of Country A is shown in Equation (5.4).

$$T_A = \begin{cases} 43.79a_A^- - 10.04 \; for \; a_A^- \in [0,1] \\ 77.54 - 43.79a_A^+ \; for \; a_A^+ \in [1,0] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.4}$$

Equation (5.4) make it possible to know fuzzy Shapley Value of Country A in different risk levels. So it will be utilized to find critical risk level for Country A.

The same calculations are done for other countries. Firstly fuzzy Shapley value for Country B is calculated and the results are shown in Equation (5.5).

$$T_B = \frac{1}{4}w(B) \oplus \frac{1}{12}(w(\{A,B\}) \oplus (-w(\{A\})) \oplus \frac{1}{12}(w(\{B,C\}) \oplus (-w(C)) \oplus \frac{1}{12}(w(\{B,D\}) \oplus (-w(D)) \oplus \frac{1}{12}(w(\{A,B,C\}) \oplus (-w(\{A,C\})) \oplus \frac{1}{12}(w(\{A,B,D\}) \oplus (-w(\{A,D\})) \oplus \frac{1}{12}(w(\{B,C,D\}) \oplus (-w(\{C,D\})) \oplus \frac{1}{4}(w(\{A,B,C,D\}) \oplus (-w(\{B,C,D\})) \tag{5.5}$$

Fuzzy Shapley Value for Country B is illustrated in Equation (5.6)

$$T_B = (-18.29, 26.75, 71.79) \tag{5.6}$$

Figure 5.2 shows graphical representation for fuzzy Shapley value of Country B. As seen in Figure 5.2 the results are plotted on the graph. α values are 0 when the Shapley Values are -18,29 or 71,79 and α value is 1 when the Shapley Value of Country B is 26,75.

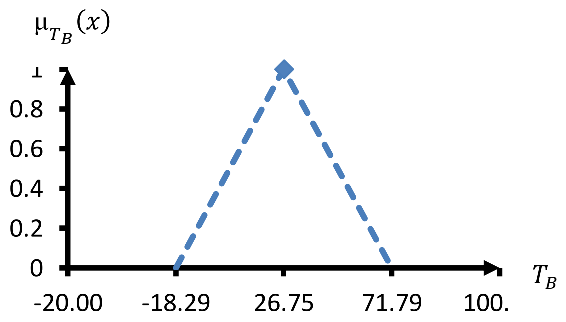

Figure 5.2: Fuzzy Shapley Value of Country B

The membership function of fuzzy Shapley value is found with geometrical operations. So the membership function for fuzzy Shapley value of Country B is demonstrated in Equation (5.7).

$$\mu_{T_B}(x) = \begin{cases} (x + 18{,}29)/45.04\, for\ x \in [-18.29, 26.75] \\ (71.79 - x)/45{,}04\ for\ \ x \in [26.75, 71.79] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.7}$$

For the purpose of finding Shapley Value of Country B in different risk levels, Equation (3.8) is implemented to problem. Equation (5.8) denotes the α-cut formulation for Country B.

$$T_B = \begin{cases} 45.04a_B^- - 18.29\ for\ a_A^-\ \in [0,1] \\ 71.79 - 45.04a_B^+\ for\ a_A^+ \in [1,0] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.8}$$

Same process is also done for Country C. Firstly the fuzzy Shapley value of Country C is calculated by using Equation (4.1). The formulation for fuzzy Shapley value of Country C is illustrated in Equation (5.9)

$$T_C = \tfrac{1}{4}w(C) \oplus \tfrac{1}{12}(w(\{A,C\}) \oplus (-w(\{A\})) \oplus \tfrac{1}{12}(w(\{B,C\}) \oplus (-w(B)) \oplus \tfrac{1}{12}(w(\{C,D\}) \oplus (-w(D)) \oplus \tfrac{1}{12}(w(\{A,B,C\}) \oplus (-w(\{A,B\})) \oplus \tfrac{1}{12}(w(\{B,C,D\}) \oplus (-w(\{B,D\})) \oplus \tfrac{1}{12}(w(\{A,C,D\}) \oplus (-w(\{A,D\})) \oplus \tfrac{1}{4}(w(\{A,B,C,D\}) \oplus (-w(\{A,B,D\})) \tag{5.9}$$

After fuzzy arithmetic operations over Equation (5.9), the fuzzy Shapley value of Country B is achieved. The result is denoted in Equation (5.10).

$$T_C = (\ -21.20,\ 23.58,\ 68.38) \tag{5.10}$$

Figure 5.3 shows membership function for fuzzy Shapley value of Country C. -21.20 and 68.38 on the x-axis are the points where α is 0. In other words highest risk intervals. On the other side 23.58 is the point where α value equals 1. This is the lowest risk value for Country C.

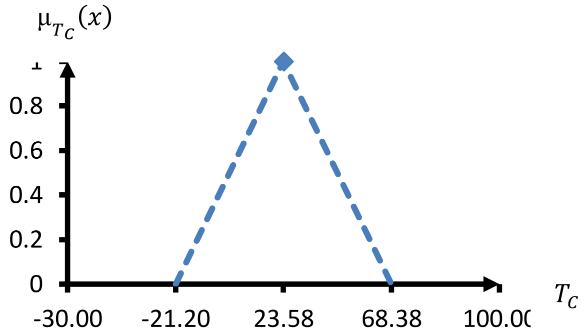

Figure 5.3: Fuzzy Shapley Value of Country C

We need to know membership function İn order to find membership of each Shapley value To do this Equation (3.9) is implemented to problem. The membership function of fuzzy Shapley value of Country C is shown in Equation (5.10).

$$\mu_{T_C}(x) = \begin{cases} (x+21.20)/44.78 \; for \; x \in [-21.20, 23.58] \\ (68.375 - x)/44{,}80 \; for \quad x \in [23.58, 68.38] \\ 0, \qquad otherwise \end{cases} \tag{5.10}$$

The α-cuts formulation for Country C is also calculated for the purpose of finding Shapley Value in different risk levels. α-cut formulation for Country C is illustrated on Equation (5.11)

$$T_C = \begin{cases} 44.78a_C^- - 21.20 \; for \; a_A^- \in [0,1] \\ 68.38 - 44.80a_C^+ \; for \; a_A^+ \in [1,0] \\ 0, \qquad otherwise \end{cases} \tag{5.11}$$

Fuzzy Shapley value of Country D is calculated with the implementation of Equation (4.1). Formula for fuzzy Shapley value of Country D is illustrated in Equation (5.12)

$$T_D = \tfrac{1}{4}w(D) \oplus \tfrac{1}{12}(w(\{A,D\}) \oplus (-w(\{A\})) \oplus \tfrac{1}{12}(w(\{B,D\}) \oplus (-w(B)) \oplus \tfrac{1}{12}(w(\{C,D\}) \oplus (-w(C)) \oplus \tfrac{1}{12}(w(\{A,B,D\}) \oplus (-w(\{A,B\})) \oplus \tfrac{1}{12}(w(\{B,C,D\}) \oplus (-w(\{B,C\})) \oplus \tfrac{1}{12}(w(\{A,C,D\}) \oplus (-w(\{A,C\})) \oplus \tfrac{1}{4}(w(\{A,B,C,D\}) \oplus (-w(\{A,B,C\})) \tag{5.12}$$

The result is shown in Equation (5.13). Previous definitions are valid for the elements of the equation.

T_D = (-33.70, 10.92, 55.54) (5.13)

The fuzzy Shapley value is plotted the graphical representation. It is illustrated on Figure 5.4.

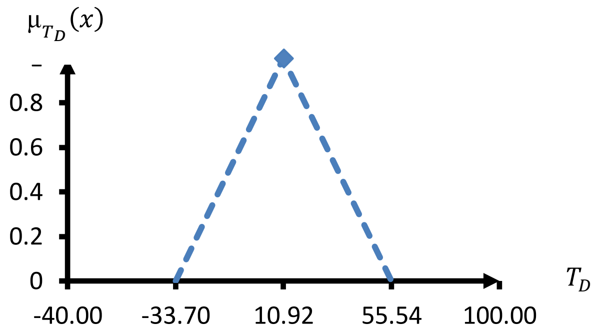

Figure 5.4: Fuzzy Shapley Value of Country D

The membership function of fuzzy Shapley value of Country D is illustrated in Equation (5.13),

$$\mu_{T_D}(x) = \begin{cases} (x + 33.70)/44.62 \; for \; x \in [-33.70, 10.92] \\ (55.54 - x)/44.62 \; for \;\; x \in [10.92,55.54] \\ 0, \qquad otherwise \end{cases} \tag{5.13}$$

Finally α-cuts formulation for Country D is also specified in Equation (5.14).

$$T_D = \begin{cases} 44.62a_D^- - 33.70 \;\; for \; a_D^- \in [0,1] \\ 55.54 - 44.62a_D^+ \; for \; a_A^+ \in [1,0] \\ 0, \qquad otherwise \end{cases} \tag{5.14}$$

For the purpose of analyzing outcomes with regards to risk levels, fuzzy Shapley values of countries should be calculated with α-cut consept. Different α levels are utilized and critical α level is determined. Group Rationality and Individual Rationality rules are considered to decide critical α levels. But firstly the fuzzy Shapley values of countries in different risk levels with 50% extension of fuzzy values are shown in Table 5.5.

Table 5.5: Fuzzy Shapley Values of Players in Different α Levels.

a $levels$	T_A			T_B			T_C			T_D		
	A_a^-	A_a^1	A_a^+	B_a^-	B_a^1	B_a^+	C_a^-	C_a^1	C_a^+	D_a^-	D_a^1	D_a^+
0	-10.04	33.75	77.54	-18.29	26.75	71.79	-21.2	23.58	68.38	-33.7	10.92	55.54
0,25	0.91	33.75	66.59	-7.03	26.75	60.53	-10	23.58	57.18	-22.55	10.92	44.39
0,50	11.86	33.75	55.645	4.23	26.75	49.27	1.19	23.58	45.98	-11.39	10.92	33.23
0,75	22.80	33.75	44.70	15.49	26.75	38.01	12.385	23.58	34.78	-0.24	10.92	22.08
1	33.75	33.75	33.75	26.75	26.75	26.75	23.58	23.58	23.58	10.92	10.92	10.92

As mentioned before outcome vector, must satisfy some rules. This vector here can be described as Shapley Value. Firstly the solution must satisfy group rationality. Group rationality for Fuzzy values can be formulated as, $(I) = \sum_{i=1}^{i=n} T_i$, where $w(I)$ is the fuzzy coalition value of all players and T_i denotes fuzzy Shapley value of player (i).

For controlling suitability of this rule for this problem, fuzzy the Shapley Values of players will be added up with fuzzy arithmetic operation. The implementation of formula to the problem is given in Equation (5.15).

$$\sum_{i=1}^{i=4} T_i = T_A \oplus T_B \oplus T_C \oplus T_D \tag{5.15}$$

$$\sum_{i=1}^{i=4} T_i = (-83.23, 95, 273.25) \tag{5.16}$$

$$w(I) = W\{A, B, C, D\} = (47.5, 95, 142.5) \tag{5.17}$$

If the results are defuzzified, equivalence of the scores can easily be seen in Equation (5.16) and Equation (5.17). Thus the problem satisfies group rationality.

Secondly, the individual rationality rule will also be investigated. Individual rationality of fuzzy games can be formulated as in Equation (5.18),

$$T_i \geq w(\{i\})\ (for\ each\ i \in N) \tag{5.18}$$

Linguistic explanation of this rule states that a coalition is reasonable for a player only if it gets higher gain than it has had when acting alone in the game. This rule is vital for this thesis. Any coalition makes sense for a country, only if the Shapley Value is higher than the power. Above all, this situation will emerge main idea of this thesis. Because, a country should take a risk on an a-cut value which verifies individual rationality. Fuzzified values of countries and Fuzzy Shapley of them will be shown in graphs for the sake of analyzing it, together.

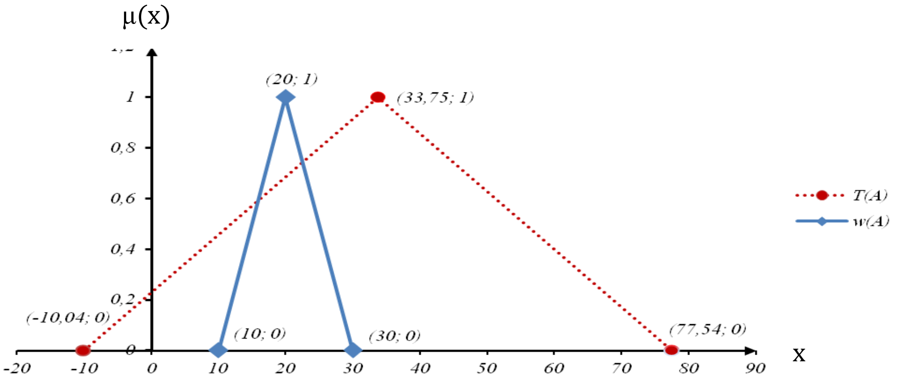

Figure 5.5: Combined Graphical Representation of *w(A)* and T_A

Figure 5.5 shows that being in any coalition may be risky for Country A. Country A should choose and a level, aiming to eliminate the risk. As seen on Figure 5.5 on the first intersection of *w(A)* and T_A, a_A^- level will provide best risk level since Shapley Value of Country A will be at least as much as its own fuzzy payoff in a possible coalition. Thus, Country A will be in any coalition only if when membership functions of *w(A)* and T_A intersect on a_A^- region, for the purpose of eliminating the risk. In other words the first intersection of *w(A)* and T_A on *y* axis is the maximum commensurable risk for Country A.

Membership function of *w(A)* is derived from Table 5.5 and illustrated in Equation (5.19).

$$\mu_{w_A}(x) = \begin{cases} (x-10)/10\ for\ x \in [10,20] \\ (30-x)/10\ for\ \ x \in [20,30] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.19}$$

The membership function of fuzzy Shapley value of Country A was also referred in Equation (5.3). On the purpose of demonstrating intersection point between $\mu_{w_A}(x)$ and $\mu_{T_A}(x)$, the fuzzy Shapley value of Country A is indicated again in Equation (5.20) as;

$$\mu_{T_A}(x) = \begin{cases} \frac{x+10{,}04}{43{,}79} for\ x \in [-10.04{,}33.75] \\ \frac{77.54-x}{43.79}\ for\ x \in [33.75{,}77.54] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.20}$$

Equation of $\mu_{w_A}(x)$ and $\mu_{T_A}(x)$ on the a_A^- side of the membership functions will give the x axis of intersection point which is also can be described as critical risk level. Then reasonable a level for Country A can be found.

$$(x-10)/10 = \frac{x+10{,}04}{43{,}79} \tag{5.21}$$

As it is understood in Equation (5.21), left sides of the triangular membership functions of $\mu_{w_A}(x)$ and $\mu_{T_A}(x)$ should be equal. The result can easily be found as $x = 15{,}93$. This result is implemented in a_A^- side of the Equation (5.21) and the critic a level is found as 0,593 in Equation (5.22).

$$\mu_{w_A}(15{,}93) = 0{,}593 \tag{5.22}$$

0,593 shows the most meaningful a level for Country A. Because if Country A reduces a level, The Shapley Value may be lower than *w(A)* and it does not suit individual rationality rule. On the other hand if Country A takes less risk The Shapley Value may stay below expectation. That's why Country A will get most advantageous position on $0{,}593$ a level risky situations. Thus, Country A should look for a coalition over $0{,}593$ a level.

Fuzzy Shapley values with 0,593 a level can be calculated with the help of Equation (5.4) and the Table 5.6 shows fuzzy Shapley values of Country A with 0,593 a Level.

Table 5.6: Fuzzy Shapley Values of Country A with 0,593 a Level

a $level$	T_A		
	A_a^-	A_a^1	A_a^+
0.593	15.93	33.75	51.57

To interpret Table 5.6, Country A can get 33,75 unit of benefit in risk free politic situations. This value is obviously higher than crisp power value of Country A. However in real environment of uncertainties risk free situations are not common situations. That's why the most reasonable point for the risk is selected. On the other hand acquisitions of Country A may be varied between 15.93 and 51.57 in most commensurable risk level.

When the same calculations are made for other countries, the following results will be achieved. Figure 5.6 illustrates both *w(B)* and T_B for the Country B.

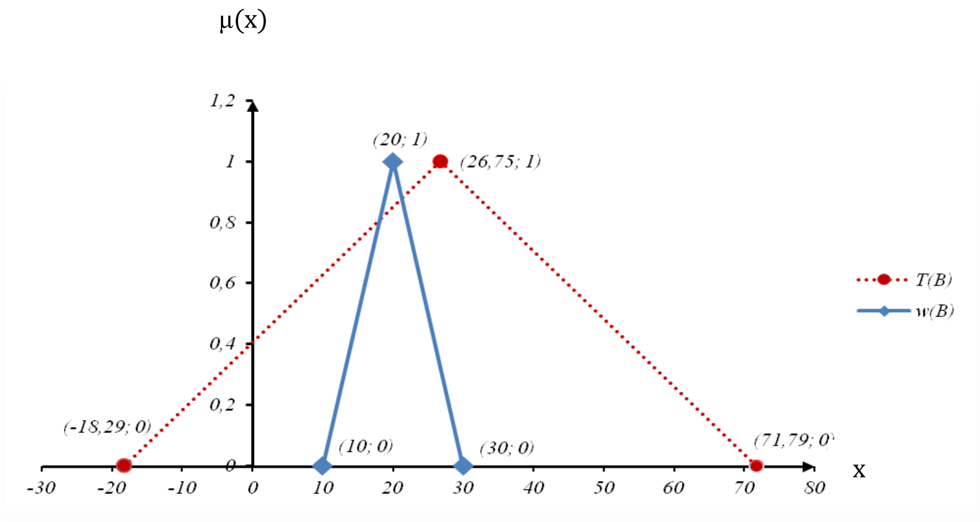

Figure 5.6: Combined Graphical Representation of *w(B)* and T_B

Membership functions of them are rewritten, so as to find intersection point of *w(B)* and T_B. So the membership function of *w(B)* is illustrated in Equation (5.23) and the membership function of T_B is illustrated in Equation (5.24).

$$\mu_{w_B}(x) = \begin{cases} (x-10)/10 \; for \; x \in [10,20] \\ (30-x)/10 \;\; for \;\; x \in [20{,}30] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.23}$$

$$\mu_{T_B}(x) = \begin{cases} (x+18{,}29)/45.04 \; for \; x \in [-18.29{,}26.75] \\ (71.79-x)/45{,}04 \;\; for \;\; x \in [26.75{,}71.79] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.24}$$

x axis of critic a level for Country B is also calculated in Equation (5.25),

$$(x-10)/10 = (x+18{,}29)/45.04 \tag{5.25}$$

$$x = 18{,}07 \tag{5.26}$$

x axis is indicated in Equation (5.26). Implementing this value in a_A^- side of the Equation (5.23) or (5.25) will deduce Critic a level for Country B and the result is in Equation (5.27).

$$\mu_{w_B}(18{,}06) = 0{,}807 \tag{5.27}$$

Country B should be in a coalition over 0,807 a level.

Equation (5.8) is used to find fuzzy Shapley values with 0,807 a level and it is denoted in Table 5.7.

Table 5.7: Fuzzy Shapley Values of Country B with 0,807 a Level

a $level$	T_B		
	$\boldsymbol{B_a^-}$	$\boldsymbol{B_a^1}$	$\boldsymbol{B_a^+}$
0.807	18.06	26.75	35.44

w(C) and T_C combined in Figure 5.7, aiming to eliminate the risk for the Country C,. As it is seen in figure, the straight line denotes *w(C)* and the dashed line is T_C.

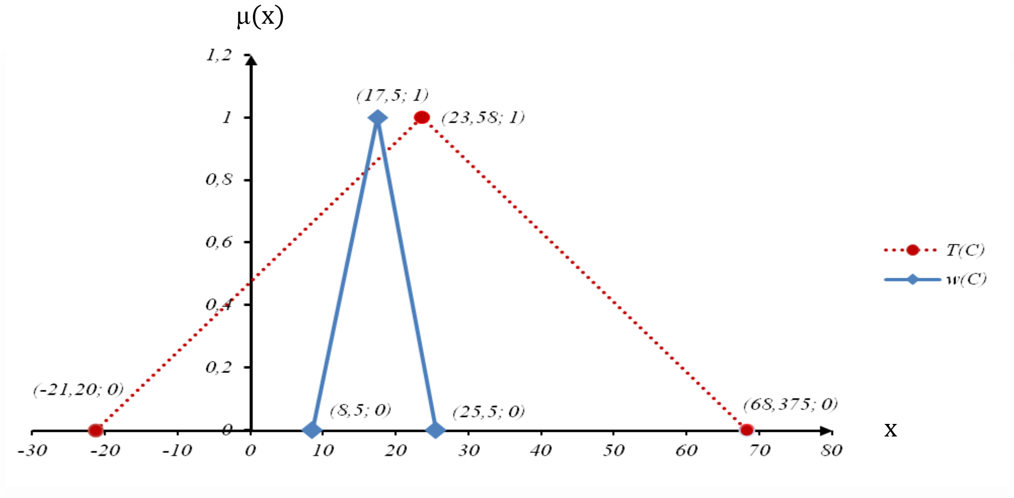

Figure 5.7: Combined Graphical Representation of *w(C)* and T_C

Membership function of *w(C)* is illustrated in Equation (5.28) and membership function of T_C is illustrated in Equation (5.29) so as to find intersection point of *w(C)* and T_C.. Then the critic risk level for the Country C can be calculated.

$$M_{w_C}(x) = \begin{cases} (x-8{,}5)/9 \, for \; x \in [8.5, 17.5] \\ (25.5-x)/8 \; for \quad x \in [17.5, 25.5] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.28}$$

The membership function of fuzzy Shapley value of Country C is,

$$\mu_{T_C}(x) = \begin{cases} (x+21.20)/44{,}78 \, for \; x \in [-21.20, 23.58] \\ (68.375-x)/44{,}80 \; for \quad x \in [23.58{,}68.38] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.29}$$

x axis of critic a level for Country C is denoted in Equation by determining first intersection of $\mu_{w_C}(x)$ and $\mu_{T_C}(x)$ in Equation (5.30).

$$(x-8{,}5)/9 = (x+21{,}2)/44{,}78 \tag{5.30}$$

$$x = 15{,}97 \tag{5.31}$$

The x axis of the critic a level for Country C is found in Equation (5.31) and critic a level is shown in Equation (5.32).

$$\mu_{w_C}(15{,}97) = 0{,}83 \tag{5.32}$$

Country C should be in a coalition over 0,83 a level. Fuzzy Shapley values with 0,83 a level is demonstrated in Table 5.8. According to this Acquisition of Country C in the game may vary from 15,9 to 31,20 and this level of a value gives the most reasonable Shapley value range for Country C.

Table 5.8: Fuzzy Shapley Values of Country C with 0,830 a Level

a level	T_C		
	C_a^-	C_a^1	C_a^+
0.83	15.97	23.58	31.20

Finally the same calculations are made for Country D. *w(D)*and T_D are illustrated in Figure 5.8.

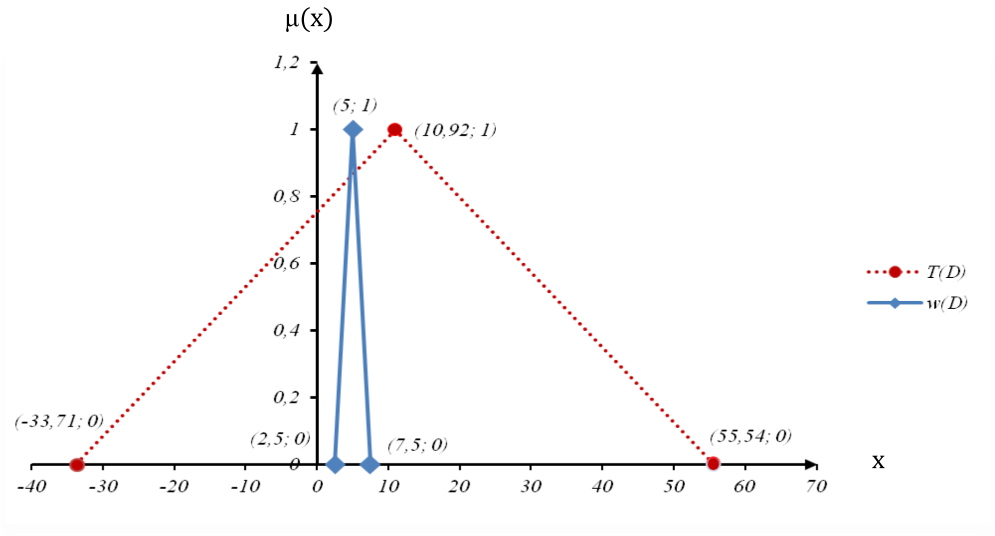

Figure 5.8: Combined Graphical Representation of *w(D)* and T_D

Country D is the last country to calculate. Intersection point of *w(D)* and T_D will be found with the help of membership functions of them. So the membership function of *w(D)* is demonstrated in Equation (5.33) and membership function of T_D is in Equation (5.34).

$$\mu_{w_C}(x) = \begin{cases} (x-2{,}5)/2{,}5 \; for \; x \in [2.5, 5] \\ (7{,}5-x)/2{,}5 \;\; for \;\; x \in [5, 7.5] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.33}$$

The membership function of fuzzy Shapley value of Country D is,

$$\mu_{T_D}(x) = \begin{cases} (x+33.70)/44.62 \; for \; x \in [-33.70, 10.92] \\ (55.54-x)/44.62 \;\; for \;\; x \in [10.92, 55.54] \\ 0, \qquad\qquad otherwise \end{cases} \tag{5.34}$$

x axis of critic a level for Country D is found with Equation (5.35) and the result is shown in Equation (5.36).

$$(x-2{,}5)/2{,}5 = (x+33{,}7)/44{,}62 \tag{5.35}$$

$$x = 4{,}65 \tag{5.36}$$

Critical a level for Country D is also calculated in Equation (5.37)

$$\mu_{w_D}(4{,}65) = 0{,}86 \tag{5.37}$$

0,86 a level and more is the reasonable risk level to be in the game for Country D. Fuzzy Shapley values of Country D with 0,86 a level is illustrated in Table 5.9.

Table 5.9: Fuzzy Shapley Values of Country D with 0,860 a Level

$a \; level$	T_D		
	$\boldsymbol{D_a^-}$	$\boldsymbol{D_a^1}$	$\boldsymbol{D_a^+}$
0.86	4.65	10.92	17.17

The problem is solved in 50% extension. If fuzzification limits of the scores change, the results will be different. Firstly maximum extension of the scores is utilized and the problem is solved accordingly. However, considering the overall scoring intervals of the experts, fuzzification can be made in a narrower range. Therefore, critical α values for countries in different extensions are calculated and illustrated in Table 5.10.

Table 5.10: Critical a Values of the Countries in Different Extensions

	Critical a Levels of the Countries			
Extension Level	**Country A**	**Country B**	**Country C**	**Country D**
10%	0	0.037	0.092	0.298
20%	0	0.519	0.547	0.649
30%	0.322	0.679	0.697	0.766
40%	0.491	0.759	0.773	0.824
50%	0.593	0.807	0.830	0.860

Table 5.10 shows that as the extension level of the scores increases, countries tend to take less risk. Because, the fuzzification of the values can causes uncertainty. If the contributions of countries to the coalition fuzzify, their incomes from the coalition will also fuzzify accordingly. In this case players may risk achieving individual rationality. So each player will tend to take less risk in order to balance the uncertainty caused by the extension. This situation is clearly seen in Table 5.10.

Country A, doesn't need to α-cut for being in coalition when the power is 10% in extension. Because as it is seen in Table 5.3, Country A has the highest contribution to coalitions. Although Country B has similar power for coalitions it should not take that much risk as Country B has less contributions to coalitions. The player who contributes more to coalitions will be less likely to lose. On the other hand power of the countries determines the reasonable risk. Contribution of Country B and Country C are close to each other. However, Country B allocates more power for coalitions. As a conclusion it can take more risk.

Proposed model shows that, determining critical α values for players to be in coalition depends on various reasons. As mentioned in previous chapters, analyzing tables and figures is considerably important. As a conclusion tables and figures of this thesis give an idea for solution of the fuzzy coalition games.

6. CONCLUSIONS AND DISCUSSIONS

In this thesis, a solution method for multiplayer fuzzy game model is proposed. The aim of the thesis is to determine the maximum risk level that players can take for a possible coalition by evaluating the dynamically changing risk situations in today's global uncertainty. The fuzzy set theory of Lotfi A. Zadeh was used in this thesis to express uncertainties and linguistic terms. Shapley value, which is widely used in solving multiplayer game models, is fuzzified in accordance with Zadeh's fuzzy sets theorem. In this way, the mathematics-based part of the game theory has been adapted to today's uncertainty and multilateral conflict environments. For this purpose, the literature is reviewed, considering that the player's payoffs are fuzzy; the fuzzy Shapley value which is proposed by Mares is studied. Group rationality and individual rationality rules, which are the characteristics of fuzzy cooperative games are applied in the method and the critical risk level for players to enter the game is determined.

The method is modeled on fictitious countries A, B, C, D that have diplomatic disagreement over the sharing of underground resources. The payoffs representing the strengths of the countries or their contribution to the problem are determined by taking the opinions of five experts. The payoffs of the countries are fuzzified with 50% extension first. The greatest difference of opinion among experts was observed in the scoring of country D. It was extended 50% for each side. This largest extension is applied for other players and coalitions. But the fuzzy Shapley values calculated for other fuzzification levels are given at the end of the thesis.

Each of the fuzzy Shapley values of the countries are calculated according to Mares' method using fuzzy arithmetic operations. Then, the accuracy of the model is verified by the control of group rationality. The problem is based on the $x_i \geq v(i)$ rule, which expresses the individual rationality feature of multiplayer games. Accordingly, it is concluded that countries will seek a diplomatic coalition if their fuzzy Shapley values are higher than or equal to the entry payoffs.

Accordingly, considering the results in 50% extension, it was concluded that country A would prefer to enter a coalition in the case of the highest risk with a value of 0.593 α. Country A's contribution to coalitions as a result of its geographical and military advantage is higher than other countries, as can be seen in Table. 5.4. On the other hand, power of Country A is relatively high. Country A is therefore more likely to make a possible coalition advantageous for itself and its partner. This verifies the result we reached in the problem. Although country B's entry power is almost the same as country A, the value of critical α is 0.807. We came to this conclusion because the contribution of country B to possible coalitions is not as high as country A. Therefore, country B may not be able to achieve individual rationality in high risk levels. This situation requires that country B takes less risk than country A. As can be seen in Table 5.4, the contribution of country C to possible coalitions is almost the same as country B. However, its payoff for the coalition, calculated with expert opinions, is less than that of country B. That’s why the critical α value for country C is 0.83. In this case, country C will enter a coalition in a less risky situation than country B. Finally, the critical α value for country D is 0.86, as country D has the lowest individual payoff and its contribution to possible coalitions is again the lowest. This means that country D is the country that will enter the coalition in the lowest risk environment.

Shapley values expressing the profits of the countries at the end of possible coalitions are shown in the thesis. It has been observed that the Shapley value of the countries has never fallen below the level, where their coalition entry power could fall due to the fuzzy extension in the proposed method. This shows that individual rationality is verified in terms of the fuzzy theorem.

Finally critical α values for countries are calculated in different extensions and illustrated in Table 5.10. It shows that more fuzzification increases the risk for players and they compensate it with entering the coalition in higher α levels (lower risk).

The results are presented to experts. The results of the game are compared with their expectations. Their common idea was the model verifies their linguistic predictions. Considering the features of country A, the experts opinion was coalition of the country A should be most valuable because of the technological and military advantage of the country A. So the country A should have valuable contribution to its partners and the game that’s why it can take more risk than other countries. In this sense our mathematical model verifies the expert’s linguistic ideas.

This new method achieved to analyze critical risk level for joining the coalition in fuzzy environment. Individual rationality is just a rule for validation of crisp cooperative games. However, it helps us for determining the critical risk level by comparing the two fuzzy membership functions. This method can have contribution to fuzzy cooperative games for analyzing the behavior of the players. On the other hand reactions of the players in different uncertainties are investigated. Today’s fuzzy environments are dynamic and full of surprises. The coalitions and coordination between sides should be followed on line, which can be possible with this method. In different risk levels, decision makers can follow the situation and they can predict how much profit they can make at different risk levels.

For future work, how the changes in the power that players will reserve for coalition in multiplayer games affect the alpha value can be examined. In addition, the amount of power players will reserve for the coalition can be found by using data mining tools. In this way, fuzzy game theory can be adapted more realistically to today's conditions. Finally, players' participation in the game can also be fuzzified. In this way, Choquet integral and the solution method of this thesis can be combined.

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APPENDIX

Opinion Collection Form

This form has been prepared to investigate the factors, determining the strength of a country and to score the powers of four simulated countries by considering these factors. In addition, possible coalitions of these countries will be scored. For this purpose, the form is divided into three parts. For each part please follow the introductions. First you will begin with Part1. After evaluating the Part 1 We will ask you to fill out the second part.

Part 1

Please write down the five factors that determine power of a country.

Part 2

Economy, Military, Technology, Population and Education are selected as five categories that determine power of a country. Features of four imaginary countries are written below.

Country A:

Country A has the largest military population among the rivals. They also has the growing technology especially in defense industry. However country A has problems with keeping their qualified individuals in the country because of the living conditions. The economy of the country is in the process of burning up however it is not irrevocable. Population is the second crowded country among others. The education of the country is not stable because perpetually the education system changes and unemployment is in the realm of possibility by reason of the unplanned number of graduated people. Despite all, Country A has always been ascendant for the region.

Country B:

After the military turmoil, Country B has relatively stable political landscape. Also this country is enthusiastic to establish good relations with countries which have strategic importance. During the turmoil period Country A supported opponents of the current government, therefore Country B is unwilling to rebuilt new relations with Country A.

Because of the strong relations with great powers, there is not difficulty for Country B to reach weapons. Owing to the archaic history and geopolitical position the country has experienced military. The education of the country is based on religion and not innovative. Also gross domestic product is one-third of Country A and same as Country C. Country B has the largest population among other countries however the technology of the country is underwhelming. Finally, Country B believes that coalition with the Country C may be more useful because Country C is the nemesis of Country A.

Country C:

As indicated above Country C has also other problems with Country A beside underground sources. That is not to say they can never solve their problems. Despite their disagreements, they both have a culture to find common way to their problems. Country C has the third largest population of four countries. Due to the economic crises, the technology of the country is underdeveloped. However the relationship with other countries and membership to the unions make the Country C advantageous for using technology. Besides, undergoing conflictions with Country A, keep military forces ready to move. Education system of Country B is well-established and stabile for years. Also culture of the Country C is ancient and inspiring to neighbor countries. Coalition with Country B may put Country A into trouble. However, by the reason of geopolitical position of Country C and Country A, the coalition of these countries may be advantageous for both of them

Country D:

Country D is still in civil war. Current government was going to lose control if Country A would not help them through military assistance. Therefore the government in Country D seems Country A as assurance of existence. Nevertheless other coalitions may also guarantee them present conditions. The economic size of the Country D is the smallest one among others. Thus Country D may be more willing to be in coalition with other countries. Country D has an advantage; it is the closest country to the underground sources. The education in Country D is the worst one. Great part of the population has difficulties to reach the proper education. Also population of the country is the smallest one. A military force of the country is divided into two groups. One of them supports current government while other is with opponents. Despite the military assistance of

Country A, most part of the Country D is still under the control of opponents. Predictably, the technology of the Country D depends on Country A. Country D has the lowest population among all these countries.

Now please Score the four countries and their possible coalitions, from 0 to 100, according to the categories above. Please attention that, the score of any coalition must be higher than the individual scores of the countries that make up that coalition or the score of the subset coalitions. For example coalition of (A, B) gets more score than individual score of both A and B. Similarly Coalition of (A,B,C) gets higher score than individual scores of A, B, C and coalition of (A,B). So the coalition of (A,B,C,D) must have the highest score. Now please fill up the table below from 0 to 100.

Scores of Countries and Possible Coalitions					
A		A,C		A,B,C	
B		A,D		A,B,D	
C		B,C		B,C,D	
D		B,D		A,C,D	
A,B		C,D		A,B,C,D	

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