# SOME RESULTS ON OPERATOR THEORY BASED ON UNBOUNDED CONVERGENCE

# OPERATÖR TEORİSİNDE SINIRSIZ YAKINSAMA TABANLI BİRTAKIM SONUÇLAR

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# ABSTRACT

## SOME RESULTS ON OPERATOR THEORY BASED ON UNBOUNDED CONVERGENCE

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Main topic of the this work is to use the notion of unbounded convergences on vector lattices to derive properties of various classes of operators and nets of operators defined between vector lattices, Banach lattices, and more generally, between locally solid vector lattices. The classes formed by *uaw*-Dunford-Pettis operators, $u\tau$-compact operators and $\mathfrak{c}$-Lotz-Räbiger nets are among those classes investigated in this work. We present several properties of these classes with the help of new perspectives provided by unbounded convergences. In addition, various examples with completely new origins are given.

First main chapter deals with *uaw*-Dunford-Pettis operators. As a result of the theory of classical Dunford-Pettis operators, it is expected that *uaw*-Dunford-Pettis operators have connections with certain classical classes of operators acting on Banach lattices. Hence, one of the aims to study *uaw*-Dunford-Pettis operators is to determine their relations with other types of operators. Further, we study domination and iteration properties of *uaw*-Dunford-Pettis operators.

The second class of operators that we investigate is the class of $u\tau$-compact operators defined between locally solid vector lattices. In this general setting, various notions related to boundedness of operators play a central role. Hence, one of the aims to study $u\tau$-compact operators is to determine the effect of boundedness on compact operators.

In the last chapter of this work, we study a generalization of norm ergodic operators. The main method is to use various versions asymptotic equivalences to study properties of Lotz-Räbiger nets defined between convergence vector lattices. Because Lotz-Räbiger nets are closely related to the classical notion of ergodic nets and ergodic operators, some of our results further apply to the particular case of classical ergodic operators.

**Keywords:** Unbounded convergence, Dunford-Pettis operator, Compact operator, Ergodic nets

# ÖZET

## OPERATÖR TEORİSİNDE SINIRSIZ YAKINSAMA TABANLI BİRTAKIM SONUÇLAR

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Bu çalışmanın temel konusu; vektör latisleri, Banach latisleri ve daha genel olarak yerel katı vektör latisleri arasında tanımlanmış olan operatör sınıfları ve operatör ağ sınıfları hakkında sınırsız yakınsama kavramını kullanarak sonuçlar elde etmektir. Bu çalışmada incelenen sınıflar arasında, *uaw*-Dunford-Pettis operatörleri, $u\tau$-kompakt operatörleri ve $\mathfrak{c}$-Lotz-Räbiger ağları bulunmaktadır. Bu sınıfların çeşitli özelliklerini, sınırsız yakınsamaların sağladığı yeni bakış açısını kullanarak sunmaktayız. Ek olarak, tamamen yeni kökenleri olan çeşitli örnekler verilmiştir.

İlk ana bölüm, *uaw*-Dunford-Pettis operatörleri ile ilgilidir. Klasik Dunford-Pettis operatörler teorisinin bir sonucu olarak, *uaw*-Dunford-Pettis operatörlerinin Banach latisleri arasında tanımlanmış bazı klasik operatör sınıflarıyla bağlantılı olması beklenir. Bu nedenle, *uaw*-Dunford-Pettis operatörlerini incelemenin amaçlarından biri diğer operatörlerle olan ilişkilerini belirlemektir. Ayrıca, *uaw*-Dunford-Pettis operatörlerinin baskınlık (dominasyon) ve iterasyon özelliklerini de incelemekteyiz.

Araştırdığımız ikinci operatör sınıfı, yerel katı vektör latisleri arasında tanımlanan $u\tau$-kompakt operatör sınıfıdır. Bu genel durumda, operatörlerin sınırlılıkları ile ilgili çeşitli kavramlar merkezi bir rol oynamaktadır. Dolayısıyla, $u\tau$-kompakt operatörleri incelemenin amaçlarından biri, sınırlılığın kompakt operatörler üzerindeki etkisini belirlemektir.

Bu çalışmanın son bölümünde, norm ergodik operatörlerin bir genelleştirilmesini inceliyoruz. Ana metod, yakınsamalı vektör latisleri arasında tanımlanmış Lotz-Räbiger ağlarının özelliklerini, asimptotik denkliklerin çeşitli türlerini kullanarak çalışmaktır. Lotz-Räbiger ağları klasik ergodik ağlar ve ergodik operatörler ile yakından ilişkili olduğundan, sonuçlarımızdan bazıları klasik ergodik operatörler için de geçerlidir.

**Anahtar Kelimeler:** Sınırsız Yakınsama, Dunford-Pettis operatörü, Kompakt operatör, Ergodik ağ

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# CONTENTS

# 1 INTRODUCTION

The notions of Dunford-Pettis operators, compact operators and ergodic operators are of central significance in the theory of operators on Banach lattices. In addition to the fact that concrete examples of these classes of operators can be explicitly given on classical Banach lattices such as $C(K)$ or $L_p(\Omega,\Sigma,\mu)$ $(1 \leq p \leq \infty)$; a physical system may induce an appropriately defined operator sharing the same topological or dynamic properties of these types of operators. Moreover, all of these classes of operators are known to admit further generalizations to more abstract settings, such as the generality of topological vector lattices.

In this thesis, we study certain special classes of operators defined between vector lattices where either the domain or the range of the operator is equipped with a convergence compatible with the underlying lattice structure. The present study presents results about unbounded absolutely weak Dunford-Pettis operators (abbreviated as *uaw*-Dunford-Pettis operator), $u\tau$-compact operators and $\mathfrak{c}$-Lotz-Räbiger nets.

Convergences and, in particular, unbounded convergences on vector lattices constitute an integral part in this work. We recall that a convergence $\mathfrak{c}$ on a vector lattice can be thought as a rule that associates a net belonging to a predefined set of nets ( which can be called the collection of $\mathfrak{c}$-convergent nets) with a point of that vector lattice in such a way that lattice operations are continuous in the sense of net characterization of continuity. Classical notions such as order and norm convergences as well as the recently defined notions such as unbounded order and unbounded norm convergences are the particular cases of convergences that this study particularly interested in.

In the following, we review some of the literature on particular cases of the notion of unbounded convergence. Since some convergences, such as the norm convergence, the topological convergence and the order convergence, have been widely used in the existing literature, their general theories and examples can be found in the standard references such as [5, 50, 54, 57, 59, 71, 76]. As remarked in [72], connections between norm and other types of order theoretical convergences have been investigated by many. Hence, our main interest lies in unbounded order, unbounded norm and unbounded absolutely weak convergences.

Sequential version of unbounded order convergence was first defined on $\sigma$-Dedekind complete vector lattices by H. Nakano in 1948, see [58]. In his inspiring paper entitled "Ergodic theorems in semi-ordered linear spaces", he generalized the individual ergodic theorem to the settings of normed semi-ordered linear spaces of Kantorovitch, see [58, Theorem 3.4]. The nomenclature unbounded order convergence, see [17, Definition 3], was initially used by R. DeMarr in 1964 in the settings of ordered vector lattices. He proved that, see [17, Theorem 1], any locally convex space over the field of real numbers can be embedded in a partially ordered linear space in such a way that the topological convergence in the locally convex space agrees with the order convergence in the ordered vector space.

The relations between the weak convergence and the unbounded order convergence were investigated by A. Wickstead in 1977, see [72]. In addition to other arguments, he presented several characterizations of Banach lattices in which weak convergence of nets implies their convergence in the sense of unbounded order, and vice versa, see [72, Theorem 5]. In [47], S. Kaplan gave characterizations of unbounded order convergence. In details, he presented two characterizations of unbounded order convergence in Dedekind complete vector lattices having weak units. He then proved of a theorem of Nakano on order boundedness of unbounded order convergent sequences, see [47, Section 3].

Recently, in [41], N. Gao and F. Xanthos studied unbounded order Cauchy nets in Banach lattices. They used these notions to characterize Banach lattices with the positive Schur property and KB-spaces. In that work, N. Gao and F. Xanthos showed that an order continuous Banach lattice has the positive Schur property if and only if a version of the Dunford-Pettis theorem holds. Moreover, they used unbounded order Cauchy sequences to extend Doob's submartingale convergence theorem to the generality of abstract martingales, see [41, Theorem 5.6]. The very same authors then published another article, see [42], on the $w^*$-representations of risk measures.

In 2014, N. Gao studied unbounded order convergence in dual spaces of Banach lattices, see [37]. In addition to other arguments, N. Gao gave various characterizations, see [37, Theorems 2.1 and 3.4], of Banach lattices in which unbounded order convergence in the dual space implies $w^*$-convergences and vice versa.

In their well-written article entitled "$Uo$-convergence and its applications to Cesàro means in Banach lattices"; N. Gao, V. Troitsky, and F. Xanthos investigated many aspects of unbounded order convergence, see [40]. One of the major achievements is the stability of the unbounded order convergence under passing to and from regular sublattices. These results allowed them to generalize several results of [37, 41]. Moreover, they used unbounded order convergence in the purpose of obtaining results about convergence of Cesàro means in Banach lattices. As a result, they derived a version of Komlós Theorem in Banach lattices, see [40, Theorem 5.9]. They further studied various aspects of Banach-Saks properties and (weakly) Banach-Saks operators in Banach lattices based on unbounded order convergence. E. Emel'yanov and M. Marabeh derived two measure-free versions of Brezis-Lieb lemma in vector lattices using unbounded order convergence in [28].

In 2017, H. Li and Z. Chen showed in [53, Proposition 2.8] that, under the additional assumption of Dedekind completeness, a vector lattice is universally complete if and only if every unbounded order Cauchy net is unbounded order convergent. This work is motivated by the case of Dedekind completions because a vector lattice is Dedekind complete if and only if every order Cauchy net is order convergent. In 2019, Y. Azouzi showed in [8, Theorem 17] that the additional assumption of the previous result was not needed in the case of universal completions.

Let us now focus on relations between lattice norms and unbounded convergences. In 2004, V. Troitsky developed the nomenclature $d$-convergence, the old name for unbounded norm convergence. He used it to study measure of non-compactness, see [67] for details. In 2016, Y. Deng, M. O'Brien, and V. Troitsky introduced the name "unbounded norm convergence" in [18]. In this work, authors studied the relations between unbounded norm convergence and other types of convergences such as unbounded order and weak convergences. Further, they showed that unbounded norm convergence is topological, see [18, Section 7]. The corresponding topology is called the unbounded norm topology. In [45], M. Kandić, M. Marabeh, and V. Troitsky exposed several properties of unbounded norm topology. For instance, they showed that the norm topology and the unbounded topology agree if and only if the Banach lattice has a strong unit, see [45, Theorem 2.3]. They further showed that the unbounded norm topology is metrizable if and only if the Banach lattice has a quasi-interior point, see [45, Theorem 3.2]. For further details, we refer to [45].

In [79], O. Zabeti introduced the notion of unbounded absolute weak convergence on Banach lattices. In his paper, he derived various relations between unbounded absolute weak convergence and other convergences.

After that, in [65, 66], M.A. Taylor investigated unbounded convergence and minimal topologies in locally solid vector lattices. Further, M. Kandić and M.A. Taylor published results on metrizability, submetrizability and local boundedness of unbounded topologies, see [43]. In [13], Y.A.M. Dabboorasad, E.Yu. Emel'yanov published a detailed survey about convergence vector lattices. All these results motivated further research. In this direction, Z. Ercan and M. Vural presented unbounded Riesz pseudonorms in [30].

We end this discussion by giving some details about the applications of unbounded convergences to operator theory. In 2018; A. Aydın, E.Yu. Emel'yanov, N. Erkurşun Özcan, and M.A.A. Marabeh published a well-written article on generalizations of compact operators to lattice normed lattices, see [6]. In [33, 34] N. Erkurşun Özcan, N.A. Gezer, and O. Zabeti published their results on Dunford-Pettis and bounded operators. In [35], N. Erkurşun Özcan and N.A. Gezer studied a generalization of Lotz-Räbiger nets to the case of unbounded convergences. Further details of these articles are presented in sequel.

The present thesis consists of four chapters.

In Chapter 2, after giving a review of notations, we give a summary of some results related to the literature on unbounded convergences. Although most of the results presented in this chapter are needed in the sequel, reader should refer to the references for a complete treatment. Beginning with the basic properties of convergences, Chapter 2 provides an outline of the theory of unbounded convergences.

In Chapter 3, we investigate *uaw*-Dunford-Pettis and *uaw*-compact operators between Banach lattices. In addition to the outline presented in Chapter 2, most of the material in Chapter 3 requires a standard background in the theory of Dunford-Pettis operators. As a result of the theory of classical Dunford-Pettis operators, the work presented in Chapter 3 shows that both *uaw*-Dunford-Pettis and *uaw*-compact operators are closely related with compact, weakly compact, $o$-weakly compact, and, $M$- or $L$-weakly compact operators. By addressing these types of relations, results of Chapter 3 fills several gaps in the literature. For example, this chapter provides tools and techniques for studying operators with respect to unbounded convergences. We present several results related to domination and iteration of *uaw*-Dunford-Pettis operators. In the last part of this chapter, we study *uaw*-compact operators. The notion of *uaw*-compact operators is closely tied to the classical compact operators. Consequently, they share some common properties. For instance, we prove that, under some conditions, the adjoint of a sequentially *uaw*-compact operator is sequentially *uaw*-compact.

In Chapter 4, we investigate $u\tau$-Dunford-Pettis and $u\tau$-compact operators on locally solid vector lattices. At first glance, these classes may seem to abstract *uaw*-Dunford-Pettis and *uaw*-compact operators to the generality of locally solid vector lattices in a direct manner. However, examples given in Chapter 4 show that this is not the case. In particular, the work presented in Chapter 4 shows that boundedness of operators plays a significant role when the underlying vector lattice is equipped with a locally solid topology. As a result, most of the material given in the beginning sections of this chapter focuses on the notion of bounded operators. We then focus on order bounded operators satisfying some additional boundedness properties. More advanced topics such as boundedly unbounded locally solid topologies are also covered.

In Chapter 5, we investigate an abstraction of Lotz-Räbiger nets. In addition to results of previous chapters, Chapter 5 requires a background in the theory of ergodic operators and iterations. Many of the results presented in this chapter depend on the fact that convergences and, in particular, unbounded convergences can be used in an appropriate manner to study asymptotic behaviours of operator nets. After defining $\mathfrak{d}$-asymptotic equivalence between two operator nets, we extend the notion of Lotz-Räbiger nets to $\mathfrak{d}$-Lotz-Räbiger nets. Since classical Lotz-Räbiger nets are closely related to the notion of $\mathscr{S}$-ergodic net of an operator semigroup, $\mathfrak{d}$-Lotz-Räbiger nets provide a new approach to study ergodic properties of operators.

# 2 PRELIMINARIES

In this chapter, we present the general background needed in this thesis. Structure of this chapter is as follows. In Section 2.1, we recall some of the basic notions related to vector and Banach lattices. In Section 2.2, we compare various types of convergences on vector and Banach lattices. Discussions given in Section 2.2 provide a summary of the fundamental properties of such convergences, which are used in the sequel to study new classes of operators.

## 2.1 Basic Concepts of Vector Lattices

Let $\leq$ be a partial order relation on a real vector space $X$. The pair $(X,\leq)$ is called an *ordered vector space* if it satisfies the following conditions: (i) $x \leq y$ implies $x+z \leq y+z$ for all $z \in X$; and (ii) $x \leq y$ implies $\lambda x \leq \lambda y$ for all $\lambda \in \mathbb{R}^+$. We put $X^+ := \{x \in X : x \geq 0\}$ for the set of nonnegative elements of $X$.

An ordered vector space $X$ is said to be *Archimedean* if for any $x, y \in X$ the relation $nx \leq y$ for $n \in \mathbb{N}$ implies $x \leq 0$. Hereafter we shall suppose that all ordered vector spaces are Archimedean.

For each $x$ and $y$ in an ordered vector space $X$ we let $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$. If $x \in X^+$ and $x \neq 0$ then we write $x > 0$.

An ordered vector space $X$ is said to be a *vector lattice* if for each $x, y \in X$ the elements $x \vee y$ or $x \wedge y$ both exist in $X$. Equivalently a vector lattice is an ordered vector space that is also a lattice. An ordered vector space $X$ is a vector lattice if and only if the supremum of every finite subset of $X$ exists.

Let $X$ be a vector lattice. Consider an arbitrary element $x \in X$. The vectors $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := (-x) \vee x$ are called the *positive part*, *negative part* and *absolute value* of $x$, respectively. Two elements $x$ and $y$ of $X$ are said to be *disjoint*, abbreviated as $x \perp y$, if $|x| \wedge |y| = 0$. *Disjoint complement* of a nonempty subset $A$ of $X$ is the subset $A^\perp$ of $X$ defined by $A^\perp := \{x \in X : x \perp a \text{ for all } a \in A\}$.

A subset $A$ of a vector lattice $X$ is *bounded from above* (*bounded from below*) if there is some $x \in X$ satisfying $a \leq x$ ($x \leq a$, respectively) for all $a \in A$. The subset $A$ of $X$ is said to be *order bounded* if it is both bounded from above and bounded from below. If $a, b \in X$ with $a \leq b$ then the subset $[a, b] := \{x \in X : a \leq x \leq b\}$ is called an *order interval* in $X$. Evidently, order intervals of $X$ are order bounded.

A linear subspace $Y$ of a vector lattice $X$ is said to be a *sublattice* of $X$ if for each $y_1$ and $y_2$ belonging to $Y$ one has $y_1 \vee y_2 \in Y$. The sublattice $Y$ of $X$ is called *order dense* in $X$ if for each $x > 0$ in $X$ there is some $0 < y \in Y$ satisfying $0 < y \leq x$. The sublattice $Y$ of $X$ is said to be *majorizing* in $X$ if for each $x \in X^+$ there exists $y \in Y$ such that $x \leq y$. Evidently, every vector lattice is an order dense and majorizing sublattice of itself.

A subset $A$ of $X$ is said to be *solid* if it follows from $a \in A$ and $|x| \leq |a|$ that $x \in A$. Therefore, a subset $A$ is solid if and only if the order interval $[-|a|, |a|]$ is contained in $A$ for every $a \in A$.

A solid vector subspace of a vector lattice $X$ is said to be an (order) *ideal* in $X$. Let $A$ be a nonempty subset of $X$. The ideal $I_A$ generated by $A$ is the smallest ideal, with respect to the partial ordering induced by the inclusion on the set of ideals of $X$, in $X$ that contains $A$. This ideal is given by

$$I_A := \{x \in X \colon \exists a_1, \dots, a_n \in A \text{ and } \lambda \in \mathbb{R}^+ \text{with } |x| \leq \lambda \sum_{j=1}^{n} |a_j|\},$$

see [5, Section 1.3] for more details. We remark that $A^\perp$ is also an ideal of $X$ that we can associate with $A$. For $x_0 \in X$ the ideal $I_{x_0}$ generated by $\{x_0\}$ is called the *principal ideal* generated by $x_0$.

Let $(\mathscr{A}, \leq)$ be a nonempty directed set. A *net* on a vector lattice $X$ is a mapping $x \colon \mathscr{A} \to X$ from the directed set $(\mathscr{A}, \leq)$ into $X$. The net $x \colon \mathscr{A} \to X$ is denoted by $(x_\alpha)_{\alpha \in \mathscr{A}}$ where $x_\alpha := x(\alpha)$ for each $\alpha \in \mathscr{A}$. If the set of indices $\mathscr{A}$ is clear from the context, we put $x_\alpha$ instead of $(x_\alpha)_{\alpha \in \mathscr{A}}$. A *subnet* of the net $(x_\alpha)_{\alpha \in \mathscr{A}}$ is any net of the form $v \colon \mathscr{B} \to X$, where $\mathscr{B}$ is itself a directed set such that there exists some $\lambda : \mathscr{B} \to \mathscr{A}$ satisfying the property that for any $\alpha \in \mathscr{A}$ there exists some $\beta_\alpha \in \mathscr{B}$ such that $\beta \geq \beta_\alpha$ implies $\lambda(\beta) \geq \alpha$.

A net $x_\alpha$ in the vector lattice $X$ is said to be *increasing*, in symbols $x_\alpha \uparrow$, if $x_\alpha \leq x_\beta$ whenever the indices $\alpha$ and $\beta$ satisfy $\alpha \leq \beta$. The symbol $x_\alpha \downarrow 0$ denotes a net decreasing to zero and its definition is analogous. In an arbitrary vector lattice $X$, an increasing net does not need to have a supremum. Indeed, in the vector lattice $c_0$ of real sequences converging to zero, denote by $x_n$ the element of $c_0$ whose first $n$ coordinates are 1 and the remaining coordinates are zero. The sequence formed by such elements is increasing but $\sup_n x_n$ does not exits in the vector lattice $c_0$.

For a net $x_\alpha$ in a vector lattice $X$, we write $x_\alpha \xrightarrow{o} x$ if $x_\alpha$ *converges* to $x$ in *order*. This means that there is a net $y_\beta$, possibly over a different index set, such that $y_\beta \downarrow 0$ and for every $\beta$ there exists $\alpha_\beta$ satisfying $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. It follows that an order convergent net has a bounded tail, and, an order convergent sequence is order bounded. For a net $x_\alpha$ and $x \in X$ we have $|x_\alpha - x| \xrightarrow{o} 0$ if and only if $x_\alpha \xrightarrow{o} x$.

A linear operator $T : X \to Y$ between two vector lattices $X$ and $Y$ is called *lattice homomorphism* if $x \wedge y = 0$ implies $T(x) \wedge T(y) = 0$. A one-to-one lattice homomorphism is called *lattice isomorphism*. A linear operator $T : X \to Y$ between vector lattices is said to be *order continuous* if $x_\alpha \xrightarrow{o} 0$ in $X$ implies $T(x_\alpha) \xrightarrow{o} 0$ in $Y$.

A subset $A$ of $X$ is called *order closed* if it follows from $x_\alpha \xrightarrow{o} x$ in $X$ for a net $x_\alpha$ in $A$ that $x \in A$. An order closed ideal is said to be a *band*. For $x_0 \in X$ the *principal band* generated by $x_0$ is the smallest band, with respect to the partial ordering induced by the inclusion on the set of bands of $X$, containing $x_0$. We denote this band by $B_{x_0}$. It follows that $B_{x_0} = \{x \in X \colon |x| \wedge n|x_0| \uparrow |x|\}$, see [5, Theorem 1.38] for details.

**Example 2.1.1.** *The vector lattice $\ell_\infty$ of all bounded real sequences is an ideal in the vector lattice $s$ of all real sequences. The space $c$ of all convergent real sequences is a sublattice of $\ell_\infty$ but not an ideal in $\ell_\infty$. The space $c_0$ is an ideal in both $c$ and $\ell_\infty$ but not a band of neither of these spaces.*

A band $B$ in a vector lattice $X$ is said to be a *projection band* if $X = B \oplus B^\perp$. The canonical projection $P_B \colon X \to X$ associated with this direct sum is called the band projection corresponding to the band $B$. If $P$ is a band projection then it is a lattice homomorphism and $0 \le P \le I$. In particular, band projections are order continuous.

A net $(x_\alpha)_{\alpha\in\mathscr{A}}$ is said to be *order Cauchy* if the double net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')\in\mathscr{A}\times\mathscr{A}}$ is order convergent to 0. A vector lattice $X$ is called *Dedekind* (or *order*) *complete* if every nonempty subset of $X$ that is bounded from above has a supremum. This implies that if $0 \le x_\alpha \uparrow \le u$ for a net $x_\alpha$ in $X$ and $u \in X$ then there is $x \in X$ such that $x_\alpha \uparrow x$. For an order bounded net $x_\alpha$ in a Dedekind complete vector lattice we have $x_\alpha \xrightarrow{o} x$ if and only if $\inf_\alpha \sup_{\beta\ge\alpha} |x_\beta - x| = 0$. Relations between order Cauchy nets and Dedekind completeness is given in Section 2.2.2.

The vector lattice $X$ is called *$\sigma$-Dedekind complete* if every countable subset $A \subseteq X^+$ which is bounded from above has a supremum in $X$. Equivalently, $X$ is $\sigma$-Dedekind complete if and only if every countable subset that is bounded from above has a supremum in $X$.

If $X$ is a vector lattice then there is a Dedekind complete vector lattice $X^\delta$ such that $X$ is a majorizing order dense sublattice of $X^\delta$. The vector lattice $X^\delta$ is called the *Dedekind* ( or *order*) *completion* of $X$ and it is unique up to a lattice isomorphism, see [5, Theorem 2.24]. The vector lattice $X$ is *universally complete* if it is both Dedekind complete and laterally complete, i.e., pairwise disjoint positive elements have a supremum in $X$. This is equivalent to saying that for a subset $A \subseteq X^+$ of a universally complete vector lattice $X$, $\sup A$ exists in $X$ whenever $A$ is bounded from above or $A$ is pairwise disjoint. It follows that every Archimedean vector lattice has a Dedekind completion $X^\delta$, a $\sigma$-Dedekind completion $X^\sigma$, and, a universal completion $X^\sharp$ which are all unique up to their defining lattice isomorphisms.

A vector lattice $X$ equipped with a norm $\|\cdot\|$ is said to be a *normed lattice* if the norm $\|\cdot\| : X \to \mathbb{R}$ is a *lattice norm*, i.e., $|x| \le |y|$ implies $\|x\| \le \|y\|$ for $x, y \in X$. Informally speaking, a range of different possibilities is available when comparing the underlying lattice and norm structures. Definition 2.1.2 highlights a few of these possibilities all of which have some importance in the subsequent chapters.

**Definition 2.1.2.** A lattice norm $\|\cdot\| : X \to \mathbb{R}$ on a vector lattice $X$ is said to be:

*(i)* order continuous, if $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$.

*(ii)* a Levi norm, if $0 \le x_\alpha \uparrow$ and $\|x_\alpha\| \le 1$ imply $\sup_\alpha x_\alpha$ exists in $X$.

*(iii)* a Fatou norm, if $0 \le x_\alpha \uparrow x$ implies $\|x_\alpha\| \uparrow \|x\|$.

*(iv)* a weak Fatou norm, if there exists some constant $K > 0$ such that $0 \le x_\alpha \uparrow x$ implies $\|x\| \le K \lim_\alpha \|x_\alpha\|$.

*(v)* an $M$-norm, if $x \wedge y = 0$ implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$.

*(vi)* $p$-additive for some $1 \le p < \infty$, if $\|x+y\|^p = \|x\|^p + \|y\|^p$ for all $x, y \in X^+$ satisfying $x \wedge y = 0$.

If a normed lattice is norm complete then it is called a *Banach lattice*. A Banach lattice $(X, \|\cdot\|)$ is said to be a *KB-space*, the short form of Kantorovich-Banach space, if it follows from $0 \le x_\alpha \uparrow$ and $\sup_\alpha \|x_\alpha\| < \infty$ for a net $x_\alpha$ that the net $x_\alpha$ is norm convergent in $X$. It follows that every *KB*-space is an order continuous Banach lattice. Therefore, a Banach lattice is a *KB*-space if and only if it has an order continuous Levi norm. Prime examples of *KB*-spaces are lattices of the form $L^p(\mu)$ for $1 \le p < \infty$. We remark that order continuity of $L^p(\mu)$ follows from the Monotone Convergence Theorem. The space $c_0$ is an order continuous Banach lattice which is not a *KB*-space. In fact, see [5, Theorem 4.60], a Banach lattice $X$ is a *KB*-space if and only if $c_0$ is not lattice embeddable in $X$.

A Banach lattice $(X, \|\cdot\|)$ is said to be an *AM*-space (*AL*-space) if the norm $\|\cdot\| : X \to \mathbb{R}$ is an $M$-norm (a $p$-additive norm for some $1 \le p < \infty$, respectively) on $X$. Every *AL*-space is, in particular, a *KB*-space.

Let $X$ be a vector lattice. An element $0 \ne e \in X^+$ is called a *strong unit* of $X$ if the principal ideal $I_e$ generated by $e$ satisfies $I_e = X$. Equivalently, the element $e$ is a strong unit of $X$ if for every $x \ge 0$ there exists $n \in \mathbb{N}$ such that $x \le ne$. An element $0 \ne e \in X^+$ is called *weak unit* if the band $B_e$ generated by $e$ satisfies $B_e = X$. It follows that an element $e$ is a weak unit if for every $x \ge 0$ one has $x \wedge ne \uparrow x$. We further recall that a positive element $e > 0$ of a normed lattice $X$ is said to be a *quasi-interior point* if $x \wedge ne \to 0$ in norm for all $x \ge 0$. By [5, Theorem 4.85], a positive element $e > 0$ is a quasi-interior point if and only if the principal ideal generated by $e$ is norm dense in $X$.

An element $a > 0$ in a vector lattice is called *atom* if for every $x$ belonging to the order interval $[0, a]$ there exists some $\lambda \in \mathbb{R}$ with $\lambda \ge 0$ such that $x = \lambda a$. This is equivalent to saying that the principal ideal $I_a$ generated by the element $a$ is one-dimensional simply because of the fact that the ideal $I_a$ is a subset of the ideal generated by $[0, a]$. It follows that the band $B_a$ generated by the atom $a$ is a projection band and that $B_a = I_a$. A vector lattice $X$ is called *atomic* if the band generated by its atoms is equal to $X$. Concrete examples of atomic Banach lattices include $c_0, c$ and $\ell_p$ for $1 \leq p \leq \infty$. The space $L^p[0,1]$ $(1 \leq p < \infty)$ is not an atomic Banach lattice.

## 2.2 Unbounded Convergences in Vector Lattices

In this section, we discuss various types of convergences. Some types of convergences result from the order structure of the underlying vector lattice. For example, the order and the relatively uniform convergences are of this type. Unbounded convergences derived from these types of convergences form the main topic of the present section.

In addition, as in the case of normed lattices (and more generally, in the case of locally convex-solid vector lattices), there exists a topological convergence induced by a linear topology. In fact, most of the spaces we are interested in posses a topological convergence. Such topological convergences were studied by many (see [46] for more on this issue), and, more information them can be found in [4, 5, 36, 57, 71]. In the present section, unbounded convergences resulting from these topological convergences are also discussed.

### 2.2.1 Order Convergence in Vector Lattices

Vector lattices have the common property that one can introduce the notion of order convergence on them. We recall from Section 2.1 that for a net $x_\alpha$ in a vector lattice $X$, we write $x_\alpha \xrightarrow{o} x$, if there is a net $y_\beta$, possibly over a different index set, such that $y_\beta \downarrow 0$ and for every $\beta$ there exists $\alpha_\beta$ satisfying $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$.

Historically speaking, various researchers have used different definitions for the order convergence. We refer reader to [3] for a detailed discussion on this issue.

**Example 2.2.1.** *Let $X$ be the Banach lattice $C[0,1]$ of continuous real valued functions on the closed unit interval $[0,1]$ with the uniform norm. Let $f_n(t) = t^n$ for $n \geq 1$ and $t \in [0,1]$. Then the sequence $f_n$ converges in order to zero even though $\|f_n\| = 1$ for every $n$.*

**Example 2.2.2.** *The proof of this example can be found in $[57, pg.9]$. Let $X$ be the Banach lattice $L^p(\mu)$ $(1 \leq p < \infty)$ of $p$-integrable functions on a measure space $(\Omega, \Sigma, \mu)$. A sequence $f_n$ in $L^p(\mu)$ is order convergent to $f \in L^p(\mu)$ if and only if there exists some $g \in L^p(\mu)^+$ such that $|f_n| \leq g$ $\mu$-almost everywhere for every $n$ and $f_n(t) \to f(t)$ for almost every $t$.*

**Example 2.2.3.** *Let $e_n$ denote the standard basis of the Banach lattice $c$ of all convergent real sequences with the uniform norm. It follows that $e_n \xrightarrow{o} 0$ but $\|e_n\| = 1$ for every $n$.*

**Remark 2.2.4.** *(Star of a convergence) A sequence $x_n$ in a vector lattice $X$ is said to order $\star$-converge to some $x \in X$ if every subsequence $x_{n_i}$ of $x_n$ contains a further subsequence*

$x_{n_{i_k}}$ *such that* $x_{n_{i_k}} \xrightarrow{o} x$. *It follows this definition that if* $x_n \xrightarrow{o} x$ *then the sequence* $x_n$ *order* $\star$*-converges to* $x$. *In general, the converse of this statement is false. Consider a* $\sigma$*-finite measure space* $(\Omega, \Sigma, \mu)$. *A sequence* $f_n$ *of measurable functions is order* $\star$*-convergent to some measurable function* $f$ *if and only if* $f_n \to f$ *in measure on every subset of finite measure, see [57, 2.6.E2] and the discussion given in [57, pg.186].*

**Remark 2.2.5.** *Although the notion of relatively uniform convergence is not needed in the sequel, we recall its definition for the sake of completeness, see [54, 71]. A net* $x_\alpha$ *in a vector lattice* $X$ *is said to relatively uniformly converge to* $x \in X$ *if there exists an element* $u \in X^+$ *such that for every* $\varepsilon > 0$ *there exists some index* $\alpha_0$ *such that* $|x_\alpha - x| \leq \varepsilon u$ *for each* $\alpha \geq \alpha_0$. *In this case, we say that the net* $x_\alpha$ *is r-convergent to* $x$, *and we write* $x_\alpha \xrightarrow{r} x$. *Relatively uniform convergence on a Banach lattice satisfies the property that every norm convergent sequence in the Banach lattice has a subsequence which is both relatively uniformly and order convergent to the norm limit of the sequence, see [71, Theorem VII.2.1]. Relatively uniform convergence is used in connection with unbounded convergences, see [7, 14, 16].*

### 2.2.2 Unbounded Order Convergence in Vector Lattices

A net $x_\alpha$ in a vector lattice $X$ is said to be *unbounded order convergent* to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in X^+$. In this case, we write $x_\alpha \xrightarrow{uo} x$, and, we say that the net $x_\alpha$ *uo*-converges to $x$.

Sequential version of unbounded order convergence was first defined on $\sigma$-Dedekind complete vector lattices by H. Nakano in 1948, see [58], under the name *individual convergence*. In [58], Nakano used individual convergence to extend individual ergodic theorem to the settings of semi-ordered linear spaces of Kantorovitch, see [58, Theorem 3.4]. The name "*uo*-convergence" was proposed initially in [17]. Several papers, see [7, 6, 16, 37, 40, 41, 72], about the *uo*-convergence and its applications in Banach lattices have been announced since then.

**Example 2.2.6.** *The uo-convergence can be regarded as an abstraction of almost everywhere convergence in* $L_p$*-spaces for* $1 \leq p < \infty$, *[40, 41]. For a measure space* $(\Omega, \Sigma, \mu)$ *and for a sequence* $f_n$ *in* $L_p(\mu)$ $(0 \leq p \leq \infty)$ *we have* $f_n \xrightarrow{uo} 0$ *if and only if* $f_n \to 0$ *almost everywhere, see [40, Remark 3.4].*

**Example 2.2.7.** *In* $c_0$ *and* $\ell_p$ $(1 \leq p \leq \infty)$ *a net* $x_\alpha$ *is uo-convergent if and only if it is pointwise convergent.*

**Example 2.2.8.** *Denote by* $e_n$ *the standard unit vectors in* $\ell_\infty$. *It follows that* $e_n \xrightarrow{o} 0$ *and* $e_n \xrightarrow{uo} 0$. *Although* $e_n \xrightarrow{uo} 0$ *in* $c_0$, *the sequence* $e_n$ *does not converge in order to zero in* $c_0$. *Therefore, a sequence which is convergent with respect to unbounded order convergence is not necessarily order convergent.*

**Example 2.2.9.** *Let $\mathbb{R}^A$ be the vector lattice, with respect to the classical pointwise order, of all real valued functions on a nonempty set $A$. A net $x_\alpha$ in $\mathbb{R}^A$ uo-converges to $x \in \mathbb{R}^A$ if and only if it converges pointwise to $x$.*

**Remark 2.2.10.** *Having various types of convergences rises the natural question about the topological and order theoretical conditions on the subset $\{x_\alpha \colon \alpha \in \Lambda\}$ formed by the elements of the net such that two such convergences mutually agree when they are restricted onto this subset. This type of discussion usually results in a characterization theorem. In this direction, see also Remark 2.2.4 and [70]. An example of this approach is presented in [72]. In that article, Wickstead starts with the observation that for any norm bounded net in the Banach lattices $c$ and $\ell_p$ ($1 < p < \infty$), weak convergence and unbounded order convergence agree. He then deduces characterizations of spaces in which weak convergence and unbounded order convergence are equivalent. Since then several theorems of this type have appeared in the literature, for example see [41, Theorem 4.7], [79, Theorem 10], and [37, Theorem 3.4].*

Almost everywhere convergence, when it is considered on the vector lattice formed by measurable functions, is not necessarily topological. The case of the vector lattice of bounded measurable real valued functions on the unit interval $[0,1]$ can be found in [60]. In view of Example 2.2.6, unbounded order convergence on a vector lattice $X$ is not necessarily topological in the sense that there might be no topology on $X$ whose topologically convergent nets agree with the *uo*-convergent nets. It is shown in [14, Theorem 1] that order convergence in infinite dimensional vector lattices is not topological.

Subsequent sections of the present chapter cover the other connections between *uo*-convergence and an arbitrary convergence. Let us focus on the weak convergence. When $X$ is a Banach lattice it is not always true that weakly null nets are *uo*-null. Following result of [72] provides a converse for this situation.

**Theorem 2.2.11.** *[72, Theorem 1] In a Banach lattice $X$ the following statements are equivalent.*

i. *If a net $x_\alpha$ in $X$ converges weakly to zero then it uo-converges to zero.*

ii. *Every order bounded net in $X$ which converges weakly to zero must order converge to zero.*

iii. *Linear span of the minimal ideals in $X$ is order dense in $X$.*

When $X$ is a Banach lattice it is not always true that *uo*-null nets are weakly null. Following result of [72] provides a converse for this situation.

**Theorem 2.2.12.** *[72, Theorem 5] In a Banach lattice $X$ the following statements are equivalent.*

i. *Every norm bounded net in $X$ which uo-converges to* 0 *converges weakly to* 0.

ii. *The space $X$ has an order continuous norm and every norm bounded disjoint sequence in $X$ converges weakly to* 0.

iii. *The spaces $X$ and $X'$, the topological dual of $X$, have order continuous norms.*

A net $x_\alpha$ in a vector lattice $X$ is said to be *uo-Cauchy* if the double net $(x_\alpha - x_\beta)_{(\alpha,\beta)}$ is *uo*-convergent to zero in $X$, see [41, Section 4]. The *uo*-Cauchy nets in $X$ are the analogues of order Cauchy nets, see [75, pg. 696]. One of the relations between unbounded order convergence and the universal completions was established in [8, Theorem 17]. In details, Y. Azouzi proved that a vector lattice $X$ is universally complete if and only if $X$ is *uo*-complete.

We further remark the following result of [65] which establishes a connection between increasing norm bounded nets and *uo*-Cauchy nets.

**Proposition 2.2.13.** *[65, Proposition 2.3] Let $X$ be a weakly Fatou Banach lattice. Then every positive increasing norm bounded net in $X$ is uo-Cauchy.*

### 2.2.3 Unbounded Norm Convergence in Normed Lattices

Let $x_\alpha$ be a net in a normed lattice $(X, \|\cdot\|)$. We write $x_\alpha \xrightarrow{\|\cdot\|} x$ if the net $x_\alpha$ converges to $x$ in norm. We say that the net $x_\alpha$ *unbounded norm converges* to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for every $u \in X^+$. In this case, we write $x_\alpha \xrightarrow{un} x$.

Unbounded norm convergence was introduced by Troitsky in [67] under the name *d*-convergence, and, further considered in [18, 44, 45]. By [18, Proposition 2.5], we have $x_\alpha \xrightarrow{un} x$ whenever $x_\alpha \xrightarrow{uo} x$ for a net $x_\alpha$ in an order continuous Banach lattice. Therefore, in the order continuous case, *uo*-convergent nets are also *un*-convergent.

**Example 2.2.14.** *Denote by $e_n$ be the standard unit vectors in $c_0$. It follows that $e_n \xrightarrow{un} 0$ in $c_0$ because positive elements of $c_0$ are in particular sequences converging to zero. By Example 2.2.8, we have $e_n \xrightarrow{uo} 0$. This situation is different in the case of $\ell_\infty$. The sequence $e_n$ uo-converges to zero in $\ell_\infty$ but it does not un-converge to zero.*

Similar to the case of norm convergence, unbounded norm convergence on a vector lattice $X$ is topological in the sense that there is a linear topology on $X$ whose topologically convergent nets agree with *un*-convergent nets. It is given in [18, Section 7] that the collection of the sets of the form $V_{u,\varepsilon} := \{x \in X\colon \ \||x| \wedge u\| \leq \varepsilon\}$ with $\varepsilon > 0$ and $u \in X^+$, forms a basis for a Hausdorff linear topology on $X$. A more general approach is given in [16] and it can also be found in Section 2.2.5.

We now focus on restrictions of *un*-convergence onto sublattices of $X$.

**Proposition 2.2.15.** *[18, Lemma 2.11] Let $X$ be a normed lattice with a quasi-interior point $e$. Then for any net $x_\alpha$ in $X$, one has $x_\alpha \xrightarrow{un} 0$ if and only if $\||x_\alpha| \wedge e\| \to 0$.*

We recall from [5, Theorem 4.85], a positive element $e > 0$ is a quasi-interior point if and only if the principal ideal generated by $e$ is norm dense in $X$. Following result of [45] provides a convenient method to check the convergence of some nets in $X$ when the elements of the net belongs to a sublattice of $X$.

**Theorem 2.2.16.** *[45, Theorem 4.3] Let $Y$ be a sublattice of a normed lattice $X$. Also let $y_\alpha$ be a net in $Y$ such that $y_\alpha \xrightarrow{un} 0$ in $Y$. The following statements hold.*

i. *If $Y$ is majorizing in $X$ then $y_\alpha \xrightarrow{un} 0$ in $X$.*

ii. *If $Y$ is norm dense in $X$ then $y_\alpha \xrightarrow{un} 0$ in $X$.*

iii. *If $Y$ is a projection band in $X$ then $y_\alpha \xrightarrow{un} 0$ in $X$.*

We recall from Section 2.1 that a vector lattice is said to be a Dedekind completion of $X$ if $X$ is lattice isomorphic to a majorizing order dense sublattice of that vector lattice.

**Corollary 2.2.17.** *[45, Corollary 4.4] If $X$ is a normed lattice and $x_\alpha \xrightarrow{un} x$ in $X$ then $x_\alpha \xrightarrow{un} x$ in the Dedekind completion $X^\delta$ of $X$.*

Theorem 4.60 of [5] provides a characterization of Banach lattices which are bands in their second topological duals. According to this theorem, a Banach lattice is a *KB*-space if and only if it is a band in its second topological dual. Following result of [45] provides an additional corollary in this direction.

**Corollary 2.2.18.** *[45, Corollary 4.5] If $X$ is a KB-space and $x_\alpha \xrightarrow{un} 0$ in $X$ then $x_\alpha \xrightarrow{un} 0$ in the second topological dual $X''$.*

**Remark 2.2.19.** *Let $X$ be a normed lattice which is also an ideal of a vector lattice $Y$. In [44], the notion of un-convergence with respect to ideal $X$ is considered. In the settings of [44], a net $x_\alpha$ in $Y$ is said to un-converge to $x \in Y$ with respect to the ideal $X$ if $|x_\alpha - x| \wedge u \xrightarrow{un} 0$ for every $u \in X^+$. This approach is further generalized to settings of convergence vector lattices, see [13] and Section 2.2.6.*

**Remark 2.2.20.** *One can study the notion of unbounded norm convergence whenever there exists a lattice norm or seminorm, or a system of these, on the vector lattice $X$. It is known that a lattice norm $\|\cdot\| : X \to \mathbb{R}$ results in a lattice seminorm $\|\cdot\|_u : X \to \mathbb{R}$ by putting $\|x\|_u := \||x| \wedge u\|$ for $x \in X$ and $u \in X^+$. This approach is used in [30] to investigate the relations between unbounded convergences and locally solid vector lattices.*

### 2.2.4 Unbounded Absolute Weak Convergences in Banach Lattices

Unbounded absolutely weak convergence, abbreviated as *uaw*-convergence, is investigated in [79] in the settings of Banach lattices. This notion plays a fundamental role in the subsequent chapters of the present work.

Let us recall the absolute weak topology on locally convex-solid vector lattices. We denote by $(X,\tau)$ a locally convex-solid vector lattice. We let $X'$ be the topological dual of $X$. The absolute weak topology on $X$, denoted by $|\sigma|(X,X')$, is the locally convex-solid topology on $X$ generated by the family of lattice seminorms $\rho_f \colon X \to \mathbb{R}$ where $\rho_f(x) = |f(x)|$ for $x \in X$ and $f \in X'$. By [4, Theorem 2.36], if $(X,\tau)$ is a Hausdorff locally convex-solid vector lattice then the absolute weak topology agrees with the weak topology, i.e. $|\sigma|(X,X') = \sigma(X,X')$; and furthermore, the weak topology $\sigma(X,X')$ is itself a locally solid topology. In particular, in the case when $X$ is a Banach lattice, the absolute weak topology on $X$ agrees with the weak topology on $X$.

Let us consider the particular case when $X$ is a Banach lattice. A net $x_\alpha$ in $X$ is said to be *unbounded absolutely weakly convergent* to $x \in X$, see [79], if $|x_\alpha - x| \wedge u \xrightarrow{w} 0$ for every $u \in X^+$. In this case, we write $x_\alpha \xrightarrow{uaw} x$ and say that the net $x_\alpha$ *uaw*-converges to $x$.

**Example 2.2.21.** *Denote by $e_n$ the standard unit vectors in $c_0$. Let $x_n := n^2 e_n$ for all $n$. It follows that $x_n \xrightarrow{uaw} 0$ but it is not absolutely weakly null.*

**Example 2.2.22.** *Consider the sequence $f_n$ in $C([0,1])$ defined by $f_n(0) = 1,\ f_n(1/n) = f_n(1) = 0$ and linear in between. Then the sequence $f_n$ is uo-null but not uaw-null.*

Following result of [79] provides a condition that allow one to compare *un*-convergence and *uaw*-convergence.

**Theorem 2.2.23.** *[79, Theorem 4] In a Banach lattice X the following statements are equivalent.*

i. *X is order continuous.*

ii. *A net $x_\alpha$ in X is un-convergent to* 0 *if and only if it is uaw-convergent to* 0.

iii. *A sequence $x_n$ in X is un-convergent to* 0 *if and only if it is uaw-convergent to* 0.

When the topological dual of the vector lattice is order continuous, following result of [79] provides a condition for norm bounded nets that allow one to compare *un*-convergence and *uaw*-convergence.

**Theorem 2.2.24.** *[79, Theorem 4] In a Banach lattice X the following statements are equivalent.*

i. *The topological dual $X'$ is order continuous.*

ii. *Every norm bounded net $x_\alpha$ in $X$ which is uaw-convergent to* 0 *converges weakly to* 0.

iii. *Every norm bounded sequence $x_n$ in $E$ which is uaw-convergent to* 0 *converges weakly to* 0.

### 2.2.5 $u\tau$-Convergence in Locally Solid Vector Lattices

Given a locally solid vector lattice $(X,\tau)$ and a net $x_\alpha$ in $X$, we write $x_\alpha \xrightarrow{u\tau} x$, see [16], if $|x_\alpha - x| \wedge u \xrightarrow{\tau} 0$ for every $u \in X^+$.

The notion of $u\tau$-convergence in locally solid vector lattices was introduced in [16] to study further properties of unbounded convergences in the settings of locally solid vector lattices. It was proved in [16] that for any locally solid topology $\tau$ on $X$ there exists an induced, in the sense given below, locally solid topology $u\tau$ on $X$ such that $u\tau$-convergence of a net agrees with the topological convergence with respect to $u\tau$. Therefore, techniques represented in [18, Section 7] are further generalized to the settings of locally solid vector lattices.

In more details, if $\{U_i\}_{i\in I}$ is a base at zero for $\tau$ consisting of solid sets, then we put

$$U_{i,u} := \{x \in X\colon\ |x| \wedge u \in U_i\}$$

for each $i \in I$ and $u \in X^+$. The collection $\{U_{i,u}\colon\ i \in I, u \in X^+\}$ is a base of neighborhoods at zero of the new locally solid topology $u\tau$ on $X$. It follows that $u\tau$-convergence is topological and generalizes the notions of unbounded norm convergence and unbounded absolute weak convergence, see Section 2.2.4, in normed lattices.

**Remark 2.2.25.** *Although the notion of multi-normed vector lattice is not needed in the sequel, we recall its definition for the sake of completeness. A multi-normed vector lattice is a locally convex-solid vector lattice $(X,\tau)$ together with an upward directed family $\{m_\lambda\}$ of lattice seminorms generating $\tau$. Various properties of the corresponding um-topology is investigated in [15].*

### 2.2.6 Convergence Vector Lattices

Convergence vector lattices are introduced in [13] to give a more systematic and abstract approach to study further properties of unbounded convergences.

A *convergence* $\mathfrak{c}$ for nets in a set $X$ is defined by the following implications related to nets over $X$:

1. If $x_\alpha \equiv x$ then $x_\alpha \xrightarrow{\mathfrak{c}} x$,

2. If $x_\alpha \xrightarrow{\mathfrak{c}} x$ then $x_\beta \xrightarrow{\mathfrak{c}} x$ for every subnet $x_\beta$ of $x_\alpha$,

where the index set of the net in discussion can be arbitrary.

Following [13], by a *convergence vector lattice* $(X,\mathfrak{c})$ we mean a vector lattice $X$ together with a convergence $\mathfrak{c}$ such that linear and lattice operations on $X$ are continuous with respect to $\mathfrak{c}$ in the sense that if $x_\alpha \xrightarrow{\mathfrak{c}} x$, $y_\beta \xrightarrow{\mathfrak{c}} y$ in $X$ and $t_\gamma \to t$ in $\mathbb{R}$ then

1. $t_\gamma x_\alpha + y_\beta \xrightarrow{\mathfrak{c}} tx+y$,
2. $t_\gamma(x_\alpha \wedge y_\beta) \xrightarrow{\mathfrak{c}} t(x\wedge y)$.

Therefore, in a convergence vector lattice lattice operations are continuous in the sense of net characterization of continuity.

**Example 2.2.26.** *If $X$ is a Banach lattice then $(X,\xrightarrow{o})$, $(X,\xrightarrow{uo})$, $(X,\xrightarrow{un})$, $(X,\xrightarrow{uaw})$ are examples of convergence vector lattices. Further examples of convergence vector lattices are given in [13].*

The following definition is given in [13], see also [44] and Remark 2.2.19 for the case of *un*-convergence. It provides a systematic way to deduce a new convergence from a convergence via an ideal of the original vector lattice.

**Definition 2.2.27.** Let $I$ be an ideal in a convergence vector lattice $(X,\mathfrak{c})$. A net $x_\alpha$ in $X$ is said to $u_I\mathfrak{c}$-converge to $x$ if $|x_\alpha - x| \wedge u \xrightarrow{\mathfrak{c}} 0$ for every $u \in I^+$. The convergence $u_I\mathfrak{c}$ is called unbounded $\mathfrak{c}$-convergence with respect to the ideal $I$.

**Example 2.2.28.** *It is given Example 2.1.1 that $c_0$ is an ideal in $\ell_\infty$. Therefore, a net $x_\alpha$ in $\ell_\infty$ satisfies $x_\alpha \xrightarrow{u_I o} x$ with $I = c_0$ if and only if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ in $\ell_\infty$ for every $u \in (c_0)^+$. Similarly, $x_\alpha \xrightarrow{u_I n} x$ with $I = c_0$ if and only if $|x_\alpha - x| \wedge u \xrightarrow{n} 0$ in $\ell_\infty$ for every $u \in (c_0)^+$ This last convergence is studied in [44, Example 2.3]. It was shown that $x_\alpha \xrightarrow{u_I n} 0$ if and only if the net $x_\alpha$ coordinate-wise converges to zero.*

Let $X$ be a normed lattice which is also an ideal of a vector lattice $Y$. In the settings of Definition 2.2.27 if we take $\mathfrak{c} = un$ and $I = X$ then we get the convergence $u_I\mathfrak{c}$ on $Y$. It is proved in [44] that the convergence $u_I\mathfrak{c}$ on $Y$ is topological.

### 2.3 Conclusions

The material presented in this chapter provides a brief summary of the literature on unbounded convergences. After presenting the concepts of unbounded order, unbounded norm and unbounded absolute weak convergences, we list some of their properties and introduce some examples. Section 2.2.2 concerns unbounded order convergence, which plays a fundamental role in the theory of convergences on vector lattices. Section 2.2.3 contains material related to unbounded norm convergence. Section 2.2.4 provides a basis for Chapter 3 of the present thesis. Similarly, Section 2.2.5 and Section 2.2.6 provide bases for Chapter 4 and Chapter 5, respectively.

# 3 *uaw*-DUNFORD-PETTIS AND *uaw*-COMPACT OPERATORS

In the present chapter, we expose the relations between unbounded absolutely weak Dunford-Pettis operators (abbreviated as *uaw*-Dunford-Pettis operators), unbounded absolutely weak compact operators (abbreviated as *uaw*-compact operators), $M$-weakly compact operators, $L$-weakly compact operators, and order weakly compact operators. Several properties of *uaw*-compact and *uaw*-Dunford-Pettis operators are studied. Moreover, we utilize some conditions on domain or range of operators to ensure us when the adjoint or the modulus of a *uaw*-compact or *uaw*-Dunford-Pettis operator has the same property. As one of the main consequences, we deduce that the square of a positive *uaw*-Dunford-Pettis ($M$-weakly compact) operator on an order continuous Banach lattice is compact. In addition, various examples are given to make the concepts and hypotheses more understandable.

Most of the results presented here can be found in the article [33] written by N. Erkurşun Özcan, N.A. Gezer, and O. Zabeti. General theory of Dunford-Pettis operators can be found in [5, 57, 63]. For the sake of completeness, a short introduction is also given in Section 3.1.

## 3.1 Some Basic Concepts on Classical Compact and Dunford-Pettis Operators

Following conceptional notions are needed in the sequel. We recall that an operator $T\colon X \to Y$ between Banach spaces is *compact* if the image $T(B_X)$ of the closed unit ball $B_X$ of $X$ has compact closure in $Y$. An operator $T\colon X \to Y$ is said to be *weakly compact* if $T(B_X)$ has a weakly compact closure in $Y$, i.e., $T$ maps norm bounded sequences into sequences having a weakly convergent subsequence.

A subset of a Banach space is relatively weakly compact if and only if it is relatively weakly sequentially compact. Generalizations of the sequential characterization of relatively (weakly) compact subsets of Banach spaces will be used in the sequel to relate compact operators with various types of convergences. This technique is also used in [7] in the settings of lattice normed vector lattices to derive results on compact-like operators. It is known that (weakly) compact, precompact and even bounded subsets, see Remark 4.2.1, of a Banach space admit such sequential characterizations, see [29, 57, 75].

If an operator $T\colon X \to Y$ is (weakly) compact then it is norm bounded. Moreover, if an operator $T\colon X \to Y$ is (weakly) compact then so is its adjoint $T'\colon Y' \to X'$, see [5, Theorem 5.23]. If $T_n\colon X \to Y$ is a sequence of compact operators (or weakly compact operators) and $\|T_n - T\| \to 0$ for some $T\colon X \to Y$ then the operator $T$ is compact (or weakly compact, respectively). If either $X$ or $Y$ is reflexive, then every operator from $X$ to $Y$ is weakly compact.

A Banach space $X$ is said to have *the Dunford-Pettis property*, see [5, Theorem 5.82] for the Brace-Grothendieck Theorem, if every weakly compact operator $T\colon X \to Y$ maps weakly compact subsets of $X$ into norm compact subsets of $Y$ where $Y$ is an arbitrary Banach space.

It follows from this definition that if two operators $T\colon X \to Y$ and $S\colon Y \to Z$ are weakly compact and the space $Y$ has the Dunford-Pettis property then the operator $ST$ is compact.

An operator $T\colon X \to Y$ is said to be *Dunford-Pettis* if $x_n \xrightarrow{w} 0$ in $X$ implies $\|Tx_n\| \to 0$. Every compact operator is Dunford-Pettis. Moreover, if $X$ has the Dunford-Pettis property then every weakly compact operator $T\colon X \to Y$ into an arbitrary Banach space $Y$ is Dunford-Pettis.

## 3.2 The Class of *uaw*-Dunford-Pettis Operators

Suppose that $E$ is a Banach lattice and that $X$ is a Banach space.

**Definition 3.2.1.** We say that an operator $T\colon E \to X$ is a *unbounded absolutely weak Dunford-Pettis operator*, abbreviated as *uaw-Dunford-Pettis*, if for every norm bounded sequence $x_n$ in $E$, $x_n \xrightarrow{uaw} 0$ implies $\|Tx_n\| \to 0$.

Evidently, *uaw*-Dunford-Pettis operators are the analogues of Dunford-Pettis operators whose definition can be found in Section 3.1. However, we shall see shortly that not all properties of Dunford-Pettis operators are satisfied by *uaw*-Dunford-Pettis operators.

We remark that *uaw*-Dunford-Pettis operators are continuous with respect to topology induced by the norm but the converse is not true. Indeed, if $T\colon E \to X$ is an *uaw*-Dunford-Pettis operator and $x_n$ is a norm null sequence in $E$ then the sequence $x_n$ is norm bounded and it satisfies $x_n \xrightarrow{uaw} 0$. It follows that $\|Tx_n\| \to 0$. For the converse, we note that the identity operator on $\ell_1$ is not *uaw*-Dunford-Pettis.

**Remark 3.2.2.** *In general, a uaw-null sequence may not be norm bounded. However, we recall that a weakly null sequence in Banach lattice should be norm bounded. An example of a uaw-null sequence which is not norm bounded can be found in [79].*

We denote by $B_{UDP}(E)$ the space of all *uaw*-Dunford-Pettis operators $T\colon E \to E$ on a Banach lattice $E$.

**Proposition 3.2.3.** *Suppose that $E$ is a Banach lattice whose dual space is order continuous and $X$ is a Banach space. In this case, every Dunford-Pettis operator $T\colon E \to X$ is uaw-Dunford-Pettis.*

*Proof.* Suppose $T$ is Dunford-Pettis and $x_n$ is a norm bounded sequence in $E$ which is *uaw*-convergent to zero. By [79, Theorem 7], the sequence $x_n$ is weakly convergent. Since $T\colon E \to X$ is Dunford-Pettis, we have $\|Tx_n\| \to 0$. This shows that $T\colon E \to X$ is *uaw*-Dunford-Pettis. □

We note that order continuity of $E'$ is essential in Proposition 3.2.3, and, it cannot be dropped. To see this, consider the identity operator $I$ on $\ell_1$. It follows from the Schur property of $\ell_1$ that the operator $I$ is Dunford-Pettis. However it can not be *uaw*-Dunford-Pettis as the *uaw*-null sequence $(e_n)_n$ formed by the standard basis of $\ell_1$ is not norm convergent to zero.

**Remark 3.2.4.** *Suppose that E is an AM-space and X is a Banach space. Since the lattice operations in E are weakly sequentially continuous [5, Theorem 4.31], and in view of Proposition 3.2.3, it can be seen that an operator $T : E \to X$ is uaw-Dunford-Pettis if and only if it is Dunford-Pettis. Indeed, if $x_n$ is a weakly null sequence in the Banach lattice E then $x_n$ is norm bounded and $x_n \xrightarrow{uaw} 0$, and hence uaw-Dunford-Pettis operators are Dunford-Pettis. Suppose further that E is an atomic order continuous Banach lattice. It follows from [57, Proposition 2.5.23] that if an operator $T : E \to X$ is uaw-Dunford-Pettis, then it is a Dunford-Pettis operator. We recall that in the Banach lattice $L_p[0,1]$ $(1 \leq p < \infty)$ lattice operations are not weakly sequentially continuous, see the example given after [57, Proposition 2.5.23].*

We now show that the classes of compact and noncompact operators differ from the class of *uaw*-Dunford-Pettis operators. Following example shows that in the case of *uaw*-Dunford-Pettis operators, the situation is different.

**Example 3.2.5.** *Let $T : \ell_1 \to \mathbb{R}$ be the compact operator defined by $T((x_n)) = \sum_{n=1}^{\infty} x_n$ for every $(x_n) \in \ell_1$. It follows by considering the standard basis of $\ell_1$ that T is not a uaw-Dunford-Pettis operator. Indeed, although the standard basis is uaw-null its image under T is not norm null. On the other hand, we recall that classical fact that every compact operator is Dunford-Pettis.*

A typical example of a Dunford-Pettis operator which is not compact is the identity operator on $\ell_1$ because of the Schur property. However this operator does not do the job for the *uaw*-case since it is not also *uaw*-Dunford-Pettis. Nevertheless, there is a good news if one considers a version of the Lozanovsky's example as it is described in [5, Page 289, Exercise 10].

**Example 3.2.6.** *Consider the operator $T : C[0,1] \to c_0$ given by*

$$T(f) = (\int_0^1 f(t)\sin t\,dt, \int_0^1 f(t)\sin 2t\,dt, \ldots)$$

*for every $f \in C[0,1]$. It follows that that T is not order bounded. Hence, by [5, Theorem 5.7], T is not compact. Denote by $(f_n) \subseteq C[0,1]$ a norm bounded sequence for which $f_n \xrightarrow{uaw} 0$ holds. It follows from [79, Theorem 7] that $f_n \xrightarrow{w} 0$ and that*

$$\|T(f_n)\| = \sup_{m \geq 1} |\int_0^1 f_n(t)\sin mt\,dt| \leq \int_0^1 |f_n(t)|dt \to 0.$$

*Hence, the noncompact operator $T$ is a uaw-Dunford-Pettis operator. We further remark that the operator $T$ is also used in [9, Example 3] and in [3, Remark 2.2] where in the former article the authors show that $T$ is boundedly unbounded $\sigma$-continuous, see [9] for more details.*

It follows that post and pre-compositions of finitely many *uaw*-Dunford-Pettis operators is again a *uaw*-Dunford-Pettis operator.

**Proposition 3.2.7.** *Suppose that $E$ is a Banach lattice. Then $B_{UDP}(E)$ is a subalgebra of the algebra $B(E)$ of continuous operators on $E$.*

*Proof.* If $T$ and $S$ are two *uaw*-Dunford-Pettis operators and $x_n$ is a norm bounded sequence satisfying $x_n \xrightarrow{uaw} 0$ then $\|TS(x_n)\| \to 0$ and $\|(T+S)x_n\| \to 0$. $\square$

### 3.3 Relationships with $L$-and $M$-Weakly Compact Operators

We recall that, see [5, Definition 5.59] for details, an operator $T\colon E \to F$ between Banach lattices $E$ and $F$ is said to be $M$-weakly compact if for every norm bounded disjoint sequence $x_n$ in $E$ one has $\|Tx_n\| \to 0$. An operator $T\colon E \to F$ is said to be $L$-weakly compact if every disjoint sequence $y_n$ in the solid hull of $T(B_E)$ is norm null, where $B_E$ denotes the closed unit ball of $E$.

**Proposition 3.3.1.** *If $T\colon E \to F$ is a uaw-Dunford-Pettis operator then $T$ is $M$-weakly compact. In particular, $T\colon E \to F$ is weakly compact.*

*Proof.* If $x_n$ is a norm bounded disjoint sequence in $E$, by [79, Lemma 2], $x_n \xrightarrow{uaw} 0$. Hence, $\|Tx_n\| \to 0$. This shows that $T\colon E \to F$ is $M$-weakly compact. By [57, Proposition 3.6.12], the operator $T\colon E \to F$ is weakly compact. $\square$

We remark that a Dunford-Pettis operator need not be $L$- or $M$-weakly compact, see [5, Page 322]. For the converse of Proposition 3.3.1, we have the following result.

**Theorem 3.3.2.** *Suppose $E$ and $F$ are Banach lattices such that either $E$ or $F$ is order continuous. Then every positive $M$-weakly compact operator $T\colon E \to F$ from $E$ into $F$ is uaw-Dunford-Pettis.*

*Proof.* Suppose $x_n$ is a bounded positive *uaw*-null sequence in $E$. Let $\varepsilon > 0$ be arbitrary. By [5, Theorem 5.60] due to Meyer-Nieberg, there is a positive $u \in E$ with $\|T(x_n) - T(x_n \wedge u)\| < \frac{\varepsilon}{2}$. Indeed, since $x_n$ is positive, it follows from [5, Theorem 1.7] that, $(|x_n| - u)^+ = x_n - x_n \wedge u$ for each $n$. First, suppose $E$ is order continuous. Since $x_n \wedge u \xrightarrow{w} 0$ and the sequence is order bounded, by [5, Theorem 4.17], we conclude that $\|x_n \wedge u\| \to 0$ so that $\|T(x_n \wedge u)\| \to 0$. This shows that when $E$ is order continuous, every positive $M$-weakly compact operator

$T: E \to F$ is *uaw*-Dunford-Pettis. For the second part, suppose $F$ is order continuous. Hence $x_n \wedge u \xrightarrow{w} 0$ results in $T(x_n \wedge u) \xrightarrow{w} 0$. We note that this sequence is order bounded so that by [5, Theorem 4.17] we further have $\|T(x_n \wedge u)\| \to 0$. We see that $\|Tx_n\| < \varepsilon$ for sufficiently large $n$, as claimed. □

**Corollary 3.3.3.** *Suppose either $E$ or $F$ is order continuous. Then every L-weakly compact lattice homomorphism from $E$ to $F$ is uaw-Dunford-Pettis.*

*Proof.* We note that a lattice homomorphism $T: E \to F$ is necessarily a positive operator. It follows that $T$ is $M$-weakly compact (for example see [5, Page 337, Exercise 4]). Conclusion follows from Theorem 3.3.2. □

**Remark 3.3.4.** *Suppose $E$ and $F$ are Banach lattices. An operator $T: E \to F$ is said to be uaw-continuous if it maps bounded uaw-null sequences to uaw-null sequences. It can be verified that every uaw-Dunford-Pettis operator is uaw-continuous but the converse is not true. The identity operator on $\ell_1$ is uaw-continuous but not uaw-Dunford-Pettis.*

In view of Remark 3.3.4, one can ask under which conditions a *uaw*-continuous operator is also *uaw*-Dunford-Pettis. We note that $L$-weakly compact operators are fruitful tools because of the following result.

**Theorem 3.3.5.** *Suppose $E$ is a Banach lattice and $F$ is an order continuous Banach lattice. Then every L-weakly compact uaw-continuous operator $T: E \to F$ from $E$ into $F$ is uaw-Dunford-Pettis.*

*Proof.* Suppose $x_n$ is a bounded positive *uaw*-null sequence in $E$. Let $\varepsilon > 0$ be arbitrary. By [5, Theorem 5.60], there is a positive $u \in F$ with $\||T(x_n)| - |T(x_n)| \wedge u\| < \frac{\varepsilon}{2}$. Indeed, it follows from [5, Theorem 1.7] that $(|Tx_n| - u)^+ = |Tx_n| - |Tx_n| \wedge u$ for each $n$. Since $Tx_n \xrightarrow{uaw} 0$, we see that $|Tx_n| \wedge u \xrightarrow{w} 0$. We note that this sequence is order bounded so that by [5, Theorem 4.17] we further have $\||Tx_n| \wedge u\| \to 0$. Therefore, $\|Tx_n\| < \varepsilon$ for sufficiently large $n$. This shows that $T: E \to F$ is *uaw*-Dunford-Pettis. □

In the following example, we show that adjoint of a *uaw*-Dunford-Pettis operator need not be *uaw*-Dunford-Pettis. We note that a similar statement holds in the case of Dunford-Pettis operators. In details, the adjoint of a Dunford-Pettis operator need not be Dunford-Pettis. Indeed, the identity operator on $\ell_1$ is Dunford-Pettis but its adjoint is not Dunford-Pettis, see [5, Section 5.4] for more details.

**Example 3.3.6.** *Consider the operator $T$ given in Example 3.2.6. We claim that the adjoint of $T$ is not uaw-Dunford-Pettis. The adjoint $T': \ell_1 \to M[0,1]$ is defined via*

$$T'(x_n)(f) = \sum_{n=1}^{\infty} x_n \left(\int_0^1 f(t) \sin nt dt\right),$$

*where $M[0,1]$ is the space of all regular Borel measures on $[0,1]$. We note that the standard basis $(e_n)_n$ of $\ell_1$ is uaw-null. For each $n \in \mathbb{N}$, put $f_n(t) = \sin nt$. Hence, it follows from*

$$\|T'(e_n)\| \geq \|T'(e_n)(f_n)\| = \int_0^1 (\sin nt)^2 dt \nrightarrow 0$$

*that the operator $T'$ is not uaw-Dunford-Pettis.*

**Remark 3.3.7.** *Observe that Example 3.3.6 can be employed to show that positivity assumption in Theorem 3.3.2 and uaw-continuity hypothesis in Theorem 3.3.5 are essential and cannot be removed. The operator $T'$ of Example 3.3.6 is not positive. Since $T$ is uaw-Dunford-Pettis, it is M-weakly-compact. By [5, Theorem 5.67], $T'$ is also M-weakly compact. However as we see from Example 3.3.6, it is not uaw-Dunford-Pettis. Furthermore, [5, Theorem 5.67] convinces us that $T'$ is also L-weakly compact. We claim that $T'$ is not uaw-continuous. We note that $e_n \xrightarrow{uaw} 0$. For every $n \in \mathbb{N}$, consider $f_n(t) = \sin nt$. Also, since the sequence $(\sin n)_n$ is dense in $[-1,1]$, we can choose sufficiently large $n \in \mathbb{N}$ with $\sin n > \frac{1}{4}$. Suppose $\delta_1$ is the Dirac measure at point $x_0 = 1$. Then, $(T'(e_n) \wedge \delta_1)(\sin nt) > \frac{1}{4}$.*

### 3.4 Domination and Iteration of *uaw*-Dunford-Pettis Operators

We recall that an operator $T: E \to X$ from a Banach lattice $E$ into a Banach space $X$ is $o$-weakly compact if $T$ maps order intervals of $E$ into relatively weakly compact subsets of $X$, see [57, Section 3.4]. Compatible with [5, Theorem 5.91 and Corollary 5.92] and [79, Lemma 2], we have the following result.

**Proposition 3.4.1.** *Every uaw-Dunford-Pettis operator $T: E \to X$ from a Banach lattice $E$ into a Banach space $X$ is o-weakly compact.*

*Proof.* If $x_n$ is an order bounded disjoint sequence in $E$ then $x_n$ is norm bounded. Hence, we have $x_n \xrightarrow{uaw} 0$ for such sequence. It follows from $\|Tx_n\| \to 0$ and [5, Theorem 5.57] that $T$ is $o$-weakly compact. □

In Section 3.3 we give various examples of *uaw*-Dunford-Pettis operators. In view of Proposition 3.4.1, they are also $o$-weakly compact.

**Proposition 3.4.2.** *Square of a uaw-Dunford-Pettis operator carries order intervals into norm totally bounded sets.*

**Remark 3.4.3.** *We briefly recall the general domination problem, see [1, 2, 5]. Consider two operators $T, S\colon X \to Y$ between vector lattices $X$ and $Y$ such that $T$ is positive. We say that the operator $S$ is dominated by $T$ if $|S(x)| \leq T(|x|)$ for every $x \in X$. Let us restrict ourselves to the case where the operator $T$ satisfies a certain property. It is then natural to ask what further can be deduced about the operators dominated by $T$. This general question is called* *the domination problem. Because of its relations with the order structure, this problem arises in the theory of operators on vector and Banach lattices. In this direction, Dodds-Fremlin theorem, see [5, Theorem 5.20], says that if $T\colon X \to Y$ is a positive compact operator where $X$ and $Y$ are Banach lattices such that $X'$ and $Y$ are order continuous, then every operator $S\colon X \to Y$ dominated by $T$ is compact.*

Now, we have the following.

**Theorem 3.4.4.** *Suppose $E$ is a Banach lattice, and, $T$ is a positive uaw-Dunford-Pettis operator on $E$. Let $S$ be a positive operator on $E$ dominated by $T^2$. Then, the operator $S^2$ is compact.*

*Proof.* By Proposition 3.3.1 and Proposition 3.4.1, the operator $T$ is both $o$-weakly compact and $M$-weakly compact. Moreover, by Proposition 3.4.2, the operator $T^2$ maps order intervals into norm totally bounded sets. Conclusion follows from [5, Page 338, Exercise 13]. $\square$

Observe that since the identity operator on $\ell_1$ is Dunford-Pettis, we can not expect compactness of any power of $T$. However, the following result is surprising.

**Corollary 3.4.5.** *Suppose $E$ is a Banach lattice. For every positive uaw-Dunford-Pettis operator $T$ on $E$, the operator $T^4$ is compact.*

*Proof.* The positive operator $T^2$ is dominated by itself. It follows from Theorem 3.4.4 that $T^4$ is compact. $\square$

**Corollary 3.4.6.** *Suppose $E$ is a Banach lattice. The identity operator on $E$ is uaw-Dunford-Pettis if and only if $E$ is finite dimensional.*

*Proof.* Suppose that the identity operator on $E$ is *uaw*-Dunford-Pettis. By Corollary 3.4.5, it is compact. This yields that $E$ is finite dimensional. Suppose $E$ is finite dimensional. Hence, $E$ is atomic and reflexive. Therefore, every *uaw*-null sequence in $E$ is weakly null so that norm null. This means that the identity operator on $E$ is *uaw*-Dunford-Pettis. $\square$

**Proposition 3.4.7.** *Suppose $E$ is an order continuous Banach lattice. Let $T$ be a positive uaw-Dunford-Pettis operator on $E$. If an operator $S$ satisfies $0 \leq S \leq T$, then the operator $S^2$ is compact. In particular, square of a positive uaw-Dunford-Pettis operator is compact.*

*Proof.* By Proposition 3.4.1, $T$ is $o$-weakly compact. This means that the order bounded set $T[0,x]$ is relatively weakly compact in $E$. By [5, Theorem 4.17], the set $T[0,x]$ is relatively compact in $E$. By using [5, Page 338, Exercise 13], we conclude that if a positive operator $S$ dominated by $T$, then the square of $S$ is a compact operator. $\square$

Furthermore, in view of Theorem 3.3.2, we get the following important result.

**Corollary 3.4.8.** *Square of a positive $M$-weakly compact operator on an order continuous Banach lattice $E$ is compact.*

**Remark 3.4.9.** *For the uaw-convergence, we have $x_\alpha \xrightarrow{uaw} x$ in a Banach lattice $E$ if and only if $|x_\alpha - x| \xrightarrow{uaw} 0$ in $E$, see [79, Lemma 1]. In some cases, this observation allows one to reduce null nets with respect to uaw-convergence to positive null nets with respect to uaw-convergence.*

In Example 3.3.6 we showed that adjoint of a *uaw*-Dunford-Pettis operator is not necessarily *uaw*-Dunford-Pettis. In the next example, we show that adjoint of a non *uaw*-Dunford-Pettis operator can be *uaw*-Dunford-Pettis.

**Example 3.4.10.** *Consider the operator $T\colon \ell_1 \to L^2[0,1]$ defined by $T(x_n) = (\sum_{i=1}^{\infty} x_n)\chi_{[0,1]}$ for all $x_n \in \ell_2$ where $\chi_{[0,1]}$ denotes the characteristic function of $[0,1]$. The operator $T$ is compact but it is not uaw-Dunford-Pettis. Its adjoint $T'\colon L^2[0,1] \to \ell_\infty$ is compact, and hence, it is Dunford-Pettis. By Proposition 3.2.3, we conclude that it is uaw-Dunford-Pettis.*

**Remark 3.4.11.** *One may verify that every positive operator which is dominated by a positive uaw-Dunford-Pettis operator is again uaw-Dunford-Pettis. Therefore, if $T$ is an operator whose modulus is uaw-Dunford-Pettis, it follows that $T$ is also uaw-Dunford-Pettis. Furthermore, a remarkable theorem of Kalton-Saab, see [5, Theorem 5.90], asserts that if the range space is order continuous, then we can deduce the former statement in the case of Dunford-Pettis operators. Hence, this point can be considered as an advantage for uaw-Dunford-Pettis operators.*

Let us now investigate closedness properties of $B_{UDP}(E)$. We already observed in Proposition 3.2.7 that $B_{UDP}(E)$ is a subalgebra of $B(E)$.

**Proposition 3.4.12.** *Let $E$ be a Banach lattice. The algebra $B_{UDP}(E)$ is a closed in $B(E)$.*

*Proof.* Suppose $T_m$ is sequence of *uaw*-Dunford-Pettis operators which is convergent to the operator $T$. We show that $T$ is also *uaw*-Dunford-Pettis. Assume that $x_n$ is a bounded *uaw*-null sequence in $E$. Given any $\varepsilon > 0$, there is an $m_0$ such that $\|T_m - T\| < \frac{\varepsilon}{2}$ for each $m > m_0$. Fix an $m > m_0$. For sufficiently large $n$, we have $\|T_m(x_n)\| < \frac{\varepsilon}{2}$. Therefore,

$$\|T(x_n)\| < \|T_m - T\| + \|T_m(x_n)\| < \varepsilon$$

for sufficiently large $n$. This implies that the operator $T$ is *uaw*-Dunford-Pettis. □

As the following example shows the closed algebra $B_{UDP}(E)$ of all *uaw*-Dunford-Pettis operators is not order closed in $B(E)$.

**Example 3.4.13.** *Put $E = c_0$. Suppose $P_n$ is the projection onto the n-th first components. For every n, the operator $P_n$ is a finite rank operator so that it is Dunford-Pettis. By Proposition 3.2.3, $P_n$ is uaw-Dunford-Pettis for all n. Also, $P_n \uparrow I$, where I denotes the identity operator on $E$. However, I is not uaw-Dunford-Pettis as the standard basis $(e_i)_{i=1}^{\infty}$ is uaw-null but not norm convergent to zero.*

**Remark 3.4.14.** *It is a natural question to ask whether the algebra $B_{UDP}(E)$ has a lattice structure or not. This can be reduced as follows. When does the modulus of a uaw-Dunford-Pettis operator exists, and, is it again uaw-Dunford-Pettis? In general, the answer to this question is not affirmative. Consider [5, Example 5.6] which is due to Krengel. Observe that the space E mentioned there, is a Dedekind complete order continuous Banach lattice whose dual is again order continuous. The operator $T$, see [5, Example 5.6] for its definition, is compact so that Dunford-Pettis. By Proposition 3.2.3, it is uaw-Dunford-Pettis. The sequence $\hat{x}_n$ is disjoint so that by [79, Lemma 2], it is uaw-null. However, as we see in the example $|T|(\hat{x}_n)$ is not norm null.*

Recall that an operator $T$ between vector lattices $E$ and $F$ is said to preserve disjointness if $x \perp y$ in $E$ implies $Tx \perp Ty$ in $F$. For the following result, we remark that Example 3.2.6 provides an example of a *uaw*-Dunford-Pettis operator which is not order bounded.

**Theorem 3.4.15.** *Suppose E is a Banach lattice. Let T be an order bounded uaw-Dunford-Pettis operator. If T preserves disjointness then T possesses a modulus $|T|$ which is uaw-Dunford-Pettis.*

*Proof.* By [5, Theorem 2.40], the modulus of $T$ exists, and, it satisfies the identity $|T|(x) \leq |T(x)|$ for each positive element $x \in E$. Suppose $x_n$ is a bounded positive sequence which is *uaw*-null. By the hypothesis, $\|Tx_n\| \to 0$. Hence, $|T|(x_n)$ is also norm null in $E$. This shows that $|T|$ is *uaw*-Dunford-Pettis. □

**Remark 3.4.16.** *We observe that there is no inclusion relation between the algebra of uaw-Dunford-Pettis operators and the class of disjointness preserving operators on $E$. The identity operator on $\ell_1$ preserves disjointness but it is not uaw-Dunford-Pettis. Furthermore, consider the operator T on $C[0,1]$ defined via $T(f) = (f(0) + f(1))\mathbf{1}$, where $\mathbf{1}$ denotes the constant one function on the interval $[0,1]$. One may verify that T is a compact operator so that it is also Dunford-Pettis. By Proposition 3.2.3, it is uaw-Dunford-Pettis. However, the operator T is not disjoint preserving, as mentioned in [5, Page 117].*

**Remark 3.4.17.** *Results of Section 3.3, see Proposition 3.3.1 and Theorem 3.3.2, show that uaw-Dunford-Pettis operators are closely related to M-weakly compact operators. Therefore, in view of Theorem 3.4.15, it is natural to consider modulus of M-weakly compact and L-weakly compact operators. In this direction, we refer reader to [12]. In general, an M- or L-weakly compact operator need not to have a modulus, see [12, Theorem 2.2].*

## 3.5 The Class of *uaw*-Compact Operators

Unbounded absolutely weak compact operators are natural from the point of view of theory of compact operators. Indeed, there are many classes of operators (such as compact, weakly compact, semicompact and order weakly compact operators, and so forth) all of which share the common property that the operator maps topologically closed subsets into subsets satisfying a certain topological property.

**Definition 3.5.1.** An operator $T\colon X \to E$, where $X$ is a Banach space and $E$ is a Banach lattice, is said to be (*sequentially*) *uaw-compact* if $T(B_X)$ is relatively (sequentially) *uaw*-compact where $B_X$ denotes the closed unit ball of the Banach space $X$.

In view of the last part of Section 3.1, it is possible to state an equivalent definition for *uaw*-compact operators. In details, an operator $T\colon X \to E$ is (sequentially) *uaw*-compact if and only if for every bounded net $x_\alpha$ (respectively, every bounded sequence $x_n$) in $X$ its image under $T$ has a subnet (respectively, subsequence), which is *uaw*-convergent in $E$.

We further say that the operator $T\colon X \to E$ is *un-compact* if $T(B_X)$ is relatively *un*-compact in $E$. In [45, Section 9], some properties of *un*-compact operators are studied. A more general treatment can be found in [6, 7].

We recall from Section 2.1 that an element $0 < e \in X^+$ of a normed lattice $X$ is called a quasi-interior point if the principal ideal $I_e$ generated by $e$ is norm dense in $X$. The element $0 < e \in X$ is a quasi-interior point if and only if for every $x \in X^+$ we have $\|x - x \wedge ne\| \to 0$ as $n$ tends to infinity.

As in [45, Proposition 9.1] and using [79, Theorem 4 and Proposition 14], we have the same conditions for *uaw*-compactness and sequentially *uaw*-compactness of an operator.

**Proposition 3.5.2.** *Let $T\colon E \to F$ be an operator between Banach lattices $E$ and $F$.*

*(i) If $F$ is order continuous and has a quasi-interior point then $T$ is uaw-compact if and only if $T$ is sequentially uaw-compact;*

*(ii) If $F$ is order continuous and $T$ is uaw-compact then $T$ is sequentially uaw-compact;*

*(iii) If $F$ is an atomic KB-space then $T$ is uaw-compact if and only if $T$ is sequentially uaw-compact.*

*Proof.* $(i)$ Since $F$ is order continuous and has a quasi-interior point, a subset of $F$ is relatively *uaw*-compact if and only if it is relatively sequentially *uaw*-compact. Hence, if $T\colon E \to F$ is sequentially *uaw*-compact then $T$ is *uaw*-compact.

$(ii)$ Since $F$ is order continuous, if $T\colon E \to F$ is *uaw*-compact then it is *un*-compact, see [79, Theorem 4]. Hence, by [45, Proposition 9.1], the operator $T$ is both *un*-compact and sequentially *un*-compact. It follows that $T$ is sequentially *uaw*-compact.

$(iii)$ Since $F$ is an atomic $KB$-space, a subset of $F$ is relatively *uaw*-compact if and only if it is relatively sequentially *uaw*-compact. Hence, if $T: E \to F$ is sequentially *uaw*-compact then $T$ is *uaw*-compact. $\square$

**Remark 3.5.3.** *One of the facts which is used in proof of [45, Proposition 9.1, (i)] is that un-topology on a Banach lattice E is metrizable if and only if E has a quasi-interior point. This result can be restated in terms of uaw-topology provided that E is order continuous. We note that order continuity is essential and can not be dropped; for instance, consider* $E = \ell_\infty$. *It is easy to see that uaw-topology and absolute weak topology agree on the unit ball* $B_E$ *of* $E$. *However* $B_E$ *is not weakly metrizable since* $E^{'}$ *is not separable. This implies that E can not be metrizable with respect to the uaw-topology.*

Similar to the case of compact and Dunford-Pettis operators, every *uaw*-compact operator is *uaw*-Dunford-Pettis. However, the following example shows that a sequentially *uaw*-compact operator need not be *uaw*-Dunford-Pettis.

**Example 3.5.4.** *The inclusion* $\ell_2 \hookrightarrow \ell_\infty$ *is weakly compact by [5, Theorem 5.24]. This operator is sequentially uaw-compact. However it is not uaw-Dunford-Pettis; because the standard basis* $(e_n)_n$ *is uaw-null but it is not norm convergent to zero.*

Also, the other implication may fail, as well. The following example shows that a *uaw*-Dunford-Pettis operator need not be *uaw*-compact. We recall the classical setting that a Dunford-Pettis operator is not necessarily compact, see [5, Section 5.4].

**Example 3.5.5.** *Consider the inclusion map* $J: L^\infty[0,1] \to L^1[0,1]$. *It follows from [5, Page 313, Exercise 7] that J is weakly compact. In fact, J is uaw-Dunford-Pettis. To see this, suppose* $f_n$ *is a norm bounded sequence which converges to zero in the uaw-topology of* $L^\infty[0,1]$. *By [79, Theorem 7], it follows that this sequence is weakly convergent. Since* $L^1[0,1] \subseteq (L^\infty[0,1])'$ *and the constant function one lies in* $L^1[0,1]$, *we conclude that* $\|f_n\|_1 \to 0$, *as claimed. However J is not uaw-compact, since the norm bounded sequence* $r_n$ *of the Rademacher functions does not have any uaw-convergent subsequence.*

Let us continue with several ideal properties.

**Proposition 3.5.6.** *Let* $S: E \to F$ *and* $T: F \to G$ *be two operators between Banach lattices* $E, F$ *and* $G$.

(i) *If T is (sequentially) uaw-compact and S is continuous then TS is (sequentially) uaw-compact.*

(ii) *If T is a uaw-Dunford-Pettis operator and S is either (sequentially) un-compact or uaw-compact then TS is compact.*

(iii) *If $T$ is uaw-Dunford-Pettis and $S$ is Dunford-Pettis then $TS$ is Dunford-Pettis.*

(iv) *If $T$ is continuous and $S$ is uaw-Dunford-Pettis, then $TS$ is uaw-Dunford-Pettis.*

*Proof.* $(i)$ We prove the results for the sequence case. For nets, the proof is similar. Suppose $(x_n) \subseteq E$ is a bounded sequence. By the assumption, the sequence $Sx_n$ is also norm bounded. Therefore, there is a subsequence $TS(x_{n_k})$ which is *uaw*-convergent.

$(ii)$ Suppose $x_n$ is a bounded sequence in $E$. There is a subsequence $x_{n_k}$ such that $S(x_{n_k}) \xrightarrow{uaw} x$ for some $x \in F$. Thus, by the hypothesis, $\|TS(x_{n_k}) - TS(x)\| \to 0$, as desired.

$(iii)$ Suppose $x_n$ is a sequence in $E$ which is weakly null. By the assumption, $\|Sx_n\| \to 0$. It follows that $Sx_n \xrightarrow{uaw} 0$. Again, this implies that $\|TS(x_n)\| \to 0$.

$(iv)$ Suppose $x_n$ is a norm bounded sequence in $E$ which is *uaw*-null. By the hypothesis, $\|Sx_n\| \to 0$ so that $\|TS(x_n)\| \to 0$, as desired. □

We denote by $K_{uaw}(E)$ and $K_{un}(E)$ the spaces of all *uaw*-compact and *un*-compact operators on the Banach lattice $E$, respectively. In general, we have $K(E) \subseteq K_{un}(E) \subseteq K_{uaw}(E)$. Indeed, if $T\colon E \to E$ is compact then for every bounded sequence $x_n$ in $E$ there exists some $y \in E$ and a subsequence $x_{n_k}$ such that $Tx_{n_k} \to y$ in norm. This implies that $Tx_{n_k} \xrightarrow{un} y$ and that $Tx_{n_k} \xrightarrow{uaw} y$.

In the next discussion, we show that not every *uaw*-compact operator is *un*-compact in the sense of [45].

**Example 3.5.7.** *The inclusion $\ell_2 \hookrightarrow \ell_\infty$ is weakly compact by [5, Theorem 5.24]. Hence, it is sequentially uaw-compact because range of the operator is an AM-space. However it is not sequentially un-compact. Since by [45, Theorem 2.3], it should be compact which is not possible.*

**Remark 3.5.8.** *The spaces $K_{un}(E)$ and $K_{uaw}(E)$ are not order closed in the classical order of the space of all continuous operators on E, as shown by [45, Example 9.3]; see also [79, Theorem 4].*

Following results are motivated by the Krengel's Theorem, see [5, Theorem 5.9].

**Theorem 3.5.9.** *Suppose $E$ is an AL-space and $F$ is a Banach lattice whose dual space is order continuous. In this case, every sequentially uaw-compact operator $T$ from $E$ into $F$ has a sequentially uaw-compact adjoint.*

*Proof.* Let $T\colon E \to F$ be a sequentially *uaw*-compact operator. For every norm bounded sequence $x_n$ in $E$, the sequence $Tx_n$ has a subsequence $Tx_{n_k}$ which is convergent in the *uaw*-topology. By [79, Theorem 7], the subsequence $Tx_{n_k}$ is weakly convergent. This implies that the operator $T$ is weakly compact. By the Gantmacher's theorem [5, Theorem 5.23], it follows that $T'$ is weakly compact. Since range of $T'$ is an *AM*-space, it is sequentially *uaw*-compact. $\square$

**Remark 3.5.10.** *We note that order continuity of $F'$ is essential and it can not be removed. For this, consider the identity operator on $\ell_1$. One may verify that it is uaw-compact because $\ell_1$ is an atomic KB-space. Therefore [45, Theorem 7.5] and [79, Theorem 4] give the desired result. However its adjoint is the identity operator on $\ell_\infty$ which is not sequentially uaw-compact.*

**Theorem 3.5.11.** *Suppose $E$ is an AL-space and $F$ is a reflexive Banach lattice. In this case, every order bounded sequentially uaw-compact operator $T$ from $E$ into $F$ has a weakly compact modulus.*

*Proof.* By Theorem 3.5.9, if $T$ is sequentially *uaw*-compact then the adjoint $T'$ is a sequentially *uaw*-compact operator. We note that $E'$ is an *AM*-space. Hence, the operator $T'$ is weakly compact and the result follows from [5, Theorem 5.35]. $\square$

**Proposition 3.5.12.** *Let $E$ be a Banach lattice whose dual space is atomic and order continuous. Also let $F$ be a Banach lattice whose dual is order continuous. Then, every (sequentially) un-compact operator $T\colon E \to F$ has a (sequentially) un-compact adjoint operator $T'\colon F' \to E'$.*

*Proof.* For any norm bounded sequence $x_n$ in $E$, the sequence $Tx_n$ has a subsequence which is *un*-convergent to zero by *un*-compactness. By [18, Theorem 6.4], it is weakly convergent. Hence, the operator $T$ is weakly compact. It follows from Gantmacher's theorem that $T'$ is weakly compact. By [45, Proposition 4.16], the operator $T'$ is *un*-compact. $\square$

## 3.6 Conclusions

Taken all together, results presented in Chapter 3 show that *uaw*-Dunford-Pettis operators have close relations with other types of operators. Most of the results of Section 3.2 utilize the order continuity of the domain of either the operator itself or its adjoint. On the other hand, results of Section 3.3 use disjoint sequences and *M*- and *L*-weakly compact operators. These classes of operators are known to play a central role in the classical theory of compact operators as well as in the theories of its generalizations, see [75, Chapter 18], [57, Chapter 3.6], and [7, Section 4]. In this direction, let us mention [11] and [12]. In the former authors showed that under some conditions on the domain and the range, the classes of *M*- and *L*-weakly compact operators between Banach lattices form a Banach lattice under the regular norm, see [11, Theorem 2.2 and 2.3]. In the latter, authors investigated the modulus of *M*- and *L*-weakly compact operators together with the positive Schur property, see [12]. In Section 3.4, the main theme is to use domination properties of operators.

In Section 3.5, we investigate *uaw*-compact operators on Banach lattices. The first part of Section 3.5 provides a complete answer to the question that under what conditions sequentially *uaw*-compact operators are *uaw*-compact. Analogues of this question may be raised in the settings of other converges as well.

In view of Chapter 4, the class of *uaw*-compact operators is a particular case of compact operators between locally solid vector lattices. To see this, let $(X,\tau_1)$ and $(Y,\tau_2)$ be locally solid vector lattices. An operator $T\colon (X,\tau_1)\to(Y,\tau_2)$ is said to be Montel if for any topologically bounded net $x_\alpha$ in $X$ there exists a subnet $x_{\alpha_\beta}$ and $y\in Y$ such that $Tx_{\alpha_\beta}\xrightarrow{\tau'} y$ in $Y$. The operator $T\colon (X,\tau_1)\to(Y,\tau_2)$ is said to be compact if there exists some zero neighborhood $U$ in $X$ such that for every net $x_\alpha$ in $U$ there exists a subset $x_{\alpha_\beta}$ and $y\in Y$ satisfying $Tx_{\alpha_\beta}\xrightarrow{\tau'} y$.

It follows that every compact operator $T\colon (X,\tau_1)\to(Y,\tau_2)$ between locally solid vector lattices is Montel. In addition, both the classes of compact and Montel operators on locally solid vector lattices generalize the compact operators on Banach lattices. Indeed, if $X$ and $Y$ are Banach lattices then a compact operator $T\colon X\to Y$ is both Montel and compact in the sense of locally solid vector lattices.

# 4 $u\tau$-DUNFORD-PETTIS AND $u\tau$-COMPACT OPERATORS

In the present chapter, our main focus consists of properties of bounded operators, $u\tau$-Dunford-Pettis and $u\tau$-compact operators defined between locally solid vector lattices. In addition to lattice structures of bounded operators, several properties of $u\tau$-Dunford-Pettis and $u\tau$-compact operators are investigated.

Most of the results presented here can be found in the article [33] written by N. Erkurşun Özcan, N.A. Gezer, and O. Zabeti. On the other hand, unbounded absolute weak Dunford-Pettis and unbounded absolute weak compact operators are studied in [34], see also Section 3 of the present work. The notions of $p$-compactness, $up$-compactness and the $um$-case on lattice normed spaces are investigated in [7, 15].

## 4.1 Some Basic Concepts and Motivation

A *topological vector space* is a vector space together with a linear topology such that the vector space operations are continuous. If $X$ is a vector lattice and $\tau$ is a linear topology on $X$ that has a base at zero consisting of solid sets then the pair $(X,\tau)$ is called a locally solid vector lattice. More details on locally solid vector lattices can be found in [4, 36, 64].

**Definition 4.1.1.** A locally solid vector lattice $(X,\tau)$ is said to have

*(i)* the Lebesgue property, if $x_\alpha \downarrow 0$ in $X$ implies $x_\alpha \xrightarrow{\tau} 0$.

*(ii)* the Levi property, if $0 \le x_\alpha \uparrow$ for a $\tau$-bounded net $x_\alpha$ then $x_\alpha \uparrow x$ for some $x \in X$.

*(iii)* the Fatou property, see [4, Chapter 4], if $\tau$ has a base at zero consisting of solid and order closed sets.

In view of Definition 2.1.2, a seminormed lattice is said to satisfy some of properties given in Definition 4.1.1, if it is a locally solid vector lattice, which is in particular a seminormed lattice, satisfying these properties.

**Example 4.1.2.** *We recall that a lattice seminorm $\rho\colon X \to \mathbb{R}$ on a vector lattice $X$ is said to have the Fatou property if $\emptyset \neq B \uparrow x$ in $X^+$ implies $\rho(x) = \sup\{\rho(y)\colon y \in B\}$. The locally solid topology induced by a Fatou seminorm on a vector lattice is a Fatou topology on $X$.*

Throughout the present chapter, we consider unbounded topology on a locally solid vector lattices. The pair $(X,\tau)$ stands for a locally solid vector lattice, whereas the pair $(Y,\tau')$ denotes a generic locally convex space. Following [16] and Section 2.2.5, we write $x_\alpha \xrightarrow{u\tau} x$ for a net $x_\alpha$ in a locally solid vector lattice $(X,\tau)$ if $|x_\alpha - x| \wedge u \xrightarrow{\tau} 0$ for all $u \in X^+$. We say that the net $x_\alpha$ is unbounded $\tau$-convergent to $x$ whenever $x_\alpha \xrightarrow{u\tau} x$. For more expositions on this notion and the related topics, see [16, 66].

The inspiring paper [68] of V.G. Troitsky provides the main motivation of this chapter. In [68], a spectral theory for bounded operators between topological vector spaces was developed. Various results on different classes of bounded operators on topological vector spaces were obtained. Among those bounded operators, the spaces of *nb*-bounded and *bb*-bounded operators were considered and many properties were investigated. In the present chapter, we combine this approach with the notion of unbounded convergence.

## 4.2 Bounded Operators Between Topological Vector Spaces

Let us recall some notions related to bounded operators between topological vector spaces. Let $X$ and $Y$ be topological vector spaces. A linear operator $T$ from $X$ into $Y$ is said to be *nb-bounded* if there is a zero neighborhood $U \subseteq X$ such that the subset $T(U)$ is bounded in $Y$. The operator $T$ is called *bb-bounded* if for each bounded set $B \subseteq X$, the subset $T(B)$ is bounded in $Y$.

We recall that if a topological vector space $X$ has a bounded neighborhood of zero then $X$ is pseudometrizable; however, even a metrizable topological vector space need not to have a bounded neighborhood of zero, see [59, Theorem 6.2.1].

**Remark 4.2.1.** *Boundedness of subsets of a topological vector space will be used in the sequel to investigate properties of compact operators. At this point, we comment that there are alternative versions of boundedness of sets in topological vector spaces. We recall that a subset $B$ of a topological vector space $(X,\tau)$ is said to be (topologically) bounded (also denoted by $\tau$-bounded) if for every zero neighborhood $U$ in $X$ there exists some $\lambda > 0$ such that $\lambda B \subseteq U$. However, mathematicians such as Kolmogorov, Mazur and Orlicz used a different version of boundedness of sets. In their approach, a subset $B$ of a topological vector space $X$ is said to be bounded if for any sequence $t_n$ of scalars satisfying $t_n \to 0$ and any sequence $x_n$ in $B$ one has $t_n x_n \to 0$ in $X$.*

It is evident that the notions of *nb*-bounded and *bb*-bounded operators are not equivalent. However in a normed space, these concepts have the same meaning (see [68, 77, 59] for more details on this topic).

The class of all *nb*-bounded operators on a topological vector space $X$ is denoted by $B_n(X)$. This space is equipped with the topology of uniform convergence on some zero neighborhood, see [68, Par. 2.21]. That is to say, a net $S_\alpha$ of *nb*-bounded operators on $X$ converges to zero on some zero neighborhood $U \subseteq X$ if for any zero neighborhood $V \subseteq X$ there is an index $\alpha_0$ such that $S_\alpha(U) \subseteq V$ for each $\alpha \geq \alpha_0$.

The class of all *bb*-bounded operators on $X$ is denoted by $B_b(X)$. It is equipped with the topology of uniform convergence on bounded sets, see [68, Par. 2.16]. In details, a net $S_\alpha$ of *bb*-bounded operators on $X$ uniformly converges to zero on a bounded set $B \subseteq X$ if for any zero neighborhood $V \subseteq X$ there is an index $\alpha_0$ with $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$.

**Example 4.2.2.** *Let $\rho \colon X \to \mathbb{R}$ be a seminorm on a linear space $X$. In this case it follows from [59, Theorem 5.7.4] that an operator $T \colon X \to X$ belongs to $B_n(X)$ if and only if $T$ is continuous with respect to linear topology on $X$ induced by $\rho$.*

The class of all continuous operators on $X$ is denoted by $B_c(X)$ and is equipped with the topology of equicontinuous convergence, see [68, Par. 2.18]. That is to say, a net $S_\alpha$ of continuous operators on $X$ converges equicontinuously to zero if for each zero neighborhood $V$ there is a zero neighborhood $U$ such that for every $\varepsilon > 0$ there exists an index $\alpha_0$ with $S_\alpha(U) \subseteq \varepsilon V$ for each $\alpha \geq \alpha_0$.

See [68] for more information on these classes of operators. In general, we have $B_n(X) \subseteq B_c(X) \subseteq B_b(X)$; and when $X$ is *locally bounded*, i.e., when $X$ has a bounded zero neighborhood, they coincide.

## 4.3 $u\tau$-Dunford-Pettis and Unbounded Compact Operators

Suppose $(X,\tau)$ is a locally solid vector lattice and $(Y,\tau')$ is a topological vector space. Throughout solidness hypothesis of zero neighborhoods of $X$ are needed for the construction of the $u\tau$-topology on $X$. Also, we are interested in the cases where $(Y,\tau')$ is in particular either locally convex or locally solid.

**Definition 4.3.1.** An operator $T \colon (X,\tau) \to (Y,\tau')$ is said to be *$u\tau$-Dunford-Pettis* if for every $\tau$-bounded net $x_\alpha$ in $X$, $x_\alpha \xrightarrow{u\tau} 0$ implies $T(x_\alpha) \xrightarrow{\tau'} 0$ in $Y$.

We denote by $DP_{u\tau}(X,Y)$ the linear space generated by $u\tau$-Dunford-Pettis operators.

**Example 4.3.2.** *Let $X$ be a normed lattice. In this case, $T \in DP_{un}(X)$ if and only if for every norm bounded net $x_\alpha$ one has $Tx_\alpha \to 0$ in norm whenever $x_\alpha \xrightarrow{un} 0$.*

In topological vector space settings, there are different and non-equivalent notions for bounded operators, see Section 4.2 for details. The same is also true for compact operators.

**Definition 4.3.3.** ([68, Section 7]) A linear operator $T$ on a topological vector space $X$ is said to be

*(i)* *n-compact*, if there is a zero neighborhood $U \subset X$ such that $T(U)$ is relatively compact.

*(ii)* *b-compact* (sometimes called Montel), if for each bounded set $B \subseteq X$ the set $T(B)$ is relatively compact.

As remarked in [68, Section 7], in the generality of topological vector spaces, if $T\colon X\to X$ is $n$-compact then it is both $b$-compact and $nb$-bounded. Also, if $T\colon X\to X$ is $b$-compact then it is $bb$-bounded, in the sense of Section 4.2.

As a result, we first observe that $n$-compactness (or $b$-compactness) of an operator implies its $nb$-boundedness ($bb$-boundedness, respectively). This implication is known to be false in the general settings of [7].

Consequently, there are two variants of unbounded compact operators on locally solid vector lattices.

**Definition 4.3.4.** Let $(X,\tau)$ be a locally solid vector lattice, and, $(Y,\tau')$ be a topological vector space. An $nb$-bounded operator $T\colon (Y,\tau')\to(X,\tau)$ is called *$nu\tau$-compact* if there exists a zero neighborhood $U\subseteq Y$ such that the set $T(U)$ is $u\tau$-relatively compact in $X$.

We denote by $K_{nu\tau}(Y,X)$ the linear space generated by $nu\tau$-compact operators from $Y$ into $X$.

**Example 4.3.5.** *Let $X$ be a Banach lattice, and, denote by $\tau$ the locally convex-solid topology generated by its norm. It follows that $T\in K_{nu\tau}(X)$ if and only if $T\in K_{un}(X)$, see Section 3.5 and [45, Section 9].*

**Definition 4.3.6.** Let $(X,\tau)$ be a locally solid vector lattice, and, $(Y,\tau')$ be a topological vector space. A $bb$-bounded operator $T\colon (Y,\tau')\to(X,\tau)$ is said to be *$bu\tau$-compact* if for every bounded set $B\subseteq Y$, the set $T(B)$ is $u\tau$-relatively compact in $X$.

We denote by $K_{bu\tau}(Y,X)$, the class of all $bu\tau$-compact operators from $Y$ into $X$. An analogue of Example 4.3.5 can also be given for $bu\tau$-compact operators.

In many cases, it is useful to consider sequential versions of these operators. In details, an operator $T\colon (X,\tau)\to(Y,\tau')$ is said to be *sequentially $u\tau$-Dunford-Pettis* if for every $\tau$-bounded sequence $x_n$ in $X$, $x_n\xrightarrow{u\tau}0$ implies $T(x_n)\xrightarrow{\tau'}0$ in $Y$. A variant of $u\tau$-Dunford-Pettis operators is investigated in [34] and in Section 3.

An $nb$-bounded operator $T\colon (Y,\tau')\to(X,\tau)$ is said to be *sequentially $nu\tau$-compact* if there exists a zero neighborhood $U\subseteq Y$ such that for every sequence $x_n$ in $U$ the sequence $T(x_n)$ has a $u\tau$-convergent subsequence in $X$.

**Example 4.3.7.** *Let $X$ be a Banach lattice, and, denote by $\tau$ the locally convex-solid topology generated by its norm. In view of Example 4.3.5 and [45, Section 9], and operator $T\colon X\to X$ is sequentially $nu\tau$-compact if and only if $T$ is sequentially un-compact.*

Similarly, a $bb$-bounded operator $T$ is called *sequentially $bu\tau$-compact* if for every sequence $x_n$ in $B$, where $B$ is a bounded set in $Y$, the sequence $T(x_n)$ has a $u\tau$-convergent subsequence in $X$; see [34, 45] for more information on these notions.

We denote by $DP_{u\tau}(X)$, $K_{nu\tau}(X)$, and $K_{bu\tau}(X)$, the space of all $u\tau$-Dunford-Pettis, $nu\tau$-compact, and $bu\tau$-compact operators on a locally solid vector lattice $X$, respectively.

## 4.4 Vector Lattices of Order Bounded Topologically Bounded Operators

**Remark 4.4.1.** *It is known that every order bounded operator from a Banach lattice to a normed lattice is continuous,. However, an order bounded operator between locally solid vector lattices need not to be continuous. Suppose $X$ is $\ell_\infty$ with the weak topology and $Y$ is $\ell_\infty$ with the usual norm topology. Consider the identity operator from $X$ into $Y$. This operator is order bounded but not continuous due to comparison of weak and norm topologies on $\ell_\infty$. In addition, an order bounded operator between locally solid vector lattices need not to be nb-bounded or bb-bounded. Let $X$ be $\mathbb{R}^{\mathbb{N}}$, the space of all real sequences with the product topology and the pointwise ordering. The identity operator on $X$ is order bounded but not nb-bounded. Finally, suppose $X$ is $c_{00}$ with pointwise ordering and the usual norm topology. Then the operator $T$ on $X$ which maps every $x_n$ into $(nx_n)$ is order bounded but certainly not bb-bounded. Let us note the example given in [59, Example 8.8.8]. Suppose $X$ is the Hilbert space $L_2[-\pi,\pi]$. Then the identity operator $I\colon (X,\sigma(X,X')) \to (X,\|\cdot\|)$ is bb-bounded but not continuous.*

It is natural that we can not expect order properties from bounded operators between topological vector spaces but there are good news if we restrict our attention to order bounded topologically bounded operators between locally solid vector lattices.

**Lemma 4.4.2.** *Suppose $X$ is a Dedekind complete locally solid vector lattice with Fatou topology. Let $B_n^b(X)$ be the space of all order bounded nb-bounded operators on $X$. Then $B_n^b(X)$ is a vector lattice.*

*Proof.* It suffices to prove that for an operator $T \in B_n^b(X)$ we have $T^+ \in B_n^b(X)$. By the Riesz-Kantorovich formula, we have

$$T^+(x) = \sup\{T(u)\colon 0 \le u \le x\}$$

for each $x \in X^+$. Since $T$ is *nb*-bounded, there is a zero neighborhood $U \subseteq X$ such that $T(U)$ is bounded. It follows from the definition of bounded sets that for arbitrary order closed zero neighborhood $V$, there is a positive $\gamma$ with $T(U) \subseteq \gamma V$. Therefore, for each $x \in U^+$, we have $T(x) \in \gamma V$. It follows from the solidness of $U$ and order closedness of $V$ that $T^+(x) \in \gamma V$. Thus, we see that $T^+(U)$ is also bounded. Hence, $T^+$ is an *nb*-bounded operator and $T^+ \in B_n^b(X)$. $\square$

**Lemma 4.4.3.** *Suppose $X$ is a Dedekind complete locally solid vector lattice with Fatou topology. Let $B_c^b(X)$ be the space of all order bounded continuous operators on $X$. Then $B_c^b(X)$ is a vector lattice.*

*Proof.* It suffices to prove that for an operator $T \in B_c^b(X)$ we have $T^+ \in B_c^b(X)$. By the Riesz-Kantorovich formula (see [5, page 15]), we have

$$T^+(x) = \sup\{T(u)\colon 0 \le u \le x\}$$

for each $x \in X^+$. For an arbitrary zero neighborhood $V \subseteq X$, choose a zero neighborhood $U$ satisfying $T(U) \subseteq V$. Therefore, for each $x \in U^+$, we have $T(x) \in V$. It follows from solidness of $U$ and order closedness of $V$ that $T^+(x) \in V$. Thus, we see that $T^+(U) \subseteq V$. Hence, $T^+$ is a continuous operator and $T^+ \in B_c^b(X)$. □

**Lemma 4.4.4.** *Suppose $X$ is a Dedekind complete locally solid vector lattice with Fatou topology. Let $B_b^b(X)$ be the space of all order bounded bb-bounded operators on $X$. Then $B_b^b(X)$ is a vector lattice.*

*Proof.* It suffices to prove that for an operator $T \in B_b^b(X)$ we have $T^+ \in B_b^b(X)$. By the Riesz-Kantorovich formula, we have

$$T^+(x) = \sup\{T(u)\colon 0 \le u \le x\}$$

for each $x \in X^+$. Suppose $V \subseteq X$ is an arbitrary zero neighborhood. Fix bounded set $B \subseteq X$. Without loss of generality, we can assume that $B$ is solid, otherwise, consider the solid hull $sol(B)$ of $B$ which is indeed bounded. There exists $\gamma > 0$ such that $T(B) \subseteq \gamma V$. Therefore, for each $x \in B^+$, $T(x) \in \gamma V$, so that $T^+(x) \in \gamma V$ using solidness of $B$ and order closedness of $V$. Thus, we have $T^+(B) \subseteq \gamma V$ and the conclusion follows. □

**Remark 4.4.5.** *In view of [68, Proposition 2.3], we have $B_n^b(X) \subseteq B_c^b(X) \subseteq B_b^b(X)$ whenever $X$ is a Dedekind complete locally solid vector lattice with Fatou topology. In general, the collection of all order bounded operators can be quite different than the vector lattice $B_n^b(X)$. If $X$ is not locally bounded then the identity operator $I\colon X \to X$ does not belong to $B_n^b(X)$ though it is order bounded.*

The following results extend [78, Theorem 6 and Theorem 7] to a more general setting.

**Theorem 4.4.6.** *Suppose $X$ is a Dedekind complete locally solid vector lattice with Fatou topology. Then $B_n^b(X)$ is locally solid with respect to the uniform convergence topology on some zero neighborhood.*

*Proof.* Let $T \in B_n^b(X)$ and $x \in X^+$. By the Riesz-Kantorovich theorem, we have

$$T^+(x) = \sup\{T(u)\colon 0 \le u \le x\}$$

for each $x \in X^+$. Suppose $T_\alpha$ is a net of order bounded *nb*-bounded operators that converges uniformly on some zero neighborhood $U \subseteq X$ to an operator $T$ in $B_n^b(X)$, see Section 4.2.

Choose arbitrary zero neighborhood $V \subseteq X$. Fix $x \in U^+$. We recall that for two subsets $A$ and $B$ in a vector lattice, we have $\sup(A) - \sup(B) \leq \sup(A - B)$. Thus,

$$\sup\{T_\alpha(u) : u \in X^+, u \leq x\} - \sup\{T(u) : u \in X^+, u \leq x\}$$

$$\leq \sup\{(T_\alpha - T)(u) : u \in X^+, u \leq x\}$$

for each such $x \in U^+$ and $\alpha$. There exists an $\alpha_0$ such that $(T_\alpha - T)(U) \subseteq V$ for each $\alpha \geq \alpha_0$. Therefore, using the order closedness of the zero neighborhood $V$ and solidness of the zero neighborhood $U$, we have

$$T_\alpha{}^+(x) - T^+(x) \leq (T_\alpha - T)^+(x) \in V$$

for each such $x \in U^+$. The result follows from [4, Theorem 2.17]. □

**Remark 4.4.7.** *In view of Theorem 4.4.6, one can ask if the space of all order bounded nb-bounded operators, i.e. the space $B_n^b(X)$, is closed with respect to topology of uniform convergence on some zero neighborhood. An answer to this question can be found in [68, Example 2.22].*

**Theorem 4.4.8.** *Suppose $X$ is a Dedekind complete locally solid vector lattice with Fatou topology. Then $B_c^b(X)$ is locally solid with respect to the equicontinuous convergence topology.*

*Proof.* The proof is similar to that of Theorem 4.4.6. Let $T \in B_c^b(X)$ and $x \in X^+$. By the Riesz-Kantorovich theorem, we have

$$T^+(x) = \sup\{T(u), 0 \leq u \leq x\}$$

for each $x \in X^+$. Suppose $T_\alpha$ is a net of order bounded continuous operators which is convergent equicontinuously to an operator $T$ in $B_c^b(X)$, see Section 4.2. Choose arbitrary zero neighborhood $V \subseteq X$. There exists zero neighborhood $U \subseteq X$ such that for each $\varepsilon > 0$ we have $(T_\alpha - T)(U) \subseteq \varepsilon V$ for sufficiently large $\alpha$. Fix $x \in U^+$. We recall again that for two subsets $A, B$ in a vector lattice, we have $\sup(A) - \sup(B) \leq \sup(A - B)$. Thus,

$$\sup\{T_\alpha(u) : u \in X^+, u \leq x\} - \sup\{T(u) : u \in X^+, u \leq x\}$$

$$\leq \sup\{(T_\alpha - T)(u) : u \in X^+, u \leq x\}$$

for each such $x \in U^+$ and $\alpha$. Therefore, using the order closedness of zero neighborhood $V$ and solidness of zero neighborhood $U$, we have

$$T_\alpha{}^+(x) - T^+(x) \leq (T_\alpha - T)^+(x) \in \varepsilon V.$$

The result follows from [4, Theorem 2.17]. $\square$

**Theorem 4.4.9.** *Suppose $X$ is a Dedekind complete locally solid vector lattice with Fatou topology. Then $B_b^b(X)$ is locally solid with respect to the uniform convergence topology on bounded sets.*

*Proof.* Let $T \in B_b^b(X)$ and $x \in X^+$. By the Riesz-Kantorovich theorem, we have

$$T^+(x) = \sup\{T(u), 0 \leq u \leq x\}$$

for each $x \in X^+$. Suppose $T_\alpha$ is a net of order bounded $bb$-bounded operators that converges uniformly on bounded sets to an operator $T$ in $B_b^b(X)$, see Section 4.2. Fix a bounded set $B \subseteq X$. In view of Lemma 4.4.4, we can assume that $B$ is solid. Choose arbitrary zero neighborhood $V \subseteq X$. Fix $x \in B^+$. Thus,

$$\sup\{T_\alpha(u) : u \in X^+, u \leq x\} - \sup\{T(u) : u \in X^+, u \leq x\}$$

$$\leq \sup\{(T_\alpha - T)(u) : u \in X^+, u \leq x\}$$

for each such $x \in B^+$ and $\alpha$. There exists an $\alpha_0$ such that $(T_\alpha - T)(B) \subseteq V$ for each $\alpha \geq \alpha_0$. Therefore, using the order closedness of zero neighborhood $V$ and solidness of bounded set $B$, we have

$$T_\alpha{}^+(x) - T^+(x) \leq (T_\alpha - T)^+(x) \in V.$$

The result follows from [4, Theorem 2.17]. $\square$

**Remark 4.4.10.** *In view of Remark 4.4.7, one can ask if the collections $B_c^b(X)$ and $B_b^b(X)$ are closed with respect to equicontinuous convergence and uniform convergence on bounded sets, respectively. In this direction, we refer reader to [68, Lemma 2.17 and 2.19].*

## 4.5 Results on Unbounded Compact Operators

In this section, we investigate some conditions for which these spaces of operators agree.

**Remark 4.5.1.** *Let $(X, \tau)$ be a locally solid vector lattice. It follows that $K_n(X) \subseteq K_{nu\tau}(X) \subseteq B_n(X)$ and $K_b(X) \subseteq K_{bu\tau}(X) \subseteq B_b(X)$, see Section 4.3 for the definitions. Indeed, if $T \in K_n(X)$ then there is a zero neighborhood $U \subseteq X$ such that $T(U)$ is $\tau$-relatively compact. Hence, the set $T(U)$ is $u\tau$-relatively compact. It follows that the operator $T$ is $nu\tau$-compact. The inclusion $K_{nu\tau}(X) \subseteq B_n(X)$ follows from the fact that every $nu\tau$-compact operator is nb-bounded, see Definition 4.3.4. If $T \in K_b(X)$ then $T(B)$ is $\tau$-relatively compact for every bounded subset $B$ of $X$. Hence, $T(B)$ is $u\tau$-relatively compact. Thus, $T \in K_{bu\tau}(X)$. The inclusion $K_{bu\tau}(X) \subseteq B_b(X)$ follows from Definition 4.3.6. We recall that a topological vector space is said to have the Heine-Borel property if every closed and bounded subset of it* *is compact. If $X$ has the Heine-Borel property then $K_n(X) = B_n(X)$ and $K_b(X) = B_b(X)$, see [77, Proposition 2.5 and Remark 2.6]*

In the following, we consider some ideal properties for these spaces of operators.

**Proposition 4.5.2.** *Let $S\colon (X,\tau) \to (Y,\tau')$ and $T\colon (Y,\tau') \to (Z,\tau'')$ be two operators between locally solid vector lattices $(X,\tau)$, $(Y,\tau')$ and $(Z,\tau'')$.*

i. *If $T$ is $nu\tau$-compact and $S$ is nb-bounded then $TS$ is $nu\tau$-compact.*

ii. *If $T$ is $bu\tau$-compact and $S$ is bb-bounded then $TS$ is $bu\tau$-compact.*

iii. *If $T$ is a $u\tau$-Dunford-Pettis operator and $S$ is $bu\tau$-compact then $TS$ is b-compact.*

iv. *If $T$ is continuous and $S$ is $u\tau$-Dunford-Pettis, then $TS$ is $u\tau$-Dunford-Pettis.*

*Proof.* $(i)$. Suppose $U \subseteq X$ and $V \subseteq Y$ are zero neighborhoods such that $S(U)$ is bounded in $Y$ and $T(V)$ is $u\tau''$-relatively compact in $Z$. There is some positive $\gamma$ with $S(U) \subseteq \gamma V$, so that $TS(U) \subseteq \gamma T(V)$. This implies that $TS(U)$ is $u\tau''$-relatively compact.

$(ii)$. Fix a bounded set $B \subseteq X$. Since $S(B)$ is bounded in $Y$, by assumption, $TS(B)$ is $u\tau''$-relatively compact in $Z$.

$(iii)$. Suppose $x_\alpha$ is a bounded net in $X$. There is a subnet $y_\beta$ such that $S(y_\beta) \xrightarrow{u\tau'} y$ for some $y \in Y$. Thus, by the hypothesis, $T(S(y_\beta)) \xrightarrow{\tau''} T(S(y))$, as desired.

$(iv)$. Suppose $x_\alpha$ is a bounded $u\tau$-null net in $X$. By the assumption, $S(x_\alpha) \xrightarrow{\tau'} 0$. By the assumption, $T(S(x_\alpha))$ is topologically null. □

Suppose $(X,\tau)$ is a locally solid vector lattice in which every convergent net is eventually bounded. This property is satisfied in many known cases including metrizable spaces, normed spaces, spaces equipped with weak topology, and in particular, when we consider sequences. In this case, one may verify that every $u\tau$-Dunford-Pettis operator is continuous but the converse is not true, in general. To see this consider the identity operator on $\ell_1$ when it is equipped with the norm topology and see the comment given before Remark 3.2.4.

We recall that the topology $\tau$ on the locally solid vector lattice $X$ is said to be unbounded if $\tau = u\tau$ (see [66], Definition 2.7). We will consider a notion named "boundedly unbounded" for a locally solid topology $\tau$.

**Definition 4.5.3.** A locally solid topology $\tau$ on a vector lattice $X$ is said to be *boundedly unbounded* if $\tau = u\tau$ in every bounded subset of $X$.

We note that boundedly unboundedness and unboundedness differ in general. Consider $X = c_0$ together with the absolute weak topology $\tau = |\sigma|(X,X')$. Using [79, Theorem 7], we conclude that $\tau$ is boundedly unbounded but not unbounded since the sequence $(x_n) \subseteq X$ defined via $x_n = (0, \ldots, 0, n, 0, \ldots)$ where $n$ is in the $n^{th}$-place is $u\tau$-null but not $\tau$-null. In fact, it can be seen from [79, Theorem 7] that absolute weak topology on a Banach lattice $X$ is boundedly unbounded if and only if $X'$ is order continuous. We have the following.

**Theorem 4.5.4.** *Suppose that $X$ is a locally solid vector lattice. In this case, $DP_{u\tau}(X) = B_c(X)$ if and only if $X$ is boundedly unbounded.*

*Proof.* Suppose $X$ is boundedly unbounded. Every bounded $u\tau$-null net is $\tau$-null because $\tau = u\tau$ on bounded subsets of $X$. This means that the identity operator on $X$ is $u\tau$-Dunford-Pettis. So, by Proposition 4.5.2, we see that $DP_{u\tau}(X) = B_c(X)$. Conversely, suppose $DP_{u\tau}(X) = B_c(X)$. Therefore, the identity operator on $X$ lies in $DP_{u\tau}(X)$. Hence, $X$ is boundedly unbounded. □

When we focus on norm topology, we obtain more familiar results. Let $B(X)$ denote the linear space of all continuous operators on $X$, i.e., $B(X) = B_c(X)$. We recall from Section 2.1 that a positive nonzero element of a vector lattice is called a strong unit if the principal ideal generated by that element is equal to that vector lattice.

**Proposition 4.5.5.** *Suppose that $X$ is a normed lattice. In this case, $DP_{un}(X) = B(X)$ if and only if $X$ has a strong unit.*

*Proof.* Suppose $X$ has a strong unit. By [45, Theorem 2.3], the *un*-topology and the norm topology agree on $X$. It follows that $DP_{un}(X) = B(X)$. For the converse, assume that $DP_{un}(X) = B(X)$. In this case, the identity operator $I$ lies in $DP_{un}(X)$. Thus, every norm bounded *un*-null net is norm null. The result follows from [45, Lemma 2.1 and Lemma 2.2]. □

**Theorem 4.5.6.** *Suppose that $X$ is a locally solid vector lattice. One has $K_{bu\tau}(X) = B_b(X)$ if and only if $X$ is atomic and has both the Levi and Lebesgue properties.*

*Proof.* Suppose $X$ is an atomic locally solid vector lattice with Levi and Lebesgue properties. Then by [15, Theorem 6], the identity operator is $bu\tau$-compact. So, by Proposition 4.5.2, $K_{bu\tau}(X) = B_b(X)$. For the other directions, suppose $K_{bu\tau}(X) = B_b(X)$. Therefore, the identity operator lies in $K_{bu\tau}(X)$. Therefore, every bounded subset of $X$ is $u\tau$-relatively compact. The result follows from [15, Theorem 6]. □

As a corollary of the above theorems, we have the following result.

**Corollary 4.5.7.** *Suppose $X$ is a Banach lattice. Then, $K_{un}(X) = B(X)$ and $K_{uaw}(X) = B(X)$ if and only if $X$ is an atomic KB-space.*

**Remark 4.5.8.** *We do not know whether $K_{nu\tau}(X) = B_n(X)$, in general. A sufficient condition is that $X$ has the Heine-Borel property, see Remark 4.5.1. However, this condition is not necessary as $X = \ell_1$ does not have the Heine-Borel property; nevertheless $K_{nu\tau}(X) = B_n(X)$ by Corollary 4.5.7.*

In the following, we investigate whether unbounded Dunford-Pettis operators or unbounded compact operators are topologically closed with respect to the induced topologies from corresponding classes of bounded operators. Also, we consider order closedness property for them.

The class of all $nu\tau$-compact ( $bu\tau$-compact) operators are not closed in the corresponding class of bounded operators, respectively. However, the class of *un*-compact operators is closed; see [45, Proposition 9.2]. In addition, neither of these spaces are order closed. The case of sequentially *un*-compact operators is discussed in [45, Example 9.5].

**Example 4.5.9.** *Assume that $X$ is $c_0$ with the norm topology and $Y$ is $c_0$ with the weak topology. Let $P_n$ be the projection on the first n-components from $X$ into $Y$. Each $P_n$ is compact in both unbounded senses. In addition, the sequence $P_n$ converges uniformly on the unit ball to the identity operator $I$ from $X$ into $Y$. The operator $I$ is neither $nu\tau$-compact nor $bu\tau$-compact since the sequence $x_n$ defined via n one terms at first and null in the sequel is neither norm convergent nor weak convergent in $c_0$ by Dini Theorem [5, Theorem 3.52]. Since $c_0$ is boundedly unbounded, it is not also uaw-convergent. Finally, we note that $P_n \uparrow I$. Hence, these classes of compact operators are not order closed.*

**Proposition 4.5.10.** *Let $X$ be a locally solid vector lattice. Then $DP_{u\tau}(X)$ is a closed subalgebra of $B_c(X)$.*

*Proof.* One can see easily that $DP_{u\tau}(X)$ is an algebra. Suppose $S_\alpha$ is a net of $u\tau$-Dunford-Pettis operators which is uniformly convergent equicontinuously, see Section 4.2, to the continuous operator $S$. Let $W \subseteq X$ be an arbitrary zero neighborhood. There is a zero neighborhood $V \subseteq X$ with $V + V \subseteq W$. Then there exists a zero neighborhood $U \subseteq X$ such that for each $\varepsilon > 0$ there is an index $\alpha_0$ with $(S_\alpha - S)(U) \subseteq \varepsilon V$ for each $\alpha \geq \alpha_0$ so that $S(U) \subseteq S_\alpha(U) + \varepsilon V$. Assume $x_\beta$ is a bounded $u\tau$-null net in $X$. Find positive scalar $\gamma$ with $(x_\beta) \subseteq \gamma U$. Corresponding to $\varepsilon = \frac{1}{\gamma}$, we have $(S_\alpha - S)(U) \subseteq \frac{1}{\gamma} V$ for sufficiently large $\alpha$ so that $(S_\alpha - S)(x_\beta) \subseteq V$. Fix an $\alpha$. There exists a $\beta_0$ with $S_\alpha(x_\beta) \subseteq V$ for each $\beta \geq \beta_0$. This concludes that $S(x_\beta) \subseteq W$ for sufficiently large $\beta$. □

Note that we have seen in [34, Example 2.24], also in Example 3.4.13, unbounded absolute weak Dunford-Pettis operators are not order closed, in general.

**Remark 4.5.11.** *One can see directly that if a positive operator $T$ is dominated by a positive $u\tau$-Dunford-Pettis operator $S$, then $T$ is also $u\tau$-Dunford-Pettis. This does not hold for un-*

*bounded compact operators as shown by [45, Example 9.7]. So, from the former statement, we conclude that if $T$ and $S$ are two positive $u\tau$-Dunford-Pettis operators, so is $T \vee S$.*

## 4.6 Conclusions

In Section 4.2, we presented three different abstractions about bounded operators between topological vector spaces; namely, *nb*-bounded, *bb*-bounded and continuous operators. All of these notions are known to generalize different properties of classical norm bounded operators between normed spaces. Historically speaking, these notions are well-motivated and details can be found in [59, 64, 68]. Furthermore, some results presented in Section 4.2 will be used as a starting point of Chapter 5 of the present thesis, for instance see Example 5.1.2. In Section 4.3, we focused on locally solid and locally convex-solid vector lattices in the purpose of using the induced unbounded topology. We observed that both $nu\tau$-compact and $bu\tau$-compact operators satisfy a corresponding boundedness assumption, which were given in Section 4.2. This is motivated by the fact that the classical compact operators are norm bounded. In Section 4.4, we focused on those order bounded operators which additionally satisfy some topological boundedness properties. As locally solid vector lattices are particular cases vector lattices, this approach can be continued with other important classes of operators between vector lattices. We observed in Remark 4.4.7 and Remark 4.4.10 that in addition to vector lattice structure of these operators some topological properties can also be studied. Main focus of Section 4.5 were $u\tau$-Dunfordm-Pettis, $nu\tau$-compact and $bu\tau$-compact operators.

# 5 UNBOUNDED ASYMPTOTIC EQUIVALENCES OF OPERATOR NETS

A Lotz-Räbiger net is a net of operators acting on a Banach space and satisfying certain properties, see [31, 25, 20, 22, 23, 24, 52, 62]. It is closely related to the notion of $\mathscr{S}$-ergodic net of an operator semigroup, see [49, Chapter 2.2], and to the notion of $M$-sequences of [62]. Strong asymptotic equivalence is used in [20] to investigate the properties of both Lotz-Räbiger and martingale nets on Banach spaces. In the present chapter, we define various equivalence relations on the collection $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ of all $\Lambda$-indexed nets of $\mathfrak{c}$-continuous operators on a convergence vector lattice $(X,\mathfrak{c})$ in the purpose of deriving properties of unbounded Lotz-Räbiger and unbounded martingale nets. The unified approach given in [20] is utilized in the settings of unbounded topology to obtain results on classes of unbounded Lotz-Räbiger and unbounded martingale nets on $X$.

Most of the results presented here can be found in the article [35] written by N. Erkurşun Özcan, N.A. Gezer.

Structure of this chapter is as follows. In Section 5.1, we recall the notion of Lotz-Räbiger net and some of its relations with ergodic operators. In Section 5.2, after stating various definitions, we give examples which are needed in the sequel. In particular, the notions of $\mathfrak{c}$-Markov, $\mathfrak{d}$-martingale and $\mathfrak{d}$-Lotz-Räbiger nets are introduced. Prime examples of these notions can be obtained from the classical cases by setting $\mathfrak{d} = \mathfrak{c} = n$. In Section 5.3, we discuss general properties of $\mathfrak{d}$-asymptotic equivalence relation. We conclude from Proposition 5.3.5 that two operator nets are asymptotically equivalent in the unbounded sense if they are asymptotically equivalent in the classical sense, and in addition, if their orbits satisfy certain topological properties. In Section 5.4, we prove several properties on $\mathfrak{d}$-convergent and $\mathfrak{d}$-bounded operator nets. A new class of operators, called $\mathfrak{d}$-limit of a net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ of operators is deduced. These operators are denoted by $\hat{\mathfrak{T}}^{\mathfrak{d}}$. As $\mathfrak{d}$ varies over convergences on $X$, one obtains different operators $\hat{\mathfrak{T}}^{\mathfrak{d}}$. In Section 5.5, we study the notion of $un$-abelian nets and their relations to $\mathfrak{d}$-convergent and $\mathfrak{d}$-bounded operator nets. The approach given in [20] is used in the settings of convergences. In Section 5.6, we prove basic properties of $\mathfrak{d}$-Lotz-Räbiger nets and $\mathfrak{d}$-martingale nets. One of the purposes of this section is to obtain results generalizing some results of [31, 25, 20, 22, 23, 24, 31, 52, 62] to the settings of convergences. In Section 5.7, we state results regarding $o$-bounded $o$-Lotz-Räbiger nets.

Throughout the present chapter, $\Lambda$ denotes a nonempty directed set.

## 5.1 Basic Concepts Related to Lotz-Räbiger Nets on Banach Spaces

Definition 5.1.1, see below, has been introduced in [62, Definition 2.1] under the name $M$-net. It was motivated by the notion of $\mathscr{S}$-ergodic net of an operator semigroup and by the notion of $M$-sequences, see [62]. In this thesis, following the conventions of [22, 23, 24, 31], we will use the nomenclature Lotz-Räbiger nets instead of $M$-nets.

**Definition 5.1.1.** A net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ of operators acting on a Banach space $X$ is said to be a Lotz-Räbiger net (abbreviated as an *LR*-net) if

*(i)* the net $\mathfrak{T}$ of operators is uniformly bounded, i.e., $\sup_{\lambda\in\Lambda}\|T_\lambda\| < \infty$,

*(ii)* $\lim_\lambda T_\lambda(T_\mu - I)x = 0$ for all $x \in X$ and $\mu \in \Lambda$,

*(iii)* $\lim_\lambda (T_\mu - I)T_\lambda x = 0$ for all $x \in X$ and $\mu \in \Lambda$,

where $I$ stands for the identity operator on $X$.

Various examples of Lotz-Räbiger nets can be found in [25, 20, 22, 23, 24, 31, 52, 62]. The following examples are needed in the sequel.

**Example 5.1.2.** *Every concrete example of an $\mathscr{S}$-ergodic net on a Banach space, in the sense of [49, Chapter 2.2] and of [62, Example 2.2.d] is an example of a Lotz-Räbiger net. It is instructive to recall this important notion. Let $\mathscr{S}$ be a semigroup of norm bounded operators $T\colon X \to X$ where $X$ is a Banach space. A uniformly bounded net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ of operators on $X$ is said to be (a two sided) $\mathscr{S}$-ergodic net for the semigroup $\mathscr{S}$ if*

(i) *$T_\lambda x$ belongs to the closed convex hull of the orbit $\mathscr{S}x$ for every $x \in X$ and every $\lambda \in \Lambda$,*

(ii) *$\lim_\lambda T_\lambda(I - T)x = 0$ for all $x \in X$ and $T \in \mathscr{S}$,*

(iii) *$\lim_\lambda (I - T)T_\lambda x = 0$ for all $x \in X$ and $T \in \mathscr{S}$,*

*where $I$ stands for the identity operator on $X$. If the norm limits are replaced by the weak limits then $\mathfrak{T}$ is said to be a weakly $\mathscr{S}$-ergodic net associated with the semigroup $\mathscr{S}$. For more details on the case when $X$ is a topological vector space, we refer reader to [49, Chapter 2.2]. In this general case, the uniform boundedness assumption can be replaced by an equi-continuity assumption, see Section 4.2.*

**Example 5.1.3.** *This example is a particular case of Example 5.1.2. Let $T\colon X \to X$ be a norm bounded operator on a Banach space $X$ such that $n^{-1}T^n \to 0$ strongly and that the sequence $A_n^T = \frac{1}{n}\sum_{k=0}^{n-1} T^k$ of Cesàro averages of $T$ is uniformly bounded. Then $\mathfrak{T} = (A_n^T)_{n=1}^\infty$ is an LR-net with the index set $\Lambda = \mathbb{Z}^+$. In the settings of Example 5.1.2, if one takes $\mathscr{S}$ as the discrete semigroup generated by $T$, i.e., $\mathscr{S} = \{T^k\colon k \geq 0\}$, then the sequence $A_n^T$ can be regarded as an $\mathscr{S}$-ergodic net for the semigroup $\mathscr{S}$.*

**Definition 5.1.4.** ([19, Section 8.4]) Let $(X,\tau)$ be a topological vector space. A continuous operator $T\colon X \to X$, in the sense of Section 4.2, is said to be *mean ergodic* if

$$P_T(x) = \tau\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} T^k(x)$$

exists for every $x \in X$.

By [62, Theorem 1.2], every power bounded operator on a reflexive Banach space $X$ is mean ergodic. In particular, if $X$ is a Hilbert space and $T\colon X \to X$ is a contraction then the operator $T$ is mean ergodic. In this case, the conditions of Example 5.1.3 are satisfied, i.e., Cesàro averages of $T$ is uniformly bounded and $n^{-1}T^n \to 0$ strongly. Conversely, if $X$ is a Banach space and $T\colon X \to X$ is mean ergodic then Cesàro averages of $T$ is uniformly bounded and $n^{-1}T^n \to 0$ strongly, see [19, Lemma 8.16].

## 5.2 Preliminary Definitions Related to $\mathfrak{d}$-Lotz-Räbiger and $\mathfrak{d}$-Martingale Nets

Let $X$ be a vector lattice. Following [10, 13] and Section 2.2.6, we say that $\mathfrak{c}$ is a convergence for nets over $X$ if the linear and lattice operations on $X$ are continuous with respect to $\mathfrak{c}$. The pair $(X,\mathfrak{c})$ is said to be a convergence vector lattice.

Let $J$ be an order dense ideal, see [1] and [5] for more on this notion, in the vector lattice $(X,\mathfrak{c})$. Following [13], we write $x_\lambda \xrightarrow{u_J\mathfrak{c}} x$ for a net $x_\lambda \in X$ if $|x_\lambda - x| \wedge u \xrightarrow{\mathfrak{c}} 0$ for all $u \in J^+$. In the case $J = X$, we write $u\mathfrak{c}$ instead of $u_J\mathfrak{c}$.

Denote by $L(X,\mathfrak{c})$ the algebra of all $\mathfrak{c}$-continuous linear operators $T\colon (X,\mathfrak{c}) \to (X,\mathfrak{c})$. These operators are precisely those linear maps that send $\mathfrak{c}$-convergent nets to $\mathfrak{c}$-convergent nets. It is known that not all convergences on $X$ yield a linear topology on $X$, see [41, 45] and Section 2.2 for more information.

Two nets $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ and $\mathfrak{S} = (S_\lambda)_{\lambda\in\Lambda}$ in the algebra $L(X)$ of bounded operators in a normed lattice $X$ are said to be *strongly asymptotically equivalent* if $(T_\lambda - S_\lambda)(x) \xrightarrow{n} 0$ for all $x \in X$. This is the classical definition of strong asymptotic equivalence. Asymptotic equivalences are known to be useful and extremely important tools to study infinite behaviors of strongly convergent operator nets and continuous semigroups.

In the present chapter, we study operator nets with respect to two arbitrary onvergences $\mathfrak{c}$ and $\mathfrak{d}$ on $X$. Our main focus is the case when $\mathfrak{d} = \mathfrak{c}$ or $\mathfrak{d} = u_J\mathfrak{c}$ for an order dense ideal $J \leq X$.

Let $\Lambda$ be a partially ordered set. We denote by $\mathbf{T}$ the constant $\Lambda$-indexed net at $T \in L(X,\mathfrak{c})$. When we use this notation for constant operator nets, the partially ordered index set $\Lambda$ is understood from the context.

We assume that all convergences $\mathfrak{c}$ have the property $T_1$ in the sense that the constant net $x_\lambda$ on the singleton $\{x\} \subset X$ satisfies $x_\lambda \xrightarrow{\mathfrak{c}} x$. Two convergences are said to be *equivalent* if they have exactly the same convergent nets. Because $\mathfrak{c}$ has property $T_1$, two examples of $\Lambda$-indexed operator nets are $\mathbf{1}$ and $\mathbf{0}$. The former is the constant net at the identity operator of $(X,\mathfrak{c})$ and the latter is the constant net at the zero operator on $(X,c)$.

**Example 5.2.1.** *Even in the case of a topological $T_1$ convergence $\mathfrak{c}$, a $\mathfrak{c}$-convergent net need* *not have unique limits. An example of the form $\mathfrak{c} = u_J n$ on $L^p(\mu)$ ($1 \le p < \infty$) with $\mu$ being a finite measure can be found in [44, Example 1.3]. We note that this agrees with $u_J n$ being $T_1$ because the constant net $x_\lambda$ on the singleton $\{x\} \subset L^p(\mu)$ still satisfies $x_\lambda \xrightarrow{u_J n} x$.*

Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ and $\mathfrak{S} = (S_\lambda)_{\lambda\in\Lambda}$ be two $\Lambda$-indexed operator nets belonging to collection $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ of all $\Lambda$-indexed $\mathfrak{c}$ to $\mathfrak{c}$ continuous operator nets on $(X,\mathfrak{c})$ for some convergence $\mathfrak{c}$ on $X$. The collection $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$, when it is equipped with the pointwise product

$$(\mathfrak{T}\cdot\mathfrak{S})_\lambda = T_\lambda \circ S_\lambda$$

for $\mathfrak{T},\mathfrak{S} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ and $\lambda \in \Lambda$, forms both a real algebra, in the usual sense, and a monoid with unit element $\mathbf{1}$.

**Example 5.2.2.** *Let $\Lambda$ be a singleton and $\mathfrak{c} = uaw$ for a Banach lattice $X$. In this case, $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ can be identified with the collection $L(X,uaw)$ of all $T\colon X \to X$ satisfying $T(x_\alpha) \xrightarrow{uaw} 0$ whenever $x_\alpha \xrightarrow{uaw} 0$. Evidently, the linear space $L(X,uaw)$ is closed under pre- and post-compositions. Similarly, consider the case $\mathfrak{c} = uo$, the unbounded convergence on the vector lattice $X$. It follows that $T \in L(X,uo)$ if and only if $T$ is unbounded order continuous precisely in the sense of [9, Definition 1].*

**Example 5.2.3.** *As $\mathfrak{c}$ varies over all convergences on $X$, elements of the algebras $L(X,\mathfrak{c})$, and hence the algebras $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$, are quite different. Consider the vector lattice $X = \ell_\infty$ of all bounded real sequences. Let $e_n$ be the standard unit vectors. If $\mathfrak{c} = uo$ then any operator $T \in L(X,\mathfrak{c})$ should satisfy $T(e_n) \xrightarrow{uo} 0$ because $e_n \xrightarrow{uo} 0$. However, if $\mathfrak{c} = un$ then the sequence $e_n$ is not un-null.*

Denote by $X'_c$ the linear space of $\mathfrak{c}$ to norm continuous linear functionals on $X$. We remark that functionals satisfying various continuity conditions are also used in [28, 55] in connection with a measure free version of Brezis-Lieb lemma. We further note that *uo*-continuity of order bounded functionals are discussed in [9, Theorem 1].

**Definition 5.2.4.** Let $\mathfrak{d}$ be an arbitrary convergence, possibly different than $\mathfrak{c}$, on $X$. A net $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is said to be $\mathfrak{d}$-*convergent* if for every $x \in X$ the net $T_\lambda(x)$ $\mathfrak{d}$-converges to some $y \in X$.

In particular, if $\mathfrak{c}$ and $\mathfrak{d}$ are both equivalent to the same norm convergence of a Banach lattice structure on $X$, then $\mathfrak{T}$ is $\mathfrak{d}$-convergent if and only if $\mathfrak{T}$ is strongly stable, i.e., pointwise norm convergent.

**Example 5.2.5.** *Let $X$ be a vector lattice and $T\colon X \to X$ an order continuous operator satisfying o-$\lim_k T^k(x) = 0$ for all $x \in X$. In particular, uo-$\lim_k T^k(x) = 0$ for all $x \in X$. Thus, $\mathfrak{T} = (T_k)_{k\ge 1}$ with $T_k = T^k$ for $k \ge 1$ satisfies $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ with $\mathfrak{c} = o$, $\Lambda = \mathbb{N}$, and, moreover $\mathfrak{T}$ is a uo-convergent net.*

**Example 5.2.6.** *Let $X$ be a Banach lattice, $\Lambda = \{t \in \mathbb{R} : t \geq 0\}$ and let $\mathfrak{T} = (T_t)_{t\geq 0}$ be a strongly continuous semigroup. If $\mathfrak{T}$ is strongly stable then un-*$\lim_t T_t(x) = 0$ *for every $x \in X$. Hence, $\mathfrak{T}$ is un-convergent.*

Following definition is very useful for our purposes. We say that the net $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is $\mathfrak{d}$-*bounded* in $L(X,c)$ if there exists a positive operator $S \in L(X,\mathfrak{c}) \cap L(X,\mathfrak{d})$ such that $|T_\lambda| \leq S$ for every $\lambda \in \Lambda$. Let us remark the obvious fact that we assume $|T_\lambda|$ exists whenever $\mathfrak{T}$ is $\mathfrak{d}$-bounded for the convergence $\mathfrak{d}$, see [5] for details on modulus of an operator, for all $\lambda \in \Lambda$.

An operator net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ of positive operators is said to be a $\mathfrak{c}$-*Markov net* if there exists a weak unit $x_0 \in X^+$ and a strictly positive $\mathfrak{c}$-continuous functional $x_0' \in X_c'$ such that $T_\lambda(x_0) = x_0$ and $T_\lambda'(x_0') = x_0'$ for all $\lambda \in \Lambda$.

**Definition 5.2.7.** Two nets $\mathfrak{T}, \mathfrak{S} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ are said to be $\mathfrak{d}$-*asymptotically equivalent* if

$$(T_\lambda - S_\lambda)(x) \xrightarrow{\mathfrak{d}} 0$$

for all $x \in X$. We put $\mathfrak{T} \approx_\mathfrak{d} \mathfrak{S}$ whenever $\mathfrak{T}$ and $\mathfrak{S}$ are $\mathfrak{d}$-asymptotically equivalent.

The notions of $\mathfrak{d}$-martingale and $\mathfrak{d}$-*LR*-nets are well motivated, see [25, 20, 22, 23, 52, 62] for their applications. Since operator nets may not be uniformly bounded in the settings of convergences, uniformly bounded $n$-martingale and $n$-*LR*-nets are martingale and *LR*-nets in the sense of [20, 23], respectively. In particular, any concrete example of *LR*-nets is an example of a $\mathfrak{d}$-*LR*-net with $\mathfrak{d} = n$. Converse is false even in the case of $\mathfrak{d} = n$, see Example 5.6.7 for details.

**Definition 5.2.8.** A net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is said to be a $\mathfrak{d}$-*martingale net* on the vector lattice $X$ if

$$\mathbf{T}_\mu \cdot \mathfrak{T} \approx_\mathfrak{d} \mathbf{T}_\mu \approx_\mathfrak{d} \mathfrak{T} \cdot \mathbf{T}_\mu$$

for all $\mu \in \Lambda$.

See Example 5.6.1 for a concrete example of a $\mathfrak{d}$-martingale net where positive projections together with a $T_1$-convergence $\mathfrak{d}$ are used. We recall the standing convention that $\mathbf{T}_\mu$ denotes the constant $\Lambda$-net at $T_\mu \in L(X,\mathfrak{c})$.

**Definition 5.2.9.** A net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is said to be a $\mathfrak{d}$-*Lotz-Räbiger net* (or $\mathfrak{d}$-*LR*-net for short) on the vector lattice $X$ if

$$\mathbf{T}_\mu \cdot \mathfrak{T} \approx_\mathfrak{d} \mathfrak{T} \approx_\mathfrak{d} \mathfrak{T} \cdot \mathbf{T}_\mu$$

for all $\mu \in \Lambda$.

### 5.3 Properties of $\mathfrak{d}$-Asymptotic Equivalence

Recall from [13] that a convergence $\mathfrak{d}$ on the vector lattice $X$ is said to be Lebesgue if $x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{\mathfrak{d}} 0$. Equivalently, the convergence $\mathfrak{d}$ is said to be Lebesgue if every order convergent net in $X$ is $\mathfrak{d}$-convergent in $X$. Examples of Lebesgue convergences include the norm convergence on an order continuous Banach lattice, the topological convergence on a locally solid vector lattice having the Lebesgue property, see Definition 4.1.1, and the order convergence itself which need not to be topological.

**Proposition 5.3.1.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ and $\mathfrak{S} = (S_\lambda)_{\lambda\in\Lambda}$ be $\Lambda$-indexed nets belonging to $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$. Let $\mathfrak{d}$ be a convergence on $X$. Denote by $J$ an order dense ideal in $X$.*

i. *If $\Gamma$ denotes a subnet and $\mathfrak{T} \approx_{u_J\mathfrak{c}} \mathfrak{S}$ then $(T_\gamma)_{\gamma\in\Gamma} \approx_{u_J\mathfrak{c}} (S_\gamma)_{\gamma\in\Gamma}$ in $\mathscr{N}_\Gamma(L(X,c))$. In particular, this is the case when $J = X$.*

ii. *If $\mathfrak{T} \approx_{\mathfrak{d}} \mathfrak{S}$ and $\mathfrak{T}' \approx_{\mathfrak{d}} \mathfrak{S}'$, not to be confused with adjoints, then $t\mathfrak{T}+\mathfrak{T}' \approx_{\mathfrak{d}} t\mathfrak{S}+\mathfrak{S}'$ for any scalar $t$. In particular, if $\mathfrak{d} = u_J\mathfrak{c}$ then $\mathfrak{T} \approx_{u_J\mathfrak{c}} \mathfrak{S}$ and $\mathfrak{T}' \approx_{u_J\mathfrak{c}} \mathfrak{S}'$ imply that $t\mathfrak{T}+\mathfrak{T}' \approx_{u_J\mathfrak{c}} t\mathfrak{S}+\mathfrak{S}'$ for any scalar $t$.*

iii. *$\mathfrak{T} \approx_{un} \mathfrak{S}$ if and only if $|(T_\lambda - S_\lambda)(x)| \xrightarrow{un} 0$ for all $x \in X$.*

iv. *If $T_\lambda(x) \xrightarrow{un} y$ for some $x, y \in X$ and $\mathfrak{T} \approx_{un} \mathfrak{S}$ then $\|y\| \le \liminf_\lambda \|S_\lambda(x)\|$.*

v. *The set*
$$R_{\approx_{\mathfrak{d}}} := \{R \subseteq \mathscr{N}_\Lambda(L(X,\mathfrak{c})) \times \mathscr{N}_\Lambda(L(X,\mathfrak{c}))\colon R\circ \approx_{\mathfrak{d}} = \approx_{\mathfrak{d}} \circ R\},$$
*which in particular contains all equivalence relations on $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ that commute with $\mathfrak{d}$-asymptotic equivalence, is nonempty, and, it forms a semigroup under composition.*

vi. *$\mathfrak{T} \approx_o \mathfrak{S}$ implies $\mathfrak{T} \approx_{\mathfrak{c}} \mathfrak{S}$ whenever $\mathfrak{c}$ is a Lebesgue convergence on $X$.*

*Proof.* $(i)$. Suppose that $(T_\lambda - S_\lambda)(x) \xrightarrow{u_J\mathfrak{c}} 0$ for all $x \in X$. For any subnet $\Gamma$ of $\Lambda$, one has $(T_\gamma - S_\gamma)(x) \xrightarrow{u_J\mathfrak{c}} 0$ where the limit is over $\gamma \in \Gamma$.

$(ii)$. Let $x \in X$ be such that $(T_\lambda - S_\lambda)(x) \xrightarrow{\mathfrak{d}} 0$ and $(T'_\lambda - S'_\lambda)(x) \xrightarrow{\mathfrak{d}} 0$. It follows that $(tT_\lambda - T'_\lambda - tS_\lambda - S'_\lambda)(x) = t(T_\lambda - S_\lambda)(x) - (T'_\lambda - S'_\lambda)(x) \xrightarrow{\mathfrak{d}} 0$ as each term tends to 0. Linear and lattice operations are continuous with respect to $\mathfrak{d}$. Particular case follows from the fact that if $J \le X$ is an order dense ideal of $X$ then $\mathfrak{d} = u_J\mathfrak{c}$ is indeed a convergence on $X$.

$(iii)$. The results follows from definitions, see Section 2.2.3 and Definition 5.2.7. In details, $\mathfrak{T} \approx_{un} \mathfrak{S}$ if and only if for every $x \in X$ one has $|(T_\lambda - S_\lambda)(x)| \wedge u \xrightarrow{n} 0$ for all $u \in X^+$.

$(iv)$. If $T_\lambda(x) \xrightarrow{un} y$ and $\mathfrak{T} \approx_{un} \mathfrak{S}$ then $S_\lambda(x) \xrightarrow{un} y$. To see this, we note that

$$|S_\lambda(x) - y| \wedge u \le |S_\lambda(x) - T_\lambda(x)| \wedge u + |T_\lambda(x) - y| \wedge u$$

for every $x \in X$, $\lambda \in \Lambda$ and $u \in X^+$. Each term of the right side converges to zero in norm. Thus by [18, Lemma 2.8], one has $\|y\| \leq \liminf_\lambda \|S_\lambda(x)\|$.

$(v)$. It is easy to see that the set of all equivalence relations on $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ that commute with $\approx_\mathfrak{d}$ is nonempty. Composition of two such relations, see [29, Section 8.1], commutes with the relation $\approx_\mathfrak{d}$.

$(vi)$. One has $\mathfrak{T} \approx_o \mathfrak{S}$ if and only if $(T_\lambda - S_\lambda)(x) \xrightarrow{o} 0$ for all $x \in X$. Because $\mathfrak{c}$ is Lebesgue convergence on $X$, one has $(T_\lambda - S_\lambda)(x) \xrightarrow{\mathfrak{c}} 0$. Hence, result follows. $\square$

**Remark 5.3.2.** *Let X ba Banach lattice. In view of the fact that $un = u_J n$ with $J = X$, it is natural to ask if statement $(iv)$ of Proposition 5.3.1 can be generalized to the case where J is not order dense in X. Because of the non Hausdorff cases, the answer of this question is negative.*

In the following proposition, we give a list of implications of equivalences as the convergence $\mathfrak{c}$ varies over previously known convergences. The notations *so* and *ru* stand for sequential order convergence and relative uniform convergence, respectively. We recall from [16] that $x_\lambda \xrightarrow{ru} x$ in a vector lattice $X$ if there exists some $u \in X^+$ such that for every $n \in \mathbb{N}$ there exists some $\lambda_n$ such that $|x_\lambda - x| \leq \frac{1}{n}u$ for every $\lambda \geq \lambda_n$.

**Proposition 5.3.3.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ and $\mathfrak{S} = (S_\lambda)_{\lambda\in\Lambda}$ be $\Lambda$-indexed nets belonging to $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$.*

i. $\mathfrak{T} \approx_{ru} \mathfrak{S}$ *if and only if* $\mathfrak{T} \approx_o \mathfrak{S}$ *whenever X is a Lebesgue and complete metrizable locally solid vector lattice.*

ii. $\mathfrak{T} \approx_{so} \mathfrak{S}$ *if and only if* $\mathfrak{T} \approx_n \mathfrak{S}$ *whenever X is a Banach lattice of countable type which is lattice isomorphic to $c_0$ and $\Lambda = \mathbb{N}$.*

*Proof.* See [13] and the references therein. In the case of Lebesgue and complete metrizable locally solid vector lattice a net $x_\lambda$ relatively uniformly converges to some $x \in X$ if and only if $x_\lambda$ order converges to $x \in X$. It follows that if we let $\mathfrak{c} = ru$ where $ru$ denotes the relatively uniformly convergence then the result follows. If $X$ is a Banach lattice of countable type and lattice isomorphic to $c_0$ then sequential order convergence agrees with the norm convergences. $\square$

A subset $S$ of a Banach lattice $X$ is almost order bounded, see [75, Chapter 18], if for every $\varepsilon > 0$ there exists some $u \in X^+$ such that $\|(|x| - u)^+\| < \varepsilon$ for all $x \in S$. Equivalently, a subset $S$ of $X$ is almost order bounded if and only if for every $\varepsilon > 0$ there exists $u \in X^+$ such that $S \subseteq [-u,u] + \varepsilon B_X$ where $B_X$ denotes the closed unit ball of $X$.

**Proposition 5.3.4.** *Let $X$ be a Banach lattice, and $\mathfrak{T}, \mathfrak{S} \in \mathscr{N}_\Lambda(L(X,n))$. Denote by $\mathscr{O}(x) = \{(T_\lambda - S_\lambda)(x) \colon \lambda \in \Lambda\}$ the set of differences of orbits of $x \in X$. Let $\mathfrak{d}$ be a convergence on $X$ which is not necessarily topological. The following cases*

i. *$\mathscr{O}(x)$ is order bounded for all $x \in X$.*

ii. *$\mathscr{O}(x)$ is un-totally bounded for all $x \in X$.*

iii. *$\mathscr{O}(x)$ is almost order bounded for all $x \in X$.*

iv. *$\mathscr{O}(x)$ is norm bounded for all $x \in X$.*

v. *$\mathscr{O}(x)$ is (relatively) sequentially compact for all $x \in X$.*

vi. *$\mathscr{O}(x)$ is relatively countably compact for all $x \in X$.*

vii. *$\mathscr{O}(x)$ is relatively un-(weakly) countably compact for all $x \in X$.*

viii. *$\mathscr{O}(x)$ has conditional $\mathfrak{d}$-Bolzano-Weierstrass property for all $x \in X$.*

*all induce an equivalence relation on the algebra $\mathscr{N}_\Lambda(L(X,n))$ by setting $\mathfrak{T} \approx \mathfrak{S}$ if and only if $\mathscr{O}(x)$ satisfies the given property for every $x \in X$. Moreover, regarding the induced relations, $(i)$ implies $(iii)$, $(iii)$ implies $(iv)$, and, $(v)$ implies $(vi)$.*

*Proof.* Proofs of $(i), (iv), (v)$ and $(vi)$ are similar to proofs given below.

$(ii)$. To show that $\mathfrak{T} \approx \mathfrak{S}$ and $\mathfrak{S} \approx \mathfrak{W}$ imply $\mathfrak{T} \approx \mathfrak{W}$ with $\mathfrak{W} = (W_\lambda)_{\lambda \in \Lambda}$, let $V_{\varepsilon,u} = \{x \in X \colon \| |x| \wedge u \| < \varepsilon\}$ be a zero neighborhood in the *un*-topology with $\varepsilon > 0$ and $u \in X^+$. In [45], these neighborhoods for the *un*-topology are introduced. Because $\{(T_\lambda - S_\lambda)(x) \colon \lambda \in \Lambda\}$ and $\{(S_\lambda - W_\lambda)(x) \colon \lambda \in \Lambda\}$ are *un*-totally bounded, one has

$$\{(T_\lambda - W_\lambda)(x) \colon \lambda \in \Lambda\} \subseteq \bigcup_{i=1}^{m} T_{\lambda_i}(x) - W_{\lambda_i}(x) + 2V_{\varepsilon,u}$$

for some $\lambda_1, \dots, \lambda_m \in \Lambda$. It follows from Theorem 3.1 of [5] that the set $\{(T_\lambda - W_\lambda)(x) \colon \lambda \in \Lambda\}$ is *un*-totally bounded. Thus, $\mathfrak{T} \approx \mathfrak{W}$ holds.

$(iii)$. Algebraic sums of almost order bounded sets and their nonempty subsets are almost order bounded, see [75, Chapter 18]. Hence, let $\mathfrak{W} = (W_\lambda)_{\lambda \in \Lambda}$. It follows from $(T_\lambda - W_\lambda)(x) = (T_\lambda - S_\lambda)(x) + (S_\lambda - W_\lambda)(x)$ for $x \in X$, $\lambda \in \Lambda$. Hence, for each $x \in X$,

$$\{(T_\lambda - W_\lambda)(x) \colon \lambda \in \Lambda\} \subseteq \{(T_\lambda - S_\lambda)(x) \colon \lambda \in \Lambda\} + \{(S_\lambda - W_\lambda)(x) \colon \lambda \in \Lambda\}$$

where each of the sets in the right side is almost order bounded. Therefore, $\mathfrak{T} \approx \mathfrak{W}$.

$(vii)$. A set $S \subset X$ is relatively *un*-countably compact if and only if every sequence in $S$ has an *un*-cluster point in $X$. Given a sequence $x_n$ in the set $\{(T_\lambda - W_\lambda)(x) \colon \lambda \in \Lambda\}$ there exists sequences $x'_n$ in $\{(T_\lambda - S_\lambda)(x) \colon \lambda \in \Lambda\}$ and $x''_n$ in $\{(S_\lambda - W_\lambda)(x) \colon \lambda \in \Lambda\}$ with *un*-cluster points, such that $x_n = x'_n + x''_n$. Thus the sequence $x_n$ has an *un*-cluster point.

A set $S \subset X$ is relatively *un*-weakly countably compact if and only if every sequence in $S$ has an *un*-weak cluster point in $X$.

$(viii)$. A set $S \subset X$ has conditional $\mathfrak{d}$-Bolzano-Weierstrass property if every infinite subset of $S$ has a $\mathfrak{d}$-accumulation point in $X$, see [75]. Algebraic sum of sets having conditional $\mathfrak{d}$-Bolzano-Weierstrass property has conditional $\mathfrak{d}$-Bolzano-Weierstrass property. A nonempty subset of set having conditional $\mathfrak{d}$-Bolzano-Weierstrass property has conditional $\mathfrak{d}$-Bolzano-Weierstrass property. $\square$

If $\mathfrak{T}, \mathfrak{S} \in \mathscr{N}_\Lambda(L(X,n))$ with $\Lambda = \mathbb{N}$ and $\mathfrak{T} \approx_n \mathfrak{S}$ then the set $\{(T_n - S_n)(x) \colon n \in \mathbb{N}\}$ is relatively $n$-compact, and hence, almost order bounded. See Proposition 5.3.5 for an application of almost order bounded sets. In view of Proposition 5.3.5, one can consider different equivalences as given in Proposition 5.3.4.

**Proposition 5.3.5.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda \in \Lambda}$ and $\mathfrak{S} = (S_\lambda)_{\lambda \in \Lambda}$ be $\Lambda$-indexed nets belonging to $\mathscr{N}_\Lambda(L(X,n))$ where $X$ is a Banach lattice. If $\mathfrak{T}$ and $\mathfrak{S}$ are $n$-asymptotically equivalent then $\mathfrak{T} \approx_{un} \mathfrak{S}$. Conversely, if $\mathfrak{T} \approx_{un} \mathfrak{S}$ and the set $\mathscr{O}(x) = \{(T_\lambda - S_\lambda)(x) \colon \lambda \in \Lambda\}$ is almost order bounded for all $x \in X$ then $\mathfrak{T}$ and $\mathfrak{S}$ are $n$-asymptotically equivalent.*

*Proof.* Suppose that the nets $\mathfrak{T}$ and $\mathfrak{S}$ are $n$-asymptotically equivalent. It follows that for every $x \in X$ one has $(T_\lambda - S_\lambda)(x) \xrightarrow{n} 0$. Hence, $(T_\lambda - S_\lambda)(x) \xrightarrow{un} 0$ for every $x \in X$. Conversely, if $\mathfrak{T} \approx_{un} \mathfrak{S}$ then $(T_\lambda - S_\lambda)(x) \xrightarrow{un} 0$ for every $x \in X$. Because $\{(T_\lambda - S_\lambda)(x) \colon \lambda \in \Lambda\}$ is almost order bounded, it follows from [18, Lemma 2.9] that $(T_\lambda - S_\lambda)(x) \xrightarrow{n} 0$. Hence, the nets $\mathfrak{T}$ and $\mathfrak{S}$ are $n$-asymptotically equivalent. $\square$

**Remark 5.3.6.** *Let $X$ be a Banach lattice. We recall from [75, Chapter 18] that a subset $S$ of $X$ is almost order bounded if and only if $|S|$ is almost order bounded. Further, if $S$ is totally bounded (or order bounded) in $X$ then $S$ is almost order bounded. Therefore, in the settings of Proposition 5.3.5, if either the set $|\mathscr{O}(x)|$ is almost order bounded in $X$ for all $x \in X$, or, $\mathscr{O}(x)$ is totally bounded (or order bounded) for every $x \in X$ then $\mathscr{O}(x)$ is almost order bounded. We note that by [75, Ex. 122.7], in $\ell_p$ $(1 \leq p < \infty)$ every almost order bounded set is totally bounded. Therefore, in Proposition 5.3.5 it suffices to have $\mathscr{O}(x)$ to be totally bounded for every $x \in X$ if $X = \ell_p$ $(1 \leq p < \infty)$. Similarly, in an AM-space with a strong norm unit, every almost order bounded set is order bounded. Therefore, in Proposition 5.3.5 it suffices to have $\mathscr{O}(x)$ to be order bounded for every $x \in X$ whenever $X$ is an AM-space with a strong norm unit.*

**Proposition 5.3.7.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ and $\mathfrak{S} = (S_\lambda)_{\lambda\in\Lambda}$ be $\Lambda$-indexed nets belonging to $\mathcal{N}_\Lambda(L(X,n))$ where $X$ is an order continuous Banach lattice. If $\mathfrak{T}$ and $\mathfrak{S}$ are o-asymptotically equivalent then $\mathfrak{T} \approx_{uo} \mathfrak{S}$. Conversely, if $\mathfrak{T} \approx_{uo} \mathfrak{S}$ and the set $\mathcal{O}(x) = \{(T_\lambda - S_\lambda)(x)\colon \lambda \in \Lambda\}$ is almost order bounded for all $x \in X$ then $\mathfrak{T}$ and $\mathfrak{S}$ are o-asymptotically equivalent.*

*Proof.* Proof is similar to that of Proposition 5.3.5 but utilizes the result [41, Proposition 3.7]. Suppose that the nets $\mathfrak{T}$ and $\mathfrak{S}$ are $o$-asymptotically equivalent. It follows that for every $x \in X$ one has $(T_\lambda - S_\lambda)(x) \xrightarrow{o} 0$. Hence, $(T_\lambda - S_\lambda)(x) \xrightarrow{uo} 0$ for every $x \in X$. Conversely, if $\mathfrak{T} \approx_{uo} \mathfrak{S}$ then $(T_\lambda - S_\lambda)(x) \xrightarrow{uo} 0$ for every $x \in X$. Because $\{(T_\lambda - S_\lambda)(x)\colon \lambda \in \Lambda\}$ is almost order bounded, it follows from [41, Proposition 3.7] that $(T_\lambda - S_\lambda)(x) \xrightarrow{o} 0$. Hence, the nets $\mathfrak{T}$ and $\mathfrak{S}$ are $o$-asymptotically equivalent. $\square$

**Proposition 5.3.8.** *Suppose that $\mathfrak{T}, \mathfrak{S} \in \mathcal{N}_\Lambda(L(X,\mathfrak{c}))$ with $X = L_1(\mu)$, $\mathfrak{c} = \|\cdot\|_1$. If $\mathfrak{T} \approx_{un} \mathfrak{S}$ and $\mathcal{O}(f) = \{(T_\lambda - S_\lambda)(f)\colon \lambda \in \Lambda\}$ is relatively weakly compact for every $f \in L_1(\mu)$ then $\mathfrak{T} \approx_n \mathfrak{S}$.*

*Proof.* By the Dunford-Pettis theorem, a subset of $L_1(\mu)$ is relatively weakly compact if and only if it is almost order bounded. By Proposition 5.3.5, the result follows. $\square$

**Proposition 5.3.9.** *Let $X$ be an order continuous Banach lattice. Also let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ and $\mathfrak{S} = (S_\lambda)_{\lambda\in\Lambda}$ be un-asymptotically equivalent nets belonging to $\mathcal{N}_\Lambda(L(X,\mathfrak{c}))$. For every $x \in X$ there exists an increasing sequence $\lambda_k$ in $\Lambda$ depending on $x$, $\mathfrak{T}$ and $\mathfrak{S}$ such that $(T_{\lambda_k} - S_{\lambda_k})(x) \xrightarrow{un} 0$ and $(T_{\lambda_k} - S_{\lambda_k})(x) \xrightarrow{uo} 0$.*

*Proof.* This follows from [18, Corollary 3.5]. Indeed, for each fixed $x \in X$, we have a net $(T_\lambda - S_\lambda)(x)$ in $X$ satisfying $(T_\lambda - S_\lambda)(x) \xrightarrow{un} 0$. An application of [18, Corollary 3.5] to the net $(T_\lambda - S_\lambda)(x)$ gives the result. $\square$

We end this section by recalling some standard terminology. Group of units of the algebra $\mathcal{N}_\Lambda(L(X,\mathfrak{c}))$ for a convergence $\mathfrak{c}$ is the set

$$U(\mathcal{N}_\Lambda(L(X,\mathfrak{c}))) = \{\mathfrak{T} \in \mathcal{N}_\Lambda(L(X,\mathfrak{c}))\colon \mathfrak{T}\cdot\mathfrak{S} = \mathfrak{S}\cdot\mathfrak{T} = \mathbf{1} \text{ for some } \mathfrak{S} \in \mathcal{N}_\Lambda(L(X,\mathfrak{c}))\}.$$

Elements of $U(\mathcal{N}_\Lambda(L(X,\mathfrak{c})))$ are called units of $\mathcal{N}_\Lambda(L(X,\mathfrak{c}))$.

We say that a net $\mathfrak{T} \in \mathcal{N}_\Lambda(L(X,\mathfrak{c}))$ is $\mathfrak{d}$-*unit* if $\mathfrak{T}$ is $\mathfrak{d}$-asymptotically equivalent to a unit of $\mathcal{N}_\Lambda(L(X,\mathfrak{c}))$. In Example 5.3.10, we give concrete examples of $o$-units. In Proposition 5.5.11 we use the notion of $\mathfrak{d}$-units.

**Example 5.3.10.** *Let $E$ be a Dedekind complete vector lattice and $F$ be a non-empty set. Consider the Dedekind complete vector lattice $X = E^F$. For every $\phi\colon F \to F$ the linear maps $T_\phi\colon E^F \to E^F$ defined by $T_\phi(x) = x \circ \phi$ are lattice homomorphisms. If $\phi$ is a bijection* *then $T_\phi$ is a lattice automorphism. It is a well-known fact, see [5, Theorem 2.21], that a lattice homomorphism is order continuous if and only if its kernel is a band. In particular, if $\phi\colon F \to F$ is a bijection then $T_\phi$ is an order continuous lattice automorphism. For every $\phi\colon F \to F$, the algebra $\mathcal{N}_\Lambda(L(X,o))$ contains the constant $\Lambda$-indexed operators $\mathbf{T}_\phi$, and hence, it contains the algebra generated by such operators. Clearly, $\mathbf{T}_\phi \cdot \mathbf{T}_{\phi_2} = \mathbf{T}_{\phi_1\phi_2}$. When $\phi\colon F \to F$ is a bijection, $\mathbf{T}_\phi \in U(\mathcal{N}_\Lambda(L(X,\mathfrak{c})))$. Every such $\mathbf{T}_\phi$ is an o-unit of the algebra $\mathcal{N}_\Lambda(L(X,\mathfrak{c}))$. Consider the case $\Lambda = \mathbb{Z}$. If $\phi\colon F \to F$ is a bijection then the net $\mathfrak{S} = (S_k)_{k\in\mathbb{Z}}$ where $S_k := T_\phi^k$ for $k \in \mathbb{Z}$ belongs to $\mathcal{N}_\Lambda(L(X,\mathfrak{c}))$ with $\Lambda = \mathbb{Z}$.*

## 5.4 $\mathfrak{d}$-Convergent, $\mathfrak{d}$-Bounded and $\mathfrak{d}$-Markov Operator Nets

We recall from [13] that a convergence $\mathfrak{d}$ on the vector lattice $X$ is Lebesgue if $x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{\mathfrak{d}} 0$.

**Theorem 5.4.1.** *Let $\mathfrak{d}$ be a Lebesgue convergence on a Dedekind complete vector lattice $X$. If a $\mathfrak{d}$-closed ideal $I \leq X$ is $\mathfrak{T}$-invariant for some $\mathfrak{T} = (T_{\lambda'})_{\lambda'\in\Lambda} \in \mathcal{N}_\Lambda(L(X,\mathfrak{c}))$ consisting of order bounded operators $T_{\lambda'}$ then $I$ is $\mathfrak{S}$-invariant for every $\mathfrak{S} = (S_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{N}_\Lambda(L(X,\mathfrak{c}))$ with $S_\lambda \in \mathrm{Band}(T_{\lambda'}\colon \lambda' \in \Lambda) \leq L_b(X)$ for all $\lambda \in \Lambda$.*

*Proof.* It suffices to show that if $S$ belongs to the band generated by $\{T_{\lambda'}\colon \lambda' \in \Lambda\}$ in the space of order bounded operators $L_b(X)$, see [5, Section 1], then $S$ leaves $I$ invariant.

Suppose that the statement is true for the ideal $I'$ generated by $\{T_{\lambda'}\colon \lambda' \in \Lambda\}$ in $L_b(X)$. Then there is an increasing net $T_\alpha \in I'$ such that $0 \leq T'_\alpha \uparrow |S|$. This implies, see Theorem 1.19 of [5], that for every $x \in I$ one has $T_\alpha|x| \uparrow S|x|$. Because $\mathfrak{d}$ is a Lebesgue convergence, $T_\alpha|x| \xrightarrow{\mathfrak{d}} |S||x|$. Since $T_\alpha|x| \in I$ and $I$ is $\mathfrak{d}$-closed, $|S||x| \in I$. Since $I$ is solid, $|Sx| \leq |S||x| \in I$ implies $Sx \in I$.

Suppose that $S$ belongs to $I'$. There exists $T_{\lambda'_1}, \dots, T_{\lambda'_n}$ such that $|S|$ is bounded by $\Sigma_{i=1}^n t_i|T_{\lambda'_i}|$ for some positive scalars $t_i \in \mathbb{R}$. Thus, $|S(x)| \in I$ whenever $x \in I$. ☐

**Remark 5.4.2.** *In Theorem 5.4.1, we considered a $\mathfrak{d}$-closed ideal $I$ of a convergence vector lattice $(X,\mathfrak{c})$ where $\mathfrak{d}$ is an arbitrary (and hence not necessarily topological) convergence on $X$. One of the main motivations for using topological properties in the settings of non-topological convergences is based on the notion of order closed, see [5, page 33] and [76, Section 15], subsets of $X$. The importance of this approach lies in the usage of order closed sets in Fatou topologies, see [4, Chapter 4] and [36, Section 23].*

Following result is motivated from elementary topology. It can be used to check if an operator net $\mathfrak{T}$ is $\mathfrak{d}$-convergent for a convergence $\mathfrak{d}$ by passing to a Cauchy-equivalent convergence $\mathfrak{d}'$.

**Proposition 5.4.3.** *Suppose that two $T_1$-convergences $\mathfrak{d}'$ and $\mathfrak{d}''$ on $X$ are Cauchy-equivalent, i.e. their Cauchy nets are equal. Suppose further that $X$ is both $\mathfrak{d}'$ and $\mathfrak{d}''$ complete. In* *this case, an operator net $\mathfrak{T}$ is $\mathfrak{d}'$-convergent, see Definition 5.2.4, if and only if it is $\mathfrak{d}''$-convergent.*

*Proof.* Result follows from the fact that if the net $T_\lambda(x)$ is $\mathfrak{d}'$-convergent then it is $\mathfrak{d}'$-Cauchy. Since the convergences $\mathfrak{d}'$ and $\mathfrak{d}''$ have the same Cauchy nets, the net $T_\lambda(x)$ is $\mathfrak{d}''$-Cauchy. By the completeness assumption, the net $T_\lambda(x)$ is $\mathfrak{d}''$-convergent. □

**Proposition 5.4.4.** *Let $\mathfrak{T}$ be a $\mathfrak{c}$-convergent net in $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ for $\mathfrak{c} = o$. Then the net $\mathfrak{T}$ is $\mathfrak{d}$-convergent for every Lebesgue convergence $\mathfrak{d}$ on $X$.*

*Proof.* By assumption, for every $x \in X$ there is some $y \in X$ such that $T_\lambda(x) \xrightarrow{o} y$. Because $\mathfrak{d}$ is a Lebesgue convergence, see [13], this implies that $T_\lambda(x) \xrightarrow{\mathfrak{d}} y$. □

We combine Theorem 5.4.1 and Proposition 5.4.4.

**Corollary 5.4.5.** *Let $\mathfrak{d}$ be a Lebesgue convergence on a Dedekind complete vector lattice $X$. If a $\mathfrak{d}$-closed ideal $I \leq X$ is $\mathfrak{T}$-invariant for some $o$-convergent $\mathfrak{T} = (T_{\lambda'})_{\lambda' \in \Lambda} \in \mathscr{N}_\Lambda(L(X,o))$ then for every $\mathfrak{d}$-convergent $\mathfrak{S} = (S_\lambda)_{\lambda \in \Lambda} \in \mathscr{N}_\Lambda(L(X,o))$ with $S_\lambda \in \mathrm{Band}(T_{\lambda'} \colon \lambda' \in \Lambda) \leq L_b(X)$ for all $\lambda \in \Lambda$ one has $\mathfrak{d}$-$\lim_\lambda S_\lambda(x) \in I$ for all $x \in I$.*

*Proof.* By Lemma 1.54 of [5], the operator $T_{\lambda'}$ is order bounded for each $\lambda' \in \Lambda$. Because the net $\mathfrak{S} = (S_\lambda)_{\lambda \in \Lambda}$ is $\mathfrak{d}$-convergent, and, the ideal $I$ is $\mathfrak{S}$-invariant by Theorem 5.4.1, $\mathfrak{d}$-$\lim_\lambda S_\lambda(x) \in I$ for all $x \in I$. □

The notion of $\mathfrak{d}$-convergence of operator nets induces a natural bi-linear map which is related to asymptotic behavior of $\mathfrak{T}$ and iterated limits. Denote by $\mathscr{N}_{\Lambda,\mathfrak{d}}(L(X,\mathfrak{c}))$ the space of all $\mathfrak{d}$-convergent operator nets in $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$.

Given an $x \in X$, let us write

$$\hat{\mathfrak{T}}^{\mathfrak{d}}(x) = \mathfrak{d}\text{-}\lim_{\lambda \to \infty} T_\lambda(x)$$

for $\mathfrak{T} \in \mathscr{N}_{\Lambda,\mathfrak{d}}(L(X,\mathfrak{c}))$. In this case, we say that the operator $\hat{\mathfrak{T}}^{\mathfrak{d}}$ is the $\mathfrak{d}$-limit of the operator net $\mathfrak{T}$. We write $\hat{\mathfrak{T}}$ instead of $\hat{\mathfrak{T}}^{\mathfrak{d}}$ if the convergence $\mathfrak{d}$ is clear from the context. We recursively define $\hat{\mathfrak{T}}^{(k)}(x) = \hat{\mathfrak{T}} \circ \hat{\mathfrak{T}}^{(k-1)}(x)$ with $\hat{\mathfrak{T}}^{(1)} = \hat{\mathfrak{T}}$ for $k \geq 1$ and $x \in X$. Thus, $\hat{\mathfrak{T}}^{(k)}$ equals to nothing but the classical $k$-th iteration of $\hat{\mathfrak{T}}$. If we need to emphasize the fact that $k$-th iteration of the $\mathfrak{d}$-limit of $\mathfrak{T}$ depends on $\mathfrak{d}$, we write $\hat{\mathfrak{T}}^{\mathfrak{d},(k)}$ for $\hat{\mathfrak{T}}^{(k)}$.

**Example 5.4.6.** *Let $X$ be a Banach lattice with $\mathfrak{d} = n$. It follows from the classical Banach-Steinhaus theorem that if the set $\{T_k(x)\}$ is weakly bounded for every $x \in X$ and it is norm Cauchy as $x$ varies over a dense subset of $X$ then the operator sequence $T_k$ converges pointwise on $X$ to a norm continuous linear operator $T \colon X \to X$. Therefore, one has $\hat{\mathfrak{T}} = T$ and $\hat{\mathfrak{T}}^{(k)} = T^k$ for all $k \geq 1$ where $\mathfrak{T} = (T_k)$, $\Lambda = \mathbb{N}$ and $\hat{\mathfrak{T}}$ is the $n$-limit of $\mathfrak{T}$.*

**Example 5.4.7.** *Let $X$ be a Dedekind complete vector lattice with $\mathfrak{d} = o$. A net $(T_\lambda)_{\lambda\in\Lambda}$ of order bounded operators on $X$ decreases to an order bounded operator $T \in L_b(X)$ if and only if $T_\lambda(x)$ decreases to $T(x)$ for all $x \in X$. Therefore, one has $\hat{\mathfrak{T}} = T$ and $\hat{\mathfrak{T}}^{(k)} = T^k$ for all $k \geq 1$ where $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ and $\hat{\mathfrak{T}}$ is the o-limit of $\mathfrak{T}$.*

**Example 5.4.8.** *Let $T\colon X \to X$ be a mean ergodic operator on a Banach lattice $X$. Hence, $\mathfrak{d} = n$. The sequence of Cesàro averages $\mathfrak{T} = (\frac{1}{n}\sum_{k=0}^{n-1} T^k)_{n=1}^{\infty}$ converges strongly for every $x \in X$ to the mean ergodic projection $P\colon X \to \mathrm{Fix}(T)$. Therefore, one has $\hat{\mathfrak{T}} = P$ and $\hat{\mathfrak{T}}^{(k)} = P$ for all $k \geq 2$ where $\hat{\mathfrak{T}}$ is the n-limit of $\mathfrak{T}$.*

**Theorem 5.4.9.** *Let $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ be a $\mathfrak{d}$-convergent net for some convergence $\mathfrak{d}$ on $X$.*

i. *For every $k \geq 1$, the map $\hat{\mathfrak{T}}^{\mathfrak{d},(k)}\colon X \to X$ is linear. If $f \in X'_\mathfrak{c}$ is a functional and $\hat{\mathfrak{T}}^{\mathfrak{d}}, (\hat{\mathfrak{T}}^{\mathfrak{d}})' \in L(X,\mathfrak{c})$ then $f \circ \hat{\mathfrak{T}}^{\mathfrak{d},(k)} = (\hat{\mathfrak{T}}^{\mathfrak{d},(k)})'(f) \in X'_c$.*

ii. *For every $k \geq 1$, the map defined by $\mathscr{Q}_{\mathfrak{d},k}(\mathfrak{T}) = \hat{\mathfrak{T}}^{\mathfrak{d},(k)}$ on the space $\mathscr{N}_{\Lambda,\mathfrak{d}}(L(X,\mathfrak{c}))$ of $\mathfrak{d}$-convergent operator nets satisfies $\mathscr{Q}_{\mathfrak{d},k}(t\cdot\mathfrak{T}) = t^k \cdot \hat{\mathfrak{T}}^{\mathfrak{d},k}$ for $t \in \mathbb{R}$ and*

$$\mathscr{Q}_{\mathfrak{d},k}(\mathfrak{T}+\mathfrak{S}) = \mathfrak{d}\text{-}\lim_{\lambda\to\infty} T_\lambda(\mathscr{Q}_{\mathfrak{d},k-1}(\mathfrak{T}+\mathfrak{S})(x)) + \mathfrak{d}\text{-}\lim_{\lambda\to\infty} S_\lambda(\mathscr{Q}_{\mathfrak{d},k-1}(\mathfrak{T}+\mathfrak{S})(x))$$

*for every $x \in X$ and $\mathfrak{d}$-convergent nets $\mathfrak{T}$ and $\mathfrak{S}$.*

iii. *There exists a chain*

$$\mathrm{Fix}(\mathfrak{T}) \leq \cdots \leq \hat{\mathfrak{T}}^{\mathfrak{d},(k)}(X) \leq \hat{\mathfrak{T}}^{\mathfrak{d},(k-1)}(X) \leq \cdots \leq \hat{\mathfrak{T}}^{\mathfrak{d},(2)}(X) \leq \hat{\mathfrak{T}}^{\mathfrak{d}}(X)$$

*of subspaces of $X$ for the convergence $\mathfrak{d}$.*

*Proof.* $(i)$. Because

$$\hat{\mathfrak{T}}^{\mathfrak{d}}(t\cdot x+y) = \mathfrak{d}\text{-}\lim_{\lambda\to\infty} T_\lambda(t\cdot x+y) = t\cdot\mathfrak{d}\text{-}\lim_{\lambda\to\infty} T_\lambda(x) + \mathfrak{d}\text{-}\lim_{\lambda\to\infty} T_\lambda(y)$$

for all $x,y \in X$ and $t \in \mathbb{R}$, the operator $\hat{\mathfrak{T}}^{\mathfrak{d},(1)}\colon X \to X$ is linear. By induction, $\hat{\mathfrak{T}}^{\mathfrak{d},(k)}\colon X \to X$ is linear.

$(ii)$. The space $\mathscr{N}_{\Lambda,\mathfrak{d}}(L(X,\mathfrak{c}))$ of all $\mathfrak{d}$-convergent nets in $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ forms a linear space. Indeed, if $\mathfrak{T}$ and $\mathfrak{S}$ are $\mathfrak{d}$-convergent operator nets then so is the net $t\cdot\mathfrak{T}+\mathfrak{S}$. The formula $\mathscr{Q}_{\mathfrak{d},k}(t\cdot\mathfrak{T}) = t^k\cdot\hat{\mathfrak{T}}^{\mathfrak{d},k}$ for $t \in \mathbb{R}$ and

$$\mathscr{Q}_{\mathfrak{d},k}(\mathfrak{T}+\mathfrak{S}) = \mathfrak{d}\text{-}\lim_{\lambda\to\infty} T_\lambda(\mathscr{Q}_{\mathfrak{d},k-1}(\mathfrak{T}+\mathfrak{S})(x)) + \mathfrak{d}\text{-}\lim_{\lambda\to\infty} S_\lambda(\mathscr{Q}_{\mathfrak{d},k-1}(\mathfrak{T}+\mathfrak{S})(x))$$

follows from the definition of $\mathscr{L}_{\mathfrak{d},k}$.

$(iii)$. Recall that $\mathrm{Fix}(\mathfrak{T})$ denotes the intersection of fixed spaces of $T_\lambda$, that is

$$\mathrm{Fix}(\mathfrak{T}) = \bigcap_{\lambda\in\Lambda} \ker(I - T_\lambda).$$

It is clear that $\mathrm{Fix}(\mathfrak{T})$ is a subspace of $\hat{\mathfrak{T}}^{(k)}(X) \le X$ for all $k \ge 1$. The spaces $\hat{\mathfrak{T}}^{(k)}(X)$ are not necessarily $\mathfrak{T}$-invariant. To show that $\hat{\mathfrak{T}}^{(k)}(X) \le \hat{\mathfrak{T}}^{(k-1)}(X)$ for $k \ge 2$, let $x \in \hat{\mathfrak{T}}^{(k)}(X)$. There is a $y \in X$ such that $\hat{\mathfrak{T}}^{(k)}(y) = x$. Put $y' = \hat{\mathfrak{T}}(y)$. Hence, $x \in \hat{\mathfrak{T}}^{(k-1)}(y')$. □

**Corollary 5.4.10.** *Let $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ be an un-convergent net on an order continuous Banach lattice $X$. For every $x \in X$ there exists an increasing sequence $\lambda_k$ depending on $x$ and $\mathfrak{T}$ such that the limit $uo$-$\lim_{k\to\infty} T_{\lambda_k}(x)$ exists.*

*Proof.* See [18, Corollary 3.5] and Proposition 5.3.9. □

**Corollary 5.4.11.** *Let $\mathfrak{d}$ be a Lebesgue convergence on a Dedekind complete vector lattice $X$. If a $\mathfrak{d}$-closed ideal $I \le X$ is $\mathfrak{T}$-invariant for some $\mathfrak{T} = (T_{\lambda'})_{\lambda'\in\Lambda} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ consisting of order bounded operators $T_{\lambda'}$ then for every $\mathfrak{d}$-convergent $\mathfrak{S} = (S_\lambda)_{\lambda\in\Lambda} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ with $S_\lambda \in \mathrm{Band}(T_{\lambda'} \colon \lambda' \in \Lambda)$ for all $\lambda \in \Lambda$ one has $\hat{\mathfrak{S}}^{(k)}(I) \le I$ for all $k \ge 1$.*

*Proof.* Follows from Theorem 5.4.1. □

We recall from Section 5.2 that a net $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is said to be $\mathfrak{d}$-*bounded* in $L(X,c)$ if there exists a positive operator $S \in L(X,\mathfrak{c}) \cap L(X,\mathfrak{d})$ such that $|T_\lambda| \le S$ for every $\lambda \in \Lambda$.

**Remark 5.4.12.** *In view of Remark 3.4.3, if a net $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is $\mathfrak{d}$-bounded for a convergence $\mathfrak{d}$ on the vector lattice $X$, then there exists a positive $S \in L(X,\mathfrak{c})$, which is furthermore $\mathfrak{d}$-continuous, such that the operator $T_\lambda$ is dominated by $S$ for every $\lambda \in \Lambda$. Indeed, because $\mathfrak{T}$ is $\mathfrak{d}$-bounded, $|T_\lambda|$ exists for every $\lambda \in \Lambda$. It follows from $|T_\lambda(x)| \le |T_\lambda|(|x|) \le S(|x|)$ for every $x \in X$ that $T_\lambda$ is dominated by $S$. Let us consider the particular case $\mathfrak{c} = o$ and $\mathfrak{d} = o$. In this case, the net $\mathfrak{T}$ is o-bounded if and only if each $T_\lambda$ is order continuous, and, there exists a positive and order continuous operator $S$ such that $|T_\lambda| \le S$ for every $\lambda \in \Lambda$.*

**Proposition 5.4.13.** *Suppose that $\mathfrak{T} \in \mathscr{N}_{\Lambda,o}(L(X,o))$, the algebra of o-convergent and o-continuous operator nets, is an o-bounded net of lattice homomorphisms where $X$ is a Dedekind complete vector lattice.*

i. *If $x \in \hat{\mathfrak{T}}^{uo,(k)}(X)$ then $x^+, x^-, |x| \in \hat{\mathfrak{T}}^{uo,(k)}(X)$ for every $k \ge 1$.*

ii. *If $\mathfrak{T}$ is uo-convergent and $y \in \hat{\mathfrak{T}}^{uo}(X)$ with $y \ge 0$ then there exists some $x \in X^+$ such that $\hat{\mathfrak{T}}^{uo}([0,x]) \subseteq [0,y]$.*

*Proof.* $(i)$. Consider the case $k = 1$. If $x \in \hat{\mathfrak{T}}^{uo}(X)$ then there is some $y \in X$ such that $T_\lambda(y) \xrightarrow{uo} x$. It follows from [41, Lemma 3.1] and the fact that each $T_\lambda$ is a lattice homomorphism, $T_\lambda(y)^+ = T_\lambda(y^+) \xrightarrow{uo} x^+$. Hence, we have $x^+, x^-, |x| \in \hat{\mathfrak{T}}^{uo,(1)}(X)$ for every $x \in \hat{\mathfrak{T}}^{uo}(X)$. For the general case, we use induction. Suppose that the statement is true for $k$. If $x \in \hat{\mathfrak{T}}^{uo,(k+1)}(X)$ then there is some $y \in \hat{\mathfrak{T}}^{uo,(k)}(X)$ such that $T_\lambda(y) \xrightarrow{uo} x$. It follows that $y^+, y^-, |y| \in \hat{\mathfrak{T}}^{uo,(k)}(X)$ and that $T_\lambda(y)^+ = T_\lambda(y^+) \xrightarrow{uo} x^+$. Hence, we have $x^+, x^-, |x| \in \hat{\mathfrak{T}}^{uo,(k+1)}(X)$ for every $x \in \hat{\mathfrak{T}}^{uo,(k+1)}(X)$.

$(ii)$. If $y \in \hat{\mathfrak{T}}^{uo}(X)$ and $y \geq 0$ then there is some $x' \in X$ with $x' \geq 0$ such that $T_\lambda(x') \xrightarrow{uo} y$. Let $x \in [0, x']$. Because $T_\lambda$ is a lattice homomorphism, $T_\lambda(x) \leq T_\lambda(x')$ for all $\lambda \in \Lambda$. As $\mathfrak{T}$ is *uo*-convergent, there exists some $y' \in [0, y]$, see [41, Lemma 3.1], such that $T_\lambda(x) \xrightarrow{uo} y'$. Hence, the result follows. $\square$

**Proposition 5.4.14.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda \in \Lambda}$ be a net such that*

$$T_\lambda = \sup\{T_{\lambda'} \colon \lambda' \leq \lambda\}$$

*for all $\lambda \in \Lambda$ where $X$ is a Dedekind complete Banach lattice.*

i. *If $\mathfrak{T}$ is uo-convergent then $T_\lambda(x) \uparrow \mathfrak{d}\text{-}\lim_{\lambda \to \infty} T_\lambda(x)$ with $\mathfrak{d} = uo$ for all $x \in X$.*

ii. *If $\mathfrak{T}$ is un-convergent then $T_\lambda(x) \xrightarrow{n} \mathfrak{d}\text{-}\lim_{\lambda \to \infty} T_\lambda(x)$ with $\mathfrak{d} = un$ for all $x \in X$.*

*Proof.* It follows from $T_\lambda = \sup\{T_{\lambda'} \colon \lambda' \leq \lambda\}$ for $\lambda \in \Lambda$ that $(T_\lambda(x))_{\lambda \in \Lambda}$ is an increasing net in $X$ for every $x \in X$. Because $\mathfrak{T}$ is *uo*-convergent, $\hat{\mathfrak{T}}^{uo}(x)$ exists and belongs to $X$ for every $x \in X$. Similarly, $\hat{\mathfrak{T}}^{un}(x)$ exists and belongs to $X$ whenever $\mathfrak{T}$ is *un*-convergent. By [45, Lemma 1.2], results follow. $\square$

We now focus on $\mathfrak{c}$-Markov nets where $\mathfrak{c}$ is a convergence on the vector lattice $X$. A functional $x' \in X'_{\mathfrak{c}}$ is said to be strictly positive, see [5, page 190], if $x'(x) > 0$ for all $x \in X^+$. We recall from Section 5.2 that an operator net $\mathfrak{T}$ of positive operators is said to be $\mathfrak{c}$-Markov if there exists a weak unit $x_0 \in X^+$ and a strictly positive $\mathfrak{c}$-continuous functional $x'_0 \in X'_c$ such that $T_\lambda(x_0) = x_0$ and $T'_\lambda(x'_0) = x'_0$ for all $\lambda \in \Lambda$.

In Example 5.4.15, we give a concrete example of $o$-Markov operator nets.

**Example 5.4.15.** *Let $\mathscr{P} = (P_n)_{n \geq 1}$ be an abstract bistochastic filtration on an order continuous Banach lattice $X$, see [41, Section 5] for details. It follows from definition that there exists a weak unit $x_0 \in X$ and a strictly positive order continuous functional $x'_0 \in X'_o$ such that $P_n(x_0) = x_0$ and $P'_n(x'_0) = x'_0$ for all $n \geq 1$. Hence, $\mathscr{P} = (P_n)_{n \geq 1}$ is an o-Markov operator sequence.*

**Remark 5.4.16.** *Let $X$ be a Banach lattice and $\mathfrak{c} = n$. Every uniformly bounded n-Markov operator net is Markov in the sense of [32, Definition 3]. Evidently, the converse of this statement is not correct.*

Taken together, Example 5.4.15 and Remark 5.4.16 suggest that $\mathfrak{c}$-Markov net is a notion intermediate between abstract bistochastic filtrations and Markov operator nets. We remark that a $\mathfrak{c}$-Markov operator net consists of positive operators by definition.

**Proposition 5.4.17.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ be an o-Markov operator net on a vector lattice. For every $\lambda \in \Lambda$ and $k \geq 1$ the operator $T_\lambda^k\colon X \to X$ is strictly positive and order continuous. In particular, if $\mathfrak{T}$ is o-Markov then $\mathfrak{T}^k$, the k-fold product of $\mathfrak{T}$ with itself, belongs to the algebra $\mathscr{N}_\Lambda(L(X,o))$ for every $k \geq 1$.*

*Proof.* As $T_\lambda \geq 0$ for every $\lambda \in \Lambda$, we have $T_\lambda^k \geq 0$ for every $k \geq 1$ and $\lambda \in \Lambda$. Suppose $T_\lambda^k(x) = 0$ for some $x \geq 0$. Then $x_0'(x) = ((T_\lambda')^k x_0')(x) = x_0'(T_\lambda^k(x)) = 0$ because $T_\lambda'(x_0') = x_0'$. Since $x_0'$ is a strictly positive functional, it follows that $x = 0$. This shows that $T_\lambda^k$ is strictly positive. Put $x_\alpha \downarrow 0$ and $0 \leq z \leq T_\lambda(x_\alpha)$. Then $0 \leq x_0'(z) \leq (T_\lambda' x_0')(x_\alpha) = x_0'(x_\alpha) \downarrow 0$ by order continuity of $x_0'$. Hence, $T_\lambda$ is order continuous for every $\lambda \in \Lambda$. This imply that $T_\lambda^k$ is order continuous for every $k \geq 1$. In particular, $T_\lambda^k \in L(X,o)$ and $\mathfrak{T}^k \in \mathscr{N}_\Lambda(L(X,o))$ for $k \geq 1$. □

**Proposition 5.4.18.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ be an o-Markov operator net on a normed lattice $X$. Let $k \geq 1$ be a fixed integer. The following statements are equivalent.*

i. $x \in \ker(T_\lambda^k - I_X)$ *for a quasi-interior point* $x > 0$.

ii. $(T_\lambda')^k$ *is strictly positive.*

*Proof.* Suppose $(T_\lambda')^k x' = 0$ for some $x' \geq 0$. Then

$$x'(x) = x'(T_\lambda^k(x)) = ((T_\lambda')^k(x'))(x) = 0.$$

Since $x$ is a quasi-interior point, $x' = 0$. Conversely, let $x \in \ker(T_\lambda^k - I_X)$ be such that $x \geq 0$ and $x \neq 0$. Also let $x' > 0$ so that $(T_\lambda')^k(x') > 0$. It follows that $x'(x) = x'(T_\lambda^k x) = ((T_\lambda')^k(x'))(x) > 0$. Hence, by [5, Theorem 4.85], $x$ is a quasi-interior point. □

Proposition 5.4.19 is successfully used in [32].

**Proposition 5.4.19.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda} \in \mathscr{N}_\Lambda(L(X,n))$ be an n-Markov operator net on a normed lattice $X$. Then there exists a norm complete $\tilde{X}$ such that each $T_\lambda$ extends uniquely to a positive contraction $\tilde{T}_\lambda : \tilde{X} \to \tilde{X}$.*

*Proof.* The space $N(x_0') = \{x \in X\colon x_0'(|x|) = 0\}$ is a norm closed order ideal in $X$. Because each $T_\lambda$ is positive and $T_\lambda'(x_0^*)|x| = x_0'(T_\lambda|x|)$, the order ideal $N(x_0')$ is $\mathfrak{T}$-invariant. For each $T_\lambda$ denote by $T_\lambda'$ the norm bounded operator that completes the commutative diagram where $\pi\colon X \to X/N(x_0')$ is the canonical quotient map. Then each $T_\lambda'$ extends to a positive contraction $T_\lambda'\colon \tilde{X} \to \tilde{X}$ where $\tilde{X}$ is the norm completion of $X/N(x_0')$. □

**Proposition 5.4.20.** *Let $x_\alpha$ be an order bounded net in a $\mathfrak{c}$-complete convergence vector lattice $(X,\mathfrak{c})$ and let $x' \in X'_{\mathfrak{c}}$ be strictly positive. In this case, $x_\alpha \xrightarrow{\mathfrak{c}} 0$ if and only if $x'(|x_\alpha|) \to 0$.*

*Proof.* If $x_\alpha \xrightarrow{\mathfrak{c}} 0$ then $|x_\alpha| \xrightarrow{\mathfrak{c}} 0$ as the lattice operations are continuous on the convergence vector lattice $(X,\mathfrak{c})$. Because $x'$ is $\mathfrak{c}$ to norm continuous, result follows. Conversely, given a net $x_\alpha$ we write $x_\alpha = x_\alpha^+ - x_\alpha^-$ where both $x_\alpha^+$ and $x_\alpha^-$ are non-negative. It follows from $0 \le x'(x_\alpha^\pm) \le x'(|x_\alpha|)$ that $x'(|x_\alpha|) \to 0$ implies $x'(x_\alpha^\pm) \to 0$. Hence, $x'(x_\alpha^+ - x_\alpha^-) \to 0$ as the net $x_\alpha$ is order bounded. □

An important case happens in the case of sequences of operators. If $\mathfrak{T} = (T_n)_{n\ge1}$ is a $\mathfrak{c}$-Markov operator sequence then we write

$$\mathscr{M}_i(\mathfrak{T}) := \{(x_n)_{n\ge1} \colon (\delta_{i1}I - T_n)(x_m) = (-1)^{i+1}x_n \text{ if } m \ge n\}$$

for $i = 0,1$. The letter $\mathscr{M}$ reminds the relationship with martingales, see Example 5.4.21. We further put

$$\mathscr{S}\mathscr{M}_i(\mathfrak{T}) := \{(x_n)_{n\ge1} \colon x_m \in T_m(X) \text{ and } (\delta_{i1}I - T_n)(x_m) \ge (-1)^{i+1}x_n \text{ if } m \ge n\}$$

for $i = 0,1$. For an application of these sets of sequences see Proposition 5.6.5.

**Example 5.4.21.** *Let $\mathscr{P} = (P_n)$ be an abstract bistochastic filtration on a Banach lattice, see [41] for details. Recall that $P_nP_m = P_mP_n = P_{n\wedge m}$ for all $m,n \ge 1$. By Example 5.4.15, sequence of operators $\mathscr{P}$ is o-Markov. It readily follows that $\mathscr{M}_0(\mathscr{P})$ and $\mathscr{S}\mathscr{M}_0(\mathscr{P})$ are the sets of all martingales and submartingales relative to filtration $\mathscr{P} = (P_n)$, respectively.*

Recall that a subset $A \subset X$ is said to be $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$-invariant if $T_\lambda(A) \subseteq A$ for every $\lambda \in \Lambda$.

The space of linear $\mathfrak{T}$-equivalences $W_A(\mathfrak{T})$ of a $\mathfrak{T}$-invariant set $A \subseteq X$ consists of all $\mathfrak{c}$-continuous linear isomorphisms $S \colon X \to X$ for which $S(T_\lambda(x)) = T_\lambda(S(x))$ for all $x \in A$ and $\lambda \in \Lambda$. It readily follows that $W_A(\mathfrak{T})$ is a sub-algebra of the algebra $L(X,\mathfrak{c})$ for every $\mathfrak{T}$-invariant set $A$.

**Example 5.4.22.** *Suppose that $\mathscr{P} = (P_n)_{n=1}^\infty$ is an o-Markov operator net, and that, $(x_n)_{n\ge1} \in \mathscr{M}_0(\mathscr{P})$ is a martingale such that the linear span A of $(x_n)_{n\ge1}$ is $\mathscr{P}$-invariant, see Example 5.4.21. For every o-Markov operator net $\mathscr{P}$, there is such a martingale. Thus, there exists a corresponding sub-algebra $W_A(\mathscr{P})$ of $L(X,o)$.*

Recall that given a set $A \subset X$, the set $\omega_{\mathfrak{d}}(A)$ of all $\omega$-limits of $A$ with respect to convergence $\mathfrak{d}$ consists of those elements of $x \in X$ for which there exists a sequence $x_k \in A$ and a cofinal sequence $\lambda_k$ in $\Lambda$ such that $T_{\lambda_k}(x_k) \xrightarrow{\mathfrak{d}} x$. We assume that the net $\mathfrak{T}$ has a cofinal sequence. It is clear that for any $T_1$-convergence $\mathfrak{d}$, $\omega_{\mathfrak{d}}(x) = \{x\}$ for every $x \in \mathrm{Fix}(\mathfrak{T})$.

**Proposition 5.4.23.** *Let $\mathfrak{T} = (T_\lambda)_\Lambda \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ be an operator net such that each $T_\lambda$ is $\mathfrak{d}$-continuous for some fixed convergence $\mathfrak{d}$ on $X$. Suppose that for every $\alpha,\beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $T_\alpha \circ T_\beta(x) = T_\gamma(x)$ for every $x \in X$ and $\gamma \geq \alpha \vee \beta$. Then the sets $\{T_\lambda(x)\colon x \in X\}$ and $\omega_\mathfrak{d}(A) \subseteq X$ for arbitrary $A \subset X$ are $\mathfrak{T}$-invariant.*

*Proof.* The set $\{T_\lambda(x)\colon x \in X\}$ is clearly $\mathfrak{T}$-invariant. Let us verify that $\omega_\mathfrak{d}(A) \subseteq X$ is $\mathfrak{T}$ invariant. Let $y \in \omega_\mathfrak{d}(A)$ so that there exists a sequence $x_k \in A$ and a cofinal sequence $\lambda_k$ in $\Lambda$ such that $T_{\lambda_k}(x_k) \xrightarrow{\mathfrak{d}} y$. Because $T_\lambda$ is $\mathfrak{d}$-continuous, $T_\lambda(T_{\lambda_k}(x_k)) \xrightarrow{\mathfrak{d}} T_\lambda(y)$. By the given property of $\mathfrak{T}$, we write $T_\lambda \circ T_{\lambda_k}(x_k) = T_{\lambda_k'}(x_k)$. Hence, $T_\lambda(y) \in \omega_\mathfrak{d}(A)$. The sequence $\lambda_k'$ is again cofinal in $\Lambda$ because $\lambda_k' \geq \lambda_k$. □

**Proposition 5.4.24.** *Let $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ be an operator net.*

i. *If $\mathfrak{T}$ is $\mathfrak{d}$-convergent and $A \subseteq X$ is a $\mathfrak{d}$-closed $\mathfrak{T}$-invariant subset then $\hat{\mathfrak{T}}^{\mathfrak{d},(k)}(A) \subseteq A$ for every $k \geq 1$.*

ii. *If $\mathfrak{T}$ is $\mathfrak{c}$-Markov and $A \subseteq X$ is a $\mathfrak{c}$-closed and $\mathfrak{T}$-invariant subset then $A \cap (x')^{-1}(t)$ is $\mathfrak{T}$-invariant for some $x' \in X_c'$ and $t \in \mathbb{R}$.*

*Proof.* $(i)$. Recall that $\hat{\mathfrak{T}}^{\mathfrak{d},(k)}$ denotes the $k$-th iteration of the $\mathfrak{d}$-limit of the operator net $\mathfrak{T}$. Because $A \subset X$ is $\mathfrak{d}$-closed and $\mathfrak{T}$-invariant,$\hat{\mathfrak{T}}^{\mathfrak{d},(k)}(A) \subseteq A$ for every $k \geq 1$.

$(ii)$. The operator net $\mathfrak{T} = (T_\lambda)_\lambda$ is $\mathfrak{c}$-Markov if there exists a weak unit $x_0 \in X^+$ and a strictly positive $\mathfrak{c}$-continuous functional $x_0' \in X_c'$ such that $T_\lambda(x_0) = x_0$ and $T_\lambda'(x_0') = x_0'$ for all $\lambda \in \Lambda$. Let us take $x' = x_0'$. Because $x'$ is strictly positive, $x' \neq 0$. Let $t > 0$. The set $A \cap (x')^{-1}(t)$ is $\mathfrak{c}$-closed. If $x \in A \cap (x')^{-1}(t)$ then $x'(T_\lambda(x)) = (T_\lambda'(x'))(x) = t$ for all $\lambda \in \Lambda$. Hence, $A \cap (x')^{-1}(t)$ is $\mathfrak{T}$-invariant. □

### 5.5 The Space of Asymptotic Commutators

In the settings of the present paper, a commutator in the algebra $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is an element of the form $[\mathfrak{T},\mathfrak{S}] := \mathfrak{T}\cdot\mathfrak{S} - \mathfrak{S}\cdot\mathfrak{T}$ for some $\mathfrak{T},\mathfrak{S} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$. We note the fact that the notion of classical commutators goes back to works of Dedekind and Frobenious.

**Definition 5.5.1.** Two nets $\mathfrak{T}$ and $\mathfrak{S}$ are said to be $\mathfrak{d}$-asymptotically commutative if the commutator $\mathfrak{T}\cdot\mathfrak{S} - \mathfrak{S}\cdot\mathfrak{T}$ satisfies $\mathfrak{T}\cdot\mathfrak{S} - \mathfrak{S}\cdot\mathfrak{T} \approx_\mathfrak{d} \mathbf{0}$. In this case, we say that the nets $\mathfrak{T}$ and $\mathfrak{S}$ commute $\mathfrak{d}$-asymptotically. Following [20], we further say that a net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ is $\mathfrak{d}$-abelian if $\mathfrak{T}$ and the constant net $\mathbf{T}_\mu = (T_\mu)_{\lambda\in\Lambda}$ commute $\mathfrak{d}$-asymptotically for all $\mu \in \Lambda$.

**Example 5.5.2.** *Let $T\colon X \to X$ be a norm bounded mean ergodic operator on Banach lattice $X$. The sequence of Cesàro means $A_k^T = k^{-1}\Sigma_{j=0}^{k-1}T^j$ is both n-convergent and n-abelian. Indeed, because $T\colon X \to X$ is mean ergodic, the norm limit $\lim_{k\to\infty} A_k^T(x)$ exists for every*

*$x \in X$. Thus, the sequence of Cesàro means $A_k^T$ is n-convergent. Also, because $A_k^T T = TA_k^T$ for every $k \geq 1$ we have $A_k^T A_l^T = A_l^T A_k^T$ for every $k,l \geq 1$. It follows that the sequence of Cesàro means $A_k^T$ is n-abelian.*

We first present a technique to produce new $\mathfrak{d}$-abelian operator nets from a given $\mathfrak{d}$-abelian operator net. Let $R\colon (X,\mathfrak{d}) \to (Y,\mathfrak{d}')$ be a $\mathfrak{d}$ to $\mathfrak{d}'$ continuous operator. Following [21, Section 1.1.13], we put

$$\pi_R(T_\lambda)(x,y) = (T_\lambda(x), R(x) - R \circ T_\lambda(x) + y)$$

for every $(x,y) \in X \times Y$ and $\lambda \in \Lambda$. In the space $X \times Y$, we say that a net is a null net if coordinates of each element of the net tends to zero with respect to the corresponding convergences. In the classical case, see [21, Proposition 1.1.18], if an operator $T$ is mean ergodic then the operator $\pi_R(T)$ is mean ergodic.

**Theorem 5.5.3.** *If a net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is $\mathfrak{d}$-abelian for some convergence $\mathfrak{d}$ on $X$ and $R\colon (X,\mathfrak{d}) \to (Y,\mathfrak{d}')$ is $\mathfrak{d}$ to $\mathfrak{d}'$ continuous then the net $(\pi_R(T_\lambda))_{\lambda\in\Lambda}$ is $(\mathfrak{d},\mathfrak{d}')$-abelian with respect to coordinate-wise convergence.*

*Proof.* Let $(x,y) \in X \times Y$ be arbitrary. We will show that the net

$$(\pi_R(T_\lambda)\pi_R(T_\mu) - \pi_R(T_\mu)\pi_R(T_\lambda)(x,y))_{\lambda\in\Lambda}$$

is a null net for all $\mu \in \Lambda$ with respect to coordinate-wise convergence. We have, after cancellations,

$$(\pi_R(T_\lambda)\pi_R(T_\mu) - \pi_R(T_\mu)\pi_R(T_\lambda))(x,y) = ([T_\lambda, T_\mu](x), R[T_\mu, T_\lambda](x))$$

whose right side is independent of $y \in Y$. Because $\mathfrak{T}$ is $\mathfrak{d}$-abelian and $R$ is $\mathfrak{d}$ to $\mathfrak{d}'$ continuous, we have the result. $\square$

**Theorem 5.5.4.** *If a net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is $\mathfrak{c}$-abelian and $\mathfrak{c}$-convergent then $\hat{\mathfrak{T}}^{\mathfrak{c},(k)}(X)$ is $\mathfrak{T}$-invariant.*

*Proof.* This follows by induction on $k$. For the case $k = 1$, let $x \in \hat{\mathfrak{T}}^{\mathfrak{c}}(X)$ so that there exists some $y \in X$ with $\hat{\mathfrak{T}}^{\mathfrak{c}}(y) = x$. Denote by $\mathfrak{S}_1$ and $\mathfrak{S}_2$ the nets $\mathbf{T}_\mu \cdot \mathfrak{T}$ and $\mathfrak{T} \cdot \mathbf{T}_\mu$, respectively. It follows from $\mathfrak{c}$-continuity and hypothesis that

$$T_\mu(\hat{\mathfrak{T}}^{\mathfrak{c}}(y)) = \hat{\mathfrak{S}}_1^{\mathfrak{c}}(y) = \hat{\mathfrak{S}}_2^{\mathfrak{c}}(y). \tag{5.5.1}$$

Thus,

$$T_\mu(x) = T_\mu(\hat{\mathfrak{T}}^{\mathfrak{c}}(y)) = \hat{\mathfrak{T}}^{\mathfrak{c}}(T_\mu(y)).$$

In particular, $T_\mu(x) \in \hat{\mathfrak{T}}^{\mathfrak{c}}(X)$. Because the index $\mu$ is arbitrary, $\hat{\mathfrak{T}}^{\mathfrak{c}}(X)$ is $\mathfrak{T}$-invariant. Suppose the statement is true for the case $k$. For any $x \in \hat{\mathfrak{T}}^{k+1}(X)$ there exists some $y \in \hat{\mathfrak{T}}^{k}(X)$ such that $\hat{\mathfrak{T}}^{\mathfrak{c}}(y) = x$, see Theorem 5.4.9. It follows that $T_\mu(y) \in T_\mu(\hat{\mathfrak{T}}^{(k)}(X)) \leq \hat{\mathfrak{T}}^{(k)}(X)$. From Equation 5.5.1, the result follows. □

**Example 5.5.5.** *Let $T: X \to X$ be a norm bounded mean ergodic operator on Banach lattice $X$. Denote by $\mathfrak{T} = (A_k^T)_{k=1}^{\infty}$ the sequence of Cesàro means of $T$. By Example 5.5.2, $\mathfrak{T}$ is both n-convergent and n-abelian. Hence, Theorem 5.5.4 applies. The operator $\hat{\mathfrak{T}}^n = P$ where $P\colon X \to \mathrm{Fix}(T)$ equals to the corresponding ergodic projection, see Yosida's Theorem given in [21, Theorem 1.1.9]. Clearly, $\hat{\mathfrak{T}}^{(k)} = P$ for all $k \geq 1$. The space $\hat{\mathfrak{T}}(X)$ is equal to Fix$(\mathfrak{T})$ which is $A_k$-invariant for every $k \geq 1$.*

Recall from Section 5.3 that $U(\mathscr{N}_\Lambda(L(X,\mathfrak{c})))$ stands for the space of units of the algebra $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$. If $\mathfrak{d}$ is a Lebesgue convergence on $X$, then $x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{\mathfrak{d}} 0$. The next results considers the case $u_J\mathfrak{d}$.

**Proposition 5.5.6.** *Let $\mathfrak{T}$ and $\mathfrak{S}$ be $\Lambda$-indexed nets belonging to $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$. If $\mathfrak{T}$ and $\mathfrak{S}$ commute $\mathfrak{d}$-asymptotically then they commute $u_J\mathfrak{d}$-asymptotically. Similarly, if $\mathfrak{T}$ is $\mathfrak{d}$-abelian then $\mathfrak{T}$ is $u_J\mathfrak{d}$-abelian. Further, for every $\mathfrak{T} \in U(\mathscr{N}_\Lambda(L(X,\mathfrak{c})))$ there is a unit $\mathfrak{S} \in U(\mathscr{N}_\Lambda(L(X,\mathfrak{c})))$ such that $\mathfrak{T}$ and $\mathfrak{S}$ commute $u_J\mathfrak{d}$-asymptotically.*

*Proof.* Suppose that $\mathfrak{T}$ and $\mathfrak{S}$ commute $\mathfrak{d}$-asymptotically. This means that

$$T_\lambda S_\lambda(x) - S_\lambda T_\lambda(x) \xrightarrow{\mathfrak{d}} 0$$

for every $x \in X$. Hence, $T_\lambda S_\lambda(x) - S_\lambda T_\lambda(x) \xrightarrow{u_J\mathfrak{d}} 0$. The net $\mathfrak{T}$ is $\mathfrak{d}$-abelian if and only if $\mathfrak{T}$ and $\mathbf{T}_\mu$ commute $\mathfrak{d}$-asymptotically for all $\mu \in \Lambda$. In particular, $\mathfrak{T}$ and $\mathbf{T}_\mu$ commute $u_J\mathfrak{d}$-asymptotically for all $\mu \in \Lambda$. Hence, the net $\mathfrak{T}$ is $u_J\mathfrak{d}$-abelian. For the last part, observe that for every net $\mathfrak{T} \in U(\mathscr{N}_\Lambda(L(X,\mathfrak{c})))$ there is some $\mathfrak{S}$ depending on $\mathfrak{T}$ such that $\mathfrak{T}\cdot\mathfrak{S} - \mathfrak{S}\cdot\mathfrak{T} = \mathbf{0}$. This implies that $\mathfrak{T}$ and $\mathfrak{S}$ commute $u_J\mathfrak{d}$-asymptotically. □

**Example 5.5.7.** *Let $T: X \to X$ be a norm bounded mean ergodic operator on Banach lattice $X$. Denote by $\mathfrak{T} = (A_k^T)_{k=1}^{\infty}$ the sequence of Cesàro means of $T$, see Example 5.5.2. For every order dense ideal $J \leq X$, the operator sequence $A_k^T$ is $u_J n$-abelian.*

It follows from Proposition 5.3.1 that the set

$$W_{\mathfrak{d}} = \{\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))\colon \mathfrak{T} \approx_{\mathfrak{d}} \mathbf{0}\}$$

is a vector subspace of the algebra $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$. Following example shows that in the case of asymptotic equivalence obtained from unbounded convergence $u_J\mathfrak{d}$ the subspace $W_{u_J\mathfrak{d}}$ is not an algebraic ideal with respect to standard multiplication in $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$, in general.

**Example 5.5.8.** *Consider the Banach lattice $X = C_0(\mathbb{R}_+)$ of continuous functions on $\mathbb{R}_+$ that vanish at infinity with the classical supremum norm. Hence, $\mathfrak{d} = n$. Let us put $T(f) = f(t+1)$ for $f \in X$ and consider the net $\mathfrak{T} = (T_n)_{n\geq 0}$ with $T_n = T^n$ obtained via itineraries of $T$. In this case, the subspace $W_{un}$ of $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is not an algebraic ideal.*

Following example shows that some well-known operator classes can be used to produce elements in $W_{\mathfrak{d}}$.

**Example 5.5.9.** *Let $T\colon X \to X$ be an operator on a Banach lattice $X$ with $\|T\| < 1$. We consider the case of sequence of operators, hence $\Lambda = \mathbb{N}$. The discrete semigroup generated by $T$ belongs to $W_n$. Similarly, suppose that $\sigma(T) \subset \mathbb{D}$ for some norm bounded operator $T\colon X \to X$. It follows from the classical spectral radius formula that there exists $c < 1$ and $N \geq 0$ such that $\|T^n(x)\| \leq c^n \|x\|$ for every $x \in X$ and $n \geq N$. Then the discrete semigroup generated by $T$ belongs to $W_n$.*

The nets $\mathfrak{T}$ and $\mathfrak{S}$ commute $u_J\mathfrak{d}$-asymptotically if and only if the commutator $\mathfrak{T}\cdot\mathfrak{S} - \mathfrak{S}\cdot\mathfrak{T}$ belongs to the subspace $W_{u_J\mathfrak{d}}$ of $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$. It follows that a net $\mathfrak{T}$ is $u_J\mathfrak{d}$-abelian if and only if $\mathfrak{T}\cdot\mathbf{T}_\mu - \mathbf{T}_\mu\cdot\mathfrak{T} \in W_{u_J\mathfrak{d}}$ for all $\mu \in \Lambda$.

**Proposition 5.5.10.** *A net $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is $\mathfrak{d}$-abelian if and only if the net $\mathbf{1} - \mathfrak{T}$ is $\mathfrak{d}$-abelian. Same result also holds if $\mathfrak{d} = u_J\mathfrak{c}$. In particular, the net $(I_X - (I_X - T)^k)_{k=0}^{\infty}$ is $\mathfrak{c}$-abelian for every $T \in L(X,\mathfrak{c})$.*

*Proof.* It suffices to show that

$$[\mathbf{1} - \mathfrak{T}, \mathbf{1} - \mathbf{T}_\mu] = (\mathbf{1} - \mathfrak{T})\cdot(\mathbf{1} - \mathbf{T}_\mu) - (\mathbf{1} - \mathbf{T}_\mu)\cdot(\mathbf{1} - \mathfrak{T}) \in W_{\mathfrak{d}}$$

whenever $\mathfrak{T}$ is $\mathfrak{d}$-abelian. The given net equals to $\mathfrak{T}\cdot\mathbf{T}_\mu - \mathbf{T}_\mu\cdot\mathfrak{T}$ which is a commutator. If $\mathfrak{T}$ is $\mathfrak{d}$-abelian then then $\mathfrak{T}\cdot\mathbf{T}_\mu - \mathbf{T}_\mu\cdot\mathfrak{T} \in W_{\mathfrak{d}}$. If $\mathbf{1} - \mathfrak{T}$ is $\mathfrak{d}$-abelian then $\mathfrak{T}$ is $\mathfrak{d}$-abelian.

For the particular case, let $T \in L(X,\mathfrak{c})$ be arbitrary. We consider the discrete semigroup $\{T^k\}$ generated by $T$ as a sequence of operators. The discrete semigroup generated by $T$ is clearly $\mathfrak{c}$-abelian. Thus, the discrete semigroup generated by $I_X - T$ is $\mathfrak{c}$-abelian. From the above proof, $(I_X - (I_X - T)^k)_{k=0}^{\infty}$ is $\mathfrak{c}$-abelian. □

**Proposition 5.5.11.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ and $\mathfrak{S} = (S_\lambda)_{\lambda\in\Lambda}$ be $\Lambda$-indexed nets belonging to $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$. If $\mathfrak{T}$ is $\mathfrak{d}$-abelian and $\mathfrak{T} \approx_{\mathfrak{d}} \mathfrak{S}$ then $\mathfrak{S}\cdot\mathbf{T}_\mu - \mathbf{T}_\mu\cdot\mathfrak{S} \in W_{\mathfrak{d}}$ for all $\mu \in \Lambda$. In particular, if a $\mathfrak{d}$-unit $\mathfrak{T}$ is $\mathfrak{d}$-abelian then there exists a net $\mathfrak{S} \in U(\mathscr{N}_\Lambda(L(X,\mathfrak{c})))$ such that $\mathbf{T}_\mu\cdot\mathfrak{S} - \mathfrak{S}\cdot\mathbf{T}_\mu \in W_{\mathfrak{d}}$ for all $\mu \in \Lambda$.*

*Proof.* For every $\mu \in \Lambda$, one has

$$\mathfrak{S}\cdot\mathbf{T}_\mu - \mathbf{T}_\mu\cdot\mathfrak{S} = (\mathfrak{S} - \mathfrak{T})\cdot\mathbf{T}_\mu + \mathfrak{T}\cdot\mathbf{T}_\mu - \mathbf{T}_\mu\cdot(\mathfrak{S} - \mathfrak{T}) - \mathbf{T}_\mu\cdot\mathfrak{S}$$

which is $\mathfrak{d}$-asymptotically equivalent to $\mathbf{0}$ by Proposition 5.3.1.

If $\mathfrak{T}$ is an $\mathfrak{d}$-unit then there is some $\mathfrak{S} \in U(\mathscr{N}_\Lambda(L(X,\mathfrak{c})))$ such that $\mathfrak{T} \approx_{\mathfrak{d}} \mathfrak{S}$. Because $\mathfrak{T}$ is $\mathfrak{d}$-abelian, one has $\mathbf{T}_\mu \cdot \mathfrak{S} - \mathfrak{S} \cdot \mathbf{T}_\mu \in W_{\mathfrak{d}}$ for all $\mu \in \Lambda$. □

**Proposition 5.5.12.** *Let $\mathfrak{T}$, $\mathfrak{S}$ and $\mathfrak{W}$ be $\Lambda$-indexed nets belonging to $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$. If the net $\mathfrak{W}$ commutes $\mathfrak{d}$-asymptotically with both $\mathfrak{T}$ and $\mathfrak{S}$ then $\mathfrak{W}$ commutes $\mathfrak{d}$-asymptotically with the commutator of $\mathfrak{T}$ and $\mathfrak{S}$. In this case, commutator operation $[,]\colon \mathscr{N}_\Lambda(L(X,\mathfrak{c})) \times \mathscr{N}_\Lambda(L(X,\mathfrak{c})) \to \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is a bilinear map which is associative up to $\approx_{\mathfrak{d}}$ as a binary operation.*

*Proof.* Let us consider the commutator $\mathfrak{W}'$ of $\mathfrak{W}$ and $\mathfrak{T} \cdot \mathfrak{S} - \mathfrak{T} \cdot \mathfrak{S}$. It follows that $\mathfrak{W}' = \mathfrak{T} \cdot \mathfrak{S} \cdot \mathfrak{W} - \mathfrak{W} \cdot \mathfrak{T} \cdot \mathfrak{S} - \mathfrak{S} \cdot \mathfrak{T} \cdot \mathfrak{W} - \mathfrak{W} \cdot \mathfrak{T} \cdot \mathfrak{S}$. Because $\mathfrak{W}$ commutes with both $\mathfrak{T}$ and $\mathfrak{S}$ up to $\approx_{\mathfrak{d}}$, it follows from Proposition 5.3.1 that $\mathfrak{W}' \approx_{\mathfrak{d}} \mathbf{0}$.

In order to show associativity up to $\approx_{\mathfrak{d}}$, we compute the corresponding difference. It follows that difference of associativity rule gives the net $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$ where

$$\mathfrak{A}_1 = \mathfrak{T} \cdot \mathfrak{S} \cdot \mathfrak{W} - \mathfrak{S} \cdot \mathfrak{T} \cdot \mathfrak{W} - \mathfrak{W} \cdot \mathfrak{T} \cdot \mathfrak{S} + \mathfrak{W} \cdot \mathfrak{S} \cdot \mathfrak{T}$$

and

$$\mathfrak{A}_2 = -\mathfrak{T} \cdot \mathfrak{S} \cdot \mathfrak{W} + \mathfrak{T} \cdot \mathfrak{W} \cdot \mathfrak{S} + \mathfrak{S} \cdot \mathfrak{W} \cdot \mathfrak{T} - \mathfrak{W} \cdot \mathfrak{T} \cdot \mathfrak{S}.$$

The net $\mathfrak{A}$ reduces to

$$-\mathfrak{S} \cdot \mathfrak{T} \cdot \mathfrak{W} - \mathfrak{T} \cdot \mathfrak{S} \cdot \mathfrak{W} + \mathfrak{T} \cdot \mathfrak{S} \cdot \mathfrak{W} + \mathfrak{S} \cdot \mathfrak{T} \cdot \mathfrak{W}.$$

Because $\mathfrak{W}$ commutes with both $\mathfrak{T}$ and $\mathfrak{S}$ up to $\approx_{\mathfrak{d}}$, the last net $\mathfrak{A}$ obtained is $\mathfrak{d}$-asymptotically equivalent to the zero net $\mathbf{0}$. Hence, in this case associativity holds up to $\approx_{\mathfrak{d}}$. □

The linear space $W_{\mathfrak{d}}$ has a distinguished subspace $W_{\mathfrak{d},ab}$ generated by all commutators of the form $\mathfrak{T} \cdot \mathbf{T}_\mu - \mathbf{T}_\mu \cdot \mathfrak{T}$ as $\mathfrak{T}$ runs through $\mathfrak{d}$-abelian nets and $\mu$ runs through $\Lambda$.

Nontrivial subspaces of $W_{u_J\mathfrak{d},ab}$ are important for the purposes the present paper. We first observe that $\mathscr{N}_\Lambda(Z(L(X,\mathfrak{d}))) \leq Z(\mathscr{N}_\Lambda(L(X,\mathfrak{d})))$ where $Z$ denotes the classical center. For every $\mathfrak{T} \in Z(\mathscr{N}_\Lambda(L(X,\mathfrak{d})))$ the corresponding commutators $\mathfrak{T} \cdot \mathbf{T}_\mu - \mathbf{T}_\mu \cdot \mathfrak{T}$ are equal to $\mathbf{0}$. Hence, they do not generate nontrivial subspaces of $W_{u_J\mathfrak{d},ab}$.

We define

$$Z_{\mathfrak{T},\mathfrak{d}} = \{\mathfrak{S} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))\colon \mathfrak{S} \text{ is } \mathfrak{d}\text{-abelian and } \mathfrak{T} \cdot \mathfrak{S} - \mathfrak{S} \cdot \mathfrak{T} = 0\}$$

for all $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$. Denote by $Z_{\mathfrak{T}} W_{\mathfrak{d},ab}$ the linear subspace of $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ generated by $\mathfrak{S} \cdot \mathbf{S}_\mu - \mathbf{S}_\mu \cdot \mathfrak{S}$ as $\mathfrak{S}$ runs through $Z_{\mathfrak{T},\mathfrak{d}}$ and $\mu$ runs through $\Lambda$.

**Proposition 5.5.13.** *Let $X$ be a Banach lattice and $\Lambda$ be an index set, as above. Then one has the inclusion*

$$W_{\mathfrak{d},ab} \leq W_{\mathfrak{d}} \leq \mathscr{N}_{\Lambda}(L(X,\mathfrak{c}))$$

*of linear spaces. For every $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ belonging to $\mathscr{N}_{\Lambda}(L(X,\mathfrak{c}))$ the inclusions*

$$\mathfrak{T} + Z_{\mathfrak{T}} W_{\mathfrak{d},ab} \leq \mathfrak{T} + W_{\mathfrak{d},ab} \leq \mathfrak{T} + W_{\mathfrak{d}} \leq \mathscr{N}_{\Lambda}(L(X,\mathfrak{c}))$$

*and*

$$\mathbf{T}_\mu + Z_{\mathfrak{T}} W_{\mathfrak{d},ab} \leq \mathbf{T}_\mu + W_{\mathfrak{d},ab} \leq \mathbf{T}_\mu + W_{\mathfrak{d}} \leq \mathscr{N}_{\Lambda}(L(X,\mathfrak{c}))$$

*of cosets hold for all $\mu \in \Lambda$.*

*Proof.* Because $W_{\mathfrak{d},ab}$ is generated by commutators of the form $\mathfrak{T}\cdot\mathbf{T}_\mu - \mathbf{T}_\mu\cdot\mathfrak{T}$ where $\mathfrak{T}$ is $\mathfrak{d}$-abelian and $\mu \in \Lambda$, each commutator $\mathfrak{T}\cdot\mathbf{T}_\mu - \mathbf{T}_\mu\cdot\mathfrak{T}$ is $\mathfrak{d}$-asymptotically equivalent to $\mathbf{0}$. It follows that $W_{\mathfrak{d},ab}$ is indeed a subspace of $W_{\mathfrak{d}}$.

Let $\mathfrak{T} \in \mathscr{N}_{\Lambda}(L(X,\mathfrak{c}))$ and consider a generator $\mathfrak{S}\cdot\mathbf{S}_\mu - \mathbf{S}_\mu\cdot\mathfrak{S}$ of $Z_{\mathfrak{T}} W_{\mathfrak{d},ab}$ with $\mathfrak{S} \in Z_{\mathfrak{T},\mathfrak{d}}$ and $\mu \in \Lambda$. Because $\mathfrak{S}$ is $\mathfrak{d}$-abelian, the commutator $\mathfrak{S}\cdot\mathbf{S}_\mu - \mathbf{S}_\mu\cdot\mathfrak{S}$ is $\mathfrak{d}$-asymptotically equivalent to $\mathbf{0}$. This implies that $Z_{\mathfrak{T}} W_{\mathfrak{d},ab}$ is a linear subspace of $W_{\mathfrak{d},ab}$ for all $\mathfrak{T} \in \mathscr{N}_{\Lambda}(L(X,\mathfrak{c}))$. Remaining statements follow easily from these facts. □

### 5.6 $\mathfrak{d}$-LR and $\mathfrak{d}$-Martingale Nets on Convergence Vector Lattices

The approach given in [20] can be utilized in the settings of Banach lattices and $u_J\mathfrak{c}$-asymptotic equivalences to define the notion of unbounded martingale nets on Banach lattices.

**Example 5.6.1.** *Let $\mathscr{P} = (P_n)$ be an abstract bistochastic filtration on a Banach lattice, see [41] for details. Recall that $P_nP_m = P_mP_n = P_{n\wedge m}$ for all $m,n \geq 1$. Then $\mathscr{P}$ is a $\mathfrak{d}$-martingale net with $\Lambda = \mathbb{N}$ on $X$ for every $T_1$-convergence $\mathfrak{d}$ on $X$. To see this, we have to show $\mathbf{P}_m\cdot\mathscr{P} - \mathbf{P}_m \approx_{\mathfrak{d}} \mathbf{0}$ and $\mathscr{P}\cdot\mathbf{P}_m - \mathbf{P}_m \approx_{\mathfrak{d}} \mathbf{0}$ for all $m \geq 1$. The first one follows from $(P_mP_k - P_m)(x) \xrightarrow{\mathfrak{d}} 0$ because $P_mP_k = P_m$ whenever $k \geq m$. Similarly, $(P_kP_m - P_m)(x) \xrightarrow{\mathfrak{d}} 0$ as $P_kP_m = P_m$ whenever $k \geq m$.*

**Proposition 5.6.2.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ be a $\Lambda$-indexed $\mathfrak{d}$-abelian net belonging to $\mathscr{N}_{\Lambda}(L(X,\mathfrak{c}))$ such that $\mathbf{T}_\mu\cdot\mathfrak{T} \approx_{\mathfrak{d}} \mathbf{0}$ for all $\mu \in \Lambda$. The coset*

$$\mathbf{T}_\mu + W_{\mathfrak{d}} \in \mathscr{N}_{\Lambda}(L(X,\mathfrak{c}))/W_{\mathfrak{d}}$$

*equals to the zero coset for all $\mu \in \Lambda$ if and only if $\mathfrak{T}$ is an $\mathfrak{d}$-martingale net.*

*Proof.* Let $\mathfrak{T}$ be an $\mathfrak{d}$-abelian net such that $\mathbf{T}_\mu + W_{\mathfrak{d}} = 0$ in $\mathscr{N}_{\Lambda}(L(X,\mathfrak{c}))/W_{\mathfrak{d}}$ for all $\mu \in \Lambda$. Thus, $\mathbf{T}_\mu \in W_{\mathfrak{d}}$. Because $\mathfrak{T}$ is $\mathfrak{d}$-abelian, $\mathbf{T}_\mu\cdot\mathfrak{T} - \mathfrak{T}\cdot\mathbf{T}_\mu \in W_{\mathfrak{d},ab}$. It follows from $W_{\mathfrak{d},ab} \leq W_{\mathfrak{d}}$, see Proposition 5.5.13, that $\mathbf{T}_\mu \approx_{\mathfrak{d}} \mathbf{0}$ and $\mathbf{T}_\mu \cdot \mathfrak{T} - \mathfrak{T} \cdot \mathbf{T}_\mu \approx_{\mathfrak{d}} \mathbf{0}$ for all $\mu \in \Lambda$. Hence, $\mathbf{T}_\mu \cdot \mathfrak{T} \approx_{\mathfrak{d}} \mathbf{T}_\mu \approx_{\mathfrak{d}} \mathfrak{T} \cdot \mathbf{T}_\mu$. Conversely, $\mathbf{T}_\mu \cdot \mathfrak{T} \approx_{\mathfrak{d}} \mathbf{0}$ for all $\mu \in \Lambda$ implies that $\mathbf{T}_\mu + W_{\mathfrak{d}}$ equals to zero coset. □

**Proposition 5.6.3.** *If $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ is an $\mathfrak{c}$-LR-net then $\hat{\mathfrak{T}}^{\mathfrak{c}}(X) \leq \mathrm{Fix}(\mathfrak{T})$. In particular, the complex given in Theorem 5.4.9 has length one.*

*Proof.* If $y \in \hat{\mathfrak{T}}^{\mathfrak{c}}(X)$ then there exists $x \in X$ with $\hat{\mathfrak{T}}^{\mathfrak{c}}(x) = y$. By definition of $\mathfrak{c}$-*LR*-nets, $\mathfrak{d}\text{-}\lim_{\lambda\to\infty}(T_\mu - 1)T_\lambda(x) = 0$ with $\mathfrak{d} = \mathfrak{c}$. Because $\mathbf{T}_\mu - \mathbf{1} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ and limit is over the index $\lambda$, one has

$$\mathfrak{d}\text{-}\lim_{\lambda\to\infty}(T_\mu - 1)\circ T_\lambda(x) = (\widehat{\mathbf{T}_\mu - \mathbf{1}})(\hat{\mathfrak{T}}(x)) = (\widehat{\mathbf{T}_\mu - \mathbf{1}})(y) = 0$$

with $\mathfrak{d} = \mathfrak{c}$ for all $\mu \in \Lambda$. It follows that $T_\mu y = y$ for all $\mu \in \Lambda$. □

**Proposition 5.6.4.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ be a $\Lambda$-indexed $\mathfrak{d}$-abelian net belonging to $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))$ such that $\mathbf{T}_\mu \cdot \mathfrak{T} \approx_{\mathfrak{d}} \mathbf{0}$ for all $\mu \in \Lambda$. The coset*

$$\mathfrak{T} + W_{\mathfrak{d}} \in \mathscr{N}_\Lambda(L(X,\mathfrak{c}))/W_{\mathfrak{d}}$$

*equals to the zero coset if and only if $\mathfrak{T}$ is an $\mathfrak{d}$-LR-net.*

*Proof.* Let $\mathfrak{T}$ be an $\mathfrak{d}$-abelian net such that $\mathfrak{T} + W_{\mathfrak{d}}$ equals to zero coset in $\mathscr{N}_\Lambda(L(X,\mathfrak{c}))/W_{\mathfrak{d}}$. Because $\mathfrak{T}$ is $\mathfrak{d}$-abelian, $\mathbf{T}_\mu \cdot \mathfrak{T} - \mathfrak{T} \cdot \mathbf{T}_\mu \in W_{\mathfrak{d},ab}$. It follows from $W_{\mathfrak{d},ab} \leq W_{\mathfrak{d}}$, see Proposition 5.5.13, that $\mathfrak{T} \approx_{\mathfrak{d}} \mathbf{0}$ and $\mathbf{T}_\mu \cdot \mathfrak{T} - \mathfrak{T} \cdot \mathbf{T}_\mu \approx_{\mathfrak{d}} \mathbf{0}$ for all $\mu \in \Lambda$. Hence, $\mathbf{T}_\mu \cdot \mathfrak{T} \approx_{\mathfrak{d}} \mathfrak{T} \approx_{\mathfrak{d}} \mathfrak{T} \cdot \mathbf{T}_\mu$. Conversely, if $\mathfrak{T}$ is an $\mathfrak{d}$-*LR*-net and $\mathbf{T}_\mu \cdot \mathfrak{T} \approx_{\mathfrak{d}} \mathbf{0}$ for all $\mu \in \Lambda$ then $\mathfrak{T}$ belongs to $W_{\mathfrak{d}}$. □

The following result is similar to that of [20, Proposition 2]. Recall from Section 5.4 that for a $\mathfrak{c}$-Markov operator sequence $\mathfrak{T} = (T_n)_{n\geq 1}$ we put

$$\mathscr{M}_i(\mathfrak{T}) := \{(x_n)_{n\geq 1} \colon (\delta_{i1}I - T_n)(x_m) = (-1)^{i+1}x_n \text{ if } m \geq n\}$$

for $i = 0, 1$.

**Proposition 5.6.5.** *A net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ is an $u_J\mathfrak{d}$-martingale net if and only if $\mathbf{1} - \mathfrak{T}$ is an $u_J\mathfrak{d}$-LR-net. Furthermore, if $\mathfrak{T}$ is a sequence of operators which is a $\mathfrak{c}$-Markov $u_J\mathfrak{c}$-martingale net then $\mathscr{M}_0(\mathfrak{T}) = \mathscr{M}_1(\mathbf{I} - \mathfrak{T})$ and $\mathscr{S}\mathscr{M}_0(\mathfrak{T}) = \mathscr{S}\mathscr{M}_1(\mathbf{I} - \mathfrak{T})$.*

*Proof.* It follows from Proposition 5.5.10 that $\mathfrak{T}$ is $u_J\mathfrak{c}$-abelian if and only if $\mathbf{1} - \mathfrak{T}$ is $u_J\mathfrak{c}$-abelian. If $\mathfrak{T}$ is an $u_J\mathfrak{c}$-*LR*-net then $(\mathbf{1} - \mathbf{T}_\mu) \cdot (\mathbf{1} - \mathfrak{T}) - (\mathbf{1} - \mathbf{T}_\mu) = -\mathfrak{T} + \mathbf{T}_\mu \cdot \mathfrak{T}$ which is $u_J\mathfrak{c}$-asymptotically equal to zero. Hence, $\mathbf{1} - \mathfrak{T}$ is an $u_J\mathfrak{c}$-martingale net. □

**Example 5.6.6.** *Being an un-martingale operator net or unLR-net is invariant under passing to an equivalent lattice norm on the Banach lattice $X$. More precisely, if $\mathfrak{T}$ is an un-martingale net (un-LR-net) and $\|\cdot\|_2$ is a lattice norm on $X$ that is equivalent to the original norm on $X$, then $\mathfrak{T}$ is again an un-martingale net (un-LR-net) on $(X, \|\cdot\|_2)$.*

**Example 5.6.7.** *Let $\mathfrak{d} = n$, and $T$ be a norm bounded operator on a Banach lattice $X$. We do not assume that $T$ is contraction. The sequence of Cesàro means $A_k^T = k^{-1}\Sigma_{j=0}^{k-1} T^k$ is an n-LR-net on $X$ with $\Lambda = \mathbb{N}$. If $T$ is a contraction then the sequence $A_k^T$ is both n-LR-net and LR-net, in the sense of [25, 20, 22, 23]. Suppose that $T$ is mean ergodic so that $(A_k^T)_{k=1}^{\infty}$ is strongly convergent. If $\mathfrak{d}$ is a convergence on $X$ for which n-convergent nets are also $\mathfrak{d}$-convergent then $(A_k^T)_{k=1}^{\infty}$ is $\mathfrak{d}$-convergent.*

## 5.7 An Application to $o$-Bounded $o$-$LR$-Nets

Fixed point sets of operator nets and semigroups have been studied by many researchers. A major problem in this theory is the characterization of operator nets and operator semigroups whose actions are equivalent to actions of well-known operator semigroups. Throughout present section denote by $\mathfrak{d}$ a convergence on $X$.

Let $\mathfrak{T} = (T_\lambda)_{\lambda \in \Lambda}$ be a net in $\mathcal{N}_\Lambda(L(X,\mathfrak{c}))$. Recall that we denote by $\mathrm{Fix}(\mathfrak{T})$ the intersection of fixed spaces of $T_\lambda$, that is

$$\mathrm{Fix}(\mathfrak{T}) = \bigcap_{\lambda \in \Lambda} \ker(I - T_\lambda).$$

Two vectors $x_1, x_2 \in X$ are said to be $(\mathfrak{T},\mathfrak{d})$-equivalent if $T_\lambda(x_1 - x_2) \xrightarrow{\mathfrak{d}} 0$. The set of all vectors of $X$ which are $(\mathfrak{T},\mathfrak{d})$-equivalent to a given vector $x \in X$ is independent of $x$. We put $x_1 \equiv_{(\mathfrak{T},\mathfrak{d})} x_2$ for $x_1, x_2 \in X$ if $x_1$ and $x_2$ are $(\mathfrak{T},\mathfrak{d})$-equivalent. It follows that if $T_\lambda(x) \xrightarrow{\mathfrak{d}} y$ for some $y \in \mathrm{Fix}(\mathfrak{T})$ then $x \equiv_{(\mathfrak{T},\mathfrak{d})} y$.

**Proposition 5.7.1.** *If $\mathfrak{T} = (T_\lambda)_{\lambda \in \Lambda} \in \mathcal{N}_\Lambda(L(X,\mathfrak{c}))$ is an $\mathfrak{d}$-LR-net then $x \equiv_{(\mathfrak{T},\mathfrak{d})} T_\mu(x)$ for every $\mu \in \Lambda$. Conversely, a $\mathfrak{d}$-abelian net $(T_\lambda)_{\lambda \in \Lambda}$ for which $x \equiv_{(\mathfrak{T},\mathfrak{d})} T_\mu(x)$ for every $\mu \in \Lambda$ and $x \in X$ is an $\mathfrak{d}$-LR-net on the convergence vector lattice $(X,\mathfrak{c})$.*

*Proof.* If $\mathfrak{T} = (T_\lambda)_{\lambda \in \Lambda}$ is an $\mathfrak{d}$-*LR*-net then $\mathfrak{T} \approx_{\mathfrak{d}} \mathfrak{T} \cdot \mathbf{T}_\mu$ for all $\mu \in \Lambda$. Thus, $T_\lambda(T_\mu(x) - x) \xrightarrow{\mathfrak{d}} 0$ for each fixed $\mu \in \Lambda$. This shows that $x \equiv_{(\mathfrak{T},\mathfrak{d})} T_\mu(x)$ for all $\mu \in \Lambda$. Conversely, $x \equiv_{(\mathfrak{T},\mathfrak{d})} T_\mu(x)$ for all $x \in X$ implies that $\mathfrak{T} \approx_{\mathfrak{d}} \mathfrak{T} \cdot \mathbf{T}_\mu$. Because $\mathfrak{T}$ is $\mathfrak{d}$-abelian, $\mathbf{T}_\mu \cdot \mathfrak{T} - \mathfrak{T} \cdot \mathbf{T}_\mu \in W_{\mathfrak{d}}$ where $W_{\mathfrak{d}}$ is the space $W_{\mathfrak{d}} = \{\mathfrak{T} \colon \mathfrak{T} \approx_{\mathfrak{d}} 0\}$ which is defined in Section 5.5. It follows that $\mathfrak{T}$ is an $\mathfrak{d}$-*LR*-net. $\square$

**Proposition 5.7.2.** *Let $\mathfrak{T} \in \mathcal{N}_\Lambda(L(X,o))$ be an o-LR-net consisting of lattice automorphisms of the vector lattice $X$. If $x \equiv_{(\mathfrak{T},o)} 0$ then for every $y \in \mathrm{Fix}(\mathfrak{T})$ one has $T_\mu(x) \vee y \equiv_{(\mathfrak{T},o)} y^+$ and $T_\beta(x) \wedge y \equiv_{(\mathfrak{T},o)} y^-$ for every $\mu \in \Lambda$.*

*Proof.* Because $x \equiv_{(\mathfrak{T},o)} 0$, one has $T_\lambda(x) \xrightarrow{o} 0$. There exists a net $y_\lambda$ in $X$ such that $y_\lambda \downarrow 0$ and $|T_\lambda(x) \vee T_\lambda(y) - y \vee 0| \leq y_\lambda$. Hence, $x \vee y \equiv_{(\mathfrak{T},o)} y^+$ holds. Equivalently, $T_\lambda(x \vee y - y \vee 0) \xrightarrow{o} 0$. Similarly, one can show that $x \wedge y \equiv_{(\mathfrak{T},o)} y^-$. Finally, we note that $x \equiv_{(\mathfrak{T},o)} 0$ implies $T_\mu(x) \equiv_{(\mathfrak{T},o)} 0$ for all $\mu \in \Lambda$. $\square$

According to definition given in Section 5.2, an $o$-bounded $o$-*LR*-net $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,o))$ is an operator net for which each $T_\lambda \in L(X,o)$ for every $\lambda \in \Lambda$. Because $\mathfrak{T}$ is $o$-bounded, there exists an order bounded and order continuous $S$ such that $|T_\lambda| \leq S$. By Ogasawara Theorem, see [5, Theorem 1.57], $|T_\lambda|$ is order continuous for every $\lambda \in \Lambda$.

**Proposition 5.7.3.** *Let $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,o))$ be an o-bounded o-LR-net on a Dedekind complete vector lattice $X$. The linear subspace* $\mathrm{Fix}(\mathfrak{T})$ *is order closed.*

*Proof.* Denote by $S\colon X \to X$ the $o$-continuous positive operator for which $|T_\lambda| \leq S$ for all $\lambda \in \Lambda$. Let $x \in X$ belong to order closure of $\mathrm{Fix}(\mathfrak{T})$ so that there exists nets $x_\alpha$ in $\mathrm{Fix}(\mathfrak{T})$ and $e_\alpha \downarrow 0$ in $X$ such that $|x - x_\alpha| \leq e_\alpha$ for all $\alpha$. For every $\lambda \in \Lambda$, one has

$$|T_\lambda x - x| \leq |T_\lambda||x_\alpha - x| + |x_\alpha - x| \leq S(e_\alpha) + e_\alpha$$

in which right hand side of the inequality order converges to 0 by the order continuity of $S$. $\square$

**Proposition 5.7.4.** *Let $\mathfrak{T} \in \mathscr{N}_\Lambda(L(X,o))$ be an o-bounded o-LR-net on a Dedekind complete vector lattice $X$. A vector $x$ belongs to* $\mathrm{Fix}(\mathfrak{T})$ *if and only if there exists an $\lambda' = \lambda'(x) \in \Lambda$ such that $T_\lambda(x) = x$ for every $\lambda \geq \lambda'$.*

*Proof.* Because $\mathfrak{T}$ is an $o$-*LR*-net there exists a net $f_\lambda \downarrow 0$ in $X$ and $\lambda_0 \in \Lambda$ such that $|(I - T_\mu)T_\lambda|(x) \leq f_\lambda$ for every $\lambda \geq \lambda_0$. It is also given that $T_\lambda(x) \xrightarrow{o} x$. There exists a net $e_\lambda \downarrow 0$ and $\lambda_1 \in \Lambda$ such that $|T_\lambda(x) - x| \leq e_\lambda$ for every $\lambda \geq \lambda_1$. Denote by $S\colon X \to X$ the $o$-continuous positive operator for which $|T_\lambda| \leq S$ for all $\lambda \in \Lambda$. It follows that

$$|T_\mu(x) - x| \leq |T_\mu x - T_\mu T_\lambda x| + |T_\mu T_\lambda x - T_\lambda x| + |T_\lambda x - x| \leq S e_\lambda + f_\lambda + e_\lambda$$

for every $\lambda \geq \lambda_0 \vee \lambda_1$. Right side converges to 0 as $S$ is order continuous. $\square$

Order continuous operators on order ideals play an important role in Veksler's Theorem, see [5, Theorem 1.65]. Given an ideal $B$ of $X$ and an operator $T \in L_b(X)$, let us write

$$T_B(x) := \sup T(B \cap [0,x])$$

for $x \in X^+$. It is known that the operator $T_B$ is a component of $T$ in $L_b(X)$, see [5, Chapter 2] for details.

**Theorem 5.7.5.** *Let $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ be a positive o-bounded net on a Dedekind complete vector lattice $X$. Let $B$ be a $\mathfrak{T}$-invariant band of $X$ such that $B \le \cup_{\lambda\in\Lambda}(I-T_\lambda)(X)$. The net $\mathfrak{T}$ is an o-LR-net on $B$ if and only if the net $\mathfrak{T}_B := (T_{\alpha,B})_{\lambda\in\Lambda}$ is an o-LR-net on $X$.*

*Proof.* Denote by $S\colon X \to X$ the $o$-continuous positive operator for which $|T_\lambda| \le S$ for all $\lambda \in \Lambda$. The operator $S_B$ is order continuous on $X$. For every $\lambda \in \Lambda$, $|T_\lambda| \le S$ implies that $|T_{\lambda,B}| \le S_B$. Let us show that $o\text{-}\lim_\lambda T_{\lambda,B}(I - T_{\mu,B})x = 0$ for every $x \in X^+$. Since $X$ is Dedekind complete and $B$ is $\mathfrak{T}$-invariant, we can write $x - T_{\mu,B} = x_1 - T_{\mu,B}x + x_2$ for some $x_1 \in B$ and $x_2 \in B^\perp$. Furthermore,

$$T_{\alpha,B}(x - T_{\mu,B}x) = T_{\alpha,B}(x_1 - T_{\mu,B}x)$$

holds. It follows from $B \le \cup_{\lambda\in\Lambda}(I-T_\lambda)(X)$ that $o\text{-}\lim_\lambda T_{\lambda,B}(x_1 - T_{\mu,B}x) = 0$. Let us show that $o\text{-}\lim_\lambda (I - T_{\mu,B})T_{\lambda,B}x = 0$ for every $x \in X^+$. Since $T_{\lambda,B}x \in B$ and $T_{\mu,B} = T_\mu$ on $B$, one has $o\text{-}\lim_\lambda (I - T_{\mu,B})T_{\alpha,B}(x) = 0$. It follows that $\mathfrak{T}_B$ is an $o$-$LR$-net on $X$.

Converse implication is easy because if $\mathfrak{T}_B$ is an $o$-$LR$-net then $T_{\lambda,B} = T_\lambda$ on $B$ for every $\lambda \in \Lambda$. $\square$

**Proposition 5.7.6.** *Let $\mathfrak{T} \in \mathcal{N}_\Lambda(L(X,o))$ be a directed upward o-bounded o-LR-net on a Dedekind complete vector lattice $X$. Then $\sup_\lambda T_\lambda : X \to X$ is a retract onto the space* $\mathrm{Fix}(\mathfrak{T})$. *Moreover, $x \equiv_{(\mathfrak{T},o)} \sup_\lambda T_\lambda x$ for every $x \in X^+$.*

*Proof.* Because $\mathfrak{T}$ is directed upward, $(\sup_\lambda T_\lambda)x = \sup_\lambda T_\lambda x$ defines an operator on $X$ such that $\sup_\lambda T_\lambda x = o\text{-}\lim_\lambda x$ holds. It follows from

$$T_\mu \sup_\lambda T_\lambda x - \sup_\lambda T_\lambda x = o - \lim_\lambda (T_\mu - I)T_\lambda x = 0$$

for $\mu \in \Lambda$ that image of $\sup_\lambda T_\lambda$ equals to $\mathrm{Fix}(\mathfrak{T})$. The operator $\sup_\lambda T_\lambda$ acts as identity on $\mathrm{Fix}(\mathfrak{T})$. Therefore, $\mathrm{Fix}(X)$ is a retract of $X$. Given $x \in X$, it follows from $o\text{-}\lim T_\lambda(x) \in \mathrm{Fix}(\mathfrak{T})$ that $x \equiv_{(\mathfrak{T},o)} \sup_\lambda T_\lambda x$. $\square$

**Proposition 5.7.7.** *Let $\mathfrak{T} \in \mathcal{N}_\Lambda(L(X,o))$ be an o-bounded o-LR-net on a Dedekind complete vector lattice $X$. Denote by $X_n^\sim$ the order continuous dual of $X$, see [5]. If $\mathfrak{T}$ is o-convergent then* $\mathrm{Fix}(\mathfrak{T})$ *separates* $\mathrm{Fix}(\mathfrak{T}') \cap X_n^\sim$ *for the adjoint operator net $\mathfrak{T}' = (T'_\lambda)_{\lambda\in\Lambda}$.*

*Proof.* Let us put $Q(x) := o - \lim_\lambda T_\lambda(x)$ for all $x \in X$. With respect to notations of Section 5.4, $Q(x) = \hat{\mathfrak{T}}^o(x)$. Let $f \in \mathrm{Fix}(\mathfrak{T}') \cap X_n^\sim$ be an order continuous functional on $X$ such that $f(x) \ne 0$ for some $x \in X$. It follows from order continuity of $f$ that

$$f(Q(x)) = o - \lim_\lambda f(T_\alpha(x)) = o - \lim_\lambda (T'_\lambda f)(x) = f(x).$$

Hence, $\mathrm{Fix}(\mathfrak{T})$ separates the space $\mathrm{Fix}(\mathfrak{T}') \cap X_n^\sim$. $\square$

## 5.8 Conclusions

In this chapter, we extended the notion of Lotz-Räbiger net to allow for a framework not covered by the definition of Lotz-Räbiger nets on Banach spaces in [25, 20, 22, 23, 24, 52, 62]. A $\mathfrak{d}$-Lotz-Räbiger net $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ is a net of operators satisfying the conditions of Definition 5.2.9 and it is formed by $\mathfrak{c}$-continuous operators $T_\lambda \colon X \to X$ on a convergence vector lattice $(X, \mathfrak{c})$ where $\mathfrak{d}$ is an arbitrary convergence, possibly different than $\mathfrak{c}$, on $X$. We accomplished this by using $\mathfrak{d}$-asymptotic equivalence, which is an equivalence relation on the collection of operator nets and which can be used as a useful tool for understanding the behavior of orbits of $\mathfrak{T} = (T_\lambda)_{\lambda\in\Lambda}$ as $\mathfrak{d}$ varies over convergences on $X$. We then show that $\mathfrak{d}$-Lotz-Räbiger nets generalize certain properties of Lotz-Räbiger nets. Many of the results presented in this chapter depend on the fact that convergences and, in particular, unbounded convergences can be used in an appropriate manner to study asymptotic behaviors of operator nets. This provides an example to the fact that unbounded convergences can be combined with the results of ergodic theory to get new theorems.

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