MORTALITY MODELLING WITH RENEWAL PROCESS AND OPTIMAL HEDGING STRATEGY UNDER BASIS RISK

ÖLÜMLÜLÜĞÜN YENİLEME SÜRECİ İLE MODELLENMESİ VE BAZ RİSKİ ALTINDA OPTİMAL KORUNMA STRATEJİSİNİN BULUNMASI

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Submitted to

Graduate School of Science and Engineering of Hacettepe University

as a Partial Fulfillment to the Requirements

for the Award of the Degree of Doctor of Philosophy

in Actuarial Sciences

2020

To my wonderful grandmother Bedia…

ABSTRACT

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January 2020, 122 pages

In this thesis, we address the risks that are related to the random residual lifetime of insureds. These risks could be classified as catastrophic mortality risk and longevity risk. Catastrophic mortality risk represents the sudden increases in mortality rates which means that the insurance companies or pension plans would have to make sudden payments to many policyholders. While catastrophic mortality risk describes the shorter lifetime than anticipated of an individual or a group, its counterpart longevity risk represents the uncertain evolution in mortality rates. When a group or an individual live longer than anticipated, insurance companies or pension plans would make annuity payments longer than expected. Since the catastrophic mortality risk and longevity risk could cause serious financial consequences, management of these risks is important for insurance companies and pension plans.

Catastrophic mortality risk often causes transitory jumps on the mortality curve. Several stochastic mortality models have been developed to capture these jump effects. To the best of author' knowledge, all these jump models in the actuarial literature assume that the mortality jumps occur once a year, or they used a Poisson process for their jump frequencies. Due to their low probability and high-impact nature, the timing and the frequency of future catastrophic events and hence mortality jumps are unpredictable, however, the history of events could give information about their future occurrences. In this thesis, a new approach for the modelling of the frequency of catastrophic mortality risk is introduced and a specification of the Lee-Carter model using a renewal process is proposed. The history of events can be included in jump modelling by using this process.

We perform several statistical tests on the inter-arrival times data of the catastrophic events to show that the renewal process could be used for jump frequencies. For this purpose, first, we detect outliers on the mortality time index. The statistical tests are applied to the inter-arrival times of these detected outliers. According to the test results, we can use the lognormal renewal process to model jump frequencies for all selected countries.

Longevity risk is another risk factor that we examined in this thesis. We use index-based longevity swaps to hedge this risk. Index-based securities have many advantages. In such capital market solutions, it is possible to transfer the longevity risk to capital markets at lower costs. However, the potential differences between hedging instruments and pension or annuity portfolio cause longevity basis risk. Furthermore, we extended the proposed mortality model to incorporate longevity basis risk. We modelled reference population's mortality by using the proposed mortality model and then the portfolio's mortality is modelled by using the information of the reference population. According to our analysis, the common age effect is important for both populations.

Since the longevity-linked derivatives are traded in the over-the-counter markets, an insurer or a pension plan can be exposed to counterparty default risk. In this thesis, we provide a hedging framework for longevity basis risk in the context of collateralization.

We assume that both parties are posting the collateral and they re-hypothecate it to increase the benefits of this transaction.

We build hypothetical pension plan and index-based longevity swap transaction to show the effects of collateralization and risk reduction level. Our analysis present that bilateral collateral posting increases longevity basis risk reduction level and hedge effectiveness.

Keywords: Catastrophic mortality risk, renewal process, longevity basis risk, collateralization, bilateral collateral posting, hedge effectiveness.

ÖZET

ÖLÜMLÜLÜĞÜN YENİLEME SÜRECİ İLE MODELLENMESİ VE BAZ RİSKİ ALTINDA OPTİMAL KORUNMA STRATEJİSİNİN BULUNMASI

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Ocak 2020, 122 sayfa

Bu tezde, sigortalı bireylerin kalan yaşam sürelerindeki belirsizlikten kaynaklanan riskler ele alınmıştır. Bu riskler katastrofik ölümlülük riski ve uzun ömürlülük riski olarak sınıflandırılabilir. Katastrofik ölümlülük riski, ölüm oranlarında meydana gelen ani artışları ifade etmektedir. Bu durum sigorta şirketlerinin birçok poliçe sahibine anlık ödemeler yapması anlamına gelmektedir. Katastrofik ölümlülük riski bireylerin beklenilenden daha kısa süre yaşaması riskini ifade ederken uzun ömürlülük riski bireylerin ölüm oranlarındaki gelişimin belirsizliğini göstermektedir. Bir grup ya da bir bireyin beklenenden daha uzun süre yaşaması durumunda sigorta şirketleri veya emeklilik planları da anüite ödemelerini beklenenden daha uzun süre yapmak zorunda kalmaktadırlar. Bu sebeple katastrofik ölümlülük riski ve uzun ömürlülük riski ciddi finansal kayıplara sebep olmaktadır ve bu risklerin yönetimi sigorta şirketleri ve emeklilik planları için büyük önem taşımaktadır.

Katastrofik ölümlülük riski, ölüm eğrileri üzerinde kısa süreli geçici sıçramalara sebep olmaktadır. Ölüm eğrisi üzerindeki bu sıçrama etkilerini modelleyebilmek amacı ile stokastik ölümlülük modelleri geliştirilmiştir. Bilindiği kadarı ile aktüerya literatüründe sıçramaları dahil eden ölümlülük modelleri, sıçramaların ya yılda bir kez meydana geldiğini ya da sıçrama sıklıklarının Poisson sürecine sahip olduğunu varsaymaktadır. Ölümlülük sıçramalarının meydana gelme olasılıklarının düşük olması ve yüksek şiddetli bir yapıya sahip olmaları nedeniyle ortaya çıkma zamanları tahmin edilememektedir; Ancak olayların geçmişi, gelecekteki meydana gelme olasılıkları hakkında bilgi vermektedir. Bu tezde, katastrofik ölümlülük sıçramalarının sıklığının modellenmesi için yeni bir yaklaşım geliştirilmiş ve yenileme süreci kullanılarak Lee-Carter ölümlülük modelinin farklı bir versiyonu önerilmiştir. Yenileme süreci kullanılarak olayların geçmişinin sıçrama sıklıklarının modellenmesine dahil edilmesi sağlanmıştır.

Yenileme sürecinin sıçrama sıklıklarının modellenmesinde kullanılabileceğinin gösterilebilmesi amacı ile birkaç istatistiksel test uygulanmıştır. Bu amaçla, öncelikle uç değer analizi yöntemi kullanılarak ölümlülük zaman indeksindeki uç değerler tespit edilmiştir. Bu uç değerlerin meydana gelmeleri arasındaki geçen zamanlara istatistiksel analizler yapılmıştır. Test sonuçlarına göre sıçrama frekanslarının modellenmesinde lognormal yenileme sürecinin kullanılmasının seçilen bütün ülkeler için uygun olacağı görülmüştür.

Bu tezde ele alınan bir başka risk faktörü ise uzun ömürlülük riskidir. Bu riskten korunma sağlamak amacı ile indekse bağlı uzun ömürlülük swapları kullanılmıştır. Ancak korunma araçları ve emeklilik veya sigortacı portföylerinin ölümlülük yapıları arasında farklılıklar ortaya çıkabilmektedir. Portföylerin ölümlülükleri arasındaki farktan kaynaklanan bu risk uzun ömürlülük baz riski olarak adlandırılmaktadır. Bu riskten korunma sağlamak ve modellemeye dahil edebilmek amacı ile önerilen ölümlülük modeli baz riski çerçevesinde genişletilmiştir. Önerilen model kullanılarak korunma aracının bağlı olduğu popülasyonun ölümlülüğü modellenmiştir. Daha sonra buradan elde edilen bilgiden de

faydalanılarak korunma sağlanan portföyün ölümlülüğü modellenmiştir. Analiz sonuçlarına göre ortak yaş etkisinin her iki popülasyon için önemli olduğu görülmüştür.

Ölümlülüğe bağlı türev ürünler tezgâh üstü piyasalarda işlem gördüğü için bir sigorta şirketi ya da emeklilik planı karşı taraf temerrüt riski ile karşı karşıya kalmaktadır. Bu tezde, teminatlandırma bağlamında uzun ömürlülük baz riski için bir riskten korunma çerçevesi önerilmektedir. Önerilen çerçevede her iki tarafın da teminat verdiği ve teminatlandırma işleminin faydasını arttırmak amacı ile verilen teminatların yeniden kullanımının da yapıldığı varsayılmaktadır.

Teminatlandırma işleminin etkilerini ve risk azalım seviyesini göstermek amacı ile varsayımsal bir emeklilik planı oluşturulmuş ve indekse bağlı uzun ömürlülük swap işlemi ele alınmıştır. Analiz sonuçları karşılıklı teminat alımının uzun ömürlülük baz riskinde önemli bir azalma sağladığını ve riskten korunma etkinliğini arttırdığını göstermektedir.

Anahtar Kelimeler: Katastrofik ölümlülük riski, yenileme süreci, uzun ömürlülük baz riski, teminatlandırma, karşılıklı teminat alımı, korunma etkinliği.

ACKNOWLEDGEMENT

First of all, I would like to express my sincere appreciation and thanks to my supervisor Prof. Dr. Kasırga Yıldırak for his support and guidance throughout this thesis.

I would like to express my deepest gratitude to my co-advisor, Assoc. Prof. Dr. Şule Şahin for her continuous encouragement, patience, support, insights and motivations over the years. She has always been there for me and I know she will always be there. I feel very lucky to have such precious co-advisor.

My sincere thanks also goes to my committee chair Prof. Dr. Sevtap Kestel, and committe members Prof. Dr. Fatih Tank, Assoc. Prof. Dr. Könül Bayramoğlu Kavlak and Assist. Prof. Dr. Başak Bulut Karageyik for their valuable participation and suggestions.

Special thanks to all my friends for their friendship and support.

I would like to thank my dad Vedat, mum Betül and sister Pelin for their unconditional love and continuous support.

Finally, my husband Onur Özen deserves my deepest gratitude for his support, endless love, patience and the trust in me.

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1. INTRODUCTION

Insurance companies and pension plans are constantly exposed to risks that have an impact on their long-term financial liabilities, including inflation risks, interest-rate risks, longevity risks, and catastrophic mortality risks. “Catastrophic mortality risk” describes dramatic increase in mortality rates over certain periods of time. This risk might be caused by catastrophic events like pandemics or wars. These events are infrequent; however, their occurrences could cause many death claims. Hence, insurance companies or pension plans have to make sudden payments to many policyholders. It is therefore important to manage their catastrophic mortality risk exposure for insurers or pension plans.

The number of catastrophic events has risen in the last four decades. According to the World Disasters Report in 2016, rising global temperatures are causing global climate changes that increase the frequency of natural disasters. The timing and the severity of future catastrophic events are unknown; however, the history of such events could give information about their future occurrences. In this thesis, a new approach for the modelling of the frequency of catastrophic mortality risk is introduced and a specification of the Lee-Carter model using a renewal process is proposed. With this process, the history of events can be included in the modelling process. To the author's best knowledge, this is the first such approach to be presented in the actuarial literature.

Another uncertainty for human mortality level is longevity risk that is the uncertain evolution of mortality recorded at adult and elderly ages. Mortality rates improved rapidly in the past decades with the improvements in hygienic and medical techniques. For example, the life expectancy of Canadian female and male new-borns was approximately 66 and 62 years in 1940, respectively, while in 2011, new-born females and males could respectively be expected to be lived to 84 and 80. For developed countries such as Japan and Canada, dramatic increases in life expectancy have been recorded. Although such increases in life expectancy throughout the 20th century may be regarded among the greatest achievements of human society, the uncertainty related to this increase in life expectancy has effects on the financial strength of the insurance industry. The uncertainty of longevity improvements has increased pressures on annuity providers and pension plans since they might end up paying the annuity benefits for longer periods of time than they expect [47]. As uncertainty regarding future mortality could cause important financial implications for individuals, annuity providers, pension plans, and social insurance program, the hedging of longevity risk is also an important consideration for both pension plan providers and companies offering life insurance.

By definition, there are two important aspects involved in longevity risk: the uncertainty underlying human mortality and the adverse financial consequences for providers of pension plans and for insurance companies. It is important that both aspects be addressed while studying longevity risk. An appropriate stochastic mortality model can accurately measure the underlying uncertainty. For risk management purposes, we also need to investigate how we can efficiently manage adverse financial consequences.

Various solutions have been presented to both manage and mitigate longevity risk. Reinsurance is a common and effective strategy for protection against large losses. However, due to the high costs of reinsurance, annuity and pension providers are limited in the extent to which they apply this strategy. Another approach is product diversification solutions, including the natural hedging strategy discussed by Cox and Lin [46] or the reinsurance swap presented by Lin and Cox [97]. These solutions have several advantages. For example, a liquid market is not required, and they can also be arranged at lower transaction costs. It is possible for insurance companies to diversify their products, life insurances, and annuities optimally for the hedging of longevity risk [123]. However, natural hedging's effectiveness depends on the mortality rates' age-specific distributions. Although Cox and Lin [46] indicated that natural hedging is an effective hedging strategy, the usage of this strategy is restricted in a way that involves the adjustment of the sales volume of annuity products and life insurance in order to maintain the proportion of liability [137].

Capital market solutions represent another hedging method; they include mortality securitization with the application of mortality- or longevity-linked securities such as longevity bonds or longevity swaps. In such capital market solutions, it is possible for an insurance company to transfer its mortality or longevity risk exposure to capital markets. Securitization has remained an important financial innovation for pension plan providers and life insurance companies since 1988. This is because securitization could serve to increase the value of a firm by decreasing the agency and transaction costs, taxation, regulation, and informational asymmetries [44]. That is why capital market strategies are adopted for the hedging of longevity risk in this thesis.

The mortality must be modelled for the assessment of longevity risk and the valuation of longevity-linked derivatives. While modelling and forecasting mortality, it is essential to use an appropriate mortality model. This model can be applied for quantifying the risk and providing a foundation for pricing and reserving. Due to the inadequacy of the quality and size of portfolio data, a reference population index is commonly used by hedgers. Therefore, mortality risk trading usually entails two different populations: the first is affiliated with the portfolio of the hedger, while the other is linked to the hedging instrument. As an example of this, we may consider the Swiss Re mortality bond, which was issued by Swiss Re in 2003. That bond was associated with a broad population mortality index; meanwhile, the hedger's exposure was linked to some insured lives [139]. Population basis risks are risks that are associated with differences in the experience of mortality between different populations of individuals who are associated with hedging instruments and the populations of individuals who are associated with the underlying exposure. Here, a multi-population mortality model is required for measuring the basis risk and for modelling mortality.

Several multi-population mortality models have recently been presented. For example, Carter and Lee [29] proposed the so-called joint-*k* model, which models the mortality dynamics for two different populations by applying the same time-varying mortality index for both. Li and Lee [92] and Li and Hardy [91] proposed an augmented common factor model and the co-integrated Lee-Carter model. Furthermore, Venter and Şahin [127] extended the joint mortality modelling in the Bayesian shrinkage context. A number of single-population mortality models containing jump effects have been developed and broadly applied, while Zhou et al. [139] have been the only authors to date to consider a two-population model incorporating transitory jump effects. It is necessary to consider mortality jump risks in the modelling of mortality securitization as they pose an important problem for a life insurer's solvency. The model of Zhou et al. [139] may be viewed as a generalization of that of Chen and Cox [34], who applied the two-population Lee-Carter model for the modelling of mortality. In the model of Lee and Carter [84], central mortality rates are modelled in such a way as to be correlated log-linearly with a time-dependent mortality index as well as adjusted for age-specific effects via the use of two sets of age-dependent coefficients. Since extreme events and mortality improvements have different effects in different age groups, adjustments for age groups are necessary. With this approach, the model is able to successfully capture overall mortality trends as well as the age-specific changes occurring for different age groups [34]. Since mortality should be forecast accurately, we need a period- and age-dependent mortality model. For this purpose, the specification of the Lee-Carter mortality model is used for mortality jump modelling.

Another risk factor that an insurer could be exposed to while trading longevity risk is the counterparty default risk. Since longevity-linked instruments are traded over the counter, there is always counterparty default risk existing. The previous financial crisis and historical experiences showed that counterparty default risk often leads to crucial losses for companies. Hence, a counterparty default risk mitigation tool should be considered. ISDA [71] indicates that the most credit-enhancing way is posting collateral regarding the value of longevity-linked security.

This thesis has a twofold aim. First, the history of catastrophic events will be included in the mortality modelling process and a realistic modelling approach will thus be provided. For this reason, the renewal process will be introduced for mortality jump frequencies and it will be shown that the renewal process fits the data best. To the author's knowledge, no other study in the actuarial literature to date has used the renewal process for jump frequencies.

The second aim is the introduction of collateralization for counterparty default risk within the framework of longevity basis risk management. This thesis provides a hedging framework in the context of collateralization and further rehypothecates the posted collateral to increase the benefits of this transaction and to provide a meaningful risk reduction. Again, to the best of the author's knowledge, this hedging framework is the first study of its kind in the actuarial literature.

The organization of the thesis is as follows. In the next chapter, we will identify the outliers that cause mortality jumps in the time series of mortality and we will also examine the causes of the outliers and their effects on human mortality. As the first step, we will apply an outlier detection method to the mortality time index and detect outliers on the mortality curve. In this chapter, a building block is constructed to model mortality jumps.

In Chapter 3, we will model transitory mortality jumps with the renewal process. In the first step, we will perform statistical analysis to show that the renewal process can be used for mortality jump frequencies. Afterwards, a specification of the Lee-Carter model with mortality jumps will be proposed. We will then price a hypothetical bond to show the impact of the renewal process on bond prices.

In Chapter 4, we will review longevity hedging products. In this chapter, the stakeholders of the longevity risk market will be introduced, followed by a comparison of the index versus customized hedging and a presentation of their advantages and disadvantages. The steps for constructing a hedge for longevity basis risk will also be described in this chapter.

In Chapter 5, steps for the building a two-population mortality model will be described. The existing literature for multi-population mortality models are examined. The relative approach is used for constructing the two-population mortality model. After modelling the mortality rates, future mortality rates are obtained by using the semi-parametric bootstrapping method.

In Chapter 6, we will present the general hedging framework for longevity basis risk. Then we construct a hedging framework for longevity basis risk in the presence of counterparty default risk and collateralization. A hypothetical pension plan will be used to show the collateralization. Furthermore, optimal recovery rates will be obtained for hedging and the effect of the collateralization on hedge effectiveness will be examined.

In the final section, conclusions and suggestions for future research will be presented.

2. DETECTION OF OUTLIERS IN MORTALITY TIME INDEXES

2.1. Introduction

Interrupting phenomena have been a trending topic in the analysis of time series with the study of mortality trends. The main objective is finding the interruptions that are not consistent with normal trends of the mortality index. For instance, catastrophic events, pandemics, or terrorist attacks could bring about an immeasurable number of deaths. In recent years, for example, the avian flu in 2006 and the Ebola virus in 2014 caused approximately 1 million deaths. These interruptive events create spikes, which have a short effect, in the time series of mortality trends. In the statistical literature, they are thus referred to as outliers [88].

A variety of sources cause outliers, such as nonrepetitive exogenous events in a mortality index. By examining these outliers, significant information about mortality shocks that have an impact on time series could be discovered. The presence of outliers could be misleading in the analysis of time series of mortality [65]. Moreover, knowledge about the frequency, timing, and size of outliers could assist researchers in their efforts to forecast how time series of mortality might behave in the case of the occurrence of events of a similar interruptive nature. Thus, detecting these outliers is important for both model estimation and forecasting [88].

Several methods have been developed for analysing outliers in a time series. In 1972, Fox introduced innovative outliers and additive outliers. An iterative approach for identifying outliers was proposed by Tsay [124]. Chang et al. [32] developed likelihood ratio criteria to test for existing outliers for both criteria and types to distinguish them, and they introduced an iterative procedure to estimate time-series parameters in ARIMA models. A partially graphical method based on the mapping of time series into multivariate Euclidean space was proposed by Gather et al. [60]. More recently, Galeano and Pena [61] dealt with outliers in seasonal ARIMA models. Unfortunately, not many studies have examined the existence of outliers in historical mortality trends. Lee and Carter [84] considered the influenza pandemic of 1918 as being anomalous and they dealt with it by applying an intervention model. However, outlier analysis is commonly used in stochastic investment modelling. A detailed discussion of it may be found in the works of Chan [30] and Chan and Wang [31].

Identifying the types and the locations of outliers is a common way to deal with them. Effective methods for finding the locations of outliers and for the estimation of the effects of largely isolated outliers have been considered by a few researchers; however, in this regard, some unresolved issues remain

- The existence of outliers might lead to inappropriate modelling.
- Certain outliers might fail to be detected as a result of a masking effect.
- Outliers present in a time series could create bias in the parameter estimates and therefore could have an impact on the efficiency of outlier detection.

Chen and Liu [37] proposed an iterative estimation method to solve these problems. In this chapter, we use the proposed method of Chen and Liu [37] to detect the outliers in the mortality index. Our aim here is to find the outliers that cause spikes in the mortality curve and construct a background for the jump modelling process. We do not consider the outlier adjustment problem.

The main aim of this chapter is to provide a basis for modelling mortality jump frequency by identifying the locations of outliers. Here, we will only focus on the additive outliers that have an immediate and one-time short effect on the mortality index. With this goal in mind, time-series analysis is performed for a mortality index obtained from the Lee-Carter model. We will seek the additive outliers of the mortality index and then the detected outliers will be matched with the events that might have caused them. Mortality data for the US, the UK, Switzerland, France, and Italy will be used here.

In the next subsection, the events that cause the outliers are introduced. Afterwards, in Section 2.3, mortality data are described. Section 2.4 provides the definition and properties of the Lee-Carter mortality model. In Section 2.5, definitions are given for four types of outliers and then the detection procedure is explained. Finally, in Section 2.6, Chen and Liu's method is applied to find the outliers of the mortality of the US, the UK, Switzerland, France, and Italy. The conclusions of this chapter are presented in Section 2.7.

2.2. Sources of Outliers

There are several factors that cause outliers in a mortality time index. They may be classified as follows [59]:

- Miscalculation of claim levels
- Random statistical fluctuations
- Misestimation of mortality trends
- Catastrophic events

Catastrophic events are the most important of these factors. Since catastrophic events might be the source of substantial increase in losses within a narrow time period, they present a threat to insurers and pension plans. In this section, the possible reasons for these outliers are presented.

2.2.1. Catastrophic Events

Catastrophic events may be defined as events that cause significant and typically abrupt suffering or damage; in other words, they are disasters. Catastrophic events could be summarized as follows.

2.2.1.1. Wars

Throughout history, millions of military personnel and civilians have died because of wars. The estimated numbers of deaths due to the 20th century's deadliest wars are shown in Table 2.1. Estimating the losses that arise from wars, however, is quite difficult due to unreliable data, the multitude of different causes of deaths, and historians' varying views. The estimates in the table are constructed to show the magnitude of these events.

As a result, life insurers want to have war exclusion clauses in their policy contracts to avoid paying out claims that are related to wars [118].

Table 2.1: Five Deadliest Wars

Years	War	Estimated Number of Deaths
1914-1918	World War I	15,000,000
1917-1922	Russian Civil War	9,000,000
1928-1937	1st Chinese Civil War	5,000,000
1939-1945	World War II	66,000,000
1960-1975	2nd Indo-China War	4,200,000

Source: CRED [49].

2.2.1.2. Natural Disasters

A natural disaster is another event that causes significant loss of human lives and destruction [49]. Earthquakes, floods, extreme temperatures, cyclones, wildfires, volcanic activities, and droughts are all examples of natural disasters. Table 2.2 shows the 5 deadliest natural disasters of the 20th century. It can be understood from the table that floods and droughts are typically the deadliest natural disasters, and these events occur most often in developing countries. Since the populations of such countries are often large, the effect on the life insurance industry could be even larger [68].

Table 2.2: Five Deadliest Natural Disasters

Years	Country	Natural Disaster	Estimated Number of Deaths
1928	China	Drought	3,000,000
1931	China	Flood	3.700,000
1942	India	Drought	1,500,000
1943	Bangladesh	Drought	1,900,000
1959	China	Flood	2,000,000

Source: CRED [49].

2.2.1.3. Transport, Industrial, and Other Accidents

Examples of industrial accidents are leaks or spills of toxic chemicals and various types of explosions. Accidents in boat, air, road, and rail transport are some examples of transport accidents. Fires and structural collapses are other types of accidents. Table 2.3 presents the 5 deadliest transport and industrial accidents since 1900.

It should also be noted here that a nuclear accident could cause increased mortality rates over a longer time period. For instance, the Chernobyl disaster of 1986 was the world's worst nuclear power plant accident in history and caused up to 50 deaths in the immediate fire and explosion. However, it is estimated that it may have also caused between 4,000 and 985,000 subsequent cancer deaths in the years since 1986 [133].

Table 2.3: Five Deadliest Transport and Industrial Accidents

Year	Country	Accident	Estimated Number of Deaths
1923	Japan	Fire	3,800
1956	Colombia	Explosion	2,700
1984	India	Gas leak	2,500
1987	Philippines	Boat accident	4,000
2002	Senegal	Boat accident	1,860

Source: CRED [49].

2.2.1.4. Terrorist Attacks

Terrorism is planned, politically motivated violence against non-combatant targets perpetrated by agents or subnational groups [109]. Because data on terrorist attacks are obtained from open sources rather than government collection programs, the data are incomplete. The deadliest terrorist attack in history was the September 11 terrorist attack of 2001, in which approximately 3,000 people died. Other terrorist attacks such as bombings and hijackings have resulted in many deaths. Moreover, biological weapons also cause substantial numbers of deaths [68]. The deadliest terrorist attacks in recent times are shown in Table 2.4.

Table 2.4: Five Deadliest Terrorist Attacks

Year	Country	Accident	Estimated Number of Deaths
1978	Iran: Abadan	Mujahideen-I-Khalq	430
1994	Rwanda: Gikoro	Hutus	1,180
2001	United States: NYC Washington	Al-Qaeda	3,000
2004	Nepal: Bedi	Communist Party of Nepal	518
2007	Iraq: Sinjar	Islamic State of Iraq	430

Source: National Counterterrorism Center [109].

2.2.2. Disease

2.2.2.1. AIDS and SARS

Human immunodeficiency virus (HIV) leads to acquired immunodeficiency syndrome (AIDS). immune system's cells are infected with this virus that weakens it. The average time to develop AIDS for an HIV-infected individual is 10-15 years [135]. The World Health Organization considers HIV as a pandemic. The deadliest outbreak of HIV occurred more than four decades ago and caused approximately 60 million people to be infected, nearly 30 million of whom died due to AIDS-related causes. However, therapy and HIV prevention efforts have become increasingly available, leading to decreases in AIDS-related deaths and HIV infections [77].

Severe acute respiratory syndrome (SARS) is thought to be a virus that has crossed over from animals to humans. The SARS coronavirus is the cause of this serious respiratory illness [112]. In 2002, the first SARS outbreak occurred in China, and the last one occurred in 2003. During this period a total of 8,096 SARS cases caused 774 deaths across 26 countries. As long as the SARS coronavirus continues to exist in wildlife species, the possibility of another SARS epidemic remains [132].

2.2.2.2. Pandemics

A pandemic could be defined as a geographically widespread outbreak of an infectious disease. In such a case, the disease may spread around the whole world, infecting a substantial proportion of the global population. The World Health Organization [134] has stated that a pandemic could be triggered by the following conditions being met:

- An agent that has been absent from the human population for a long time or is new could cause a global disease outbreak;
- A human could be affected directly by the agent and acquire a serious illness;
- The agent could spread sustainably and efficiently among humans.

On the contrary, an epidemic is a disease that affects people in a more specific geographical region [113]. Throughout history, several notable pandemics have occurred, such as smallpox, plague, and tuberculosis [108]. Furthermore, diseases such as anthrax, Crimean-Congo haemorrhagic fever, avian influenza, Ebola, human monkeypox, H1N1 influenza virus or other influenzas, Hendra virus infection, and tularemia could be considered as possible sources of future pandemics [135].

2.2.3. Influenza Pandemics

An influenza pandemic is the spread of an infectious viral disease that causes a large number of deaths. The future occurrence of an influenza pandemic is inevitable, since these viruses may mutate, leaving the human immune system unable to recognize them [111].

Although seasonal influenza epidemics are an annually occurrence, in general, influenza pandemics are infrequent and unpredictable. In the last century, 4 pandemics occurred: the Spanish flu, the Asian flu, the Hong Kong flu, and the H1N1 flu. These are summarized in Table 2.5.

Table 2.5: Four Influenza Pandemics

Year	Country	Disease	Estimated Deaths
2009-2010 (H1N1 flu)	Mexico	H1N1	18,500
1968-1969 (Hong Kong flu)	China	H3N2	1,000,000
1957-1958 (Asian flu)	China	H2N2	1,000,000-2,000,000
1918-1919 (Spanish flu)	Global	H1N1	50,000,000-100,000,000

Source: WHO [135].

2.3. Data Description

In this thesis, mortality data for the US, the UK, Switzerland, France, and Italy are used. We need both the central death rates and exposed to risk data to be able to fit the Lee-Carter model and conduct outlier analysis.

The US mortality data originate from the National Center for Health Statistics (NCHS) for the years of 1900-2017 and for all ages. The UK, Swiss, and French data originate from the Human Mortality Database (HMD); they are from the period of 1922-2016 and again for all ages. The Italian mortality data are also obtained from the HMD for the period of 1922-2014 and for all ages.

2.4. The Lee-Carter Model

This model describes the logarithm of central death rates in the following way:

$$\ln(m_{x,t}) = a_x + b_x k_t + e_{x,t} \tag{2.1}$$

Here, a_x is an age-specific component; the time-varying k_t parameter summarizes the general level mortality level; and the other age-specific parameter, b_x, explains how slowly or how rapidly mortality varies for each age as the mortality index changes. $e_{x,t}$ is an error term reflecting age-specific influences that are not captured by the model.

The Lee-Carter model is an overparameterized model. For obtaining a unique solution, a_x is taken to be the arithmetic mean of $\ln(m_{x,t})$ overtime and the sums of b_x and k_t are respectively normalized to unity and zero. Because the parameters on the right-hand side of Equation (2.1) are not observable, using the method of ordinary least squares to fit the model would be impossible. A two-stage estimation procedure is used to overcome this problem and this procedure gives the exact solution. We can apply the singular value decomposition (SVD) method to the matrix of $\ln(m_{x,t}) - a_x$ for obtaining the estimates of b_x and k_t as the first step. As the second step, the time-varying terms are iteratively reestimated given the values of a_x and b_x, which come from the first step. This makes the actual sum of death at time *t* equal to the implied sum of deaths at time *t.*

$$D_t = \sum_x (P_{x,t}(\exp(a_x + b_x k_t))),$$

where D_t represents the actual sum of deaths at time *t*, and $P_{x,t}$ signifies the population composing age group *x* at time *t*. In the original Lee-Carter model, the autoregressive integrated moving average (ARIMA) model is applied for the modelling of the dynamics of k_t [89].

Several alternative methods have also been presented in the actuarial literature to estimate parameters of the Lee-Carter model. For instance, Brouhns et al. [23] proposed a regression-type model with Poisson assumption for number of deaths (D_t). This strategy was found to have very good statistical properties when numbers of deaths had a Poisson distribution. However, this assumption may lead to incorrect inferences when D_t includes outliers as a result of pandemics, wars, or other catastrophic events [88]. In addition, as indicated by Brouhns et al. [23], the estimation results from both methods are almost the same. Therefore, we adopt the SVD method to obtain the parameters.

The two-stage SVD procedure is thus implemented for the historical mortality data of the US, the UK, Switzerland, Italy, and France for the relevant time periods. We obtain the fitted a_x and b_x values given in Table 2.6 and the time-varying mortality index k_t in Figure 2.1. Parameter values of a_x show that mortality demonstrates an upward trend in general; the younger ages reveal lower rates of mortality while the older ages reveal higher mortality. Parameter b_x reflects mortality's tendency at age x to change as the general level of the mortality index is changing. As we can see from Table 2.6, the values of b_x are positive, indicating that mortality decreases for all ages. The decreasing trend of time-varying mortality index k_t highlights the improvement of mortality with time for all countries. Moreover, sudden increases that cause mortality jumps in the 1910s and 1970s may be seen in Figure 2.1, which will be discussed in later sections.

Table 2.6: Estimated Values of a_x and b_x from the Lee-Carter Model

	US		UK		Switzerland		Italy		France	
Age Group	a_x	b_x	a_x	b_x	a_x	b_x	a_x	b_x	a_x	b_x
<1	-3.593	0.146	-4.022	0.147	-4.204	0.133	-3.662	0.151	-3.947	0.152
1-4	-6.450	0.192	-6.698	0.193	-7.033	0.165	-6.520	0.206	-6.828	0.172
5-14	-7.401	0.153	-7.877	0.152	-7.846	0.136	-7.621	0.136	-7.781	0.135
15-24	-6.399	0.096	-7.037	0.106	-6.892	0.094	-6.785	0.106	-6.677	0.109
25-34	-6.085	0.097	-6.737	0.097	-6.678	0.094	-6.542	0.099	-6.344	0.103
35-44	-5.575	0.082	-6.105	0.077	-6.194	0.085	-6.061	0.079	-5.793	0.078
45-54	-4.858	0.061	-5.208	0.063	-5.341	0.073	-5.276	0.057	-5.036	0.057
55-64	-4.095	0.051	-4.285	0.051	-4.442	0.067	-4.414	0.046	-4.230	0.053
65-74	-3.317	0.048	-3.367	0.047	-3.514	0.066	-3.475	0.047	-3.492	0.057
75-84	-2.483	0.043	-2.459	0.041	-2.533	0.056	-2.470	0.043	-2.552	0.053
>85	-1.661	0.029	-1.589	0.026	-1.575	0.031	-1.549	0.029	-1.597	0.032

2.5. Analysis of Outliers

After fitting the Lee-Carter model, we can proceed to the outlier analysis of the mortality time index, k_t. Two key issues should be considered in analysing the outliers of time-series data:

- Searching for the outliers' types and locations (i.e., outlier detection problem)
- Incorporating the outliers' effects into a model for obtaining improved parameter estimates of an underlying time-series model (i.e., outlier adjustment problem)

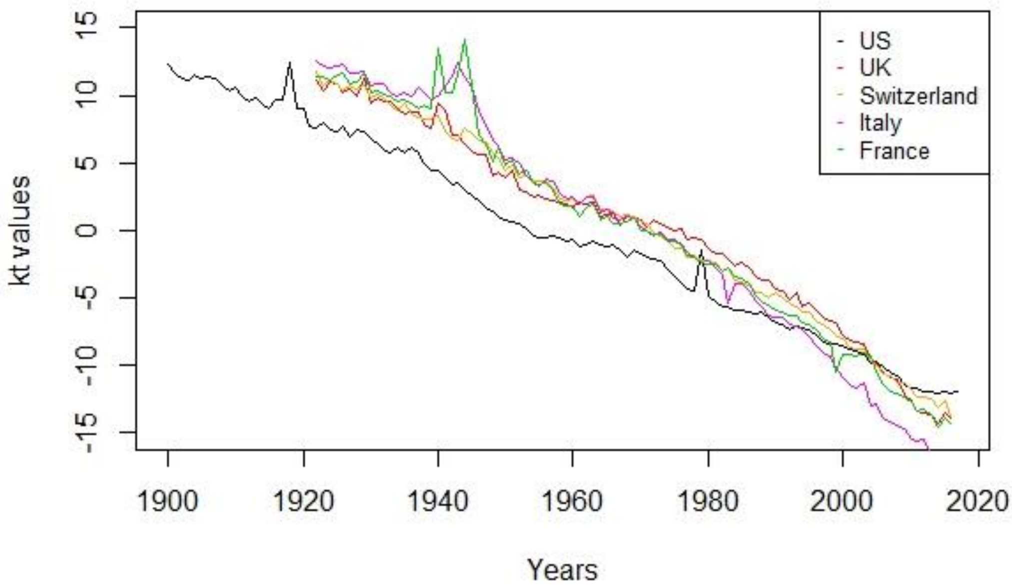

Figure 2.1: Estimation of k_t for All Countries

Since we want to incorporate the timing and frequency of outliers into our mortality modelling process, only the first issue is considered in this thesis. The problem of outlier detection in time series was addressed first in the work of Fox [58], who adopted likelihood ratio testing for detecting outliers [88]. Chen and Liu [37] later presented an augmented iterative method for use in the joint estimation of both outlier effects and model parameters.

This thesis employs the method of Chen and Liu [37], which is based on outliers' effects on estimated residuals for the detection of outliers. We will restrict the analysis to the necessary points for detecting the outliers. The first step in the process of detecting outliers is specifying the correct ARIMA model to be used for a mortality time index that is free of outliers.

2.5.1. Outlier Models in Time Series

It is possible to apply the adopted method to general seasonal and nonseasonal ARIMA processes. Let z_t be a time series that follows an ARIMA process without outliers:

$$\theta(B)(1-B)^d z_t = \phi(B)a_t \tag{2.2}$$

where $Bz_t = z_{t-1}$; a_t is normally distributed white noise. A time series that is subject to the influences of a nonrepetitive event could be described as follows:

$$z_t^* = z_t + wL(B)I_t(t_1), \tag{2.3}$$

where z_t follows the general ARIMA process as described in Equation (2.2), w is the magnitude of the outlier, and $I_t(t_1) = 1$ if $t = t_1$ and $I_t(t_1) = 0$ otherwise. $I_t(t_1)$ is an indicator function here for the occurrence of outlier effects, while t_1 is the outlier's possible location, and w and $L(B)$ respectively signify the magnitude and the dynamic pattern of the outlier effect. If we know the dynamic pattern and the location of an outlier, then model (2.3) is the intervention model from the work of Box and Tiao [17]. Here we assume the estimation problem where neither dynamic pattern nor location is known. The applied approach aims to classify outliers' impacts into four categories with the imposing of a special structure on $L(B)$. These categories include additive outliers (AOs), innovational outlier (IOs), temporary changes (TCs), and level shifts (LSs), respectively defined as follows [37]:

$$\text{AO:} \quad L(B) = 1, \tag{2.4}$$

$$\text{IO:} \quad L(B) = \phi(B) / \theta(B), \tag{2.5}$$

$$\text{TC:} \quad L(B) = 1/(1-\delta B), \tag{2.6}$$

and

$$\text{LS:} \quad L(B) = 1/(1-B). \tag{2.7}$$

A discussion on structure and the nature of these outliers may be found in the works of Fox [58], Tsay [124], and Chen and Tiao [38].

2.5.2. Impact of Outliers on the Time-Series

The impacts of outliers on observed time series are model-independent, except for that of an IO. In addition, LS and AO are two boundary cases of TCs, for which $\delta = 0$ and $\delta = 1$. The effect of a TC will exponentially decay according to a dampening factor, δ, on the time series at a given time. In practice, the value of the dampening factor lies between 0.6 and 0.8 [99]. However, Chen and Liu [37] recommended that $\delta = 0.7$ be used to identify a TC. The LSs generate step changes in the series that are abrupt and permanent. For a time series, all observations are affected by an IO beyond time *T* via the memory of the underlying outlier-free process. In the case of AOs, the outlier exerts one short and immediate effect on the time series [37].

Using Equation (2.5), in the event that an IO occurs at $t = t_1$, this outlier's effect on z_{t_1+k}, for $k \geq 0$, equals $w\psi_k$, where w represents initial effect and ψ_k is the *k*th coefficient of the $\psi(B)$ polynomial, where:

$$\psi(B) = \phi(B) / \theta(B)$$

$$= \{\psi_0 + \psi_1(B) + \psi_2 B^2 + ...\}, \quad \psi_0 = 1.$$

A graphical illustration of the effects of the outliers is given in Figure 2.2[1]. If time series z_t is subject to *m* outliers, Equation (2.3) becomes the following:

$$z_t^* = z_t + \sum_{j=1}^{m} w_j L_j(B) I_t(t_j), \tag{2.8}$$

where $L_j(B) = \phi(B) / \theta(B)$ for IOs, $L_j(B) = 1$ for AOs, $L_j(B) = 1/(1-B)$ for LSs, and $L_j(B) = 1/(1-\delta B)$ for TCs at $t = t_j$ [37].

2.5.3. Detection of Outliers

The first step in the process of outlier detection is the specification of the appropriate ARIMA model to be applied for an outlier-free series.

[1] Figure 2.2 is taken from the study of Li and Chan [88], and *T* represents the outlier time in the figure.

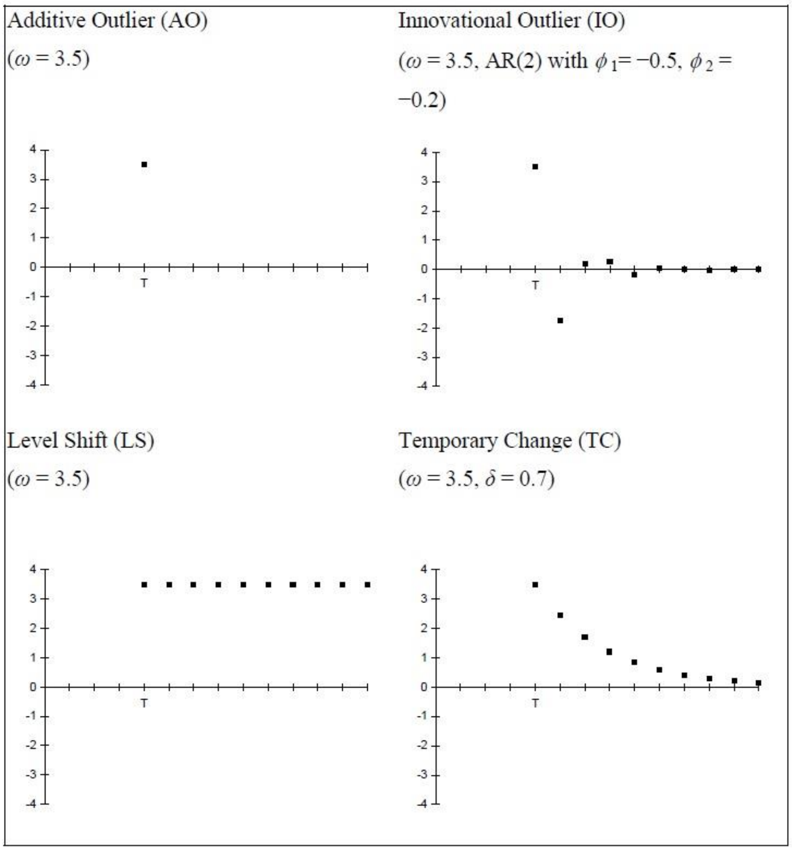

Figure 2.2: Types of Outliers.

Chen and Liu [37]'s method is used for the detection of outliers. They assumed that time-series parameters are known and that the series is being observed from $t=-J$ to $t=n$. Here, J is an integer that is larger than $p+q$, where p and q are the orders of polynomials $\theta(B)$ and $\phi(B)$. They defined the $\pi(B)$ polynomial as a first step:

$$\pi(B)=(\theta(B)(1-B)^{d})/(\phi(B))=1-\pi_1 B-\pi_2 B_2-...$$

Here, the π_j weights for j beyond a large J will equal 0, because $\phi(B)$'s roots are all outside of the unit circle. The estimated residuals $\hat{e}_t$ may then be written as follows:

$$\hat{e}_t=\pi(B)z_t^{*},\quad \text{for} \quad t=1,2,\ \ldots \tag{2.9}$$

If the estimated residuals are rearranged for the outlier types, then we have

$$\text{AO:}\quad \hat{e}_t=w\pi(B)I_t(t_1)+a_t \tag{2.10}$$

$$\text{IO:}\quad \hat{e}_t=wI_t(t_1)+a_t \tag{2.11}$$

$$\text{TC:} \quad \hat{e}_t = w(\pi(B)/(1-\delta \mathrm{B}))I_t(t_1) + a_t \tag{2.12}$$

and

$$\text{LS:} \quad \hat{e}_t = w(\pi(B)/(1-\mathrm{B}))I_t(t_1) + a_t \tag{2.13}$$

Alternatively, Equations (2.10) - (2.13) can be rewritten as follows:

$$\hat{e}_t = wx_{it} + a_t, \quad t = t_1, t_1+1, \ldots, n, \quad \text{and} \quad i = 1, 2, 3, 4, \tag{2.14}$$

where $x_t = 0$ for $t < t_1$, $x_t = 1$, and for all $k \geq 1$, $x_{1(t_1+k)} = 0$, $x_{2(t_1+k)} = -\pi_k$, $x_{3(t_1+k)} = 1 - \sum_{j=1}^{k} \pi_j$, and $x_{4(t_1+k)} = \delta^k - \sum_{j=1}^{k-1} \delta^{k-1} \pi_j - \pi_k$. For the effect of the outlier at $t = t_1$, using the least squares estimate makes it possible to express the following:

$$\hat{w}_{AO}(t_1) = \frac{\sum_{t=t_1}^{n} \hat{e}_t x_{2t}}{\sum_{t=t_1}^{n} x_{2t}^2},$$

$$\hat{w}_{IO}(t_1) = \hat{e}_{t_1},$$

$$\hat{w}_{TC}(t_1) = \frac{\sum_{t=t_1}^{n} \hat{e}_t x_{4t}}{\sum_{t=t_1}^{n} x_{4t}^2},$$

$$\hat{w}_{LS}(t_1) = \frac{\sum_{t=t_1}^{n} \hat{e}_t x_{3t}}{\sum_{t=t_1}^{n} x_{3t}^2},$$

Here it should be noted that, for the last observation, $t = t_1$, $\hat{w}_{AO}(n) = \hat{w}_{IO}(n) = \hat{w}_{LS}(n) = \hat{w}_{TC}(n) = \hat{e}_n$. Hence, it is not possible to distinguish the type of an outlier if that outlier occurs at the end of a time series [37].

A possible method for detecting outliers was discussed by Chang et al. [32], who found the maximum value of standardized statistics for outlier effects:

$$\hat{\tau}_{AO}(t_1) = \left(\sum_{t=t_1}^{n} x_{2t}^2 \right)^{1/2} \left(\hat{w}_{AO}(t_1) / \hat{\sigma}_a \right),$$

$$\hat{\tau}_{IO}(t_1) = \hat{w}_{IO}(t_1) / \hat{\sigma}_a,$$

$$\hat{\tau}_{TC}(t_1) = \left(\sum_{t=t_1}^{n} x_{4t}^2 \right)^{1/2} \left(\hat{w}_{TC}(t_1) / \hat{\sigma}_a \right),$$

$$\hat{\tau}_{LS}(t_1) = \left(\sum_{t=t_1}^{n} x_{3t}^2 \right)^{1/2} \left(\hat{w}_{LS}(t_1) / \hat{\sigma}_a \right). \quad (2.15)$$

These statistics follow an approximately normal distribution for a given location. The final test statistic is given below:

$$\eta_t = \max\{ |\hat{\tau}_{IO}(t)|, |\hat{\tau}_{AO}(t)|, |\hat{\tau}_{LS}(t)|, |\hat{\tau}_{TC}(t)| \}.$$

If $\max_t \eta_t = |\hat{\tau}_{t_p}(t_1)| > C,$ C being a predetermined critical value, then the type of an outlier detected at t_1; t_p is AO, IO, TC, or LS. For a reasonable level of sensitivity, in this thesis it is assumed that *C*=2.5, as per the recommendation of Liu and Hudak [99].

2.5.4. Estimation of the Residual Standard Deviation σ_a

We must estimate σ_a to obtain the outliers' test statistics as given by Equation (2.15). The detection of outliers might be sensitive to this estimate. Therefore, if outliers exist and if the usual sample standard deviation is used, σ_a could be overestimated. There are a few methods that provide a better estimation of σ_a. The first one is the median absolute deviation (MAD) method, the second one is the α% trimmed method, and the last one is the omit-one method.

In this study we employ the MAD estimate of residual standard deviation, which is expressed as follows:

$$\hat{\sigma}_a = median\{ |\hat{e}_t - \bar{e}| \} \times 1.483,$$

Here, $\bar{e}$ represents the estimated residual median [5]. Using this method allows us to decrease the possibility of misdetection due to inflated estimates of residual standard deviation. If the outliers' locations are specified and their impacts are also estimated, then we can calculate σ_a on the basis of the adjusted residuals' sample standard deviation.

2.6. Outlier Detection for Countries

In this section, a step-by-step illustration of the detection of outliers is provided for the mortality indexes, k_t s, which are obtained from the mortality data of five countries.

Step I: To implement the outlier detection process, we need to specify the order of the underlying outlier-free ARMA process for the mortality index. We use the Box and Jenkins method to identify the order of the model and the mortality time-series index, k_t, is modelled as AR(1), which is a more appropriate order for the mortality indexes for the countries. Furthermore, Leng and Peng [85] showed the consistency of the dynamics of the mortality index if k_t follows an AR(1) model, but an AR(p) model leads to a different conclusion and gives inconsistent results [94]. Please see Figure 2.3. The ACF figure demonstrates the AR effect on the time series and PACF shows the order of AR.

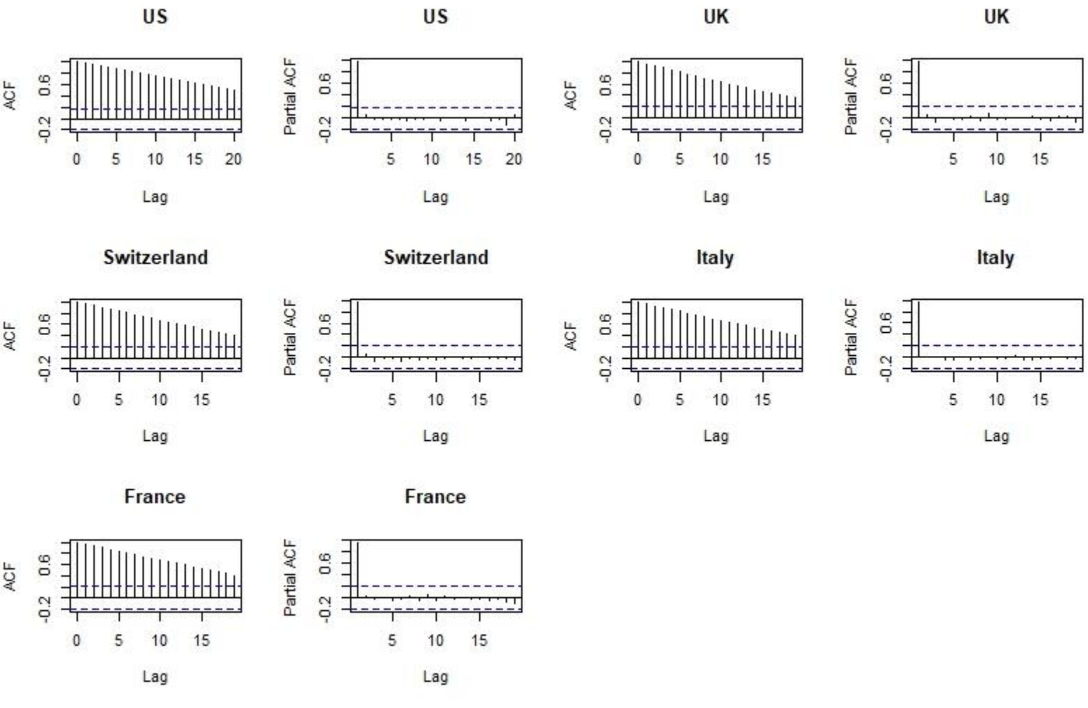

Figure 2.3: ACF and PACF for the Countries

Then the parameter of the AR process is estimated from the mortality time series. The parameter values and their standard errors are shown in Table 2.7 for all countries.

Table 2.7: Parameters of AR Process

	US	UK	Switzerland	Italy	France
	0.9982	0.9988	0.9992	0.9989	0.9963
s.e.	0.0025	0.0018	0.0011	0.0014	0.0049

Step II: By using the AR(1) model, we obtain the residuals. As we mentioned before, we only focus on additive outliers because of our mortality jump modelling purposes. For $t = 1,\ldots n$, the test statistic $\hat{\tau}_{AO}(t)$ is computed by using the estimated residuals obtained from the model. Let $\eta_t = \max\{|\hat{\tau}_{AO}(t)|\}$, and if $\max_t \eta_t = |\hat{\tau}_{AO}(t_1)| > C$, then it means that there is an outlier at time t_1.

According to this detection procedure, we compute the outliers, which signify jumps in the mortality curve, in the mortality time series for all countries. The years of detected outliers' years for the countries and their test statistic values are shown in Table 2.8.

For the US, in 1901 the occurrence of an extreme heat wave was the source of 9,500 deaths. That heat wave stretched across half of both June and July along the eastern coast of the United States and, more than 100 years later, is still considered as one of the worst heat waves of history. In the 1918-1919 period, the additive outliers are the consequence of the Spanish flu pandemic. Effects of World War I and the pandemic could be the reason for the outliers in 1920 and 1921. In the 1937-1941 period, floods and hurricanes caused many deaths. In 1974 and 1975, 148 tornadoes occurred across 13 states. In 1978, six natural disasters occurred in the US: a firestorm, tropical storm, cold wave, hurricane, blizzard, and tornado. The 1980 heat wave, though less devastating than that of 1901, nevertheless claimed at least 1,700 lives.

For the UK, the number of deaths in 1929 was almost double that of 1928. A possible cause is the Great Depression, which had a four-year effect on UK mortality. The Second World War could be the reason for the outlier in 1941. In 1950, South and North Korea went to war. That three-year war included thousands of UK conscripts; in particular, the

English and Welsh populations lost notable numbers of healthy, younger individuals. This war could be the reason for the outlier in 1952. Floods, outbreaks, a flu pandemic, and severe winter weather could have caused the outlier in 2009.

Table 2.8: Years of Detected Outliers and Test Statistic Values

US												
Year	1901	1918	1919	1920	1921	1937	1938	1941	1974	1975	1978	1980
Test statistic	-2.6	6.9	-8.3	-3.3	-3.5	-3.1	-3.2	-2.5	-3.1	-2.7	7.3	-9.8
UK												
Year	1929	1939	1941	1942	1952	2009						
Test statistic	3.9	2.7	-3.8	-2.9	-2.6	-2.5						
Switzerland												
Year	1941	1949	2016									
Test statistic	-3.4	-2.7	-4.5									
France												
Year	1939	1941	1943	1945	1946	1947	1998	2005				
Test statistic	6.2	-4.7	5.9	-9.9	-6.2	-2.8	-3.7	-3.2				
Italy												
Year	1945	1946	1947	1982								
Test statistic	-2.9	-3.2	-2.6	-3.2								

The data on catastrophic events cannot be obtained separately for Switzerland, France, and Italy. In addition, they have similar geographical features and their outlier times are close to each other. For this reason, we try to analyse their jump effects based on the World Disasters Report. According to the catastrophic events around the world, general causes of outliers can be listed as follows:

- In the 1920s, natural disasters, such as floods and droughts, caused approximately 7 million people to die.
- During 1939-1945, World War II had an enormous impact on mortality around the world. This is the reason for the outliers of these countries between these years.

- In 1982, nearly 200 natural disasters occurred around the world and these natural disasters caused 8 million people to die. The deadliest of these natural disasters were floods and extreme weather.
- In 1998, approximately 300 natural disasters were recorded, the most important of which were floods, extreme weather, and earthquakes. Nearly 400 natural disasters were recorded in 2016 and the deadliest ones were floods and earthquakes.

These results are consistent with the natural disaster data.

Interim Conclusion: Outlier Analysis in the Mortality Indexes

In this chapter, we have performed an outlier detection procedure for the mortality indexes of the US, the UK, Switzerland, France, and Italy, which are obtained from the well-known Lee-Carter model. First, we present sources of the outliers. Catastrophic events, diseases, and influenza pandemics are the possible reasons for the outliers. We then explain the Lee-Carter model, followed by fitting the historical data to the model for all countries.

Secondly, we have described the outlier types and showed their effects on time series. For mortality modelling purposes, we only consider the additive outliers, which have short and immediate effects on time series. Then we perform the outlier detection procedure and obtained the outliers.

It can be inferred from this outlier analysis that the US, Switzerland, France, and Italy are more vulnerable to natural disasters such as floods and heat waves, while the UK is more vulnerable to pandemics and wars. The obtained outliers are consistent with the historical disasters. By detecting the outliers, the timing and the severity of the outliers can also be found, and in this way, consistent forecasting methods might be developed. Our main aim here is to find the location of outliers on the mortality curve and to analyse their inter-arrival times. As a result, we can model mortality in a more realistic way.

3. TRANSITORY MORTALITY JUMP MODELING WITH RENEWAL PROCESS AND ITS IMPACT ON PRICING OF CATASTROPHIC BONDS

3.1. Introduction

Insurance companies and pension plans face risks of uncertainty in future mortality. Such risk could stem from improvements in mortality or shocks such as catastrophic mortality events [68]. The latter is called "catastrophic mortality risk", which is the risk that, over short periods of time, mortality rates are much higher than expected [26]. Due to a shorter lifetime of an individual or group than expected, an insurer or a pension plan may have to make sudden pay-outs to many policyholders. Hence, seriously negative financial consequences could be experienced, including breaches in capital requirements and regulatory solvency [67]. As a result, the management of catastrophic mortality risk is fundamental for insurance companies and pension plans.

Catastrophic events, such as infectious diseases/pandemics, natural disasters, terrorist attacks, wars, and accidents, may cause sudden increases in mortality curves. These sudden increases are called mortality jumps and they were discussed in the previous section in detail. For instance, the Spanish flu virus killed 50 to 100 million people in 1918 and caused very large jumps in mortality rates. Avian flu in 2006 and the Ebola virus in 2014 also caused approximately 1 million deaths [7].

According to statistics from the Emergency Events Database (EM-DAT), frequencies, magnitudes, and durations of natural disasters have all increased since 1975. The World Disasters Report in 2016 stated that rising global temperatures caused global climate change and more natural disasters. These climate changes and natural disasters lead to catastrophic events, which have caused many diseases and deaths in recent years. In the 1970s, there were roughly 100 catastrophic events per year, and this number has consistently increased more than three times in the last decade. For the period of 1994-2013, EM-DAT records show 6,873 natural disasters causing 1.35 million deaths on average each year. Furthermore, in 2018, 348 climate-related and geophysical disaster events were recorded in the International Disaster Database, and 68 million people were reported to have been affected around the world.

The occurrence of catastrophic events could cause a large number of deaths and hence a higher rate of unexpected death claims. Consequently, financial impacts from catastrophic events for an insurer’s solvency require effective risk management to eliminate and reduce the risk [75]. In the United States, the three largest natural disasters recorded before Hurricane Katrina in 2005 caused a total insured loss of $23 billion and a few reinsurers became insolvent to pay claims [141]. Moreover, in Germany, a dramatic pandemic could generate approximately €45 billion of additional claim expenses according to the estimations of Stracke and Heinen [119]. This amount is equivalent to 100% of the German life insurance market's policyholder bonus reserves. Some experts from the field of public health predict that a new pandemic is overdue and will occur without fail because of inter-species transmission, intra-species variation, and alterations in virulence [7].

The frequency of catastrophic events and the degree to which they are accurately priced are serious concerns in managing extreme mortality risks. In recent years, catastrophic bonds have been used by insurers as tools for risk management. The first catastrophic bond was that issued by Swiss Re, called Vita I, in 2003, with the aim of reducing impacts of catastrophic events. Due to the great success of that bond, large numbers of newer catastrophic mortality bonds are now being issued (see [15], [14]). Several stochastic models are now available to capture jump effects in mortality and to value catastrophic bonds. These models have differences in the severity of mortality jumps and the type of jumps. For instance, Cox et al. [48] combined geometric Brownian motion with a compound Poisson process for the modelling of age-adjusted rates. Cox et al. [48] then modelled permanent mortality jumps by considering Poisson jump counts. Chen and Cox [34] used a normal distribution for jump severity, while Chen et al. [35] combined two types of jumps in their model. Similarly, Deng et al. [52] considered the mortality time index as a double-exponential jump process. In contrast to those studies, Liu and Li [100] investigated age patterns within the jump effects on mortality.

All of these mentioned jump models in the literature assume that mortality jumps occur once a year, or they used a Poisson process for their jump frequencies. Due to their low probability and high-impact nature, the timing and the frequency of future catastrophic events and hence mortality jumps are unpredictable [34]. On the other hand, the history of such events can give information about their future occurrences. In the Poisson process, inter-arrival times between events are taken to be independent and exponentially distributed. However, the Poisson process has a limitation arising from the memorylessness of the exponential distributions. In this thesis, the aim is to include the history of catastrophic events. One way to incorporate the history of these events is to use *duration dependence models*. Instead of a constant hazard function, these models have time-varying hazard functions. This property is important for duration analysis since the hazard function is applied for capturing the duration dependence. Hazard functions reflect the waiting times between events. For instance, an increasing hazard function represents longer waiting times between events compared to a decreasing hazard function. In these models, events are dependent such that arrival of a minimum of one event (versus the arrival of none) up to time t will influence the probability of another event's arrival in $t+\Delta t$. Thus, a link exists between the counting model and timing process. This class is known as renewal processes [78].

Winkelmann [131] was the first to derive a counting process by using the renewal process with gamma distributed inter-arrival times. Many other models were derived by using different inter-arrival times afterwards. McShane et al. used the Weibull distribution for inter-arrival times, while a log-normal distribution was used by Everson and Bradlow [56], Bradlow et al. [19], and Miller et al. [106] [8].

A new approach is proposed in this thesis for modelling the arrivals of mortality jumps. Inter-arrival time implies the time between two jumps, and we want to use the renewal process for modelling. For this purpose, we will detect jumps in the mortality time series and perform statistical tests for inter-arrival times of mortality jumps to show that we can use the renewal process as a counting process. Afterwards, we will use the Lee-Carter model together with a jump-diffusion process to model mortality, as well as the log-normal renewal process to model jump count probabilities. This model will be tested with historical data and a comparison will be performed for the goodness of fit of models with jump sizes and jump count processes for the US, the UK, Switzerland, Italy, and France. To the author's best knowledge, the use of the renewal process for jump counts is new in mortality modelling.

It can be reasonably assumed here that the renewal process exerts impacts on the pricing of catastrophic mortality bonds. To show this impact, our proposed mortality model will be used for the pricing of a catastrophic mortality bond. The pricing problem is not explicit in an incomplete market; however, it might be met by no-arbitrage methods (see [26], [34], [87] and [95]), insurance-based methods (see [35] and [130]), or economic methods [138].

The no-arbitrage approach was used often in earlier studies on the pricing of mortality-linked securities. In this method, risks' market prices cannot be uniquely identified. As a result, an arbitrary assumption is necessary for pricing. One might also use canonical valuation for the creation of a probability measure that is risk neutral. Canonical valuation could be applied without any arbitrary decision-making [139]. For this reason, we use canonical valuation for creating the aforementioned risk-neutral probability measure and for obtaining mortality risk premiums. The canonical valuation method was introduced by Stutzer [120], after which it was applied to insurance markets by Li and Ng [95], Li [87], and Chen et al. [36]. In this thesis, the Swiss Re mortality bond is used for a martingale constraint. The desired risk-neutral probability measure is identified by using this method and thus the hypothetical mortality bonds may be priced in an incomplete market.

This section is organized in the following way. Section 3.2 defines the renewal count process. Section 3.3 presents the mortality data. Section 3.4 gives the proposed model's specifications and the statistical analysis of mortality jumps. Section 3.5 demonstrates a numerical example of pricing mortality-linked security. Finally, Section 3.6 concludes.

3.2. Renewal Process

Since we want to include the history of events in our jump frequency model, we need to use a renewal process. This is a stochastic model for events occurring randomly in time, which are typically referred to as "renewals" or "arrivals". The times between the successive arrivals are taken to be independent and identically distributed, having an arbitrary distribution, and the renewal process might be applied for a foundation upon which to build more realistic models.

There are three ways to specify the renewal process. The first is by finding the joint distributions of the arrival epochs. The second is by the joint distributions of the inter-arrival times. Finally, the third is by finding joint distributions of the counting process $N(t); t \geq 0$. The simplest way for specifying the renewal process is by using the inter-arrival times, since they are iid. Because the process probabilistically begins again with each arrival period, these processes are referred to as "renewal processes".

Formally, we may take $\{X_n, n = 1, 2, ...\}$ to be the sequence of independent nonnegative random variables with the common distribution of *F*, and we further suppose $F(0) = \Pr(X_n = 0) < 1$. It is possible to express X_n as the time between the *(n-1)*st and *n*th events. Next, we may write

$$E[X_n] = \int_0^n x dF(x)$$

as the average time between successive events. Due to our assumptions of $X_n \geq 0$ and $F(0) < 1$, $0 < E[X_n] \leq \infty$ follows. We write

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i, \qquad n \geq 1,$$

where S_n represents the arrival time of the *n*th event. The number of events occurring by time *t* will equal *n*'s largest value, for which the *n*th event occurs before or at time *t*, and thus the number of events $N(t)$ by time *t* is given as follows:

$$N(t) = \sup\{n : S_n \leq t\}. \tag{3.1}$$

An important question here is to ask whether or not it is possible for an infinite number of renewals to occur within a finite amount of time. To answer this question, the strong law of large numbers is used:

$$S_n / n \to E[X_n], \quad \text{as} \quad n \to \infty.$$

Since $E[X_n] > 0$, S_n must be moving towards infinity as *n* itself moves towards infinity. Hence, it is possible for S_n to be less than or equal to *t* for only a finite number of values of *n*. Thus, applying Equation (3.1), it is seen that $N(t)$ has to be finite and may be written as follows:

$$N(t) = \max\{n : S_n \le t\}.$$

3.2.1. Distribution of N(t)

The relationship between timing and count process, where the number of renewals by time *t* will be greater than or equal to *n* if and only if the *n*th renewal occurs before or at time *t*, is used to obtain the distribution of $N(t)$. That is,

$$N(t) \ge n \quad \text{if and only if} \quad S_n \le t \tag{3.2}$$

By using the relationship between arrival times and count process, $(S_n \le t) = (N(t) \ge n)$, the distribution function of the count process might be determined in terms of the inter-arrival times' distribution function:

$$\Pr(N(t) = n) = \Pr(N(t) \ge n) - \Pr(N(t) \ge n+1) \tag{3.3}$$

$$= \Pr(S_n \le t) - (S_{n+1} \le t)$$

Since the X_i s are independent and they also have a common distribution *F*, S_n is distributed as F_n, the *n*-fold convolution of *F* with itself. Thus, we have the following thanks to Equation (3.3):

$$\Pr(N(t) = n) = F_n(t) - F_{n+1}(t).$$

Let us now assume that $m(t) = E[N(t)]$ and $m(t)$ is the renewal function. Generally, renewal theory is concerned about the determination of its properties. The relationship between $m(t)$ and F can be written as follows:

$$m(t) = \sum_{n=1}^{\infty} F_n(t). \tag{3.4}$$

Equation (3.4) can be reorganized in terms of F as follows:

$$m(t) = F(t) + \int_0^t m(t-x)dF(x).$$

The generating function is:

$$Q_t(z) = 1 + \left(1 - \frac{1}{z}\right)\sum_{n=1}^{\infty} F_n(t)z^n.$$

In general, these equations cannot be obtained explicitly because of the computation of convolutions. Another important question for the renewal process is what the behaviour of $N(t)$ will be as $t \to \infty$. We take $N(\infty) = \lim_{t\to\infty} N(t)$ to show the total number of occurring renewals, and then the following result can be obtained:

$$N(\infty) = \infty \quad \text{with probability 1.}$$

As a result, $N(t)$ approaches infinity as $t \to \infty$ as well. At the same time, it is also important to identify the rate at which $N(t)$ approaches infinity. Hence, the problem becomes a limiting problem about $N(t)/t$, which is the time-average renewal rate over $S_n / n \to E[X_n]$the interval (0, t]. This could be explained with the *strong law of large numbers for the renewal process.* We should note that S_n / n is the sample average of *n* inter-renewal intervals and it approaches $E[X_n]$ with probability 1 as $t \to \infty$, as discussed before. Let $N(S_n)$ denote the number of renewals for the period of the *n*th renewal, and let $N(S_n)/S_n$ be equal to n/S_n. Since $S_n / n \to E[X_n]$ as $n \to \infty$, one can expect that n/S_n approaches $1/E[X_n]$. Thus, $\lim_{t\to\infty} N(t)/t = 1/E[X_n]$ with probability 1 [117].

3.2.2. Computation Methods for the Renewal Process

As mentioned before, the computation of convolutions is not simple. There are several ways to compute the renewal count probabilities and the expected number of renewals. Lomnicki [103] proposed a method to compute count probabilities with the Weibull inter-arrival times and an approach built on the exponential function's expansion into Poissonian functions. McShane et al. [107] evaluated distribution probabilities by using the expansion into the powers of t. That method was also used by Jose and Abraham [78] for obtaining a counting process respectively with Mittag-Leffler and Gumbel inter-arrival times.

However, the count probabilities' convergence cannot be provided for all distributions. Let us assume that $P_0(t)$ denotes the survival function and gives the probability of the occurrence of zero events by time t. The computational techniques can be summarized as follows:

- Expanding out the exponential functions by applying series transformations to increase the speed of convergence: this approach is commonly used for Weibull renewal processes. However, it can also be utilized for other distributions.
- Monte Carlo simulations may be used for generating the renewal times up to the time t.
- Using the Laplace transform, we can obtain the survival distribution generating function, and we can convert it to the required probability's transform and invert that transform.
- Fast Fourier transform can be applied for convolutions.
- The required count probabilities can be obtained directly as convolution integrals.

The Monte Carlo approach is easy to use, and it is very useful for double-checking the results obtained from other methods. Nevertheless, its accuracy is not high. Convolutions could be done either directly or by taking the Fourier or Laplace transforms of survival distributions and inverting the results. Although accuracy cannot be guaranteed for all distributions, the use of transform methods has advantages over other methods [8]. A direct convolution method is used in the present thesis to obtain count probabilities.

3.2.3.1. Computation of Probabilities by Convolution

Before discussing the count probabilities for the renewal process, a general definition of the convolution method will be provided.

Let us take $X_1, \ldots, X_n$ to be *n* independently distributed nonnegative random variables that have common probability density function *f*. First, we consider the two-fold convolution $X_1 + X_2$,

$$f_2(t) = f_{X_1+X_2}(t) = \int_0^t f(t-s)f(s)ds.$$

It is possible to calculate the probability density function of $X_1 + \ldots + X_n$ recursively. Assuming the probability density function of $X_1 + \ldots + X_{n-1}$ is given by the (*n-1*)-fold convolution, the probability density function of $X_1 + \ldots + X_n$ is then the *n*-fold convolution, given by:

$$f_n(t) = f_{X_1+\ldots+X_n}(t) = \int_0^t f_{n-1}(t-s)f(s)ds.$$

The distribution function is obtained similarly and the distribution function of *n*-fold convolution is given by [126]:

$$F_n(t) = F_{X_1+\ldots+X_n}(t) = \int_0^t F_{n-1}(t-s)dF(s).$$

We can calculate the renewal count probabilities by using these equations. Let $R(n)$ show the probability of the *n*th event, obtained by using $R(n) = F_n(t) - F_{n+1}(t)$. An evaluation of the convolutions of the form $\int_0^t F(t-s)f(s)ds$ is now necessary. To solve this integral, we need to evaluate the following recursive relationship:

$$R_n(t) = \int_0^t F_{n-1}(t-s)f(s)ds - \int_0^t F_n(t-s)f(s)ds$$

$$= \int_0^t R_{n-1}(t-s) f(s) ds. \qquad (3.5)$$

Here it can be noted that $F_0(t) = 1$ for all t and $F_1(t) = F(t)$. This gives us $R_0(t) = F_0(t) - F_1(t) = 1 - F(t)$, which leads us to the survival function. Using Equation (3.5), it is now possible for us to compute $R_1(t)$:

$$R_1(t) = \int_0^t R_0(t) f(s) ds.$$

Finally, we can obtain $R_n(t)$ probabilities by using the recursive formula [107].

3.3. Data Description

Mortality data are used in this thesis for the US, the UK, Switzerland, France, and Italy. As explained previously, the US mortality data originate from the National Center for Health Statistics (NCHS) for the years of 1900-2017 and include all ages. The data from the UK, Switzerland, and France originate from the Human Mortality Database (HMD) for the period of 1922-2016 for all ages. Finally, the Italian mortality data also originate from the HMD, but they are for the period of 1922-2014, again for all ages.

The data are arranged in 10-year age intervals as follows: <1, 1-4, 5-14, 15-24, ..., 75-84, 85+.

3.4. Transitory Mortality Jump Modelling with Renewal Process

3.4.1. A Specification of the Lee-Carter Model

This section presents the proposed model, which is built on the original Lee-Carter model [84]. The model's details have been given in Section 2.4, and it is possible to express it as follows:

$$\ln(m_{x,t}) = a_x + b_x k_t + e_{x,t}.$$

Here, a_x, b_x, and $e_{x,t}$ have the same meanings and constraints as in the original version of the Lee-Carter model. Mortality index k_t is taken as being free of jumps; hence, changes in k_t could be understood to signify a general change in the overall mortality level (please note that this general change is not an extreme change). However, a suitable model is required for capturing the features of shape, trend, and jumps, as well as for forecasting the future mortality rates. Although the Lee-Carter model is a long-term mortality model, its time-varying mortality index should capture the short-term effects for effective mortality modelling. In the original version of the Lee-Carter model, the k_t parameters are modelled using a random walk with drift. However, it is possible to model k_t as a stochastic process to deal with the uncertainty over mortality trends. Moreover, k_t includes both negative and positive values. Hence, geometric Brownian motion does not fit the process since a negative value is not generated from the positive starting value. Therefore, we will use standard Brownian motion. We thus introduce the short-term jump effects with a diffusion process due to the existence of the transient mortality jumps in Figure 2.1. We need to choose an appropriate jump-diffusion process in order to reflect the features of time-varying mortality index k_t.

The descriptive statistics of $\Delta k_t = k_{t+1} - k_t$ indicate leptokurtic features for all countries, as shown in Table 3.1.

Table 3.1: Skewness of Δk_t for All Countries

	US	**UK**	**Switzerland**	**Italy**	**France**
Skewness	-0.598	-1.061	-1.197	-1.237	-0.427

The Δk_t distributions skew towards the left, and they have higher peaks and heavier tails than normal distributions do, as shown in Figure 3.1. Therefore, we must consider a distribution that is heavy tailed instead of a normal distribution for the jump severities.

We model k_t as a *Merton jump-diffusion model* in order to include the leptokurtic features of Δk_t. This model can be specified as follows [105]:

$$dk_t = \mu dt + \sigma W_t + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right). \tag{3.6}$$

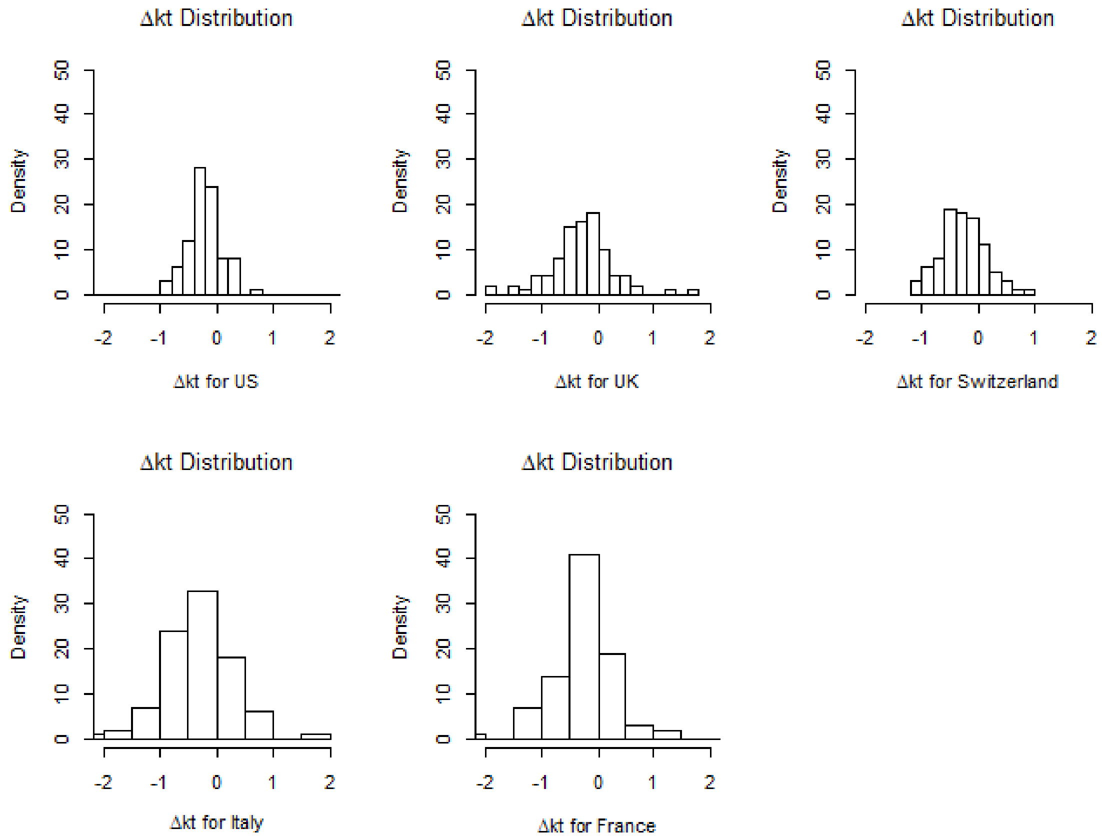

Figure 3.1: Distribution of k_t for all countries.

Here, μ and σ are constants, W_t represents standard Brownian motion and $N(t)$ represents a counting process, while V_i represents a sequence of iid nonnegative variables signifying the sizes of the jumps.

By integrating Equation (3.6), we obtain the following [52]:

$$k_t = k_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i. \qquad (3.7)$$

Here, Y_i is defined as $Y = \log(V)$.

In the original Merton model, the jump sizes, $Y = \log(V)$, are normally distributed. However, the Δk_t s have both heavier tails and higher peaks than normal distributions do for the countries examined. Thus, we consider that the jump sizes have exponential distribution. Moreover, it is assumed here that jump count variable $N(t)$ is a renewal process. Although our aim is to introduce a renewal process to model the mortality jumps, we analyse different jump sizes and count processes to compare and choose the best model. Therefore, we propose four models: normal jump and Poisson process, exponential jump and Poisson process, normal jump and renewal process, and exponential jump and renewal process. The originality of this work lies in introducing the renewal process in these models for the mortality jump counts. The jump sizes will also be compared by using normal and exponential distributions.

In order to model the count variable as a renewal process, analysis of inter-arrival times between jumps should be performed. For this reason, we use the detected jumps from Section 2.5. After obtaining the inter-arrival times between detected jumps, we can proceed to the statistical analysis of the outliers to show that we can use the renewal process for our count variable.

3.4.2. Statistical Analysis of the Outliers

Now we need to analyse the inter-arrival times of these outliers that cause jumps in the mortality curves to show that the renewal process can be used for modelling the jump frequencies. The inter-arrival times between the outliers (mortality jumps) are used for the analysis. Several statistical tests should be performed to confirm that a renewal process is appropriate for the arrivals of the jumps. The first indicator that this process is not a Poisson process is the uniformity test, and a formal statistical test of uniformity can be performed using the Kolmogorov--Smirnov test. This test is used to compare an empirical distribution function against a cumulative distribution function. All obtained p-values are lower than 0.05, meaning that we can reject the uniformity and constant mean for the process [3].

We need to analyse the inter-arrival times to check whether they are stationary, independent, and identically distributed. For this reason, we will apply the Ljung-Box test to the inter-arrival times. This test can be defined as follows:

$$Q_k = n(n+2)\sum_{d=1}^{k}\frac{r_d^2}{n-d},$$

where r_d represents the autocorrelation coefficient for the lags $1 \le d \le k$ and n represents the series' length. The Q_k statistic will be compared with a χ^2 distribution having k degrees of freedom for the purpose of testing the null hypothesis, which is "there is no correlation between inter-arrival times'". We apply the test for all lags for inter-arrival times. Based on Ljung-Box test statistics [79], the inter-arrival times are stationary and independent for all countries.

After confirming that the inter-arrival times between outliers are stationary and independent, we next need to determine the inter-arrival times' distribution. For this purpose, the inter-arrival times' properties should be considered. Since the inter-arrival time counts between the outliers are less than four, the distribution of inter-arrival times cannot be fitted for Switzerland or Italy. The estimated skewness coefficients of the inter-arrival times are 1.9, 2.1, and 2.6 for the US, the UK, and France, respectively. Since they have positive skewness, right-skewed distributions could be considered, such as Weibull, gamma, and log-normal distributions. We fit these three distributions to the inter-arrival times and the fitted results are displayed in Table 3.2.

According to this table, a log-normal distribution fits the inter-arrival times best. Since Switzerland and Italy have statistical properties similar to those of the other countries, we assume that their inter-arrival times follow log-normal distributions as well.

As a result, all of the analyses show that we can use the renewal process to model the counting process, $N(t)$. We obtain the count probabilities, which are computed by convolution techniques, for a log-normal renewal process by using "R" software. Now we can proceed to constructing and comparing the mortality models.

3.4.3. A Model with Normal Jump and Poisson Process

In the original Merton model, $N(t)$ is a Poisson process having rate λt and $Y = \log(V)$ follows a normal distribution having the following density:

Table 3.2: Fitted Results

US	Weibull Distribution	Gamma Distribution	Lognormal Distribution
Parameters	Shape=0.78, scale=6.05	Shape= 0.73, rate=0.10	Mean=1.15, sd=1.22
Log-likelihood	-31.74	-32.09	-30.24
BIC	68.72	69.35	65.71

UK	Weibull Distribution	Gamma Distribution	Lognormal Distribution
Parameters	Shape=0.62, scale=9.34	Shape= 0.51, rate=0.04	Mean=1.41, sd=1.56
Log-likelihood	-16.99	-17.25	-16.37
BIC	37.20	37.71	35.95

France	Weibull Distribution	Gamma Distribution	Lognormal Distribution
Parameters	Shape=0.62, scale=6.16	Shape= 0.50, rate=0.05	Mean=1.05, sd=1.38
Log-likelihood	-21.06	-21.73	-19.56
BIC	46.02	47.46	43.12

$$f_Y(y) = \frac{1}{\sqrt{2\pi}s} e^{-\frac{(y-m)^2}{2s^2}}, \quad -\infty < y < \infty, -\infty < m < \infty, s^2 > 0 \tag{3.8}$$

We must identify the density function of the one-period increments $r_i = \Delta k_i = k_i - k_{i-1}$ to estimate parameters and make forecasts. If we consider the one-period increment for the Merton model, conditional on event $(N(t) = n)$, we can write $X = Y_1 + Y_2 + \ldots + Y_n$, where $Y_i \sim N(m, s^2)$ are independent, and then $X \sim N(nm, ns^2)$. Then we obtain the conditional density for increments, namely the sum of $N(m, s^2)$ and $N((\mu - 0.5\sigma^2), \sigma^2)$.

By using the convolution technique, the density is found as $N((\mu - 0.5\sigma^2) + nm, \sigma^2 + ns^2)$ [63]. Then the unconditional density for a one-period increment is:

$$f(r_i) = \sum_{n=0}^{\infty} P(n) f(r_i \mid n), \tag{3.9}$$

where $P(n)=\dfrac{e^{-\lambda t}\lambda t^{n}}{n!}$ and $f(r_i \mid n)$ represents the conditional density of one-period increments, conditional on the given number of the jumps.

Let $C=k_0,k_1,...,k_t$ denote the time-varying mortality factors at equally spaced times of $t=1,2,...,T$. Then the log-likelihood of one-period increment observations is:

$$L(C;\mu,\sigma,m,s,\lambda)=\sum_{i=1}^{T}\log(f(r_i)).$$

These parameters can be estimated by applying maximum likelihood estimation (MLE).

3.4.4. A Model with Exponential Jump and Poisson Process

Δk_t has a heavier tail than a normal distribution, as shown in Figure 3.1. Thus, we assume the jump sizes follow an exponential distribution having the following density:

$$f_Y(y)=\eta e^{-\eta y}, \qquad y>0,\eta>0. \tag{3.10}$$

The count process is a Poisson process having rate λt. Conditional on event $(N(t)=n)$, we can write $X=Y_1+Y_2+\ldots+Y_n$, where $Y_i \sim \exp(\eta)$ and Y_i are independent, and then $X \sim \Gamma(n,\eta)$. The conditional density of *X* is:

$$f_X(X \mid n)=\frac{\eta^n}{(n-1)!}X^{n-1}e^{-\eta X}.$$

Now we can determine two conditional densities for the no-jump case and the n-jump case. For the no-jump case, the conditional density is:

$$f(r_i \mid 0)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(r_i-\mu+0.5\sigma^2)^2}{2\sigma^2}}.$$

For the *n*-jump case, the conditional distribution equals the independent sum of $\Gamma(n,\eta)$ and $N((\mu-0.5\sigma^2),\sigma^2)$. Applying convolution techniques, the following can be obtained:

$$f(r_i \mid n)=\int_0^{\infty}\frac{\eta^n}{(n-1)!}X^{n-1}e^{-\eta X}\,\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(r_i-X-\mu+0.5\sigma^2)^2}{2\sigma^2}}dx$$

$$= \frac{\eta^n}{(n-1)!\sqrt{2\pi}\sigma} \int_0^\infty X^{n-1} e^{-\eta X - \frac{1}{2\sigma^2}(r_i - X - \mu + 0.5\sigma^2)^2} dx.$$

The unconditional density of a one-period increment is [114]:

$$f(r_i) = P(0) f(r_i \mid 0) + \sum_{n=1}^{\infty} P(n) f(r_i \mid n). \tag{3.11}$$

Thus, the log-likelihood function becomes:

$$L(C; \mu, \sigma, \eta, \lambda) = \sum_{i=1}^{T} \log(f(r_i)).$$

3.4.5. A Model with Normal Jump and Renewal Process

Now we assume that our counting process, $N(t)$, is a renewal process, and we aim to include the history of catastrophic events to model k_t. Thus, the model has the information of the frequency of the events and we accomplish this by combining the Merton model and a renewal process. Jump sizes $Y = \log(V)$ follow normal distribution as in the original Merton model with density as follows:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}s} e^{-\frac{(y-m)^2}{2s^2}}, \qquad -\infty < y < \infty, -\infty < m < \infty, s^2 > 0$$

The count process is renewal process, and conditional on $(N(t) = n)$, we can write $X = Y_1 + Y_2 + \ldots + Y_n$, where $Y_i \sim N(m, s^2)$ and Y_i are independent. Then $X \sim N(nm, ns^2)$. Considering the jumps, we obtain the conditional density for increments, which is equal to the sum of $N(m, s^2)$ and $N((\mu - 0.5\sigma^2), \sigma^2)$. By using the convolution technique, the density is obtained as $N((\mu - 0.5\sigma^2) + nm, \sigma^2 + ns^2)$ [63]. Letting *R(n)* be the renewal process, it shows the probability of the *n*th jump. Then the unconditional density for $f(r_i)$ is:

$$f(r_i) = \sum_{n=0}^{\infty} \Pr(N(t) = n) f(r_i \mid n).$$

$C = k_0, k_1, ..., k_t$ denote the time-varying mortality factors at equally spaced times of $t = 1, 2, ..., T$. Then the log-likelihood of one-period increment observations is:

$$L(C; \mu, \sigma, m, s, \alpha, \beta) = \sum_{i=1}^{T} \log(f(r_i)).$$

3.4.6. A Model with Exponential Jump and Renewal Process

As in Section 3.4.4, we assume that the jump sizes follow exponential distribution with density as follows:

$$f_Y(y) = \eta e^{-\eta y}, \qquad y > 0, \eta > 0.$$

To estimate the parameters and to make forecasts, we need to find the density function of the one-period increments, which might be shown by $r_i = \Delta k_i = k_i - k_{i-1}$. If we consider the one-period increments for the Merton model, conditional on event $(N(t) = n)$, we might write $X = Y_1 + Y_2 + \ldots + Y_n$, where $Y_i \sim \exp(\eta)$ and Y_i are independent, and then $X \sim \Gamma(n, \eta)$. The conditional density of X is:

$$f_X(X \mid n) = \frac{\eta^n}{(n-1)!} X^{n-1} e^{-\eta X}.$$

Now we determine two conditional densities for the no-jump case and the *n*-jump case. For the no-jump case, the conditional density is:

$$f(r_i \mid 0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r_i - \mu + 0.5\sigma^2)^2}{2\sigma^2}}.$$

For the *n*-jump case, the conditional distribution equals the independent sum of $\Gamma(n, \eta)$ and $N((\mu - 0.5\sigma^2), \sigma^2)$. By using the convolution technique, we obtain:

$$f(r_i \mid n) = \int_0^\infty \frac{\eta^n}{(n-1)!} X^{n-1} e^{-\eta X} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r_i - X - \mu + 0.5\sigma^2)^2}{2\sigma^2}} dx$$

$$= \frac{\eta^n}{(n-1)!\sqrt{2\pi}\sigma} \int_0^\infty X^{n-1} e^{-\eta X - \frac{1}{2\sigma^2}(r_i - X - \mu + 0.5\sigma^2)^2} dx.$$

Our next step is deriving the unconditional density of the one-period increments. The unconditional density for $f(r_i)$ is:

$$f(r_i) = R(0)\,f(r_i \mid 0) + \sum_{n=1}^{\infty} R(n)\,f(r_i \mid n). \tag{3.12}$$

Let $C = k_0, k_1, ..., k_t$ denote the time-varying mortality factors at equally spaced times of $t = 1, 2, ..., T$. Then the log-likelihood of one-period increment observations can be given as follows:

$$L(C; \mu, \sigma, \eta, \alpha, \beta) = \sum_{i=1}^{T} \log(f(r_i)).$$

Based on the observations $C = k_0, k_1, ..., k_t$, the parameters can be estimated by maximizing the following log-likelihood function:

$$\sum_{i=1}^{T} \log\left(R(0)\,f(r_i \mid 0) + \sum_{n=1}^{\infty} R(n)\,f(r_i \mid n) \right). \tag{3.13}$$

3.4.7. Estimation Results

Solving the jump process from the diffusion components poses a significant challenge in the calibration of the underlying process. The diffusion process captures independent increments of the underlying process and jumps capture the extreme increments. A method for calibration is required for the generation of accurate parameters of large severity and low frequency for mortality jumps. In the work of Ait-Sahalia and Hansen [2], it was demonstrated that the use of MLE can be advantageous in solving jumps from diffusion. Furthermore, the process of jump-diffusion is a linear one having explicit transition density and independent increments. Hence, it is able to satisfy the requirements for complete specification of transition density in order to apply MLE. As a result, the MLE method will be used here in order to calibrate the necessary parameters.

We estimate parameters of the time-varying mortality factors for the jump-diffusion models introduced in the previous subsections for five countries. Following Cox et al.

[48], we allow maximum jump counts of 10 for a year and the estimated parameters are as given in Table 3.3.

As Table 3.3 shows, the mortality factor's expected rate of change, μ, is -0.2637 for the renewal process with exponential jumps for the US. This suggests that the mortality factor is decreasing by an annual average rate of 0.2637. μ having a negative sign here is consistent with the improvement of the US population's mortality rate with time. The annual mortality rate of change has volatility of 0.1599 for the renewal process with the exponential jump model for the US. The average severity of jumps is equal to 0.6757 $(1/\eta)$ in a year. Similar comments hold for the other countries. Additionally, significant differences exist among the means and the variances of jump frequency distributions of these two models. It can be realized here that the distribution of jump severities is important, but the process for jump frequencies is more important.

The Bayesian information criterion (BIC) is applied for model selection. Results in Table 3.3 show that the Merton model with renewal process is the best model for all countries. The reasons for this can be summarized as follows.

To begin with, the outliers present in the time-series data lead to the existence of both fat tails and a high peak within the distribution of increment Δk_t, which negates the possibility of a normal distribution. In the Lee-Carter model, outliers are treated in the same way as other points present in the evolution process of the mortality time series. For this reason, outliers increase the process's volatility, and they also lead us to overestimate standard deviation σ. In the proposed model, we apply a renewal process that is separate from the process of Brownian motion diffusion. As a result, we are able to avoid the problematic mismatching of the high peak and fat tail with normal distribution and thus the model is able to provide a better fit. Secondly, the Poisson process does not include the history of jumps. However, the history of jumps could give information about the future occurrences of jumps. Besides, the nonconstant hazard function property of the renewal process enables us to obtain more realistic models for jump frequency.

Table 3.3: Estimated Parameters for All Countries

The US	NJ with Poisson P.	EJ with Poisson P.	NJ with Renewal P.	EJ with Renewal P.
	μ =-0.2471	μ =-0.2594	μ =-0.2415	μ =-0.2637
	σ =0.1772	σ =0.1448	σ =0.1713	σ =0.1599
	m=0.0332	η =1.4874	m=0.0284	η =1.4794
	s=0.5292	λ =1.0108	s=0.4414	α =0.0190
	λ =1.006		α =0.00031	β =0.6051
			β =0.6030	
log(L)	-77.9282	-44.3847	-30.3421	-29.2499
BIC values	179.7098	107.8522	89.31	82.3532

The UK	NJ with Poisson P.	EJ with Poisson P.	NJ with Renewal P.	EJ with Renewal P.
	μ =-0.2369	μ =-0.2189	μ =-0.2389	μ =-0.2364
	σ =0.2874	σ =0.1204	σ =0.2195	σ =0.1788
	m=0.0322	η =1.4794	m=-0.0355	η =1.4963
	s=0.6124	λ =0.7435	s=0.3640	α =0.00091
	λ =0.6277		α =0.0029	β =0.6135
			β =0.6009	
log(L)	-72.8151	-45.6658	-37.4425	-36.7632
BIC values	164.3995	109.5471	102.2082	96.2958

Switzerland	NJ with Poisson P.	EJ with Poisson P.	NJ with Renewal P.	EJ with Renewal P.
	μ =-0.2468	μ =-0.2141	μ =-0.2345	μ =-0.2506
	σ =0.3626	σ =0.1641	σ =0.4128	σ =0.1887
	m=0.0413	η =1.4806	m=0.0259	η =1.4891
	s=0.1778	λ =1.0014	s=0.0581	α =0.0118
	λ =1.0472		α =0.0011	β =0.6127
			β =0.7038	
log(L)	-48.6356	-44.3310	-35.7192	-33.5098
BIC values	120.0406	106.8775	98.7617	89.7889

Table 3.3: Estimated Parameters for All Countries (Continue)

Italy	NJ with Poisson P.	EJ with Poisson P.	NJ with Renewal P.	EJ with Renewal P.
	μ=-0.2293	μ=-0.2243	μ=-0.2498	μ=-0.2232
	σ=0.3694	σ=0.0932	σ=0.3359	σ=0.1879
	m=-0.0232	η=1.4759	m=-0.0116	η=1.4858
	s=0.4399	λ=1.0942	s=0.4346	α=0.0136
	λ=1.0212		α=0.0019	β=0.6078
			β=0.6301	
log(L)	-77.9737	-56.1175	-51.0489	-47.4151
BIC values	178.6105	130.3653	129.2334	117.4931

France	NJ with Poisson P.	EJ with Poisson P.	NJ with Renewal P.	EJ with Renewal P.
	μ=-0.2057	μ=-0.2170	μ=-0.2112	μ=-0.2208
	σ=0.2246	σ=0.0970	σ=0.1863	σ=0.1384
	m=-0.0736	η=1.4993	m=-0.0353	η=1.4595
	s=0.9114	λ=1.1631	s=0.5677	α=0.0194
	λ=1.0019		α=0.0022	β=0.5524
			β=0.5304	
log(L)	108.7111	-52.2698	-46.6672	-38.6063
BIC values	240.1916	122.7551	120.6577	99.3819

We also conduct likelihood ratio test in order to compare the renewal and Poisson processes. The null and the alternative hypotheses of the test are respectively as follows:

$$H_0 = \theta = \theta_0 \quad \text{and} \quad H_1 = \theta = \theta_1.$$

Here, θ_0 signifies the Poisson process while θ_1 represents the renewal process. We can calculate likelihood ratio test statistic LR in the following way:

$$LR = 2 \times (\hat{l}_1 - \hat{l}_0),$$

where $\hat{l}_0$ represents the maximized log-likelihood of the Poisson process and $\hat{l}_0$ is the maximized log-likelihood of the renewal process. The statistics of the log-likelihood ratio test and p-values are given in Table 3.4. According to these test results, for the models with exponential jumps, all results are higher than the chi-square value, 3.84, of this test. Therefore, the null hypothesis is rejected for all cases.

Table 3.4: LR Test Statistics

	US	UK	Switzerland	Italy	France
LR	30.28	17.81	21.64	17.41	27.93
Critical Value	3.84	3.84	3.84	3.84	3.84
p-value	0.00	0.00	0.00	0.00	0.00

In order to see the difference between the Poisson process and the renewal process in terms of the expected number of mortality jumps, we can calculate the expected values for the jump frequencies by using Equation (3.4) and the parameters estimated in Table 3.3. We present the results for the expected jump frequencies in a year for the Poisson process and the renewal process models with exponential jumps in Table 3.5.

Table 3.5: Expected Jump Frequencies

Country	EJ with Poisson P.	EJ with Renewal P.
US	1.01	2.71
UK	0.74	2.43
Switzerland	1.00	1.40
Italy	1.09	1.43
France	1.16	1.90

As shown in Table 3.5, the expected jump frequencies are higher for the model with renewal process which is caused by the use of lognormal distribution and thus time-varying hazard function for the inter arrival times. Although renewal process produces higher values for all countries, the difference is particularly significant for the US and the UK.

3.5. Swiss Re Mortality Bond

In this section, we use the Swiss Re mortality bond to obtain a probability measure that is risk-neutral, which can then be used for pricing hypothetical bonds. As the first mortality-linked security, this bond was issued by the Swiss Re insurance company in December 2003. The coupons of the bond were paid quarterly at a 3-month LIBOR rate plus a spread value of 135 basis points. The bond's principal repayment is dependent on a mortality index, and the bond's purpose was to expand the insurance market's capacity for paying catastrophic losses. A special purpose vehicle, Vita Capital, was used to issue the bond, thereby allowing Swiss Re to eliminate extreme catastrophic risk from its balance sheet [34].

The bond had 3-year maturity and the issue size was $400 million. The mortality index, q_t, was obtained as the weighted average of mortality rates across five countries, for both females and males and for a range of ages. These countries were the UK, the US, France, Switzerland, and Italy. It was a principal-at-risk bond. In the event that the mortality index exceeded 1.3 times the actual 2002 base level, q_0, in any year of the bond's life, a reduced principal repayment would be received by the investors. Otherwise, the principal was repayable in full. The bond's coupon payment schedules are given by the following CP_t function:

$$CP_t = \left\{ \begin{array}{l} (LIBOR + spread) \times 400 million, t = 1, ..., T - 1 \\ (LIBOR + spread) \times 400 million + \max[0, 100\% - \sum_t Loss_t, t = T \end{array} \right\}.$$

Here, the $Loss_t$ function signifies the amount of payment lost as a result of experienced mortality [52]:

$$Loss_t = \left\{ \begin{array}{ll} 0, & q_t < 1.3q_0 \\ \dfrac{q_t - 1.3q_0}{0.2q_0}, & 1.3q_0 \le q_t \le 1.5q_0 \\ 1, & 1.5q_0 < q_t \end{array} \right\}.$$

3.5.1. Risk-Neutral Pricing

The catastrophic mortality bond market is incomplete due to the impossibility of pricing securities via the construction of a replicating portfolio in this market. Among the steps in the performance of risk-neutral valuation, a critical one is to specify a risk-neutral probability measure with which mortality bond prices might become computable in such an incomplete market. This desired risk-neutral measure could be obtained by several approaches. The no-arbitrage method is commonly used by investors. To implement this approach, estimating the distribution of future rates of mortality in the real-world probability measure is the first step. Following that, the real-world distribution must then be transformed to its risk-neutral counterpart. This process is based on the observed market prices. The mortality-linked security price could be obtained by discounting under the identified risk-neutral probability measure at a risk-free rate. Due to this approach taking actual market prices into consideration, it is challenging to apply it in mortality-linked securities markets.

Another way is using a stochastic mortality model identified in the real-world measure and then fit to historical data. The model must then be calibrated to market prices, which will yield a risk-neutral mortality process that security prices can then be calculated from. However, the market price of risk is not able to be identified uniquely with only one security price in this method. Hence, an arbitrary assumption must be made before pricing.

A distortion operator could be utilized for obtaining a probability measure that is risk neutral. Unless the assumed mortality model is kept simple, the distortion operator's parameters will not be unique if sufficient market price data are not given.

More recently, researchers have started to use a new no-arbitrage method known as canonical valuation. In this approach, a risk-neutral probability measure is defined by minimizing the Kullback--Leibler information criterion, subject to the constraint of market price. It is possible to apply this without any arbitrary decision-making about the market price of risk [138] and the canonical valuation method is thus adopted in this thesis.

The first step to implement this valuation method is generating a number of future mortality rate sample paths with equal probability from the mortality model, which is defined according to the real-world probability measure. This scenario can be obtained by bootstrap method. The generated sample paths may be understood as a collection of all of the states of nature. Hence, if M sample paths have been generated, then the following statement will provide state of nature w's probability mass function under real-world probability measure $\mathbf{P}$:

$$\Pr(w = w_j) = \pi_j = \frac{1}{M}, \qquad j = 1, 2, ..., M.$$

This probability is referred to as the empirical probability distribution. The aim is obtaining *w*'s probability distribution under risk-neutral probability measure **Q**, equivalent to **P**. Here, M=10000 is used in the calculations.

Assume that *n* distinct primary securities are included in the market, the values of which are evolving according to state of nature *w*. The *i*th primary security, $i = 1, 2, ..., M$, has the time-0 price of F_i and random discounted payoff of $f_i(w)$ at the risk-free rate. Considering π_j^*, $j = 1, 2, ..., M$, as the risk-neutral probability distribution under **Q**, the martingale constraints can be reorganized as follows:

$$E^Q[f_i(w)] = \sum_{j=1}^{M} f_i(w)\pi_j^* = F_i, \qquad i = 1, 2, ..., n. \tag{3.14}$$

If $n = M$, we can consider the market as complete. However, if $n < M$, then the market is understood to be incomplete and several risk-neutral probability measures exist to satisfy Equation (3.14). We will take Q to be the set of all of the measures equivalent to **P** and satisfying Equation (3.14); in other words, Q represents the set of all of the equivalent risk-neutral measures. For a security to be priced in an incomplete market, we must choose a martingale measure in Q. This choice could be accomplished by using the Kullback--Leibler information criterion [83], defined as follows:

$$D(Q, P) = E^P\left(\frac{dQ}{dP}\ln\frac{Q}{dP}\right) = \sum_{j=1}^{N} \pi_j^* \ln\frac{\pi_j^*}{\pi_j}.$$

Under the canonical valuation principle, risk-neutral measure Q_0 is obtained with the minimization of the Kullback--Leibler information criterion, as follows:

$$Q_0 = \arg\min_{\mathbf{Q}\in Q} D(Q, P),$$

subject to $\sum_{j=1}^{N} \pi_j^* = 1$. Equation (3.14) specifies the relevant martingale constraints. Here Q_0 will be referred to as the canonical measure. This valuation setup is also called the maximum entropy principle.

From the statistical point of view, the canonical measure is justified by the fact that it can incorporate all of the information that is contained in the prices of the m primary securities that are traded in the market and no other irrelevant or unnecessary information. It is also possible to justify the canonical measure from other geometric and economic points of view (see [57] and [95]).

Given the risk-neutral measure, a security that has the same underlying payoff structure may be priced. Let us consider a security with a payoff, discounted at the risk-free interest rate to time zero, of $g(w_j)$ in state of nature *j*. This security's price as implied by Q_0 is $\sum_{j=1}^{N} g(w_j)\pi_j^*$, where $\pi_j^*, j = 1, 2, ..., N$, represents *w*'s probability distribution under Q_0 [100].

3.5.2. Derivation of Canonical Measure

The martingale constraint of n =1 is considered here; this is on the basis of the Swiss Re mortality bond's price. The payment structure of the Swiss Re bond was summarized in Section 3.5. We might derive the risk-neutral measure on the basis of the market's actively traded mortality-linked securities, the fair prices of which are known, and then it will be possible to apply the same measure to unknown mortality-linked securities. Based on the Swiss Re mortality bond, the canonical risk measure might be expressed as follows:

$$E^Q\left(\sum_{t=2004}^{t=2006} D_t \times CP_t \times C\right), \tag{3.15}$$

where C=\$400 million, CP_t is defined as previously, and D_t represents a risk-free discount factor. We assume that coupon payments are paid annually. The risk-free interest rate is 3%, as in the work of Zhou et al. [139].

We take $V(w_j)$ to be the value of $\sum_{t=2004}^{t=2006} D_t \times CP_t \times C$ in state of nature *j* (i.e., a simulated mortality scenario). It is possible to show that *w*'s distribution under the resultant canonical measure is:

$$\hat{\pi}_j^* = \frac{\exp(\hat{\gamma}V(w_j))}{\sum_{j=1}^{N}\exp(\hat{\gamma}V(w_j))}, \qquad j = 1, 2, ..., N. \tag{3.16}$$

Here, the Lagrangian multiplier $\hat{\gamma}$ can be given as follows [95]:

$$\hat{\gamma} = \arg\min_{\gamma} \sum_{j=1}^{N} \exp(\gamma(V(w_j) - 400,000,000)).$$

In the present calculations, the applied mortality scenarios are obtained from 10,000 simulations of time-varying factor k_t for 2004-2006 on the basis of the known mortality time-varying factor of 2003. We use the Merton jump-diffusion model given in Equation (3.6) to simulate the time series of mortality with exponential jump and count processes for all countries. The mortality rates are calculated for different age groups according to the following formula: $m_{x,t} = \exp(a_x + b_x k_t)$. The standard population of the year 2000 together with the corresponding weights will be applied for computing weighted average mortality index M_t for the US. These weights are used on the basis of the technique explained in NCHS report GMWK293R. The age weights' calculation is based on exposure data for other countries and they are presented in Table 3.6. We then calculate the distribution of w under the canonical measure by using Equation (3.16) for the simulated scenarios. We use the same methodology for each mortality model and obtain the risk premiums.

Table 3.6: Age Weights for All Countries

Age Group	US	UK	Switzerland	Italy	France
<1	0.013818	0.011436	0.009806	0.009412	0.012336
1-4	0.055317	0.045715	0.040615	0.037047	0.049835
5-14	0.145565	0.126983	0.115676	0.095351	0.123835
15-24	0.138646	0.127065	0.117012	0.106701	0.129935
25-34	0.135573	0.136169	0.138486	0.150067	0.134373
35-44	0.162613	0.152864	0.167278	0.156862	0.144262
45-54	0.134834	0.127953	0.138976	0.131998	0.139694
55-64	0.087247	0.113205	0.116313	0.121131	0.102405
65-74	0.066037	0.083906	0.081884	0.103889	0.085608
75-84	0.044842	0.056444	0.054777	0.067138	0.059768
>85	0.015508	0.018263	0.019177	0.020405	0.017948

3.5.3. Pricing Hypothetical Mortality Bonds

The important point about mortality-linked securities is the premium that investors might obtain from the transaction. The premium spreads of hypothetical mortality bonds are calculated using the risk-neutral measure. The premium spread may be expressed as the premium that will compensate the investors for assuming responsibility for the extreme mortality risk.

We assume that our hypothetical bond's payment structure resembles that of the Swiss Re mortality bond. The three-year bond was issued in 2003 and written on mortality index q_t with its base level in 2003. The mortality index depends on the US, UK, Swiss, Italian, and French death rates, respectively. The index is a weighted average across the age groups based on the weights for each country. We estimate parameters for 2003 mortality rates and these parameters are used for premium calculations of the proposed model.

For attracting diverse investors, the bond is structured into two tranches with differing lower and upper strikes *M* and *U*, as illustrated in Table 3.7. The payment of each tranche is based on the Swiss Re mortality bond payment structure. The premium spreads are calculated for each tranche on the basis of the proposed model and the model with exponential jump and Poisson process.

Tranche I has both the lowest lower strike and upper strike; in other words, investors in Tranche I face the highest risk of the loss of some or all of the principal. Tranche I obviously entails the highest premium spread. Hence, the estimated premium spreads decrease while the lower and upper strikes increase. The reason for this decrease is the investors should earn less premium spread as the risk of bond reduces. As expected, the premiums that the investors earn from this transaction are higher for the trances with renewal process. This can be explained by the jump severities and jump frequencies. The risk increases with the jump frequencies and jump severities. Thus, the investors could earn more premium spreads from the tranches with the renewal process. This result is important for the investors because obtaining lower premiums than they need would cause financial problems for them.

Table 3.7: Premium Spreads of Trance I and II for All Countries

	US		UK		Switzerland		Italy		France	
	I	II	I	II	I	II	I	II	I	II
Tranche Size	$100	$100	$100	$100	$100	$100	$100	$100	$100	$100
Upper Strike U	1.04	1.12	1.04	1.12	1.04	1.12	1.04	1.12	1.04	1.12
Lower Strike M	1.02	1.10	1.02	1.10	1.02	1.10	1.02	1.10	1.02	1.10
Premium S. (Poisson P.)	45.41	43.10	65.04	63.74	50.03	47.97	21.52	20.21	73.23	71.24
Premium S. (Renewal P.)	107.53	106.34	106.09	104.80	157.51	155.72	44.14	42.85	93.52	90.93

Tranche I has both the lowest lower strike and upper strike; in other words, investors in Tranche I face the highest risk of the loss of some or all of the principal. Tranche I obviously entails the highest premium spread. Hence, the estimated premium spreads decrease while the lower and upper strikes increase. The reason for this decrease is the investors should earn less premium spread as the risk of bond reduces. As expected, the premiums that the investors earn from this transaction are higher for the trances with renewal process. This can be explained by the jump severities and jump frequencies. The risk increases with the jump frequencies and jump severities. Thus, the investors could earn more premium spreads from the tranches with the renewal process. This result is important for the investors because obtaining lower premiums than they need would cause financial problems for them.

As we can infer from the results different structure of jumps yield significantly different premium spreads. However, tranche I has the highest premiums for all cases, which means that the higher risk causes a higher risk premium.

3.6. Interim Conclusions: Transitory Mortality Jump Modelling with Renewal Process and Its Impact on Pricing of Catastrophic Bonds

This section has presented an investigation of the impacts of the history of catastrophic events on mortality modelling. We use the log-normal renewal process with exponential jumps as the counting process for transitory mortality jumps. A specification of the Lee-Carter model has been proposed, which provides a better fit for different countries. The proposed model has been applied to mortality data from the US, the UK, Switzerland, France, and Italy. This model was found to be the best for all examined countries compared to the models with normal jump with Poisson process and exponential jump with Poisson process.

We needed to perform some statistical analysis to show that we can use the renewal process for inter-arrival times of jumps. Statistical analysis of mortality jumps has shown that the inclusion of the history of events is significant for mortality modelling. Analyses have been done for the US, the UK, Switzerland, Italy, and France. Since the mortality time indexes have similar statistical properties, we can use the renewal process for jump count probabilities for all countries.

After performing the statistical tests for the renewal process, we introduced the properties of our proposed mortality model. Then we calculated and compared the premium spreads of the hypothetical bonds for all countries to show the impact of the renewal process on pricing. We conclude that the bond with the renewal process has a higher risk premium than the bond with the Poisson process.

4. A REVIEW OF LONGEVITY HEDGING PRODUCTS

4.1. Introduction

"Longevity risk" describes the risk of individuals living longer than is expected. It is a crucial financial concern for both pension plans and life insurers since they may have to make more payments than expected. Life expectancy continues to rise in association with improvements in nutrition, hygiene, medical knowledge, lifestyle, and health care. Uncertainty about future mortality improvements could have significant economic implications for annuity providers, pension providers, and social insurance programs. Although individuals have different lifetimes, longevity risk will affect all pension plans and life insurers, and hence it is not possible to diversify it with an increase in portfolio size. The hedging of longevity risk therefore has critical importance for both pension plan providers and life insurance companies [93].

Generally, three approaches are used for financial institutions to manage and mitigate their longevity risk. Reinsurance is the first of these approaches. During a reinsurance transaction, unwanted risks are transferred to a reinsurer after the paying of a risk premium. For instance, an insurer or a pension plan could buy annuities from a life office in order to transfer the risk away, or they could establish an agreement with a reinsurer to hedge the longevity risk. Traditionally, reinsurance is an effective tool to protect against large losses. However, reinsurance costs are high, and the providers of pensions and annuities will apply it only to a limited degree. The natural hedging solution is the second approach [90]. Natural hedging is a strategy for diversification that uses opposite changes in life insurances and annuities. This strategy might be feasible for large companies that have financial resources and structures to sell both kinds of policies, but the restrictions in the application of the natural hedging strategy include adjusting the sales volumes of life insurance and also that annuity products maintain the liability proportion [137].

The third approach is securitization of insurance-linked risks, which is also called capital market solutions, and securitization has gained considerable attention in the past years. Longevity- or mortality-linked derivatives and securities constitute capital market solutions. In this approach, companies have the ability to transfer their longevity exposures to capital markets at a lower cost. Securitization of insurance is a method that entails securitizing a line of business in the form of a complex bundle and then selling these securities to investors [93]. Capital market solutions provide flexibility, transparency, and additional capacity, all of which are helpful in complementing the existent insurance solutions [43]. These securities, such as longevity swaps, buy-outs, or buy-ins, constitute specially tailored transactions for specific portfolios.

The Life and Longevity Markets Association (LLMA) was founded with the collaboration of several international reinsurers, insurers, and investment banks in the UK in 2010. They aimed to improve the development of a liquid “life market” that would provide a trading platform for insurers, market investors, and reinsurers for various mortality- and longevity-linked liabilities and assets. The capital market has the potential to absorb longevity risks from pension plans and insurers in exchange for risk-adjusted returns. Additionally, market investors might want to diversify across a new market sector of longevity that is uncorrelated with traditional asset classes. Since then, the longevity risk market has shown enormous growth, both from pension plans to insurance markets and from insurers to reinsurer markets [122]. For instance, in 2011, JP Morgan issued a 10-year q-forward contract worth \pounds 70 million for the Pall pension fund. In 2014, $36.6 bn of longevity risk underwent a transfer, being shifted from pension schemes to insurers and reinsurers. Of that sum, $25.4 bn was related to only longevity transactions and this was more than twice the volume that had been written in the last three years [69].

There are two types of contracts used for securitization. The first type, customized contracts, including pension buy-ins and buy-outs, are associated with the insurer's or the pension plan's portfolio's actual mortality experience. The hedger is able to design an exact hedge to entirely eliminate the longevity risk with this type of instrument. Since the data on mortality from an individual insurer or pension plan tend to be limited, the securitization and the pricing of this type of contract are difficult, which makes the costs higher. Moreover, as these are linked to hedgers' mortality, instruments of this type have poor liquidity [91].

The second type is the index-based contract, such as index-based swaps or q-forwards, which are associated with specific national populations' mortality experiences. The first one was a q-forward contract launched by JP Morgan in 2007 with Lucida, a UK monoline insurer. Further examples are the 10-year q-forward transacted by JP Morgan with the Pall pension fund in 2011 for £70 million, or the index-based longevity swap of €12 billion between Aegon and Deutsche Bank in 2012. In index-based contracts, the payments are associated with a metric or longevity index that is based on a reference population, but not the (book) population that underlies the portfolio that is being hedged. Tradable longevity indexes are provided by the LLMA and Xpect Club Vita Index [98]. There are several advantages of index-based contracts over customized contracts, such as faster execution, greater transparency, liquidity potential, and lower costs. Additionally, they provide significant capital savings and allow effective risk management. Although index-based contracts have many advantages, their failure to eliminate the hedger's longevity risk exposures completely should also be noted [93]. Residual risk, also referred to as basis risk which will be discussed in the following chapters.

In this chapter, a review of the longevity risk transfer instruments and the stakeholders of this market are discussed. In Section 4.2, an evaluation of the longevity risk transfer market will be presented. Then, in Section 4.3, stakeholders of the longevity risk market are described. In Section 4.4, we will compare the index and the customized hedge and longevity-linked securities will be introduced. Finally, Section 4.5 will conclude.

4.2. Evaluation of the Longevity Risk Market

The longevity risk market began over a decade ago. Before that, two life insurers, Legal & General and Prudential, had been dominating the market. The total UK market size was approximately £2.7 trn. In November 2006, Paternoster launched the first buy-out of the Cuthbert Heath Family Plan, which was a small plan with 33 members. In January 2007, with Hunting PLC, Paternoster performed the first buy-in.

The longevity swap, which was the first publicly announced mortality-linked swap, was organized in April 2007 between Friends' Provident and Swiss Re. In that year, the Life Metrics Indices that includes the US, England and Wales, Germany, and the Netherlands was released by the Pension Institute and Willis Towers Watson and JP Morgan. Deutche Börse launched the Cohort Indices and Xpect Age in March 2008. These indices provide a benchmark for longevity-linked securities' trading.

A q-forward contract took place between JP Morgan and Lucida in January 2008; that was the first capital market security transaction. Canada Life then employed a longevity swap in order to hedge £500 million of its annuity book. This event was the first capital market longevity swap to be undertaken, and their longevity risk was thus transferred to investors including insurance-linked securities funds and hedge funds. In 2010, Mercer launched a pension buy-out index for the UK to examine the cost charged by companies. The cost was 44% higher than the accounting value of the liabilities, leading to a search for cheaper alternatives to transfer longevity risk, such as longevity swaps.

The LLMA was established by Aviva, Deutsche Bank, Morgan Stanley, JP Morgan, Swiss Re, AXA, and Prudential in 2010. The LLMA’s goal was the development of a liquid market to address mortality- or longevity-related risks and support the development of methodologies and consistent standards.

In 2010, Swiss Re released a longevity-spread bond with a term of eight years. The value of the bond was \textdollar50 million and Kortis Capital, as a special purpose vehicle, was used to launch it. The bond's design was orchestrated for providing the desired hedge for Swiss Re's longevity and mortality exposure. In January 2011, the first longevity swap for both active and deferred pension plan members was executed between JP Morgan and Pall. This swap's design was calculated to be able to provide the desired hedge for the longevity risk of Pall's pension liabilities. In December 2011, to find a cheaper solution for larger pension plans with liabilities above £500 million, Long Acre Life entered the market. In this solution, companies sell their pension liabilities to an insurance vehicle. In this way, they can invest and share their profits with investors. They aimed to provide a 15% return from this transaction.

In 2012, a €12 bn longevity swap was executed by Deutsche Bank for the insurer Aegon. This particular longevity swap was designed to be index-based. The index depends on the national population of the Netherlands, and the insurer could provide a hedge for its annuity book liabilities. According to the swap, floating payments were made by Deutsche Bank and Aegon paid the fixed premiums. The main objective of this swap was to reduce Aegon's regulatory capital, a goal that was successfully achieved. In June 2012, the globally largest longevity risk and pension transfer agreement was made between General Motors Co. (GM) and PICA. According to the deal, GM would transfer up to $26 bn of its pension liabilities to PICA. In the US, the total market for buy-out deals was $36 bn in the year 2012.

The first bulk annuity deal to be medically underwritten was the one executed by the UK firm Partnership in 2013. In this transaction, each member filled out a medical questionnaire, and according to their medical histories and lifestyles, accurate assessments were provided of their life expectancies. In 2013, in November, the Longevity Experience Option (LEO) was introduced by Deutsche Bank. Its maturity was 10 years. The survival rates of the bond were based on the Netherlands LLMA longevity indexes and the populations of England and Wales. It was traded over the counter. It allowed the transferring of longevity risk between insurance companies, investors, and pension funds. It aimed to provide a more liquid, cheaper alternative to bespoke longevity swaps.

The Mercer Global Pension Buy-out Index was released in 2014. Also, in 2014, the biggest single buy-out was announced by L&G in the UK. It transferred £3 bn of ICI liabilities and assets. In the UK, about £13 bn of bulk annuity agreements were introduced and the total de-risking agreement in 2014 was £35 bn, including buy-outs, buy-ins, and longevity swaps.

In 2015, Philips Pension Fund completed the largest buy-out of pension benefits of 26,000 members, a transaction valued at £2.4 bn, with PIC. This deal was furthermore conducted in combination with longevity hedging. Hannover Re then acted as the reinsurer for the longevity risk. An additional important product to be issued was equity release mortgage, which allowed individuals the option of declaring equity in their homes for the purpose of funding their retirements while avoiding any obligation of downsizing. June of 2015 also saw the launching of the Mercer Pension Risk Exchange, which provides pricing of buy-ins and buy-outs according to the data of pension plans to their clients.

In total, £8.6 bn worth of buy-outs and buy-ins with an additional £1.6 bn worth of longevity swap transactions were executed in 2016. The PGL Pension Scheme's buy-in agreement was for 4,400 pensioners with £1.2 bn with Phoenix Life and it was the largest buy-in in 2016. Moreover, streamlining and contract standardization increased in the same year.

In 2017, one of the most important deals was made between the UK pension fund of Mars & McLennan Companies and both PICA and Canada Life Reinsurance. It was a longevity swap valued at a total of £3.4 bn. In this transaction, risk was shared equally by two reinsurers and they used the ICC vehicle, which made this deal more cost-effective from the pension fund's perspective. Another longevity hedge based on an index was organized between NN Life and Hannover Re in December 2017. The agreement was designed to cover longevity trend risk for €3 bn of insurer liabilities. Hedge maturity was 20 years and NN Life's solvency capital requirements were reduced by €35 m. This deal provided an effective risk transfer for the insurer [16].

4.3. Longevity Market Stakeholders

Strategies for managing risk require that companies be managing their longevity risk in the most effective ways possible. Blake et al. [15] discussed some possible approaches to longevity risk management. For managing longevity risk, the most effective strategy involves capital market solutions, which have been focused on in this thesis. However, several conditions need to be ensured to construct a flourishing capital market (see [102]). First, sufficient exposure or participants must be provided for this market. This condition is of economic importance and hedging could not be adequately performed through the already existing securities. Additionally, an agreement that is both transparent and homogeneous is required for the market to allow sufficient exchanges to occur between agents [39]. Before discussing such securities in more detail, it will be helpful to first outline which parties might have interest in markets for longevity-linked securities.

4.3.1. Stakeholders for Longevity-Linked Securities in Markets

In this section, different classes of participants of the market are examined.

4.3.1.1. Hedgers

Hedgers are one of the stakeholders. They have exposure to longevity risk, and their aim is to offset this particular risk. For instance, life assurers gain if mortality improves, while annuity providers lose. These offsetting exposures show that life assurers and annuity providers could each hedge the longevity risk of the other party. As a second alternative, a life assurer or pension plan may decide to hedge the longevity risk with its transfer to the capital markets, or by reinsuring it.

4.3.1.2. General Investors

The institutions constituting capital markets, like hedge funds or investment banks, might have interest in accepting longevity risk exposure in the event that the expected returns are reasonable. Because the correlation with the risk factors of the financial market is not high, the positive alpha value and the low value of beta mean that longevity-linked securities are appealing investment options worth considering for diversified portfolios.

4.3.1.3. Speculators and Arbitrageurs

Speculators are short-term investors and they are trading their opinions on the directions of individual movements of security prices. When they are involved, the market is apt to be more liquid. Arbitrageurs, meanwhile, pursue gains from whatever pricing anomalies might be present within related securities. Carefully established relations for pricing are necessary between these related securities for ensuring the success of arbitrage.

4.3.1.4. Government

There are many reasons for a government to take interest in longevity-linked securities. A government may desire the promotion of markets and may want to provide assistance for financial companies that face longevity risk exposure. Governments could provide longevity bonds for the hedging of longevity risk. Such actions could serve to reduce the bankruptcy probabilities of such companies. The government also has the responsibility of being the residual risk holder if a default event occurs for private-sector pension plan funds or insurance companies.

4.3.1.5. Regulators

There are two primary goals of financial regulators: enhancing financial stability and ensuring the provision of fair deals for customers in the retail sector.

4.3.1.6. Other Stakeholders

Organized exchanges, health care providers, equity release mortgages providers, and security managers are some of the other relevant stakeholders. These parties will be able to gain benefits from novel sources of fee-based income [15].

4.4. Index versus Customized Hedge

An insurer might prefer a hedge that is free of basis risk, whereby the insured population equals the reference population, for its longevity or mortality risk. On the other hand, an insurer may use a hedge wherein the reference and insured populations are different. In index hedging, effectiveness of the hedge depends on the reference or index population. Understanding the difference between index and customized hedges is important. Furthermore, managing and measuring the basis risk in index hedges is important for capital requirements.

As discussed by Coughlan et al. [42], index hedges have important advantages over customized hedges. For instance, a customized hedge eliminates all basis risk. However, their premiums and costs are higher. On the other hand, index hedges are advantageous in terms of cost, their simple nature, and better liquidity potential. Due to advantages of index-based hedge, we adopt index-based hedging solutions in this thesis. However, index hedges do not provide a perfect hedge and they leave a residual risk.

The disadvantages and the advantages of these hedges are summarized in Table 4.1.

Table 4.1: Index vs. Customized Hedge

	Advantages	Disadvantages
Index Hedge	Lower cost	Imperfect hedge
	Lower operational costs	Basis Risk
	Shorter maturity	Base table estimation risks
	Lower counterparty risk	Roll Risk
Customized Hedge	No basis risk	More expensive
	Requires minimal monitoring	High operational costs
		Less liquidity
		Credit risks
		Less attractive for investors

Source: Coughlan [41].

4.4.1. Customized Longevity Risk Transfer Securities

Selling the annuity book's or pension plan's liabilities by using reinsurance or insurance contracts is a traditional way to deal with longevity risk. This transaction is referred to as pension buy-out. Additionally, longevity insurance and pension buy-ins have been started to use to transfer longevity risk recently. In these solutions, the hedger can fully indemnifies its risk exposure.

4.4.1.1. Pension Buy-outs

A pension buy-out is a financial asset which provides a hedge for any liabilities of a pension plan in exchange for a fixed premium. The advantage of this transaction is that the pension liabilities of trustees are fully eliminated. Pension buy-out transactions have become popular in the UK and the US since 2006. In 2016, the largest buy-out transaction was made by Legal & General.

4.4.1.2. Pension Buy-ins

Buy-ins are financial asset that provides a hedge including investment, inflation, longevity risk and interest rate of a pension plan. A buy-in transaction includes the bulk purchase of annuities that covers the characteristics of the members of the pension plan, such as gender, age and pension amount. It is similar to buy-out transactions. It provides hedge in exchange for a fixed premium.

4.4.1.3. Longevity Insurance

This solution is the insurance-based version of the longevity swaps. Only the longevity risk is transferred in this transaction. It has similar dynamics like longevity swaps that involves paying the fixed cash flows in exchange for floating cash flows. Different from the buy-ins and buy-outs, they do not provide a hedge for investment risk or other risks related to the pension plan [16].

4.4.2. Index-Based Longevity Risk Transfer Securities

Different investors will use different instruments depending on their portfolio and hedging purposes. For instance, pension buy-outs and buy-ins tend to be mostly used by insurers. On the other hand, longevity swaps are generally used by reinsurers and investment banks. Index-based longevity risk transfer instruments are examined in this section.

4.4.2.1. Longevity Bonds

Two primary types of longevity bonds exist. ``Principal-at-risk" longevity bonds are the first type, which can be represented by the Swiss Re bond. With such bonds, investors face risks of the loss of part of or even the entire principal if events related to mortality occur. The second type is the ``coupon-based" longevity bond, which can be represented by the EIB/BNP bond. With these, the coupon payments are mortality-dependent. The dependence may be variable: payments may be functions of mortality indexes, or the investor could lose all or a part of the coupon in the event that the mortality index exceeds a certain point. This type of bond could be issued in the format of annuity bonds, and there is no terminal payment for the principal since they are designed as hedge instruments. However, there are also other sorts of longevity bonds, including, for example, the repayment-of-principal type.

- Classical Longevity Bonds: The first longevity bond is that proposed by Blake and Burrows [13]. Its coupon payments are kept in proportion to the survival rate of the specified reference population, while its final payments finish upon the death of the final surviving individual from that reference cohort.
- Zero-Coupon Longevity Bonds: These bonds envisage a single coupon payment. The attraction of zero-coupon longevity bonds stems from the fact that they can offer blocks upon which tailor-made positions can be built.
- Deferred Longevity Bonds: These are bonds with deferred payment dates. Longevity bonds with deferrals could be viewed as a sort of mortality forward contract. They can take many different forms as forward contracts.
- Principal-at-Risk Longevity Bonds: A principal-at-risk bond has a similar structure to the Swiss-Re mortality bond. A pension plan or annuity provider issues these bonds via a so-called special purpose vehicle (SPV). For the beginning of the agreement, annuity providers or pension plans and investors are funding the SPV. Generally, principal and coupons are fully payable to investors. In the event that the survivor index surpasses a specific limit, however, the principal repayment to the investor will be reduced. The residual payment would be paid to a pension plan or an annuity provider [15].

4.4.2.2. Longevity Futures

If longevity bonds have a liquid market, it may be feasible to develop a futures market that will apply bond prices as the core foundation. The LIFFE Long Gilt futures contracts are the nearest equivalent example to this in the UK.

Involvement of speculators and arbitrageurs is the key issue here. Interest rate changes cause unpredictability in the prices of longevity bonds from day to day, but the changes in longevity risk emerge over long periods. According to the arbitrageurs and speculators, shifts in interest rates within the gilt market do not get mirrored correctly in the market for longevity bonds [15].

4.4.2.3. S-forwards and q-forwards

A q-forward is a mortality forward rate contract, and among the possible longevity hedging instruments, it is the simplest one [42]. For establishing a q-forward contract, the two signing parties agree on exchanging a sum that will be in proportion to the given population's actual realized mortality rate for a sum that will be in proportion to a mortality rate that is fixed. A q-forward could be seen as a swap designed for exchanging realized mortality with fixed mortality.

Another related agreement is the S-forward, which is based on a survivor index. The index is obtained by using mortality rates. In this case, longevity swaps will involve streams of S-forwards with different maturities. The first S-forward was introduced by Dowd [53] [16].

4.4.2.4. Longevity Swaps

Longevity swaps are agreements that parties will be exchanging at least one cash flow in the upcoming future on the basis of a minimum of one (random) survivor index. The structure of longevity swaps is similar to reinsurance contracts. However, there are major differences between them. The most important one is that longevity swaps are not insurance contracts; this means that they are not subject to the legal features that are in place and upheld for an insurance contract. Longevity swaps, in contrast, are held subject to the laws for securities. As an example, insurance contracts do not allow for speculation about random variables, but longevity swaps do. Moreover, a longevity swap does not require that policyholders possess insurable interest, while insurance contracts do.

Diverse forms of longevity swaps exist. A detailed discussion of this may be found in the works of Cox and Lin [45] and Dowd et al. [55].

- <u>One-Payment Longevity Swaps:</u> Among longevity swaps, a one-payment longevity swap is the most simple type. It entails exchanging one single random longevity-dependent payment with one single preset payment. Let us suppose that two firms formally agree on swapping a random amount in exchange for a preset amount at a specified future date of *t*. It is possible to interpret the preset amount as a coupon in affiliation with an implicit notional principal. Moreover, the swap agreements are specified such that the participating parties are exchanging the net differences between the two payments to keep credit risk down, and not more than that. The random amount is related to a specific reference population.

- Vanilla Longevity Swaps: One-payment longevity swaps could be regarded as central foundations for vanilla longevity swaps. With this type of swap, the parties make an agreement for exchanging some series of payments on a periodic basis until the swap's maturity. Vanilla longevity swaps share similarities with interest-rate swaps. They both involve a floating leg, which is connected with a market rate, and a fixed leg. However, there are several major differences. In an interest-rate swap, the fixed-leg payments are constant over time, whereas in a vanilla longevity swap, the fixed-leg payments will be decreasing with the passage of time together with the survivor index that is anticipated at a time of 0. Additionally, an interest-rate swap's floating leg will always be connected with some market rate, while a vanilla longevity swap's floating leg will be dependent upon a survivor index at time *t*. Interest-rate swaps are priced under zero-arbitrage conditions since the market is liquid. However, it is possible to value a vanilla longevity swap within the settings of an incomplete market.
- Other Longevity Swaps: There are also many other longevity swaps. For instance, some swaps may include exchanging one floating payment for a different one. Other types of swaps could be organized according to companies' needs, such as longevity spreads, longevity swaps across currencies, and longevity swaps with embedded properties such as options.

4.4.2.5. Advantages of Longevity Swaps

There are several advantages of longevity swaps in comparison to longevity bonds. For example, it is possible to arrange longevity swaps that have lower transaction costs. Furthermore, it is easier to cancel them than it is to cancel a bond. They offer increased flexibility, and they can be uniquely designed with the goal of suiting diverse scenarios. Longevity swaps have no requirement for a liquid market; they merely need counterparties willing to trade their views about mortality over time [15].

4.4.2.6. Uses of Longevity Swaps

- Insurance Companies' Uses: For an insurer, one possible usage of a longevity swap would be hedging the longevity risks in their life book. A longevity swap requires the matching of preset payments with the payouts of life policies that are being anticipated. An insurer could use a longevity swap to reduce their longevity risk.

  Another reason for an insurer to use a vanilla swap is to manage their exposure to risks over a reference population. It is possible for participating parties to exchange their risks over differing reference populations. For instance, one US insurance company can enter a swap agreement with a UK insurance company. Since the UK and the US longevity risks do not have perfect correlation, both companies can decrease their risks. An insurer could use a longevity swap for many other purposes. Further discussion of this topic is available in the work of Dowd et al. [55].
- Uses by Other Investors: Other financial companies, such as banks or long-term investors, are also interested in longevity swaps. These companies constantly pursue avenues for improving their expected returns based on risks. Similar to the terms for capital asset pricing models, companies try to find new investment assets that have low beta values as measured based upon the existing portfolios.
- Speculative Uses: Investors can also use longevity swaps to serve as tools that will allow speculation about longevity risks. Let us assume that a company has an opinion about future mortality. The company thinks that future mortality would be lower than the rate that is generally expected. This company thus indicates that improvement in the mortality rate will be higher than the current predictions. This company enters into a longevity swap agreement as to the preset payer. In the event that the company's beliefs are correct, the mortality rates will decline and received payments will rise in turn. Hence, this hypothetical company would profit from this swap transaction [55].

4.4.2.7. A Nascent Market in Longevity Swaps

Longevity swaps are traded over the counter (OTC). The OTC market at this point is still in its earliest stages. However, many reassurers and companies transact OTC longevity swaps, with the floating leg connected to the counterparty's realized mortality and the fixed leg being connected with some published mortality projection [15]. Some of the recent survivor longevity transactions are listed in Table 4.2[2].

Table 4.2: Recent Longevity Swap Transactions

Pension Fund or Sponsor	Provider(s)	Solution(s)	Amount	Date
Delta Llyod	RGA Re	Index-based longevity swap	€12 billion	June 2015
Aegon	Canada Life Re	Longevity swap and reinsurance	€6 billion	July 2015
Manweb	Abbey Life	Longevity swap	£1 billion	Aug. 2015
AXA France	RGA Re	Longevity swap and reinsurance	€1.3 billion	Nov. 2016
Pension Insurance	SCOR	Longevity swap and reinsurance	£1 billion	July 2017
British Airways Pension	Partner Re	Longevity swap and reinsurance	£1.6 billion	Aug. 2017

4.5. Interim Conclusion: A Review of Longevity Hedge Products

An evaluation of the longevity risk market has been examined in this section, together with the longevity risk market stakeholders. Moreover a review of the longevity risk transfer securities are presented. An investigation should be done carefully about market structure, stakeholders and risk transfer securities to increase the liquidity and effectiveness of the longevity risk market.

An important increase has been seen recently in longevity-linked securities, since they can be arranged at lower costs and provide significant capital gain. In the first subsection, the evaluation of the longevity risk transfer market is examined. Then the stakeholders of this market are presented. In final subsection, index and customized longevity risk transfer securities are described and compared. Their advantages and disadvantages are discussed. All of those mentioned products provide effective hedging; however, index-

[2] www.artemis.bm/library

based longevity swaps have more advantages than other products. They deliver higher levels of flexibility, and they can be designed uniquely to meet the needs of diverse scenarios. Hence, index-based longevity swaps were adopted as the hedging instrument for this thesis.

5. BUILDING A TWO-POPULATION MORTALITY MODEL

5.1. Introduction

Index-based longevity transactions are better at attracting increased interests both from within and from outside of the worlds of insurance and pensions. That is because these transactions offer significant capital savings and they provide management of risks effectively, as well as at decreased costs. However, as noted before, the potential differences between hedging instruments and the annuity or pension portfolios creates longevity basis risk which arises from mismatches in demographics between the “book population” (namely the portfolio's or the pension's population) and the “reference population” that is associated with the hedging instrument (namely the national population). The differences between two populations, and whether this involves two populations that are completely unlike each other or a population that is actually a subpopulation of the first one, can cause such demographic mismatches. It is possible to classify these mismatches on the basis of a few main characters [116], such as age, gender, or socioeconomic class.

If the two populations of concern possess profiles that are similar to each other in terms of these specified characters, then the basis risk will be small. If they possess profiles that are not so similar, there will then be a larger basis risk.

The sources of basis risk are as follows:

- Structuring risk arises from the difference between the payoff structure of the hedging instrument and the hedged portfolio. As an example, hedging instruments could make their payments annually, while portfolios can make monthly payments.
- Sampling risk is the risk that arises from random outcomes that occur within individual lives.
- Demographic risk arises from the socioeconomic and demographic differences between the populations of the hedging instrument and portfolio [64].

There are well-established approaches to model the structuring and sampling risk. Simulating the cash flow payoffs under the instrument and the portfolio could facilitate an assessment for structuring risk, whereas sampling risks might be evaluated by simulations of cash flows. However, an established, widely recognized method for assessing demographic basis risks do not yet exist [128]

Only a few papers have been published in the literature on quantifying the demographic basis risk and the impacts of that risk on longevity hedges' success. The lack of a suitable framework for the quantification of these risks means that it is not a simple matter to determine whether a transaction has good monetary value, or the transaction's potential impacts for the capital requirements and the overall risk profile of the insurer or the pension scheme. Previous research on quantifying basis risk have followed the framework that was constructed by Coughlan et al. [43]. In their approach, the first step for assessing longevity basis risk involves the detailed analysis of the history of both the reference and the book population with the aim of gaining a thorough comprehension of the differences in mortality between those two populations [128].

Due to these differences in portfolios, basis risk causes imperfect hedge. As discussed in [14], the demand side of the market has challenges to address measuring the basis risk in a longevity hedge, optimizing the index-based longevity hedge in the presence of longevity basis risk and building a multi-population mortality model.

In order to assess and measure basis risks, it is necessary to have a model capable of capturing the trends of mortality within the reference population and in the portfolio's population for which the risk is going to be hedged [101]. In recent years, researchers have begun to explore the basis risks between populations associated with pension plans and hedging instruments. Subsequent to those efforts, several multi-population stochastic mortality models have been proposed in the literature. Therefore, we review the existing multi-population models and address how to build a two-population mortality model to construct a hedge framework in this chapter.

In section 5.2, the notation that is used throughout this thesis is defined and a review of the existing multi-population models is presented. In section 5.3, steps for building a two-population model to measure longevity basis risk are examined. Section 5.4 concludes the chapter.

5.2. Notation

We begin with the introduction of helpful notations. Let us denote the reference population by *R*, and *B* will be used for the book population whose longevity risk is going to be hedged. An assumption is made that time will be measured in units of years, and here year *t* will refer to time interval [*t*, *t+1*]. For the reference population, D_{xt}^{R} and E_{xt}^{R} show the death counts and exposure to risk at age *x* at last birthday in year *t*. Central mortality rates for any individual of the reference population of age *x* in year *t* will be signified by m_{xt}^{R} and computed as $m_{xt}^{R} = D_{xt}^{R} / E_{xt}^{R}$. Likewise, the same values for the book population are given here as D_{xt}^{B}, E_{xt}^{B} and $m_{xt}^{B} = D_{xt}^{B} / E_{xt}^{B}$ [128].

A further assumption being made here is that the data for the reference and book populations can be different regarding specified sets of ages and specified amounts of years. For instance, we have D_{xt}^{R} and E_{xt}^{R} for consecutive ages $x = x_1, ..., x_{n_R}$ and consecutive calendar years $t = t_1, ..., t_{n_R}$ in the reference population, while D_{xt}^{B}, E_{xt}^{B} are available for ages $x_1, ..., x_{n_B}$ and calendar years $t = t_1, ..., t_{n_B}$ in the book population.

The reference population's data might be provided for a longer time frame than that of the book population, which is $n_R \geq n_B$. Moreover, the calendar years of data in a book may be provided as a subset of the comparable calendar years for the reference population, $t_{n_R} \neq t_{n_B}$. Also, the ages provided by the book might constitute a smaller portion of those that are provided for the reference population.

5.2.1. Literature Review

We need to specify an appropriate two-population model for m_{xt}^{R} and m_{xt}^{B} that has the capability of capturing the trends of mortality that are present within the reference population that supports the hedging instrument as well as those within the book population whose risk is going to be hedged in order to assess basis risk. This model should consistently and stochastically forecast the rates and trends of future mortality.

As mentioned before, several models have been developed to show the mortality evaluation of a minimum of two related populations. Generally, these models serve to expand the previous single-population models with the specification of correlations and interactions existing between populations. While the majority of research on modelling multi-population scenarios has been conducted relatively recently, the seeds of such work may be traced back to the influential paper published by Carter and Lee [29], where they introduced feasible approaches for the extension of their single-population model for differences in US mortality between men and women. Their model suggested applying independent Lee-Carter models to individual populations, and this was the first approach for multi-population models. Afterwards, the joint-κ model, based on the assumption that populations' mortality dynamics will be driven by one commonly shared time-varying factor, was developed. The third approach was based on an extension of the Lee-Carter model, applying co-integration techniques and estimating the populations jointly. The definitions of new models established on the basis of the Lee-Carter model are given below:

i. **Independent Modelling:** In this approach, mortality is modelled with the utilization of two independent Lee-Carter models. Let m_{xt}^{i} be the central death rate for population *i* in year *t* at age *x*. The model can then be expressed as follows:

$$\ln(m_{x,t}^{i}) = a_x^i + b_x^i k_t^i + e_{x,t}^i, \quad i = R, B. \tag{5.1}$$

All of those parameters hold the same meanings that they possess in the original Lee-Carter model. It is possible to estimate the model parameters with the application of singular value decomposition, the Markov chain Monte Carlo approach, or maximum likelihood estimation. For a forecast of mortality, a mortality index could be modelled using two independent ARIMA processes. It is easy to apply this method, but in this case, dependences between two populations' mortality rates are completely ignored. The basis risks would be overestimated in the event that this is the method selected to be used.

ii. **The Joint-k Model:** This model is based on the assumption of the mortality rates of both populations both being driven along by one single mortality index. This model may be expressed in the following way:

$$\ln(m^i_{x,t}) = a^i_x + b^i_x k_t + e^i_{x,t}, \quad i = R, B. \tag{5.2}$$

In the joint-k model, the mortality index is the driving force behind the changing of rates of mortality for both of the populations. Model parameters are estimated as in the previous approach. Mortality index k_t will be modelled here with the utilization of an appropriate ARIMA process. However, mortality improvements of two populations are perfectly correlated. Additionally, the common factor's presence suggests identical advancements in mortality for all populations all of the time. Hence, the assumption cannot be said to be realistic. Li and Lee [92] thus introduced a population-specific factor for this model, referred to as the "augmented common factor model".

iii. **Augmented Common Factor:** For the first approach, that of the two independent Lee-Carter models, life expectancy divergence increases in the long run. The joint-k model cannot completely resolve this issue, since discrepancy between two populations in terms of parameter b^i_x could generate divergences in the mortality predictions.

Li and Lee [92] present criteria for the divergence problem, as given below:

- $b^R_x = b^B_x$ for all x.
- k^R_t and k^B_t have identical drift terms of the ARIMA process.

Given these conditions, Li and Lee introduced a specific factor for the Lee-Carter model:

$$\ln(m^i_{x,t}) = a^i_x + b_x k_t + b^i_x k^i_t + e^i_{x,t}, \quad i = R, B \tag{5.3}$$

The $b_x^i k_t^i$ term serves to capture variations in the changing rate of mortality of population i from the long-term mortality change tendencies suggested by the common factor, $b_x k_t$. The k_t^i factors are modelled using the AR(1) process to ensure the avoidance of any divergence from the mortality projections [91].

Another modelling approach for two-population mortality is the extension of the Cairns-Blake-Dowd (CBD) mortality model for a single population [26]. A version of the CBD model for two populations and its variants were introduced by Li et al. [96]. For example, the two-population variant of the CBD model with the incorporation of quadratic effects, known as the M7 model, may be given as follows:

$$\text{logit}\, q_{x,t}^i = \kappa_t^{i,1} + (x-\bar{x})\kappa_t^{i,2} + ((x-\bar{x})^2 - \sigma_x^2)\kappa_t^{i,3} + \gamma_{t-x}^i, \quad i = R, B \tag{5.4}$$

where $\bar{x}$ represents average age and σ_x^2 is the average value of $(x-\bar{x})^2$. $\kappa_t^{i,1}$ and $\kappa_t^{i,2}$ are two stochastic processes and furthermore represent the model's two time indexes. Time index $\kappa_t^{i,1}$ reflects the level of mortality as measured at time t, while $\kappa_t^{i,2}$ denotes the model's slope and affects every age differently. γ_{t-x}^i parameter shows the cohort effect. Li et al. [96] considered three different approaches, which were presented in the work of Zhou et al. [140] to forecast future mortality rates.

The use of an age-period-cohort (APC) model with two populations was presented by Cairns et al. [25] and Dowd et al. [54]. This model may be expressed in the following way:

$$\log m_{x,t}^i = a_x^i + k_t^i + \gamma_{t-x}^i, \qquad i = R, B \tag{5.5}$$

a_x^i, k_t^i and γ_{t-x}^i are the age, period and cohort effects of the populations.

Spreads that exist between the state variables that underlie models of mortality can be modelled as a mean-reverting process for each population and this allows for differing short-term trends in rates of mortality, whereas there are parallel long-term improvements. In the work of Cairns et al. [25], a Bayesian framework was used, which allows for the estimating of the state variables that are not observable and the parameters of the stochastic process that drives them to be performed in just one stage. In another important study, Dowd et al. [54] developed a gravity approach wherein mortality rates of two populations experience attraction to one other, which is determined by a dynamic gravitational force. That force depends on the comparative sizes of the two populations in question [125].

Jarner and Kryger [74] and Cairns et al. [25] recognized the comparative value of the reference population supporting the index and the population whose longevity risk is going to be hedged. Their approach centers on the reference population at the beginning, after which the dynamics of book mortality must be given for the incorporation of characteristics from the reference population. This relative method includes the following important aspects [64]:

- It permits the mismatching of data between the book and reference population.
- The method is applicable in the typical case in which a book population is significantly smaller than a reference population.
- Models for reference populations have been studied in considerable depth and are easy to find, meaning that this aspect of the model might be considered to be thoroughly established, which allows us to instead shift our attention to informed decision-making regarding the model's book part.
- There is consistency present in the modelling of several book populations when the same reference population is used for all of them.
- Joint models may lead to an unrealistically strong correlation between the rates of mortality in two groups for different age brackets. If correlations between mortality dynamics are unrealistic, then it is not possible to claim that the analysis of the success of the hedging will have any reliability. It can be expected that multi-population models will generate a correlation between the mortality rates of two different populations that will not be perfect. In the relative approach, mortality models produce a non-perfect correlation.

There are other multi-population applications of well-known single-population models. For instance, Biatat and Currie [11] expanded the P-spline approach to encompass scenarios with two populations; previously, it had been utilized with success for cases of single populations. Hatzopoulos and Haberman [66] and Ahmadi and Li [1] applied a multivariate generalized linear model (GLM) for obtaining coherent forecasting of mortality in cases of multiple populations [128].

Existing multi-population models generally focus on two or more related populations' mortality rates in terms of the following (see [64] and [93]):

- Males and females in a specified population or country.
- National populations within different countries.
- Smokers and non-smokers in a specified population or country.
- Basis risk in terms of income, deprivation, and affluence index.

Although there are many multi-population mortality models, only a few investigate the measuring of longevity basis risks. Some of the earlier research designed for quantifying basis risk, such as that by Cairns et al. [27], Ngai and Sherris [110], and Li and Hardy [91], have applied the original framework constructed by Coughlan et al. [43].

5.3. Building a Two-Population Mortality Model

The first step in constructing the hedge is to establish a two-population mortality model in order to measure the longevity basis risk.

5.3.1. Mortality Data

All of the examples given here utilize historical UK mortality data, which were collected from the Continuous Mortality Investigation (CMI) and the Human Mortality Database (HMD). The first data represent the mortality experience of CMI assured male lives that are being hedged. The subsequent dataset is for the reference population, which provides the mortality experience of male lives in England and Wales (EW). For the reference population, a sample period from 1961 to 2016 is considered, while for the book population, the sample period comprises the years of 1961-2005. The sample age range being considered is 65 to 89.

5.3.2. Modelling the Reference Population

A relative approach is applied in this thesis, as in [64], since it has many advantages over joint modelling. However, the modelling framework of this thesis is slightly different from the original formulations that were used for the reference model. The model proposed here is a Lee-Carter model with exponential transitory jumps and renewal process. As indicated before, mortality jumps exert important impacts for mortality dynamics; therefore, it is essential that they are incorporated into the modelling process. We assume that transitory jumps are only valid for the reference population because of the quality and size of the available data for the national population. Our proposed model is given by the following:

$$\log(m_{x,t}^R) = a_x^R + b_x^R k_t^R, \tag{5.6}$$

$$k_t^R = k_0^R + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i. \tag{5.7}$$

Here, $m_{x,t}^R$ denotes the central death rate in year *t* for age *x*, a_x^R represents the age pattern of the death rates, k_t^R reflects variations that exist across time in the log mortality rates, b_x^R represents the mortality rates' sensitivity to changes in time-varying mortality index k_t^R, $W(t)$ signifies standard Brownian motion, $N(t)$ denotes the renewal process, and, finally, Y_i represents a sequence of iid exponential random variables representing the size of the jumps.

There are two identifiability constraints, which means that unique solutions exist for all of the model's parameters. These identifiability constraints are given as follows:

$$\sum_x b_x^R = 1, \qquad \sum_t k_t^R = 0.$$

We will estimate the model's parameters using the MLE method. First, reference population parameters a_x^R, b_x^R, and k_t^R are estimated. Afterwards, Equation (5.7) is used to calibrate the time-varying mortality index. We need to find the density function of the independent one-period increments, $\Delta k_i^R = r_i = k_i^R - k_{i-1}^R$, to estimate the parameters of the calibrated model.

Let $D = \{k_0, k_1, ..., k_t\}$ represent the mortality time series at times of $t = 1, 2, ..., T$, which have equal spacing. The one-period increments are iid. Unconditional density for the one-period increment $f(r_i)$ may be given as follows:

$$f(r_i) = R(0) f(r_i \mid 0) + \sum_{n=1}^{N(t)} R(n) f(r_i \mid n), \tag{5.8}$$

where $R(0)$, $R(n)$ are the probability of no jump and n jumps in the renewal process. $f(r_i \mid 0)$, $f(r_i \mid n)$ are conditional densities for a one-period increment; more specifically, they are conditional on the given numbers of jumps, which were provided in the previous section. We can write the log-likelihood of the model as follows:

$$L(D; \mu, \sigma, \eta, \alpha, \beta) = \sum_{i=1}^{T} \log(f(r_i)).$$

The fitted $a_x^R, b_x^R, \mu, \sigma, \eta, \alpha, \beta$ parameter values are shown in Table 5.1, while time-varying index k_t^R is illustrated in Figure 5.1.

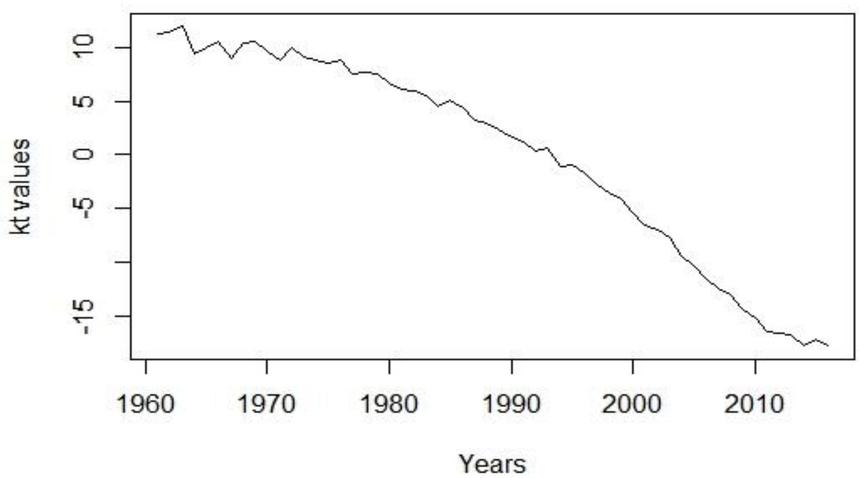

Figure 5.1: Estimated Values of k_t^R

Table 5.1: Estimated Parameters for the UK

Age	a_x	b_x	Age	a_x	b_x
60	-4.2486	0.0388	75	-2.7879	0.0356
61	-4.1505	0.0391	76	-2.6909	0.0349
62	-4.0451	0.0399	77	-2.6061	0.0335
63	-3.9482	0.0402	78	-2.5122	0.0325
64	-3.8408	0.0408	79	-2.4167	0.0314
65	-3.7472	0.0409	80	-2.3246	0.0298
66	-3.6598	0.0401	81	-2.2401	0.0278
67	-3.5517	0.0410	82	-2.1366	0.0272
68	-3.4593	0.0404	83	-2.0461	0.0257
69	-3.3607	0.0401	84	-1.9495	0.0250
70	-3.2684	0.0392	85	-1.8587	0.0233
71	-3.1758	0.0378	86	-1.7637	0.0227
72	-3.0687	0.0381	87	-1.6793	0.0213
73	-2.9749	0.0379	88	-1.5959	0.0195
74	-2.8755	0.0369	89	-1. 5088	0.0179

Jump Diffusion Parameters

μ=-0.2640	σ =0.2764	η=1.4792	α =0.0015	β=0.6173

Given the estimated parameters, the closed-form expression for the expected future central death rates can be derived as follows:

$$E[\hat{m}^R_{x,t}] = \exp(a^R_x + b^R_x(k^R_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i)). \tag{5.9}$$

5.3.3. Modelling the Book Population

With the reference population in hand, it is now time to investigate the book population's mortality dynamics. Estimating the reference population first allows us to make knowledgeable decisions regarding the model's book part, and we can also incorporate features from the reference population [128].

The dynamics of the book population's mortality are specified through the log mortality differences of the book population and the reference population. Generally, the models can be classified as Lee-Carter family, CBD family, and other models as discussed in the previous section (also see [64]). In this thesis, we compare the Lee-Carter (LC) model, the age-period-cohort (APC) model, the Cairns-Blake-Dowd (CBD) model, and common age effect models to model the book population.

Note that for all the models being compared we assume that $D_{xt}^{B} \sim Poisson(E_{xt}^{B}, q_{xt}^{B})$.

5.3.3.1. The LC Model

The dynamics of the book population are given as follows:

$$\log(m_{x,t}^{B}) - \log(m_{x,t}^{R}) = a_x^B + b_x^B k_t^B. \tag{5.10}$$

The term a_x^B denotes the difference in the book population's level of mortality compared to that of the reference population for age *x*. Thus, we can conclude that the mortality level in the book equals $a_x^R + a_x^B$.

Time index k_t^B contributes to establishing the difference that exists in the mortality trends. The b_x^B term shows us how differences in mortality for age *x* will respond if any change occurs in time index k_t^B [64].

5.3.3.2. The Common Age Effect Model

This model may be seen as an extension of the Lee-Carter model that possesses a common age effect. It can be given by the following equation:

$$\log(m_{x,t}^{B}) - \log(m_{x,t}^{R}) = a_x^B + b_x^R k_t^B. \tag{5.11}$$

The a_x^B and k_t^B parameters here are the same as in the LC model for the book population. Different from the LC model, there is a common age effect parameter, b_x^R , which is the same as for the reference model.

5.3.3.3. The APC Model

The APC model was introduced by Currie [500 and it is widely used in the literature. It can be regarded as a generalization of the LC model and a two-population version of this model may be written in the following way:

$$\log(m_{x,t}^{B}) - \log(m_{x,t}^{R}) = a_x^B + k_t^B + \gamma_{t-x}^B. \tag{5.12}$$

a_x^B, k_t^B, and γ_{t-x}^B respectively represent the age, the period, and the cohort effects of the book population [50]. The γ_{t-x}^B term is utilized here in order to account for differences that exist in the cohort effect in the two populations of interest. These parameters reflect the mortality differences between the two populations.

5.3.3.4. The CBD Model

Cairns et al. [26] introduced the following model with the aim of fitting the mortality rates:

$$\text{logit}(q_{x,t}^{B}) - \text{logit}(q_{x,t}^{R}) = \kappa_t^{B,1} + (x - \overline{x})\kappa_t^{B,2}. \tag{5.13}$$

$\kappa_t^{B,1}$ and $\kappa_t^{B,2}$ are two stochastic processes and represent the time indexes of book population. These parameters reflect the mortality differences between the two populations as in the APC model.

The analysis of the models considered in this section becomes something of a challenge due to the CBD model directly modelling one-year death rate $q_{x,t}$ while the others that are being studied here model central death rates $m_{x,t}$. In order to compare the models in a consistent way, we must introduce an additional step for the modelling of $q_{x,t}$. We transform the one-year death probabilities in the central death rates using the identity $m_{x,t} = -\log(1 - q_{x,t})$. For all mentioned models, the estimation of the parameters is done with two main steps. As indicated before, estimation of the model's reference population is to be performed first, and as the second step the estimation of the model's book population will be done, conditional on the parameters of the previously estimated reference population. Under Poisson assumption, the log-likelihood function of the book population is as follows:

$$l^B = \sum_{x,t} (D_{xt}^B \ln E_{xt}^B + D_{xt}^B \ln m_{xt}^B - E_{xt}^B m_{xt}^B - \ln(D_{xt}^B !)).$$

We estimate the parameters by applying the maximum likelihood method. The model parameters thus obtained for the book population are illustrated in Figures 5.2 and 5.3.

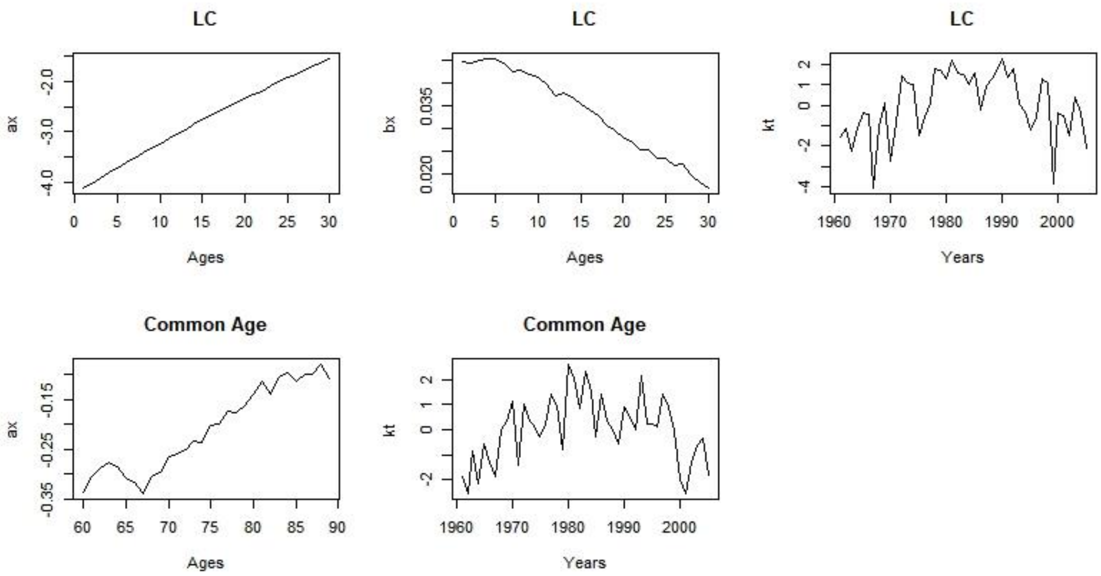

Figure 5.2: Estimated Parameters of Book Population

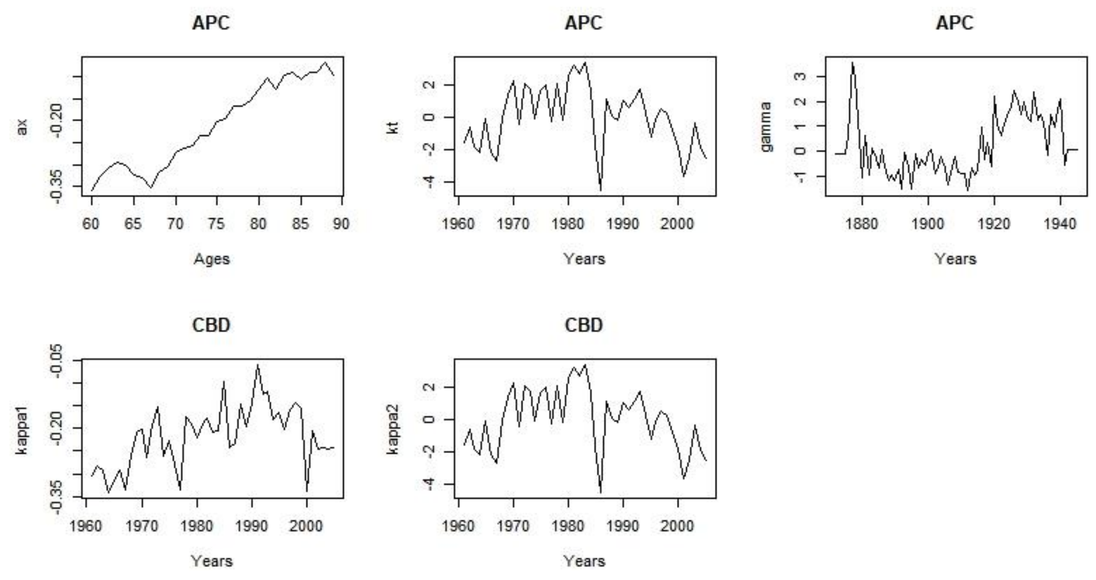

Figure 5.3: Estimated Parameters of Book Population

According to Figure 5.2 and 5.3, the a_x^B, parameter shows that the younger ages reveal lower rates of mortality while the older ages reveal higher mortality. The positive values of b_x^B demonstrate that mortality decreases for all ages. These results are valid for all a_x^B, and b_x^B parameters for all mortality models of book population. The mortality index, k_t^B, reflects the changes in mortality rates over the years for the LC, Common age and APC models. The γ_{t-x}^B parameter represents the cohort related effects in the book population. The negative values of $\kappa_t^{B,1}$ parameter in the CBD model indicate the lower mortality rates while the positive values reflect the higher mortality rates. The $\kappa_t^{B,2}$ parameter controls these lower and higher mortality rates in the CBD model for the book population.

The BIC values for fitting four models for book population mortality are given in Table 5.2. The common age effect model has the lowest BIC value according to Table 5.2. Therefore, we model the book population's mortality as in the common age effect model. According to our analysis, common age effect is important for both populations.

Table 5.2: BIC Values for the Book Population of Models

LC Model	**Common Age Effect Model**	**APC Model**	**CBD Model**
12684.89	12531.63	12809.69	12759.64

Finally, we complete the modelling framework by specifying the period's dynamics and the cohort terms, which will be used to forecast and simulate the future rates of mortality. A detailed analysis regarding the selection of the time series to be used in the dynamics can be found in the work of Li et al. [96]. This part of the thesis confines itself to focusing on the models that are more commonly applied in the literature. We assume that the two populations will experience similar improvements in the long run and thus we assume that the spread in both time indexes and cohort effects should be modelled as a stationary process.

In this thesis, the time-varying mortality indexes of the book population k_t^B are modelled as an autoregressive process of order one, or AR(1); we are thus able to write $k_t^B = \psi_0 + \psi_1 k_{t-1}^B + \xi_t$ for the LC, the common age effect, and the APC models. In the long term, the mean of k_t^B equals $\psi_0 / (1-\psi_0)$ if $|\psi_1|<1.$ The autocorrelation depends on the size of ψ_1. For additional details about the more technical aspects of time-series modelling, interested readers may refer to the work of Tsay [125].

5.3.4. Future Simulations

In evaluating the uncertainty of future outcomes and finding the optimal model to assess longevity basis risk, it is necessary to address all of the parameter errors, process errors, and model errors from a modelling or a regulatory perspective such as that of Solvency II [121]. "Parameter error" refers to the uncertainty in estimating model parameters, while "process error" arises from variations that exist within the time series and finally "model error" reflects the uncertainty that is present in the model selection.

In the literature, a number of studies have been proposed to allow for both process error and parameter error in index-based hedging. For instance, Brouhns et al. [23] used a parametric Monte Carlo simulation method for the generation of examples of model parameters following a multivariate normal distribution. Later, in a subsequent work, Brouhns et al. [22] also explored a semi-parametric bootstrapping procedure designed for the simulation of death rates from the Poisson distribution with the obtained mean equalling the observed numbers of deaths. On the other hand, Renshaw and Haberman [115] utilized fitted numbers of deaths by using the Poisson process. In another study, Koissi et al. [82] used a bootstrap method for the residuals of a fitted Lee-Carter model.

Different from the existing methods, Czado et al. [51] and Kogure et al. [81] suggested the application of Bayesian adaptations of the LC model. Li [86] quantitatively compared possible methods for simulations; according to the conclusions of that study, sampling results will all be relatively close to each other regardless of whether the method applied is parametric, semi-parametric, Bayesian, or residual bootstrapping. All of these various simulation methods possess individual advantages and disadvantages. In this thesis, the bootstrapping method of Brouhns et al. [22] has been selected due to its ability to helpfully include both parameter errors and process errors in simulating future mortality rates. The bootstrapping procedure is detailed as follows:

1. Estimation of the parameters of the LC model is performed by using original data. Once they are obtained, those estimated parameters are then applied for estimating the numbers of deaths for both the reference and the book population by $\hat{m}_{x,t}^{R} E_{x,t}^{R}$, $\hat{m}_{x,t}^{B} E_{x,t}^{B}$.
2. The new data on numbers of deaths are simulated from a binomial distribution for the book population to include the sampling risk and Poisson distribution is used for the reference population. It should be noted that it is possible to simulate the reference population's future number of lives by following the same steps; however, this is omitted here to keep the computational burden lighter, because the reference population's size is quite large for binomial assumptions to be made. The newly simulated data will then be used for estimation of the reference and book populations' mortality parameters. Incorporating this step means that the model can allow for parameter error.
3. Next, we must fit time-series processes to the new data sample's temporal model parameters, k_t^R and k_t^B, since we want to be able to simulate their future values. Furthermore, the inclusion of this step means that the model can allow for process error. k_t^R is modelled by using the proposed model and k_t^B is modelled by using AR(1).
4. We generate future mortality rate samples for all x and future t with the incorporation of the parameters obtained in step (2) and the simulated values that we gained in step (3) into $\log(m_{x,t}^{R})$ and $\log(m_{x,t}^{B})$. As a result, our set of future mortality rates will form one random future scenario.
5. We repeat steps (1) to (4) until we have produced a total of 10,000 random future scenarios.

In a previous study [64], parameter errors of the reference population were ignored. Furthermore, in that work, bootstrapping was not applied to the reference population. In this thesis, in contrast, bootstrapping is performed for both of the populations with the intentions of formalizing the total procedure.

A sample from the simulated mortality paths are shown in Figure 5.4.

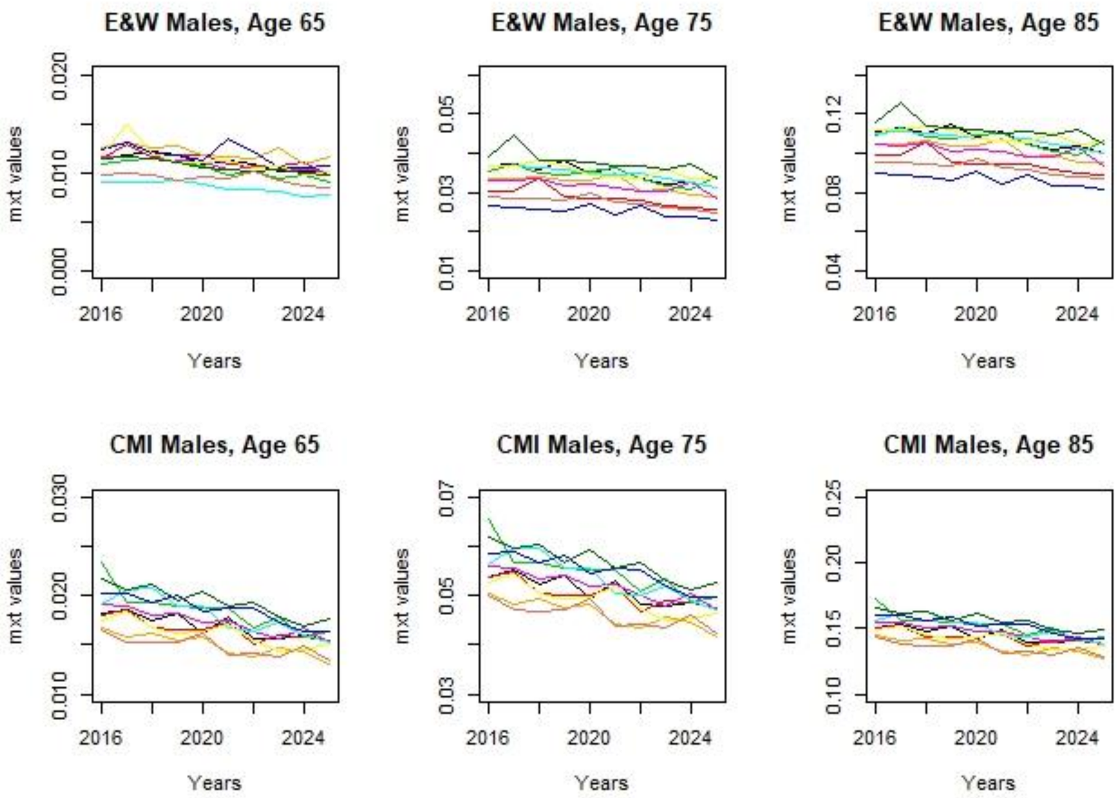

Figure 5.4: Sample Paths of $m_{x,t}$.

5.3.5. Sampling Risk

Book and reference population's finite sizes and the randomness of outcomes of individual lives cause the sampling risk. If the size of populations is infinite, the future outcomes will converge the true expected values according to the law of large numbers. Nevertheless, the size of populations is limited in reality. Although the bigger countries have very large population sizes, the annuity or pension portfolio's size is usually small. Hence the book and reference populations' outcomes will deviate randomly from their true expected values and also from each other. To reflect the effect of the portfolio size, the number of lives is simulated as:

$$l_{x+1,t+1}^{B} \sim Binomial(l_{x,t}^{B}, 1 - q_{x,t}^{B})$$

$l_{x,t}^{B}$ is the future number of lives aged x at time t of the book population. $q_{x,t}^{B}$ is the future mortality rate at age x at time t and it is simulated from the semi-parametric bootstrapping method. Simulating the number of lives of the book population by using the binomial distribution provides protection from the sampling risk [93].

5.4. Interim Conclusion: Building A Two-population Mortality Model

Steps for constructing a two-population mortality model is explained in detail in this chapter, together with the existing literature on multi-population mortality models.

An appropriate two-population model was constructed for EW male lives and CMI assured male lives to measure longevity basis risk, and we adopt the relative approach to model the populations. The reference population is modelled first, followed by the modelling of the dynamics of the book population's mortality. According to the results, the LC model with renewal process and exponential jumps and the common age effect model provide a better fit for the historical data. Moreover, the relative method allows us to use portfolios with different sizes and for different sample periods.

The bootstrap approach of Brouhns et al. [22] is applied in order to include both parameter error and process error in our simulations of future rates of mortality. We thus obtain the simulated mortality rates, taking their average values to be the best estimates for the future mortality rates. The future mortality rates are shown in Figure 5.5. The Poisson distribution is used for the simulation of the reference population's lives and the binomial distribution is used for the simulation of the book population's lives. Using the binomial distribution for the book population allows providing protection for the sampling risk.

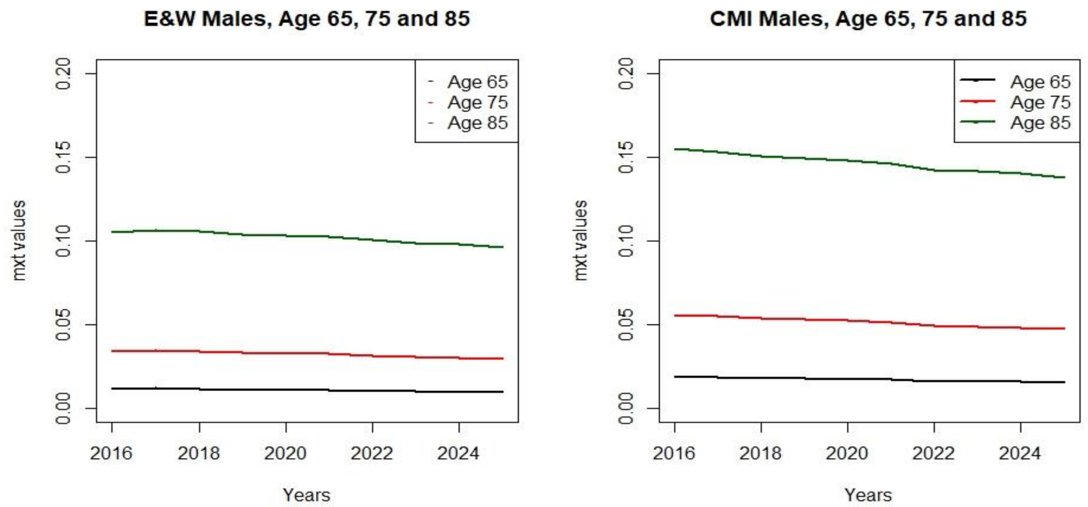

Figure 5.5: Future $m_{x,t}$ Values for Age 65, 75 and 85.

6. A HEDGING FRAMEWORK FOR LONGEVITY BASIS RISK AND COLLATERALIZATION IN INDEX-BASED LONGEVITY SWAPS

6.1. Introduction

Longevity swaps that are based on indexes are advantageous in many ways when compared to other longevity-linked securities. Their properties and advantages have already been discussed in the chapter 4. As a result of those features, significant growth has been seen in longevity swap transactions in the market of longevity-linked securities and derivatives since 2008. Capital market solutions make it possible for pension schemes and annuity providers to swap their longevity risks. In this thesis, we use index-based longevity swaps for hedging. Since longevity-linked instruments are traded OTC, each involved party will be exposed to the default risk of the counterparty. Counterparty default risk can be defined as the risk that the counterparties might not meet their obligations regarding swap payments. This risk always exists whenever an insurer or a pension plan provides hedging with longevity-linked securities or derivatives [12]. Historical experiences show that counterparty default risk often leads to significant losses, and this risk has become particularly apparent following the recent global financial crisis. Therefore, regulators have emphasized the role of credit risk mitigation tools such as clearing and collateralization for the improvement of swap contracts' credit quality [10]. The International Swap and Derivatives Association [71] indicates that the best strategy for enhancing credit is to post collateral regarding the swap contract's value.

Collateralization is a hedging strategy that includes exchanging the assets between two parties to reduce the counterparty default risk. The main idea is quite simple: securities, financial instruments or cash are passed to the counterparty to provide hedging for the default exposure. According to the ISDA, for financial institutions, nearly all swaps that are conducted will be collateralized “bilaterally” [72], which means that either of the parties will need to post collateral based on the swap's market value being either negative or positive. Recent research has shown that 73.7% of all OTC derivative trades are subject to collateralization and there are currently about 16,000 collateralized counterparties in the markets [73]. The collateralization of derivatives is performed based on the Master Swap Agreement's Credit Annex, a policy that had been introduced by the ISDA in 1994.

Bilateral collateral posting is a risk mitigator of importance for both of the parties entering into an agreement for a longevity swap. The premium that an insurer might end up paying for a swap when default risk is included and the premium to be paid when default risk is excluded is expected to be close to each other in collateralization. Amounts posted as collateral might differ according to the swap's market value. Different collateral arrangements exist, such as full collateralization and partial collateralization [12]. For instance, if a counterparty defaults, the surviving party has larger collateral inflows in the event of being in-the-money, as collateral reduces default losses and can be invested in the financial market.

In the absence of collateral, swap rates will be determined by the following factors: the best estimated probability of survival for the hedged population, and the extent of the covariation existing between the swap's floating leg and the defaultable term structure of the interest rates that the hedge supplier and the hedger will be faced with. In other words, a complete and effective longevity swap analysis must take into account the sponsor's covenant in the event that the hedger is a pension plan. If collateralization is present, the rates of the longevity swap will be additionally influenced by anticipated collateral costs, and swap valuation formulas entail a discount rate that reflects the collateral's opportunity costs [76].

Collateralization has many benefits, both privately and socially speaking (for more details, please see [9]). Collateralization decreases losses conditional on default. Whichever party received the collateral will keep the collateral, meaning that the maximal loss will be total exposure with the subtraction of any collateral that was posted. Furthermore, collateral also decreases the required regulatory capital. Transactions that are collateralized typically create a zero credit-risk weighting, which helpfully serves to make limited capital available to be used for other aims. Finally, if collateral is posted, it will decrease the odds of a counterparty defaulting [76].

This approach is different from traditional swap pricing methods. As a result of collateralization, payments by counterparties are generated between established intervals of swap dates. As cash payments such as these may bring accompanying financial benefits/costs for the receiver/payer and are an inherent part of the initiation of the swap contract, it is necessary to take them into account whenever valuing a swap.

To date, such transactions primarily included pension funds and annuity providers who desired to hedge longevity risk exposure without facing any basis risk [12]. This thesis aims to provide a hedging framework for quantifying and managing longevity basis risks when collateralization and counterparty default risk are present in the scenario. This should provide market participants (risk hedgers, regulators, etc.) with effective risk mitigation solutions and furthermore allow them to better understand the allocation between their exposure to the general population longevity trend risk and their specific population longevity risk.

This work makes two primary contributions to the existing literature on hedging longevity basis risk: i) collateralization rules are extended with the existence of longevity basis risk, and ii) the existing framework for a pension portfolio is also extended in the presence of default risk, basis risk, and collateralization. The central aim of this section is to construct a hedging framework for population basis risk and quantify its impact on hedge effectiveness. It is found here that posting collateral increases risk reduction if counterparty default risk is present. Such an extension and hedging framework is presented here for the first time in the actuarial literature to the author's best knowledge.

This section is organized in the following way. Section 6.2 will present the general basis risk hedging framework. Thereafter, Section 6.3 will construct a hedge for longevity basis risk and define the nature of index-based longevity swaps. Section 6.4 will present the bilateral collateral posting rules for index-based longevity swaps, followed by a description of the framework for modelling longevity basis risk with counterparty default risk and collateralization. Finally, Section 6.5 will conclude.

6.2. A General Hedging Framework for Longevity Basis Risk

6.2.1. Analysis of Basis Risk

A general framework will be provided in this section for the analysis of hedge effectiveness and basis risk. Analysis of basis risk needs to be arranged appropriately with hedging objectives such as time horizons, metrics, and the selected analytical methods that are to be applied.

6.2.1.1. Metrics

A lot of different metrics exist that could be utilized for providing an understanding of a longevity hedge's basis risk. Due to the complicated relationship that exists among mortality experiences across years of birth, times, and ages, it is important to analyse the historical performance of all key factors given below:

- Liability cash flows
- Mortality improvements
- Survival rates
- Liability values
- Life expectancies
- Mortality rates

Among various possible metrics, mortality rates are used most often in the assessment of basis risk, since they constitute the raw data associated with longevity. Nevertheless, performing direct comparisons between two populations' rates of mortality often generates an inaccurate picture of the situation's basis risk and of the hedge's effectiveness.

Using percentage changes in mortality rates for evaluating longevity basis risks includes the analysis of sampling variability, long-term trends, and age-bucketing.

Since a pension plan's population's survival rates state the number of survivors receiving a pension and the life expectancy correlates with the anticipated time frame in which pension payments will need to be made, these particular metrics have a closer relationship with hedge effectiveness than they do with mortality rates.

Although the metrics above may be valuable for improving our comprehension of basis risk, they are not sufficient to quantify hedge effectiveness. Although hedging studies typically focus most of their attention on eliminating variations in liability cash flows or the variation in the value of those cash flows, an effective study of basis risk must also consider the effect on cash flows and their values. The financial impacts of basis risk are mirrored directly by the metrics described here; for that reason, these metrics are understood to be suitable choices in studies aimed at evaluation of the effectiveness of longevity hedging.

The value of cash flows is of greater utility in the quantification of basis risk than the other metrics described here, but the primary disadvantage is the dependence on unique details of the benefit structure of particular annuity portfolios or pension plans. For this reason, they require complicated calculations to be performed, such as future cash flow discounting, and furthermore, these calculations need to be performed again in full every time for each new scenario. On the other hand, life expectancy and rates of survival or mortality are all independent of the benefit structure's unique details. Therefore, if suitable interpretation is performed, these are able to provide helpful knowledge about basis risk.

6.2.1.2. Time Horizon

The choice of the time horizon has critical value when basis risk is to be assessed. When longevity risk is being considered within the context of large populations, it must be understood as a risk with a cumulative trend that builds quite slowly and needs to be measured across longer time horizons. For this criterion to be met, it is necessary to evaluate the metrics across horizons that span several years at a minimum. When comparing mortality rate evolutions for two different populations, for example, the changes in their rates of mortality must be evaluated across time horizons that span multiple years.

However, as a result of using long horizons, the number of independent observations from any given historical dataset will be notably reduced. Hence, selection of a time horizon in the analysis of basis risk requires that we make a choice between, on the one hand, a horizon that will be sufficiently long to determine the trends and, on the other hand, a horizon that will be sufficiently short to offer the number of independent data points needed for an effective analysis.

6.2.1.3. Analytical Method

It is also necessary when evaluating basis risk to select an appropriate analytical method that will suit the hedging objective in terms of both time horizon and metric. With this goal in mind, it is necessary to make decisions about assorted analytical details, such as choosing to perform a comparison of the levels of some certain metric or rather comparing the changes that occur within that metric for each studied population. Once these particular analytical details are specified, we will then need to clarify our approach to the comparison of the results that we obtain from the two studied populations. Such a comparison might be performed qualitatively, e.g., by using graphs, or it might be done quantitatively with the aid of statistical analysis, e.g., by utilizing correlations.

In the analysis of longevity basis risk, the general goal is searching for a long-term and durable relationship between two selected populations. In the event that it is possible to identify such a relationship, it will then be possible in turn to calibrate suitable index-based hedging via the determination of the hedging instrument's optimized hedge ratios.

6.2.2. Hedge Calibration

Hedge calibration may be described as a process for designing a hedging instrument so as to ensure the maximization of its effectiveness in decreasing risks in light of the specified hedging objectives. Two main components are present here. The first component is the identification of both a suitable structure and the ideal features of the hedging instrument (e.g., maturity, instrument type, or the index that will be applied). The second component is ensuring optimized amount determination for the hedge with the aim of maximizing hedge effectiveness. Part of this second component includes finding the ideal “hedge ratios” for each component of the chosen hedging instrument.

As an illustration, we can take a hedging instrument that has only a single component intended for hedging a pension liability's value at some time in the future. This will be called the “hedge horizon”. Let us assume that the hedger has purchased *h* units of the hedge for each single unit of the liability: in other words, *h* is the hedge ratio. In this illustration, the hedged portfolio comprises the liability together with *h* units of the hedging instrument. We will also make the assumptions of future random liability $L = L(T)$ at time T and a hedging instrument with value $H = H(T)$, also at time *T*. Following these assumptions, the portfolio's value at time T will be $P(h) = L + hH$. Assuming that the hedger uses the variance as a tool for measuring the risk, then the optimal hedge ratio will be as follows:

$$h^* = -\rho \frac{\sigma_L}{\sigma_L}, \qquad (0.1)$$

where $\rho = Cor(L, H)$.

6.2.3. Hedge Effectiveness

In the assessment of hedge effectiveness, it is necessary to consider the chosen hedging objectives together with the type of risk that will be hedged in order to ensure the development of a suitable methodology. In choosing this methodology, one key question is whether we will be assessing the hedge effectiveness *prospectively* or *retrospectively*.

Analysing hedge effectiveness in a prospective manner entails the development of forward-focused scenarios for anticipating the future success that a hedging instrument might demonstrate. This can be done with Monte Carlo simulations of possible future pathways for mortality rates, as the hedging instrument's prospective performance can be predicted from such simulations in relation to the longevity exposure underlying the case. Here, it is necessary to take the basis risk into consideration in an explicit fashion, ensuring that simulations of future mortality rate scenarios consistently reflect observable relationships between the hedging population and the exposed population. For the best results, a stochastic mortality model with two populations must be used.

On the other hand, performing hedge effectiveness analysis retrospectively entails the use of real historical data in order to evaluate the success that a hedging instrument might have achieved in the past. For this type of effectiveness testing, basis risk is considered by incorporating the historic relationships between the mortality outcomes observed for the hedging population and for the exposed population.

After specifying our effectiveness method, we must go on to make a second choice regarding the so-called basis for comparison, which means that we will be determining the details of exactly how we will be comparing the performances of hedged exposure and unhedged exposure. A straightforward choice is possible here in terms of risk reduction degree:

$$R^2(h) = 1 - \frac{Var[P(h)]}{Var[L]}. \tag{0.2}$$

Perfect hedges would obviously reduce the risks to zero, which clearly corresponds to risk reduction of 100% [43].

6.3. Longevity Basis Risk Hedging Framework and Collateralizing

Upon obtaining the future mortality and survivor rates in the previous chapter, we need to build an effective longevity hedge for basis risk. Constructing and executing a longevity hedge that is based on an index necessitates the existence of a framework that can provide a thorough comprehension of the basis risk that exists within the scenario being examined and the calibration of the hedging instrument, as well as the evaluation of hedge effectiveness. Although the measurement of basis risk is being performed in demographic terms, hedge effectiveness should be quantified economically. The critical step in efforts to design a suitable approach to hedge effectiveness lies in the determination of the hedging objectives, which reveal the reality of the risk to be hedged and additionally call attention to the degree of risk reduction, again to be expressed in economic terms.

Determining the risk metric is the first step in constructing such a hedging framework for basis risk analysis. It may be helpful to apply different metrics in order to obtain fresh perspectives on the basis risks connected to longevity hedges. As a result of the complicated relations among experiences of mortality across ages, periods, and cohorts, an examination of the historic performance of all key metrics is also necessary. These metrics are mortality improvements, mortality and survival rates, life expectancies, liability cash flows, and liability values.

The most commonly used metric is that of mortality rates, since they function as the underlying raw data in association with longevity. However, directly performing a comparison between the mortality rates of two related populations often brings about an inaccurate view of both the basis risk and the effectiveness of the studied longevity hedge. While life expectancy and survival rates may help to resolve this disadvantage by serving as metrics for the analysis of basis risk, it needs to be stressed that none of the metrics are truly ideal for the quantification of hedge effectiveness. For this reason, it is recommended that works on basis risk be focused on the impacts on liability cash flows and/or value. Metrics such as these can be used to mirror the monetary effects on basis risk; furthermore, they are suitable metrics to apply in the evaluation of hedge effectiveness in longevity hedging [43]. Our general hedging framework steps are shown in Figure 6.1.

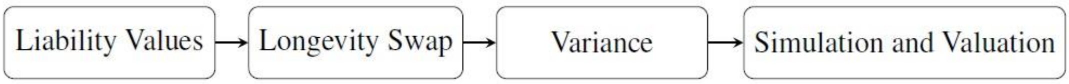

Figure 6.1: General Hedging Framework Steps

We use liability values as the hedging objective for assessing longevity basis risk. The liability to be hedged consists of a stream of uncertain future cash flows with regular payment dates payable to a population of individuals from the time of their retirement until death. The liability of the longevity risk can be represented by a set of model points, which are the age, gender, and annual benefits of a pension plan member and the age, gender, and annual benefits of the pension plan member's spouse or dependent. These factors all together are called “exposure”. For simplicity in calculations, the spouses' or dependents' benefits are not included in the analysis of this thesis.

There are two types of liabilities that we should consider in an index hedge. The first one is the liability being hedged, or in other words the current value of the set of cash flows that are promised to the population of pensioners/annuitants. This reflects the aggregate experience of mortality for the hedger's own population, which is the book population.

We assume that the hedger's own mortality has the same underlying mortality rates as those for the CMI male assured lives dataset, and we will thus be calculating the pension liability by making reference to both current and projected CMI mortality. We assume that the hedger wishes to minimize exposure to longevity risk and thus a longevity hedge is constructed with an index-based hedging instrument.

The index-based hedging instrument provides the hedge and cash flows dependent on a longevity index that is different from that associated with the exposure being hedged, which generates the second liability. The cash flows of a hedging instrument are linked with a longevity index that is dependent on the general population's mortality rather than that of a defined set of individuals. The population that is associated with a hedging instrument is referred to as the reference population and it is assumed that the underlying mortality rates are based on the EW male population. The longevity index should be set up as closely as possible to the future value of liabilities in a way that minimizes basis risk [28].

In this thesis, index-based longevity swaps are applied as hedging instrument, and they consist of floating-leg payments and fixed-leg payments. In longevity swaps, an insurer or a pension plan will receive the floating-leg payments on the basis of a reference population's rate of survival and will make fixed-leg payments of a priorly determined size, again on the basis of survival rates. We use variance risk metric to minimize the variations in the expected future cash flows of the longevity swap and the hedge effectiveness level is calculated by simulating and the valuating the liability values.

However, as indicated before, entering a swap transaction provides hedging for an insurer or pension plan, but it also introduces counterparty default risk, which is the probability that a counterparty cannot meet its obligations. Such default risk might have a negative impact on the price of swaps. Due to recent financial crises, this issue has become more important. With the effects of the financial crisis of 2008 being widely felt by companies around the world, the area of risk became the central point of attention for cautious, thorough analysis with the goal of obtaining more highly accurate measurements of risk, especially for counterparty default risk. Companies have become more cognizant of the fact that accurate estimations of risk will deliver more control of that risk's effects [6].

Due to the need of covering all losses, pension plans and insurance companies have started to develop more robust and accurate risk estimates. Moreover, Solvency II has been more effective since 2014 and insurance companies are now faced with the reality of new legislation addressing the financial buffers that they traditionally utilize for their risks. The primary aims of this new solvency regime are to achieve more realistic modelling and to thoroughly assess all types of risk that pension plans and insurance companies will face. Different from other legislation, solvency capital requirements (SCRs) are risk-based, and insurance companies are strongly advised to utilize stochastic internal models for assessing all risk factors with the utmost possible accuracy. Implementing such internal models is, however, fairly expensive, and so a scenario-based standard model has been established by the European Commission with collaborative assistance from the Committee of Insurance and Occupational Pension Supervisors (CEIOPS). All insurance companies are able to freely apply that new model when approximating their company's capital requirements [18].

The Solvency II reports (see [121]) indicate that, to maintain capital requirements in the face of counterparty default risk, the tool most often applied for risk mitigation is collateral management, and it has become increasingly crucial in derivative transactions. In this section, we will be constructing a hedging framework for quantifying the basis risk when counterparty default risk and bilateral collateral posting is present.

It seems that the work of Bauer et al. [10] is the only study in the literature that has introduced collateral management in longevity swaps. Different from their study, the aim in this thesis is to construct a hedge framework under longevity basis risk. The process as a whole might seem complicated, but detailed descriptions will be provided of the legs of the index-based longevity swap, the capital requirements, and collateral management. This should allow market participants to better understand the allocations among their exposures to the general population's longevity trend risk, their specific population's longevity risk, and their counterparty default risk. This, in turn, should allow them to accurately determine the solvency capital for their risks.

6.3.1. Index-Based Longevity Swaps

Due to their need to able to maintain an adequate level of capital or reserves and to absorb their losses, insurance companies and pension plans have started to use longevity-linked derivatives whose payoffs are based on mortality indexes. A number of mortality-dependent derivatives have been proposed recently, and one alternative is an index-based longevity swap. In a longevity swap, the payoffs of which will be tied to the outcome of at least one random survivor index, an agreement is made for the future exchanging of at least one cash flow. Longevity swaps have advantages over other mortality-linked derivatives. For instance, they are more flexible; it is possible to arrange them to have lower transaction costs; the cancellation process is simpler; and their unique designs can be freely customized in order to be applicable to many different diverse scenarios. A liquid market is not necessary for longevity swaps. What is required instead is simply that counterparties be willing to share their viewpoints on expectations for future developments in mortality [55]. The structure of longevity swaps was discussed in more detail by Lin and Cox [97].

Since longevity swaps are traded OTC, they might come with serious exposure to counterparty credit risk, which is a common problem in the trading of OTC derivatives. Typical approaches to addressing the problem of counterparty credit risk in OTC derivative markets might also be productively applied for longevity swaps. One way is to use credit insurance and credit derivatives. Insurance companies, for example, may purchase credit insurance arrangements with the aim of protecting themselves in the event that a counterparty defaults. It is also possible to complement longevity swaps with credit derivatives, a type of supplementation that guarantees payments in the case that prespecified credit events happen. However, credit insurance and derivatives have the potential to be quite costly, so they need to be chosen carefully.

Another way is to use credit enhancement strategies. These strategies include collateral agreements, re-collateralization with marking to market, credit triggers (whereby a counterparty that has suffered from a specified credit downgrade must give up its swap position and pay any remaining active debts), options for mutual termination, re-couponing (a type of cash exchange that is practiced when exposures reach a prespecified threshold, and the scheduling of payments is reset in order to return the swap value to a zero rate) [55]. Each of these methods helps companies in the management of the counterparty credit risks that are associated with the existing types of swaps, but collateralization has more advantages over the other methods and it has become a popular risk-reduction method for banks and financial institutions for many reasons, such as its reduction of capital requirements [72]. In this thesis, collateral management is used as a credit-risk management tool for this reason.

In the remainder of this section, a definition of longevity swaps is provided, together with information about how they might be priced in an incomplete market. Then default risk and collateralization are introduced. We will show how a hypothetical pension portfolio is valued in the presence of bilateral collateral posting and counterparty default risk.

6.3.2. Nature of Index-Based Longevity Swaps

Longevity swaps are used for hedging against the unpredictable nature of anticipated survival probabilities by insurance companies or pension schemes. Here we will be following a notation similar to that of Dowd et al. [55], who developed longevity swaps to hedge longevity risk.

Let us consider two companies entering into a swap agreement at time 0 for swapping a prespecified sum $K(t)$ for a random sum $F(t)$ at some time *t* in the future. Company A is an insurer, facing exposure to longevity risks in their portfolio of annuitants, while counterparty B makes an agreement for exchanging payments on a prespecified schedule whereby these repeated exchanges will last for some specified amount of time. The preset amount $K(t)$ is paid by A and the floating amount $F(t)$ is paid by B at each payment date of *t*.

Here it needs to be noted that our hypothetical companies shall only be exchanging the difference between two payments. If $K(t) > F(t)$, A pays B the amount $K(t) - F(t)$, while B's payment to A will equal $K(t) - F(t)$, in the case of $F(t) > K(t)$. Company A will need to receive payment in the event that the actual number of survivors in the company's portfolio is more than was expected, while there are no profits for the company if the actual number of survivors happens to be below an anticipated level.

The insurer's prespecified payments are thus made based on anticipated future rates of survival for a particular population selected at the beginning of the swap with the agreement of both parties, and those survival rates will be derived from the expected mortality rates of the book population. The sum that the counterparty will be paying is dependent upon the actual rates of survival of the reference population.

In this subsection, we are ignoring the default risk in the first place to focus on the individual payment structure of longevity swaps. Now we will give detailed descriptions of the legs of a longevity swap. We consider a hypothetical case study for a UK pension plan comprising only male members. As indicated before, the plan in our hypothetical study does not pay benefits for spouses or dependents for the sake of simplicity. The current date will be assumed to be the beginning of calendar year 2016. Supposing that all pensioners enrolled in our hypothetical plan are currently 65 years of age and that each pension is paying $1 annually on survival between the ages of 66 and 90 years, then we will also assume that our pension plan members share the same underlying mortality rates provided by the CMI male assured lives dataset [27]. The sponsor of the pension plan in our case study wishes to minimize exposure to longevity risk with the construction of a longevity hedge that will include index-based longevity swaps. We work with the assumption of the availability of an index-based longevity swap with a 10-year term entailing yearly payment exchanges in the de-risking market, whereby the EW male population constitutes the floating leg's reference population. On the other hand, the swap's fixed-leg payments will be based on central estimates of mortality rates in the future, taking the reference population as the base. Let r represent an interest rate that is risk-free. We use the same notation as in the work of Li et al. [93].

The present value of the pension plan's future liability, $L(t)$, equals,

$$L(t) = \sum_{t=1}^{10} l_{65+t,t}^{B} (1+r)^{-t},$$

As a floating-leg receiver, the present value of the longevity swap's future cash inflows, S(t), may be written as

$$S(t) = \sum_{t=1}^{10} ({}_t p_{65}^{R} - {}_t p_{65}^{R;\,forward})(1+r)^{-t}.$$

For this equation, we calculate random future survivor index ${}_t p_{65}^{R}$ and forward survivor index ${}_t p_{65}^{R;\,forward}$ by applying the survival probability formula, as follows:

$${}_t p_{65}^{R} = (1-q_{65,0}^{R})(1-q_{66,0}^{R})...(1-q_{65+t-1,t-1}^{R}).$$

Furthermore, the present value of the aggregate pension plan position after longevity hedging may be expressed with the following statement:

$$\sum_{t=1}^{10} l_{65+t,t}^{B}(1+r)^{-t} - w\sum_{t=1}^{10} ({}_t p_{65}^{R} - {}_t p_{65}^{R;\,forward})(1+r)^{-t}.$$

Finally, the cash outflow for the net position at individual times of $t = 1, 2, \ldots, 10$ may be signified by $l_{65+t,t}^{B} - w({}_t p_{65}^{R} - {}_t p_{65}^{R;\,forward})$. Weight w here denotes the notional amount of longevity swap necessary for successful hedging to be performed [93]. The notional amount w of the swap can be obtained in several ways. The most common way to calculate this amount is by minimizing the variance of the present value of the aggregate position. It is assumed in the present thesis that the notional amount equals 1 in order to allow us to focus on collateralizing, the topic of the next subsection.

6.4. Collateral Management

Collateral management is a hedging process between two counterparties which are exchanging the cash or financial assets to reduce the counterparty default risk. The counterparties could be pension funds, insurance companies, banks, asset managers, hedge funds and brokers. The main idea is quite simple: securities, financial instruments or cash are passed to the counterparty to provide hedging for the default exposure. The basic collateralization is illustrated in Figure 6.2 for the case that Party A is in-the-money and Party B is out-of-the-money. Collateral management began with Bankers Trust and Salomon Brothers in the 1980s. They took collateral against credit risk. They did not have legal standards and they did the majority of their calculations by hand. Derivative collateralization grew to be more common in the following decade, and the first ISDA documentation in 1994 introduced standardization conditions.

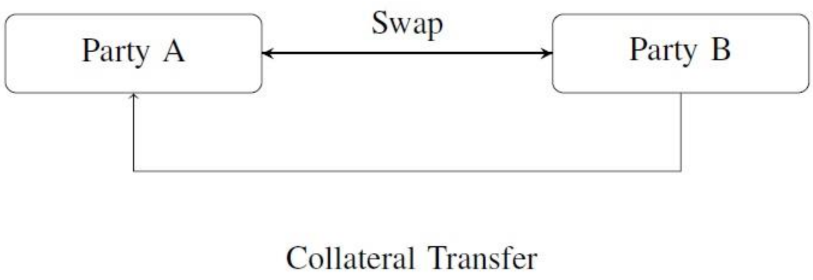

Figure 6.2: Basic Collateral Transfer

Collateral agreements outline the tolerable level of credit exposure for all involved parties. Assume that the market value of longevity swap is positive, then there is counterparty default risk arising from the swap payments. However, collateral transactions restrict the default exposure by defining the value of the collateral. If the party, that posts collateral, defaults, the counterparty would be the economic owner of the collateral. Before 2008, companies and banks assumed that large banks or companies could rarely default and hence they required collateral only for riskier and smaller customers. However, the 2008 financial crisis showed that all companies have a default risk whenever they have an obligation to make future payments. Now, collateralization is required for all companies. Although the main reason for collateralization is to reduce the default risk, there are other reasons to make collateral agreements. The reasons could be listed as follows [62]:

i. Collateralization provides capital savings and increases market liquidity.
ii. It allows doing more business by reducing the counterparty default risk.
iii. It allows making agreements with a specific party without rating restrictions.
iv. It allows obtaining the more competitive prices for derivatives.

Collateralization provides many advantages to counterparties. The primary one is the mitigation of the counterparty default risk. Holding collateral protects from the negative results of the counterparty default risk by acting as a buffer for insolvency. Another advantage is collateralization provides them to access larger markets and hence enables them to trade in larger markets. Lastly, collateralization ensures the validation of the portfolios of companies. Since collateralization has important features and advantages, the collateral transactions have shown an increase in recent years.

There are many types of collaterals used by parties. Cash, government securities, equities, covered bonds, and corporate bonds are types of collateral. However, parties most commonly use cash and government securities. The securities which are used for collateral, are different from the physical assets in terms of nature and type. For physical assets the asset value is subject to delays in the bankruptcy process, although the traders have a claim on specific assets. On the other hand, the usage of the collateral is under the control of the counterparty and it could be liquidated in the default event.

Securities that are used as collateral are traded in the collateral markets to obtain cash or other securities. The markets for collateral are important for companies. It supports their financial growth in the following ways [104]:

i. Life insurance companies could have securities which are used as collateral to borrow cash at the low interest rates and reinvest them to earn a spread.
ii. Hedge funds have portfolios of securities which are obtained by pledging the securities as collateral. By this way, they provide finance for the portfolio at a lower rate instead of unsecured borrowing.
iii. Companies could have an opportunity to lend out their cash at a low interest rate.

The posted collateral amount might differ, or full collateralization could occur, but there are still other collateral arrangements in existence. As an example, collateral might be posted only in the event that a swap value surpasses a prespecified limit; in this case, the amount above the threshold is the sum that will be posted as collateral. Conversely, perhaps only a portion of the swap value will be posted as collateral. ISDA reports indicate that the posted collateral amounts should be specified based on the swap value. This is all dependent on the details agreed upon between the participating parties of each such collateral arrangement.

Let start by determining the mechanics of collateral modelling. First, we assume that there are two default risky parties who want to enter longevity swap agreements. Both parties have the same credit rating and are subject to the same default probabilities. Furthermore, we denote their default times as τ_i and τ_c for insurer and counterparty, respectively. We follow the notation and market considerations of Brigo and Capponi [21]. We assume that portfolio time horizon $T \in \mathbb{R}^+$ is fixed and we have risk-neutral pricing model $(\Omega, G, \mathbb{Q})$ with filtration $(G_t)_{t\in[0,T]}$. We do not consider random default times, and τ_i and τ_c are the G-stopping times. F_t space, which is a right-continuous subfiltration, includes all observable market quantities and default events. Hence, we can write $F_t \subseteq G_t$. The first default to occur between the two parties may be defined in the following way:

$$\tau := \tau_i \wedge \tau_c$$

The probability of default in one year, $p^j \in [0,1]$, is taken to have a constant value over time. This value is dependent on the solvency rating or the credit of the parties and we consider $j \in$ {AAA, AA, …, B & CCC or lower}. The rating values are those that were given by the Solvency II legislation (see [121]) and they are shown in Table 6.1.

Table 6.1: One-year Probabilities of Default According to QIS5

Rating *j*	p^j
AAA	0.002%
AA	0.01%
A	0.05%
BBB	0.24%
BB	1.2%
B & CCC or lower	4.175%

Let us assume that $(C_t)_{t\geq 0}$ is the collateral process and it indicates the quantity of cash, C_t, that will be posted at each time *t* before default in reaction to shifts or fluctuations in market conditions, depending on the swap's market value. With this starting point, our analysis will be developed by taking the insurer's perspective, such that $C_t > 0$ $(C_t < 0)$ signifies that the insurer is holding (posting) collateral.

More specifically, the collateral taker is the one who holds the collateral account, and the participating parties will post or withdraw collateral during the maturity of the longevity swap agreement. The collateral provider is the other party. If $C_t > 0,$ the collateral is in favor of the insurer and the exposure is paid by the counterparty, whereas if $C_t < 0$, the collateral is in the favor of the other party instead and the exposure should be paid by the insurer.

The collateral account is taken to be a risk-free cash account for this thesis. We assume that $C_t = 0$ for all $t \leq 0$ as well as $C_t = 0$ if $t \geq \tau \wedge T$. Furthermore, we assume the collateral preceding the default of either of the parties as a fraction of the value of the longevity swap in order to obtain all possibilities, as in the work of Biffis et al. [12]:

$$C_t = (\varphi^c 1_{\{S_{t^-} \geq 0\}} + \varphi^i 1_{\{S_{t^-} < 0\}}) S_{t^-}. \tag{0.3}$$

Here, φ^i and φ^c denotes the fraction of the swap which is the posted collateral amount for insurer or counterparty and 1_H is the indicator which takes the value of unity if the event occurs, zero otherwise. We assume that S_t denotes the longevity swap value at time *t* and S_{t^-} denotes the swap value before the default. Posted collateral amount is specified as fraction of the market value of the swap at time *t*. We should note that C_t will have a negative value whenever the longevity swap is out of money from the insurer's perspective.

Bilateral collateral posting is a valuable tool for mitigating the risks for both parties entering a longevity swap. In such an arrangement, both of the swap's parties are required to post collateral. Collateral being posted in this way decreases the odds of swap payments being lost in the event of a defaulting counterparty, as the insurer will have a claim on the collateral if the swap has a positive value when looked at from the insurer's viewpoint [12].

Generally speaking, the frequency with which collateral is to be posted is uncertain when considering longevity swaps. Here we assume that collateral will be posted on a yearly basis. Moreover, we allow for rehypothecation in order to increase the benefits of collateralization in this thesis.

In this subsection, the rules for collateral posting and rehypothecation will be briefly described. We will examine the payoffs from the insurer's viewpoint and then we will examine the cases for default of the insurer, default of the counterparty, and no default by considering partial collateralizing. Afterwards, we will find the longevity basis risk reduction and hedge effectiveness.

6.4.1. Collateral Rehypothecation

We will extend the longevity basis risk hedging framework in the presence of collateral rehypothecation in this subsection. Rehypothecation is a transaction that uses the posted collateral amount as a further collateral or as an investment. Assume that swap value for an insurer is positive and the counterparty is posting collateral. If the rehypothecation is allowed, then the insurer can use the collateral as an investment or as a cash [20].

In a case of no default, the party that posts collateral will be expecting to receive the remainder of the collateral back again at the end of the swap maturity. Similarly, if a party defaults earlier, the party that posted collateral will be expecting to receive the remainder of it. In some cases, however, the party that holds collateral will have unlimited use of it up until the maturity of the swap. This usage involves several possibilities, including being able to sell the collateral on the market to a third party. Further possibilities are deciding to lend the collateral or to sell it as part of a “repo” agreement, or *rehypothecating* it. So far, the collateral amount was only invested at a risk-free rate for longevity swaps. While the interest on collateral will be partially rebated, it is possible for the benefits to be more significant in the event that the collateral can be rehypothecated, as is the case within the interest-rate swap market. This thesis aims to fill this existent gap and propose a framework with rehypothecation of collateral.

In the rehypothecation case, it is necessary that the collateral provider take into account the odds that only a fraction of the collateral might be recovered. We assume that $REC_i^{'}$ denotes the recovery fraction of the collateral rehypothecated by the defaulted insurer. Similarly, in the event that the counterparty is receiving the collateral, we take $REC_c^{'}$ to signify the recovery fraction of the collateral that is rehypothecated by the counterparty. Loss of collateral suffered by the counterparty in the case of insurer default is defined as $LGD_i^{'} = 1 - REC_i^{'}$, while $LGD_c^{'} = 1 - REC_c^{'}$ is the loss of collateral experienced by the insurer in the event of counterparty default. $REC_i (REC_c)$ denotes the recovery fraction of the swap value that will be received by the counterparty (insurer) upon the defaulting of the insurer (counterparty).

We should note that, if $REC_i^{'} = REC_c^{'}$ is taken as 1, then the collateral cannot be rehypothecated. In this case, the collateral needs to be kept into a segregated account.

Let us start by introducing some notations that will be used in this subsection. Since all of the payoffs in our scenario are being seen from the perspective of the insurer, longevity exposures will be more capital-intensive because of the fact that the insurer receives collateral if the reference mortality is lower. In contrast, the insurer posts collateral if reference mortality is higher, and so, in such a case, less capital-intensive longevity protection is being provided.

Using the notation $X^+ := \max(X, 0)$ and $X^- := \min(X, 0)$, all possible liabilities under collateralization and rehypothecation will be considered here.

1. On the insurer's default event, $t = \tau = \tau_i$, the counterparty posts collateral and the longevity swap value is paid to the insurer if $S_{t^-} \geq 0$. The exposure of counterparty is reduced by collateral and this amount is paid to the insurer. The posted collateral rehypothecated by the insurer and the $REC_i^{'}(S_t - C_t)^-$ amount is

obtained, and the remaining amount is paid to the counterparty. Then the insurer has:

$$1_{\{\tau=\tau_i\}}1_{\{S_{t^-}\geq 0\}}((S_t - C_t)^+ + REC_i^{'}(S_t - C_t)^-)$$

2. On the insurer's default event, $t=\tau=\tau_i$, if $S_{t^-}<0$, then the insurer posts collateral. The counterparty uses the collateral to reduce exposure and only a fraction of the swap is exchanged. The remaining collateral returned to the investor (if any). If the remaining collateral is not enough, then investor would have a loss for remaining value. Then the insurer has:

$$1_{\{\tau=\tau_i\}}1_{\{S_{t^-}<0\}}(-REC_i(S_t - C_t)^- - (S_t - C_t)^+)$$

3. On the counterparty's default event, $t=\tau=\tau_c$, the insurer pays the swap value to the counterparty if $S_{t^-}<0$ and the collateral was posted by the insurer. Then the insurer uses the collateral to reduce exposure. The collateral rehypothecated by the counterparty and remaining collateral is returned to the investor (if any). Then the insurer has:

$$1_{\{\tau=\tau_c\}}1_{\{S_{t^-}<0\}}(-(S_t - C_t)^- + REC_c^{'}(S_t - C_t)^+)$$

4. On the counterparty's default event, $t=\tau=\tau_c$, if $S_{t^-}\geq 0$, the counterparty posts some collateral which is used by insurer to reduce its exposure. Only a fraction of the swap is exchanged, and the remaining collateral is returned to the counterparty (if any). Then the insurer has:

$$1_{\{\tau=\tau_c\}}1_{\{S_{t^-}\geq 0\}}(REC_c(S_t - C_t)^+ + (S_t - C_t)^-)$$

The aggregated cash flows should be rearranged to determine liabilities arising from the swap payments for insurer. The liabilities of insurer can be expressed as follows:

$$\begin{aligned}
&1_{\{\tau=\tau_i\}}S_t[1_{\{S_{t^-}\geq 0\}}((1-\varphi^i)^+ + REC_i^{'}(1-\varphi^i)^-) + 1_{\{S_{t^-}<0\}}(-REC_i(1-\varphi^i)^- - ((1-\varphi^i)^+)] \\
&+1_{\{\tau=\tau_c\}}S_t[1_{\{S_{t^-}\geq 0\}}(REC_c(1-\varphi^c)^+ + (1-\varphi^c)^-) \\
&+1_{\{S_{t^-}<0\}}(-(1-\varphi^c)^- + REC_c^{'}(1-\varphi^c)^+)]. \qquad (6.4)
\end{aligned}$$

The net portfolio value should be reorganized by including collateralization. Let $L(t)$, be the liability value without hedge that is equal to $\sum_{t=1}^{T=10} {}_t p_{65}^B (1+r)^{-t}$ and let S(T) be the cash flows arising from the hedging instrument that is equal to

$$\sum_{t=1}^{10}(1+r)^{-t}\Big[1_{\{\tau=\tau_i\}} S_t [1_{\{S_{t^-}\geq 0\}}((1-\varphi^i)^+ + REC_i^{'}(1-\varphi^i)^-)$$

$$+1_{\{S_{t^-}<0\}}(-REC_i(1-\varphi^i)^- - (1-\varphi^i)^+)]$$

$$+1_{\{\tau=\tau_c\}} S_t [1_{\{S_{t^-}\geq 0\}}(REC_c(1-\varphi^c)^+ + (1-\varphi^c)^-)$$

$$+1_{\{S_{t^-}<0\}}(-(1-\varphi^c)^- + REC_c^{'}(1-\varphi^c)^+)]\Big].$$

The portfolio value after taking the hedge and collateralization at time \$t\$ can be expressed as follows:

$$E^Q[L(T)-S(T)] = \sum_{t=1}^{T=10} {}_t p_{65}^B (1+r)^{-t}$$

$$-\sum_{t=1}^{T=10} p^t \Big[1_{\{\tau=\tau_i\}} S_t [1_{\{S_{t^-}\geq 0\}}((1-\varphi^i)^+ + REC_i^{'}(1-\varphi^i)^-)$$

$$+1_{\{S_{t^-}<0\}}(-REC_i(1-\varphi^i)^- - (1-\varphi^i)^+)]$$

$$+1_{\{\tau=\tau_c\}} S_t [1_{\{S_{t^-}\geq 0\}}(REC_c(1-\varphi^c)^+ + (1-\varphi^c)^-)$$

$$+1_{\{S_{t^-}<0\}}(-(1-\varphi^c)^- + REC_c^{'}(1-\varphi^c)^+)]\Big].$$

The aim is to minimize variations in the value of unanticipated future cash flows and to find the optimal recovery values that vary depending on the collateral amount since we investigate the impact of collateral amount on the hedge. The hedged portfolio can be formulated mathematically as follows for per \$1 notion:

$$\min_{REC^* \in (0,1)} \big(Var(L(T)-S(T))\big) \quad \text{for T=1, 2, …} \tag{6.5}$$

where REC^* is the optimal recovery value. By minimizing the variance of the hedged portfolio, we obtain the optimal recovery fractions for this transaction. The variance of the hedged portfolio can be expressed as:

$$Var[L(T)-S(T)] = Var(L(T)) + Var(S(T)) - 2\,\text{cov}(L(T),S(T)) \quad (6.6)$$

$Var(L(T))$ is the variance of the pension plan and $Var(S(T))$ is the variance of the hedging instrument's cash flows. The complexity here is arising from the variance of the hedging instrument. If the Equation (6.4) is rearranged according to the market value of the swap payments, then $Var(S(T))$ can be expressed as follows:

$$Var(S(T)) = Var[S(T)(2REC_c(1-\varphi^c))]$$

Equation (6.6) can be rewritten as:

$$Var(L(T)) - 4REC_c^2(1-\varphi^c)^2 Var(S(T)) - 2REC_c(1-\varphi^c)\,\text{cov}(L(T),S(T))$$

where $\varphi \in (0,1)$ is a constant parameter that reflects the fraction of the market value of the swap. We calculate this ratio for different values of φ. Different φ values lead to different collateral amounts in the hedged portfolio. Assume that both parties are subject to the same default probabilities and they are both AA and B & CCC or lower rated companies to examine the best and the worst scenario. Moreover, we consider the basic case in which $REC_i = REC_c = REC_i^{'} = REC_c^{'}$ and $\varphi = \varphi^i = \varphi^c$ to examine the impact of collateral on the hedge for longevity basis risk. Then the optimal recovery values can be obtained as follows:

$$REC^* = \frac{\text{cov}(L(T),S(T))}{(1-\varphi)Var(S(T))}. \quad (6.7)$$

The optimal recovery rate depends on the collateral rate, covariance between liabilities and hedging instrument, and variance of the hedging instrument. We assume that r=0.03 for our calculations, and the obtained recovery ratios are shown in Table 6.2.

As we can see from the table, the recovery rates increase while the collateral amount increases for both scenarios. An increasing recovery rate means that the insurer gets more collateral and more market value of the swap in the case of default. For instance, for the AA rated companies with 75% collateral rate, the non-defaulting party could obtain 9644% of the longevity swap's market value, if the counterparty defaults.

Table 6.2: Recovery Values for Different φ Values

Collateral Rate	REC^* (AA rated)	REC^* (B & CCC or lower rated)
0	0.2411	0.0742
0.25	0.3215	0.0989
0.50	0.4822	0.1484
0.75	0.9644	0.2968

For the B & CCC or lower rated companies, if the counterparty defaults, the non-defaulting party would obtain 2968% of the market value of the longevity swap. The high difference between the default probabilities has led to these differences in recovery values. However, the important point is the non-defaulting party would obtain a fraction of the longevity swap even if the counterparty defaults and hence this transaction will reduce his exposure. Therefore, he is providing a better hedge for his longevity basis risk and counterparty default risk.

6.4.1.1. Hedge Effectiveness

Hedge effectiveness that is based on an index could be defined in terms of the extent to which the longevity risk is transferred away. A well-structured hedging position should achieve the maximum possible level of effectiveness. The remaining risk may be conceptualized as longevity basis risk. Following Coughlan et al. [43], the longevity risk reduction level for longevity hedging will be defined as follows:

$$\text{Longevity Risk Reduction} = \left(1 - \frac{risk(hedged)}{risk(unhedged)}\right) \times 100\%.$$

Here, the terms *risk(unhedged)* and *risk(hedged)* refer to the portfolio's aggregated longevity risk before and after taking the hedge. Using this metric gives us the percentage of the portfolio's initial longevity risk that is being hedged away. The most common measures for risk comprise variance, 99.5% value-at-risk, standard deviation, and 99.5% expected shortfall [93]. In this thesis, we adopt the variance risk measure to calculate the risk reduction level.

We calculate the risk reduction levels for different values of φ. The obtained values are shown in Table 6.3. As we can see from the table, the level of risk reduction increases while φ increases. For smaller values of φ, the change of risk reduction is smaller. However, when collateral is close to 1, the risk reduction level increases more for both scenarios. Even for the B & CCC or lower rated companies, the risk reduction level is 4455% for 75% collateral rate. This result leads to a situation in which more collateral provides more hedging and hence more risk reduction. These results reveal that using collateral reduces the longevity basis risk and counterparty default risk.

Table 6.3: Risk Reduction Levels for Different φ Values

Collateral Rate	Risk Reduction (AA rated)	Risk Reduction (B & CCC or lower rated)
0	0.56576	0.0275
0.25	0.56577	0.0497
0.50	0.56600	0.1484
0.75	0.78288	0.4455

6.5. Interim Conclusion: A Hedging Framework for Longevity Basis Risk and Collateralization in Index-Based Longevity Swaps

This section has provided a hedging framework to be utilized for longevity basis risk and counterparty default risk. First, the steps for constructing a hedge for longevity basis risk is presented. The framework of Coughlan et al. [43] was utilized for this purpose, which includes five steps to construct a hedge.

After that, we introduce the index-based longevity swaps and their nature. Since they are traded OTC, each party is exposed to counterparty default risk. For this reason, we have constructed a hedging framework to hedge longevity basis risk and counterparty default risk in the presence of collateralization. Step I requires that the hedging objectives be defined. Also included in this step is the definition of the hedging position and hedge horizon *T*. The liability to be hedged in this thesis is a UK pension plan with a 10-year horizon. Step II includes choosing the hedging instrument to reduce the risks. We adopt an index-based longevity swap. In Step III, we define the method to be used for assessing the hedge effectiveness. Step III has great importance since an unsuitable selection could generate inaccurate results about hedge effectiveness. Furthermore, this step also requires that we choose a risk metric, and in this thesis, variance is the risk metric that is selected.

Step IV has two stages: simulation and valuation. The simulated future mortality rates for both the reference and the book population are obtained in Subsection 5.3.4. The valuation of the hedged portfolio is shown in Subsection 6.4.1. According to the valuation results, collateralization provides an important hedging solution for longevity basis risk and counterparty default risk. Finally, in Step V, it is shown that the proposed hedging solution provides significant risk reduction.

As a result, we include collateralization into basis risk and counterparty default risk hedging solutions. To the author's best knowledge, this approach is new in the actuarial literature.

7. CONCLUSIONS AND FURTHER RESEARCH

In this thesis, we have investigated longevity risk and catastrophic mortality risk, which are critical risk factors for insurers or pension plans.

Catastrophic mortality risk arises from the uncertainty found in association with catastrophic events like natural disasters or disease pandemics. Section 2 present outlier analysis to detect these mortality jumps in mortality time indexes. Such detection is necessary to accurately model mortality jump frequencies, as well as to predict their future occurrences. In this section, a building block is provided to serve as a basis for mortality jump modelling with the detection of outliers. The locations of these detected outliers provide us the inter-arrival times of the mortality jumps.

In previous studies, most researchers assumed that mortality jumps occur once a year, or they used the Poisson process for their jump frequencies. Although the timing and the frequency of catastrophic events are unknown, the history of such events can give us information about their future occurrences. In Section 3, the renewal process is used for modelling jump frequency by considering the history of catastrophic events since the mean time between their arrivals is no longer constant. We model the time-varying mortality index as a Merton jump diffusion model. In this way, the proposed mortality model can capture the effects of mortality jumps. The working of the new model is illustrated with mortality data from the US, the UK, Switzerland, France, and Italy. For all countries, the proposed model is better at fitting the historical data than the benchmark model of Lee and Carter. In this section, hypothetical mortality bonds are also priced and we concluded that the renewal process has a significant impact on estimated prices.

In Section 4, various longevity risk products are reviewed together with the advantages and disadvantages of the index and customized hedges. Due to their advantages, index-based longevity swap are used as hedging instrument in this thesis. Since index-based longevity swaps are being considered here, longevity basis risk needs to be analysed.

Mismatches between a hedger's own liability and the hedging instrument's liabilities cause longevity basis risk. With the aim of managing and measuring the basis risk, a two-population mortality model was proposed in Section 5. A detailed analysis for constructing a two-population model is presented in this section. Our results show that the common age effect was found to be important for both populations. After specifying the appropriate model, we construct a hedging framework that is applicable for basis risk in the presence of counterparty default risk.

In Section 6, a general framework to be applied when considering longevity basis risk in the presence of counterparty default risk and hedge effectiveness is presented. Moreover, the key objectives for constructing an effective hedge are discussed. Collateralization is an effective risk management tool for counterparty default risk. Hence, collateralization was used in this section as a default risk mitigation tool. Moreover, bilateral collateral posting for a pension portfolio is described and its impact on hedge effectiveness is quantified. Our analysis has shown that collateralization provides an important risk reduction.

7.1. Further Research

Mortality can be modelled by using the outlier-adjusted data as one possibility for further research. Outlier adjustment is important for forecasting future mortality rates.

The model presented in this thesis was limited to only short-term mortality jumps. It would be possible to apply similar modelling efforts for capturing variations that occur in long-term jumps, which was discussed by Deng et al. [52]. They modelled long-term and short-term jumps together by using a double exponential jump process. Moreover, model risk is a concern of importance in the modelling of mortality, and that could also be taken into account in future research.

We have taken into account only the process and the parameter errors while simulating the future mortality rates. The simulation procedure can be extended by incorporating the model error to investigate the effects of the different models and assumptions.

In this thesis, the analyses are performed within the framework of bilateral collateral posting; however, we ignore the cost of holding collateral. Hence, this approach can be extended in future research for the cost of collateral. Moreover, we could also extend the basis risk hedging framework in the context of economic capital. In doing so, we could develop a risk metric that would measure basis risk effectively in the presence of counterparty default risk and collateralization.

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