GEOMETRIC STRUCTURES ON RIEMANN SURFACES AND REIDEMEISTER TORSION

RİEMANN YÜZEYLERİ ÜZERİNDEKİ GEOMETRİK YAPILAR VE REİDEMEİSTER TORSİYON

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ABSTRACT

GEOMETRIC STRUCTURES ON RIEMANN SURFACES AND REIDEMEISTER TORSION

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Let Σ be a closed orientable surface of genus at least 2 and $\mathrm{Rep}(\Sigma, G)$ be the smooth part of the representation variety of homomorphisms' conjugation classes from fundamental group of Σ to Lie group G.

In this thesis, the Reidemeister torsion formulas of the representations corresponding to geometric structures in two different categories, real and complex, are clearly stated that they can be calculated through the related symplectic forms.

This thesis consists of two main parts:

In the first part, real projective structures are discussed. The deformation space $\mathcal{B}(\Sigma)$ of convex real projective structures on the surface has the Goldman coordinates in the literature and this space also contains the Teichmüller space. Using these coordinates, H.C. Kim clearly expressed the Atiyah-Bott-Goldman symplectic form $\omega_{\mathrm{PSL}(3,\mathbb{R})}$ on the representation space $\mathrm{Rep}(\mathrm{PSL}(3,\mathbb{R}))$. In this part, in the light of all this information, the formula that calculates the Reidemeister torsion of representations $\mathrm{Rep}(\mathrm{PSL}(3,\mathbb{R}))$ is obtained through the symplectic form $\omega_{\mathrm{PSL}(3,\mathbb{R})}$.

In the second part, complex projective structures are considered. There is a natural holomorphic projection from $\mathcal{CP}(\Sigma)$ the space of isotopy classes of complex projective structures on the surface to the Teichmüller space. Any smooth section s of this projection yields a diffeomorhism between $\mathcal{CP}(\Sigma)$ and the cotangent bundle space $\mathrm{T}^*\mathrm{Teich}(\Sigma)$. There is the symplectic form $\omega_{\mathrm{PSL}(2,\mathbb{C})}$ on $\mathcal{CP}(\Sigma)$ which is open in $\mathrm{Rep}(\mathrm{PSL}(2,\mathbb{C}))$ and the symplectic form ω_{nat} on $\mathrm{T}^*\mathrm{Teich}(\Sigma)$. In this part, the Reidemeister torsion of the representations in $\mathcal{CP}(\Sigma)$ are expressed by ω_{nat} and $\omega_{\mathrm{PSL}(2,\mathbb{C})}$ symplectic forms thanks to considered s section is Bers, Schottky, Earle and Fuchsian section, respectively. In addition, the results are applied to 3-manifolds that its boundary consisting of closed surfaces with genus at least 2.

Keywords: Reidemeister torsion, projective structures, representation space, symplectic form, geodesic lamination, 3-manifolds.

ÖZET

RİEMANN YÜZEYLERİ ÜZERİNDEKİ GEOMETRİK YAPILAR VE REİDEMEİSTER TORSİYON

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Σ cinsi en az 2 olan kapalı yönlendirilebilir bir yüzey ve G bir Lie grubu olmak üzere $\mathrm{Rep}(\Sigma, G)$ yüzeyin temel grubundan G grubuna giden homomorfizmaların eşlenik sınıflarından oluşan temsil uzayının pürüzsüz kısmını göstersin.
Bu tezde reel ve kompleks olmak üzere iki farklı kategorideki geometrik yapılara karşılık gelen temsillerin Reidemeister torsiyonunun ilgili simplektik formlar aracılığıyla hesaplanabileceği formüller açık bir şekilde ifade edilmiştir.
Bu tez iki ana bölümden oluşmaktadır:
İlk bölümde reel projektif yapılar ele alınmıştır. Yüzey üzerindeki konveks reel projektif yapıların deformasyon uzayı $\mathcal{B}(\Sigma)$ üzerinde literatürdeki Goldman koordinatları bulunmaktadır ve ayrıca bu uzay Teichmüller uzayını da kapsamaktadır. H.C. Kim bu koordinatları kullanarak $\mathrm{Rep}(\mathrm{PSL}(3,\mathbb{R}))$ temsil uzayı üzerindeki $\omega_{\mathrm{PSL}(3,\mathbb{R})}$ Atiyah-Bott-Goldman simplektik formunu açık bir şekilde ifade etmiştir. Bu bölümde tüm bu bilgilerin ışığında, $\mathrm{Rep}(\mathrm{PSL}(3,\mathbb{R}))$ temsillerinin Reidemeister torsiyonunu hesaplayan formül $\omega_{\mathrm{PSL}(3,\mathbb{R})}$ simplektik formu aracılığıyla elde edilmiştir.
İkinci bölümde ise kompleks projektif yapılar düşünülmüştür. Yüzeydeki kompleks projektif yapıların izotopi sınıfları uzayı $\mathcal{CP}(\Sigma)$ dan Teichmüller uzayına giden doğal örten

bir izdüşüm fonksiyonu bulunmaktadır. Bu izdüşümün herhangi bir s pürüzsüz kesiti yardımıyla $\mathcal{CP}(\Sigma)$ ve $\mathrm{T}^*\mathrm{Teich}(\Sigma)$ kotanjant demeti uzayları difeomorfiktir. $\mathrm{Rep}(\mathrm{PSL}(2,\mathbb{C}))$ içinde açık olan $\mathcal{CP}(\Sigma)$ üzerinde $\omega_{\mathrm{PSL}(2,\mathbb{C})}$ ve $\mathrm{T}^*\mathrm{Teich}(\Sigma)$ uzayı üzerinde ise ω_{nat} simplektik formları bulunmaktadır. Bu bölümde, bahsedilen s kesiti sırasıyla Bers, Schottky, Earle ve Fuchsian kesiti alınarak, $\mathcal{CP}(\Sigma)$ uzayına karşılık gelen temsillerin Reidemeister torsiyonu ω_{nat} ve $\omega_{\mathrm{PSL}(2,\mathbb{C})}$ simplektik formları cinsinden ifade edilmiştir. Bunlara ek olarak, elde edilen sonuçlar sınırı cinsi en az 2 olan kapalı yüzeylerden oluşan 3-manifoldlara uygulanmıştır.

Anahtar Kelimeler: Reidemeister torsiyon, projektif yapılar, temsil uzayı, simplektik form, jeodezik laminasyon, 3-manifoldlar.

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I have respect to my all teachers. Being a teacher is like being a parent for a child. Since I am a mother, I know how much it can be difficult sometimes. Therefore, my first thank is to my all teachers, especially to my first teacher Gülnur Taşdemir and currently my last teacher, namely my supervisor Professor Yaşar Sözen. I would like to express my gratitude to my supervisor for his guidance and patience throughout my PhD. And also, I wish to thank Professor Nedim Değirmenci, Professor Ayşe Altın, Professor Özgün Ünlü, and Professor Sinem Onaran for being committee members of my Ph.D. defense and their encouragement. I am also grateful to Professor Francis Bonahon for his endless support and warm hospitality during my visit to University of Southern California. I have learned so much from him not only mathematics but also how a mathematician can be a kind and lovely human.

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Hatice Zeybek

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CONTENTS

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LIST OF SYMBOLS

G	A Lie group
$\mathfrak{g}$	The Lie algebra of G
$\pi_1(\Sigma)$	The fundamental group of Σ
$\mathrm{Teich}(\Sigma)$	Teichmüller space of Σ
$\mathbb{RP}^2$	The real projective plane
$\mathbb{CP}^1$	The complex projective line
$\mathcal{B}(\Sigma)$	The deformation space of convex $\mathbb{RP}^2$-structures on Σ
$\mathcal{CP}(\Sigma)$	The deformation space of all complex structures on Σ
$\mathrm{Hom}(\pi_1(\Sigma), G)$	The set of group homomorphisms from $\pi_1(\Sigma)$ to G
$\mathrm{Rep}(\pi_1(\Sigma), G)$	The smooth locus of $\mathrm{Hom}(\pi_1(\Sigma), G)$
$\chi(\Sigma)$	The Euler characteristic of Σ
ω_G	The Atiyah-Bott-Goldman symplectic form for Lie group G
$\mathbb{R}$	The set of real numbers
$\mathbb{Z}$	The set of integers

1 INTRODUCTION

Throughout this thesis, let Σ denote a closed orientable surface of genus at least 2 and $\mathrm{Rep}(G)$ be the smooth part of the representation variety of homomorphisms from fundamental group of Σ to Lie group G.

Reidemeister torsion is a topological invariant with many applications in several branches of mathematics and theoretical physics. In 1935, this topological invariant was first introduced by K. Reidemeister, in the paper classifying 3-dimensional lens spaces [1]. W. Franz classified the higher dimensional lens spaces extending the Reidemeister torsion [2]. In 1964, G. de Rham extended the results of Reidemeister and Franz to spaces of constant curvature +1 [3]. R.C. Kirby and L.C. Siebenmann proved that the Reidemeister torsion for manifolds is a topological invariant [4]. Then, T.A. Chapman proved the invariance for arbitrary simplicial complexes [5, 6]. Therefore, the classification of lens spaces made by Reidemeister and Franz proved to be actually topological.

In 1961, J. Milnor disproved Hauptvermutung through Reidemeister torsion by constructing two homeomorphic but combinatorially distinct finite simplicial complexes [7]. He also identified Reidemeister torsion with the Alexander polynomial which plays an important role in knot theory and links [8, 9].

In 1991, E. Witten introduced the real symplectic chain complex notion [10]. By combining this notion and Reidemeister torsion, he also computed the volume of representation space $\mathrm{Rep}(G)$. J. Dubois also used the symplectic chain complex and Reidemeister torsion together and introduced a volume element which is related to Reidemeister torsion, on a special representation space [11, 12].

$\mathrm{Hom}(\pi_1(\Sigma), G)/G$ is the orbit space of all homomorphisms from $\pi_1(\Sigma)$ to the Lie group G modulo conjugation in G has the structure of a real analytic variety. Note that this space is not necessarily Hausdorff. However, $\mathrm{Rep}(\pi_1(\Sigma), G) = \mathrm{Hom}^+(\pi_1(\Sigma), G)/G$ of all reductive representations of $\pi_1(\Sigma)$ in G is Hausdorff.

Hitchin investigated the connected components of the space $\mathrm{Rep}(\pi_1(\Sigma), G)$ for a split real semisimple Lie group G, and proved the existence of an interesting connected component not detected by characteristic classes [13]. The Hitchin component $\mathrm{Hit}_n(\Sigma)$ is a preferred component of the character variety

$$\mathrm{Hom}(\Sigma, \mathrm{PSL}(n,\mathbb{R})) = \{\text{homomorphisms } \phi : \pi_1(\Sigma) \to \mathrm{PSL}(n,\mathbb{R})\}/\mathrm{PSL}(n,\mathbb{R})$$

consisting of all group homomorphisms from the fundamental group to the Lie group $\mathrm{PSL}(n,\mathbb{R})$, up to conjugation. Note that when $n = 2$, the Hitchin component is just the Teichmüller space.

Teichmüller space $\mathrm{Teich}(\Sigma)$ is the space of isotopy classes of complex structures on Σ. It is a differentiable manifold diffeomorphic to $\mathbb{R}^{3|\chi(\Sigma)|}$. Here, $\chi(\Sigma)$ denotes the Euler characteristic of the surface. It is well-known that $\mathrm{Teich}(\Sigma)$ is a connected component of $\mathrm{Rep}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$, where $\pi_1(\Sigma)$ is the fundamental group of Σ.

As is well-known, Teichmüller space inherits three forms: ω_{WP} Weil-Petersson 2-form, $\omega_{\mathrm{PSL}(2,\mathbb{R})}$ Atiyah-Bott-Goldman symplectic form, and ω_{Thurston} Thurston real symplectic form through a maximal geodesic lamination λ [14, 15, 16]. In 1984, Goldman expressed ω_{WP} in terms of $\omega_{\mathrm{PSL}(2,\mathbb{R})}$ symplectic form [15]. On the other hand, Sözen and Bonahon expressed $\omega_{\mathrm{PSL}(2,\mathbb{R})}$ in terms of ω_{Thurston} symplectic form on the real vector space $\mathcal{Z}(\lambda;\mathbb{R})$ of transverse cocycles on λ [17].

In the literature, usually Reidemeister torsion is defined and investigated for $\mathrm{SU}(2)$, $\mathrm{PSL}(2,\mathbb{C})$, or $\mathrm{PSL}(2,\mathbb{R})$ valued representations. In 2012, Sözen showed that it can also be defined $\mathrm{PSL}(n,\mathbb{R})$ valued Hitchin representations for $n > 2$ and established a formula for Reidemeister torsion of such representations in terms of $\omega_{\mathrm{PSL}(n,\mathbb{R})}$ Atiyah-Bott-Goldman symplectic form [18].

Let $\mathcal{B}(\Sigma)$ denote the deformation space of convex real projective structures on the surface which contains the Teichmüller space. In 1990, Goldman introduced the coordinates on this space which is known as Goldman coordinates in the literature [19]. Then, Kim expressed the Atiyah-Bott-Goldman symplectic form $\omega_{\mathrm{PSL}(3,\mathbb{R})}$ on the representation space $\mathrm{Rep}(\mathrm{PSL}(3,\mathbb{R}))$ through the Goldman coordinates [20].

In section 3, we will give the Reidemeister torsion formula of representations $\mathrm{Rep}(\mathrm{PSL}(3,\mathbb{R}))$ is obtained through the symplectic form $\omega_{\mathrm{PSL}(3,\mathbb{R})}$.

Let $\mathcal{CP}(\Sigma)$ denote the space of isotopy classes of complex projective structures on the surface. A complex projective structure is also a holomorphic structure thus there is a natural holomorphic projection $p : \mathcal{CP}(\Sigma) \to \mathrm{Teich}(\Sigma)$ from this space to the Teichmüller space. Any smooth section s of p yields a diffeomorhism between $\mathcal{CP}(\Sigma)$ and cotangent bundle $\mathrm{T}^*\mathrm{Teich}(\Sigma)$. There is the symplectic form $\omega_{\mathrm{PSL}(2,\mathbb{C})}$ on $\mathcal{CP}(\Sigma)$ which is open in $\mathrm{Rep}(\mathrm{PSL}(2,\mathbb{C}))$ and the symplectic form ω_{nat} on $\mathrm{T}^*\mathrm{Teich}(\Sigma)$.

In 1996, Kawai established the relation between ω_{nat} and $\omega_{\mathrm{PSL}(2,\mathbb{C})}$ [21] considering the s section as Bers section. Then, Biswas generalized this result by considering Schottky section [22]. With the help of Earle section, it was generalized by Ares-Gastesi and Biswas [23]. In 2015, by the Fuchsian section, Loustau expressed ω_{nat} in terms of $\omega_{\mathrm{PSL}(2,\mathbb{C})}$ and ω_{WP}.

In section 4, we will give the Reidemeister torsion formula of the representations in $\mathcal{CP}(\Sigma)$ through ω_{nat} and $\omega_{\mathrm{PSL}(2,\mathbb{C})}$ symplectic forms thanks to considered s section is Bers, Schottky, Earle and Fuchsian section, respectively. Moreover, the results are applied to 3-manifolds that its boundary consisting of closed surfaces with genus at least 2.

This thesis aims to show that topological invariant Reidemeister torsion, which has many applications in several branches of mathematics also in theoretical physics, and one of the fundamental instruments of low-dimensional topology/geometry, namely geodesic laminations, can be used efficiently and effectively in the deformation spaces $\mathcal{B}(\Sigma)$ and $\mathcal{CP}(\Sigma)$ with increasing importance in low-dimensional topology/geometry.

The techniques developed in this thesis can be used in many fundamental problems. Especially, combining Reidemeister torsion and symplectic chain complex method has potential and powerful applications on certain problems with geometric significance well known by the experts, such as shedding a light on understanding the size of the space of geometric structures on a surface [10, 24].

2 PRELIMINARIES

2.1 Basic Definitions

In this section, we will give some well-known definitions. One can find more detail about the given subjects through the references.

Let us consider an n-dimensional geometric space X and a group of similarities G of X.

Definition 2.1.1 *([25])*Let $\Phi : \{\phi_i : U_i \to X\}_{i \in I}$ be a family of functions called *charts* for an n-manifold M. If Φ satisfy the following conditions then it is called an (G, X)-*atlas* for M:

- for each i, the set U_i(coordinate neighborhood) is an open connected subset of M.
- for each i, the chart ϕ_i maps U_i homeomorphically onto an open subset of X.
- M is covered by the coordinate neighborhoods $\{U_i\}_{i \in I}$.
- if U_i and U_j overlap, then the coordinate change function

$$\phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

agrees in a neighborhood of each point of its domain with an element of G.

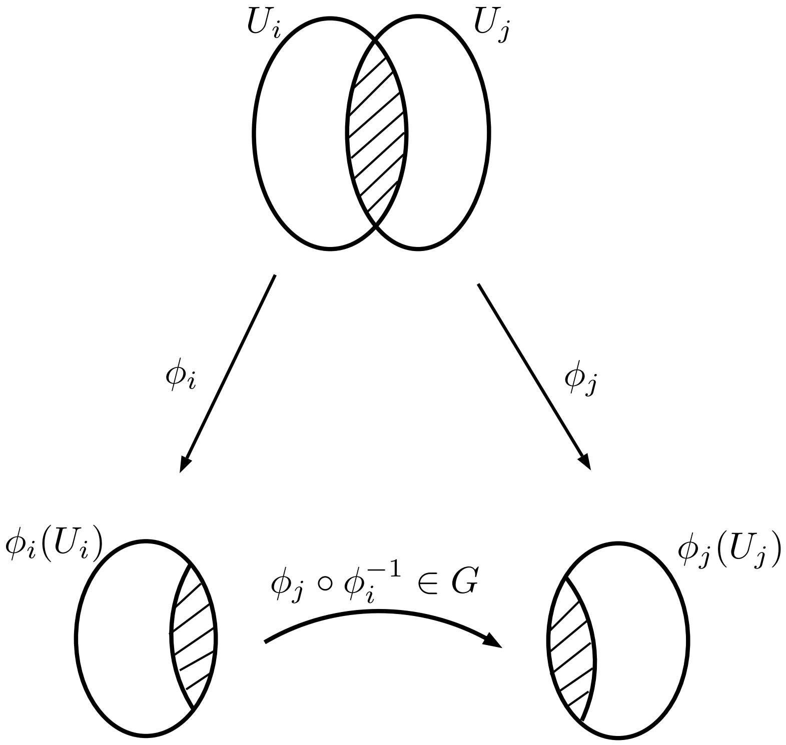


Figure 2.1: A coordinate change map

It is well-known that there is a unique maximal (G, X)-atlas for M containing Φ.

Definition 2.1.2 A maximal (G, X)-atlas for M is called an (G, X)-*structure* for an n-manifold M and an n-manifold M with an (G, X)-structure is called an (G, X)-*manifold*.

Definition 2.1.3 If the geometric space $X = \mathbb{RP}^2$ the real projective plane then this type of structure on a manifold is called a *real projective structure* and if $X = \mathbb{CP}^1$ the complex projective line then this type of structure on a manifold is called a *complex projective structure*. And also, if the geometric space $X = \mathbb{C}$ then this type of structure on a manifold is called a *complex structure*.

Definition 2.1.4 A *holomorphic function* is a complex valued function of one or more complex variables which is complex differentiable in a neighborhood of every point of its domain.

Definition 2.1.5 Let $\mathrm{Diff}^+(\Sigma)$ denote the group of orientation preserving diffeomorphisms of Σ and $\mathrm{Diff}_0^+(\Sigma)$ be the identity component of $\mathrm{Diff}^+(\Sigma)$.

- The quotient

$$\mathrm{Teich}(\Sigma) := \{ \text{ all complex structures on } \Sigma\}/\mathrm{Diff}_0^+(\Sigma)$$

is called the *Teichmüller space* of Σ. Its elements are called *marked Riemann surfaces.*

- The quotient

$$\mathcal{CP}(\Sigma) := \{ \text{ all complex projective structures on } \Sigma\}/\mathrm{Diff}_0^+(\Sigma)$$

is called the *deformation space of all complex projective structures on* Σ. Its elements are called *marked complex projective surfaces.*

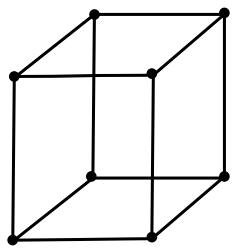
For more detail about the Teichmüller space, the reader is referred to [26, 27, 28].

In particular, a complex projective atlas is a complex atlas on Σ, namely transition functions are holomorphic. Thus, a projective structure defines an underlying complex structure. This gives a forgetful map

$$p : \mathcal{CP}(\Sigma) \to \mathrm{Teich}(\Sigma).$$

Definition 2.1.6 Let P be a polyhedron with V vertices (0-dimensional), E edges (1-dimensional), and F faces (2-dimensional). The *Euler characteristic* of P is defined

$$\chi(P) = V - E + F.$$



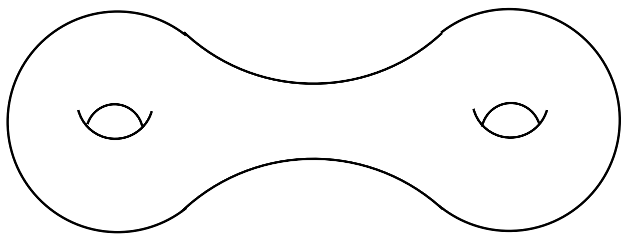
For example, let us compute the Euler characteristic of above cube in 3-dimension. There are 8 vertices, 12 edges and 6 faces in the above diagram. Therefore,

$$\begin{aligned} \chi(\text{cube}) &= V - E + F \\ &= 8 - 12 + 6 \\ &= 2. \end{aligned}$$

If Σ is a g-hole torus then we have the following formula

$$\chi(\Sigma) = 2 - 2g.$$

For example, the Euler characteristic of the following surface Σ with 2 genus equals



$$\chi(\Sigma) = 2 - 2 \cdot 2 = -2.$$

Definition 2.1.7 Let $F : M \to N$ be a smooth map and $p \in M$ be an arbitrary point. The *push-forward map*

$$F_* : \mathrm{T}_p M \to \mathrm{T}_{F(p)} N$$

yields a dual map

$$(F_*)^* : \mathrm{T}^*_{F(p)} N \to \mathrm{T}^*_p M.$$

To avoid confusion of stars, this map is expressed

$$\begin{aligned} F^* : \mathrm{T}^*_{F(p)} N &\to \mathrm{T}^*_p M \\ u &\mapsto F^* u \end{aligned}$$

and called the *pullback map* associated with F for $u \in \mathrm{T}^*_{F(p)} N$. Here, $(F^* u)(v) := u(F_* v)$ for $v \in \mathrm{T}_p M$.

Definition 2.1.8 A *covering map* is a surjective continuous map $p : \widetilde{M} \to M$ between connected, locally path connected spaces, with the property that every point $p \in M$ has a neighborhood U that is evenly covered, meaning that each component of $p^{-1}(U)$ is mapped homoeomorphically onto U by p.

For more detail about these notions, see [29].

2.2 Reidemeister Torsion of a Chain Complex

In this section, we will give the required definitions and the basic facts about one of the main notion of this thesis the Reidemeister torsion. For more information and detailed proofs, we refer the reader to [30, 31, 10] and references therein.

Let us consider a chain complex

$$C_* = C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

of a finite dimensional vector spaces over $\mathbb{F}$, where $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. This chain complex has the $p-$th homology $H_p = Z_p/B_p$ for $p = 0, \ldots, n$ and

$$\begin{aligned} B_p &= Im\{\partial_{p+1} : C_{p+1} \longrightarrow C_p\} \\ Z_p &= Ker\{\partial_p : C_p \longrightarrow C_{p-1}\}. \end{aligned}$$

Let $\mathbf{b}_p = \{b_p^1, \ldots, b_p^{m_p}\}$ and $\mathbf{h}_p = \{h_p^1, \ldots, h_p^{n_p}\}$ be the bases of the spaces B_p and H_p, respectively. By the result of the $1-$st Isomorphism Theorem, we get

$$0 \to Z_p \hookrightarrow C_p \twoheadrightarrow B_{p-1} \to 0, \tag{2.1}$$

and by the definition of H_p, we get

$$0 \to B_p \hookrightarrow Z_p \twoheadrightarrow H_p \to 0 \tag{2.2}$$

If we consider the section $l_p : H_p \to Z_p$ and the short-exact sequence (2.2) then we can get a new basis $\mathbf{b}_p \sqcup l_p(\mathbf{h}_p)$ for Z_p. After that, we take a section $s_p : B_{p-1} \to C_p$ and the short-exact sequence (2.1). Therefore, we get a new basis for C_p as $\mathbf{b}_p \sqcup l_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$.

Definition 2.2.1 Let C_p, B_p and H_p have bases $\mathbf{c}_p, \mathbf{b}_p$ and $\mathbf{h}_p$ respectively for $p = 0, 1, \ldots, n$. The *torsion of the complex* C_* with respect to bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is defined as follows

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_{p=0}^n, \{\mathbf{h}_p\}_{p=0}^n) = \prod_{p=0}^n [\mathbf{b}_p \sqcup l_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p]^{(-1)^{(p+1)}}.$$

Here, $[f, e]$ denotes the determinant of the change-base-matrix M from e to f for a vector space V with bases e and f.

Note that torsion does not depend on the bases $\mathbf{b}_p$ and the sections s_p, l_p (see [33]). This means torsion is well-defined.

Remark 2.2.2 *(Change-base-formula)* Let $\mathbf{c}'_p$ and $\mathbf{h}'_p$ be also bases for C_p and H_p, respectively. Then, one can see that

$$\mathbb{T}(C_*, \{\mathbf{c}'_p\}_{p=0}^n, \{\mathbf{h}'_p\}_{p=0}^n) = \prod_{p=0}^n \left(\frac{[\mathbf{c}'_p, \mathbf{c}_p]}{[\mathbf{h}'_p, \mathbf{h}_p]} \right)^{(-1)^p} \cdot \mathbb{T}(C_*, \{\mathbf{c}_p\}_{p=0}^n, \{\mathbf{h}_p\}_{p=0}^n).$$

Let us take a short-exact sequence of chain complexes as

$$0 \to A_* \overset{\iota}{\hookrightarrow} B_* \overset{\pi}{\twoheadrightarrow} D_* \to 0. \tag{2.3}$$

By using the Zig-Zag Lemma, we get a long-exact sequence with the length $3n+2$ as follows:

$$\begin{array}{ll} C_* : & \cdots \longrightarrow H_{p+1}(A_*) \overset{\imath_{p+1}}{\longrightarrow} H_{p+1}(B_*) \overset{j_{p+1}}{\longrightarrow} H_{p+1}(D_*) \\ & \downarrow \\ & H_p(A_*) \overset{\imath_p}{\longrightarrow} H_p(B_*) \overset{j_p}{\longrightarrow} H_p(D_*) \\ & \downarrow \\ & H_{p-1}(A_*) \overset{\imath_{p-1}}{\longrightarrow} H_{p-1}(B_*) \overset{j_{p-1}}{\longrightarrow} H_{p-1}(D_*) \\ & \downarrow \\ & \vdots \end{array} \tag{2.4}$$

where $C_{3p} = H_p(D_*)$, $C_{3p+1} = H_p(A_*)$, and $C_{3p+2} = H_p(B_*)$.

It is clear that the bases $\mathbf{h}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^B$ serve as bases for C_{3p}, C_{3p+1}, and C_{3p+2}, respectively. J. Milnor proved in [33] that the alternating product of the R-torsions of the chain complexes in (2.3) equals to the R-torsion of the chain complex (2.4). Namely,

Theorem 2.2.3 *([33]) Let $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ be bases of A_p, B_p, D_p, $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. If, moreover, $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, and $\mathbf{c}_p^D$ are compatible in the sense that $[\mathbf{c}_p^B, \mathbf{c}_p^A \oplus \widetilde{\mathbf{c}_p^D}] = \pm 1$, where $j\left(\widetilde{\mathbf{c}_p^D}\right) = \mathbf{c}_p^D$, then*

$$\begin{aligned} \mathbb{T}(B_*, \{\mathbf{c}_p^B\}_0^n, \{\mathbf{h}_p^B\}_0^n) &= \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\mathbb{T}(D_*, \{\mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^D\}_0^n) \\ &\quad \times \mathbb{T}(C_*, \{\mathbf{c}_{3p}\}_0^{3n+2}, \{0\}_0^{3n+2}). \end{aligned}$$

This result clearly yields the following sum-lemma:

Lemma 2.2.4 *Assume that A_*, D_* be two chain complexes. Assume also that $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^D$ are bases of A_p, D_p, $H_p(A_*)$, and $H_p(D_*)$, respectively. Then,*

$$\mathbb{T}(A_* \oplus D_*, \{\mathbf{c}_p^A \oplus \mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^A \oplus \mathbf{h}_p^D\}_0^n) = \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n) \cdot \mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n).$$

Definition 2.2.5 Let $C_* : 0 \to C_{2n} \overset{\partial_{2n}}{\to} C_{2n-1} \to \cdots \to C_n \to \cdots \to C_1 \overset{\partial_1}{\to} C_0 \to 0$ be a chain complex of finite dimensional real or complex vector spaces of length $2n$(n odd). For $p = 0, \ldots, 2n$, let $\omega_{p,2n-p} : C_p \times C_{2n-p} \to \mathbb{F}$ be a $\partial-$compatible anti-symmetric non-degenerate bilinear form. To be more precise,

$$\omega_{p,2n-p}\,(\partial a, b) = (-1)^{p+1}\omega_{p+1,2n-(p+1)}(a, \partial b),$$

$$\omega_{p,2n-p}(a, b) = (-1)^p \omega_{2n-p,p}(b, a).$$

Then, the triple $(C_*, \partial_*, \{\omega_{*,2n-*}\})$ is called an $\mathbb{F}$-*symplectic chain complex.*

Let us note that if C_* is a symplectic chain complex, then $[\omega_{p,2n-p}] : H_p(C_*) \times H_{2n-p}(C_*) \to \mathbb{F}$ defined by $[\omega_{p,2n-p}]([x], [y]) = \omega_{p,2n-p}(x, y)$ is an anti-symmetric and non-degenerate bilinear map.

Definition 2.2.6 Let C_* be a symplectic chain complex of length $2n$ and $\mathbf{c}_p$ be a basis of C_p, $p = 0, \ldots, 2n$. The bases $\mathbf{c}_p, \mathbf{c}_{2n-p}$ of C_p, C_{2n-p}, respectively are called $\omega-$*compatible*, if the matrix of $\omega_{p,2n-p}$ in bases $\mathbf{c}_p$, $\mathbf{c}_{2n-p}$ is the $k \times k$ identity matrix $\mathrm{I}_{k\times k}$ when $p \neq n$ and $\begin{pmatrix} 0_{l\times l} & \mathrm{I}_{l\times l} \\ -\mathrm{I}_{l\times l} & 0_{l\times l} \end{pmatrix}_{2l\times 2l}$ when $p = n$. Here, $k = \dim_{\mathbb{F}} C_p = \dim_{\mathbb{F}} C_{2n-p}$, and $2l = \dim_{\mathbb{F}} C_n$.

Clearly, every symplectic chain complex has $\omega-$compatible bases.

Suppose that C_* is a symplectic chain complex and $\mathbf{h}_p$, $\mathbf{h}_{2n-p}$ are bases of $H_p(C_*)$, $H_{2n-p}(C_*)$, respectively. Let us denote the determinant of the matrix of the non-degenerate pairing $[\omega_{p,2n-p}] : H_p(C_*) \times H_{2n-p}(C_*) \to \mathbb{F}$ in the bases $\mathbf{h}_p$, $\mathbf{h}_{2n-p}$ by

$$\Delta_{p,2n-p}(\mathbf{h}_p, \mathbf{h}_{2n-p}).$$

If there is no ambiguity, we will write $\Delta_{p,2n-p}(C_*)$ instead of $\Delta_{p,2n-p}(\mathbf{h}_p, \mathbf{h}_{2n-p})$.

The following result suggests a formula for computing R-torsion of symplectic chain complexes.

Theorem 2.2.7 *([31, 32]) If C_* is an $\mathbb{F}-$symplectic chain complex of length $2n$, and for $p = 0,\dots,2n$, $\mathbf{c}_p$ are ω-compatible bases of C_p and $\mathbf{h}_p$ are bases of $H_p(C_*)$, respectively, then the following formulas hold:*

- *If C_* is an $\mathbb{R}-$symplectic chain complex, then*

$$\mathbb{T}\left(C_*,\{\mathbf{c}_p\}_{p=0}^{2n},\{\mathbf{h}_p\}_{p=0}^{2n}\right)=\prod_{p=0}^{n-1}\Delta_{p,2n-p}(C_*)^{(-1)^p}\cdot\sqrt{\Delta_{n,n}(C_*)}^{(-1)^n}.$$

- *If C_* is a $\mathbb{C}-$symplectic chain complex, then*

$$|\mathbb{T}\left(C_*,\{\mathbf{c}_p\}_{p=0}^{2n},\{\mathbf{h}_p\}_{p=0}^{2n}\right)|=\prod_{p=0}^{n-1}|\Delta_{p,2n-p}(C_*)|^{(-1)^p}\cdot\sqrt{|\Delta_{n,n}(C_*)|}^{(-1)^n}.$$

Details and unexplained subjects can be found in [31, 32] and references therein.

2.3 Reidemeister Torsion of Representations

Let Σ be a compact surface with genus at least 2 and without boundary. Let us also denote the universal covering of Σ by $\widetilde{\Sigma}$ and the Lie group by G, the Lie algebra of G by $\mathfrak{g}$, and the non-degenerate Killing form on $\mathfrak{g}$ by B. Here, the Lie group G denotes the Lie groups $\mathrm{PSL}(3,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$.

Let $\phi:\pi_1(\Sigma)\to G$ be a homomorphism and $E_\phi=\widetilde{\Sigma}\times\mathfrak{g}/\sim$ be the associated adjoint bundle over Σ. Here, for all $\gamma\in\pi_1(\Sigma)$, $(\gamma\cdot x,\gamma\cdot t)\sim(x,t)$, γ acts in the first component as a deck transformation and in the second component by adjoint action, more precisely $\phi(\gamma)t\phi(\gamma)^{-1}$.

Let K be a cell-decomposition of Σ for which the adjoint bundle E_ϕ is trivial over each cell. Let us denote by $\widetilde{K}$ the lift of K to $\widetilde{\Sigma}$ and let

$$\mathbb{Z}[\pi_1(\Sigma)]=\left\{\sum_{i=1}^{p}m_i\gamma_i\,;m_i\in\mathbb{Z},\ \gamma_i\in\pi_1(\Sigma),\ p\in\mathbb{N}\right\}$$

be the integral group ring. Then, $C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi})$ is defined as $C_*(\widetilde{K};\mathbb{Z})\otimes\mathfrak{g}/\sim$, where, $\sigma\otimes t\sim\gamma\cdot\sigma\otimes\gamma\cdot t,\forall\gamma\in\pi_1(\Sigma)$, the action of $\pi_1(\Sigma)$ on $\widetilde{\Sigma}$ is by the deck transformation, and the action of $\pi_1(\Sigma)$ on $\mathfrak{g}$ is adjoint action.

Clearly, there is the following chain complex:

$$0 \rightarrow C_2(K;\mathfrak{g}_{\mathrm{Ad}_\phi}) \overset{\partial_2\otimes \mathrm{id}}{\longrightarrow} C_1(K;\mathfrak{g}_{\mathrm{Ad}_\phi}) \overset{\partial_1\otimes \mathrm{id}}{\longrightarrow} C_0(K;\mathfrak{g}_{\mathrm{Ad}_\phi}) \longrightarrow 0, \tag{2.5}$$

where ∂_p is the usual boundary operator. Let us denote the homology of the chain complex (2.5) by $H_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi})$. Similarly, $C^*(K;\mathfrak{g}_{\mathrm{Ad}_\phi})$ results the cohomologies $H^*(K;\mathfrak{g}_{\mathrm{Ad}_\phi})$. Recall that $C^*(K;\mathfrak{g}_{\mathrm{Ad}_\phi})$ is the set of $\mathbb{Z}[\pi_1(\Sigma)]$-module homomorphism from $C_*(\widetilde{K};\mathbb{Z})$ to $\mathfrak{g}$. The reader is refered to [30] and the references therein for more information.

Let us consider again the chain complex (2.5). Let $\{e_j^p\}_{j=1}^{m_p}$ be the generators for $C_p(K;\mathbb{Z})$. Fixing a lift $\widetilde{e}_j^p$ of e_j^p in the universal covering $\widetilde{\Sigma}$ of Σ, $j=1,\ldots,m_p$, we get a $\mathbb{Z}[\pi_1(\Sigma)]-$basis $c_p=\{\widetilde{e}_j^p\}_{j=1}^{m_p}$ for $C_p(\widetilde{K};\mathbb{Z})$. Suppose that $\mathcal{A}=\{\mathfrak{a}_k\}_{k=1}^{\dim \mathfrak{g}}$ is a B-orthonormal basis of $\mathfrak{g}$, namely, the matrix of the Killing form B in the basis $\mathcal{A}$ is the diagonal matrix $Diag(1,\overset{p}{\ldots},1,-1,\overset{r}{\ldots},-1)$, where $p+r=\dim_{\mathbb{F}}\mathfrak{g}$. Thus, $\mathbf{c}_p=c_p\otimes_\phi\mathcal{A}$ is an $\mathbb{F}-$basis for $C_p(K;\mathfrak{g}_{\mathrm{Ad}_\phi})$ and called a *geometric basis* for $C_p(K;\mathfrak{g}_{\mathrm{Ad}_\phi})$.

If $\mathbf{h_p}$ is an $\mathbb{F}$-basis of $H_p(K;\mathfrak{g}_{\mathrm{Ad}_\phi})$, $p=0,1,2$, then

$$\mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}),\{c_p\otimes_\phi\mathcal{A}\}_{p=0}^2,\{\mathbf{h}_p\}_{p=0}^2)$$

is called the *R-torsion* of the triple K, Ad_ϕ, and $\{\mathbf{h}_p\}_{p=0}^2$.

It was proved in [18] that the definition of R-torsion does not depend on $\mathcal{A}$, lifts $\widetilde{e}_j^p$, and conjugacy classes of ϕ. But the sake of completeness we shall give the proof of Proposition 2.3.1. For the independence from the cell-decomposition, the reader is referred to [31, Lemma 2.0.5].

In the following proposition, for $G=\mathrm{SL}(3,\mathbb{R})$, we also assume that ϕ is purely loxodromic representation, namely $\phi(\gamma)$ is diagonalizable in $\mathrm{SL}(3,\mathbb{R})$ for all $\gamma\in\pi_1(\Sigma)$.

Proposition 2.3.1 *([18])* $\mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}),\{c_p\otimes_\phi\mathcal{A}\}_{p=0}^2,\{\mathbf{h}_p\}_{p=0}^2)$ *is independent of* $\mathcal{A}$*, lifts* $\widetilde{e}_j^p$*, conjugacy class of* ϕ*, and the cell-decomposition* K*.*

Proof. Let $\mathcal{A}'$ be another $B-$orthonormal basis of $\mathfrak{g}$ and let T be the change-base-matrix from $\mathcal{A}'$ to $\mathcal{A}$. By using change-base-formula Remark (2.2.2), we get:

$$\begin{aligned}\mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}),\{c_p\otimes_\phi \mathcal{A}'\}_0^2,\{\mathbf{h}_p\}_0^2) &= \prod_{p=0}^{2}\left(\frac{[\mathbf{c}'_p,\mathbf{c}_p]}{[\mathbf{h}_p,\mathbf{h}_p]}\right)^{(-1)^p} \\ &\times\ \mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}),\{c_p\otimes_\phi \mathcal{A}\}_0^2,\{\mathbf{h}_p\}_0^2).\end{aligned} \tag{2.6}$$

Here,

$$\prod_{p=0}^{2}[\mathbf{c}'_p,\mathbf{c}_p]^{(-1)^p}=\prod_{p=0}^{2}((\det T)^{(-1)^p\dim C_p})^{-1}=(\det T)^{-\chi(\Sigma)}. \tag{2.7}$$

From the fact that $\mathcal{A}$ and $\mathcal{A}'$ are $B-$orthonormal basis it follows that $\det T=\mp 1$. Combining equations (2.6), (2.7), and using the fact that the Euler-characteristic $\chi(\Sigma)$ is even, $\mathcal{A}$ and $\mathcal{A}'$ will produce the same torsion. Therefore, torsion is independent of the basis $\mathcal{A}$.

We now prove the independence of the torsion from the lifts. Firstly, we can get another lift of $\{e_1^i,\dots,e_{m_i}^i\}$ taking another lift of e_1^i and leave the others the same. Let us denote this lift by $c'_i=\{\widetilde{e}_1^i\bullet\gamma,\dots,\widetilde{e}_{m_i}^i\}$. Since $\widetilde{e}_1^i\bullet\gamma\otimes t=\widetilde{e}_1^i\otimes\gamma\bullet t$, where the action in the second place is by $\mathrm{Ad}_{\phi(\gamma)}$, namely conjugation by $\phi(\gamma)$. Then, we have $c'_i\otimes\mathcal{A}=c_i\otimes\mathrm{Ad}_{\phi(\gamma)}(\mathcal{A})$. From Change-base-formula (2.2.2) and equation (2.7) it follows that

$$\begin{aligned}\mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}),\{c'_i\otimes_\phi \mathcal{A}\}_0^2,\{\mathbf{h}_i\}_0^2) &= \prod_{i=0}^{2}\left(\frac{[\mathbf{c}'_i,\mathbf{c}_i]}{[\mathbf{h}_i,\mathbf{h}_i]}\right)^{(-1)^i} \\ &\quad\times\mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}),\{c_i\otimes_\phi \mathcal{A}\}_0^2,\{\mathbf{h}_i\}_0^2) \\ &= (\det T)^{-\chi(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}))} \\ &\quad\times\mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}),\{c_i\otimes_\phi \mathcal{A}\}_0^2,\{\mathbf{h}_i\}_0^2).\end{aligned}$$

Here, T is the matrix of $\mathrm{Ad}_{\phi(\gamma)}:\mathfrak{g}\longrightarrow\mathfrak{g}$ with respect to basis $\mathcal{A}$. The fact that $\phi(\gamma)\in\mathfrak{g}$ it follows that $\phi(\gamma)$ can be diagonalizable. To be more precisely, there exist a $Q=Q(\gamma)\in G$ so that $Q\phi(\gamma)Q^{-1}=D=Diag(\lambda_1,\dots,\lambda_N)$, where λ_i's are eigenvalues of $\phi(\gamma)$. Using this, we obtain the following equality:

$$\mathrm{Ad}_{\phi(\gamma)}=\mathrm{Ad}_{Q^{-1}DQ}=(\mathrm{Ad}_Q)^{-1}\circ\mathrm{Ad}_D\circ\mathrm{Ad}_Q.$$

From this fact, once we fix a basis for $\mathfrak{g}$, to compute the determinant of $\mathrm{Ad}_{\phi(\gamma)}$ it suffices to find the determinant of Ad_D. To do this, since the determinant of Ad_D is independent of basis of $\mathfrak{g}$, we will consider the following basis:

$$\mathfrak{sl}(2,\mathbb{C}) \to \begin{cases} E_{12}, \\ E_{21}, \\ E_{11} - E_{22}, \end{cases}$$

and

$$\mathfrak{sl}(n,\mathbb{R}) \to \begin{cases} E_{kj}, & k \neq j, \\ E_{kk} - E_{k+1,k+1}, & 1 \leq k \leq n-1. \end{cases}$$

Here, E_{ij} denotes the matrix with 1 in the ij entry and 0 elsewhere. Thus, by using given basis one can easily find the determinant of Ad_D and see that this matrix has determinant 1. Finally, since $\det T = 1$ then we have the same torsion.

Independence of conjugacy class of ϕ: If ϕ, ϕ' are conjugate represantation, then the corresponding twisted chains and cochains are isomorphic. Therefore, ϕ and ϕ' will produce the same torsion.

This finishes the proof of Proposition 2.3.1.

$\square$

Since R-torsion of representations is invariant under subdivision, instead of $\mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}), \{\mathbf{c}_p\}_0^2, \{\mathbf{h}_p\}_0^2)$ we can write $\mathbb{T}(\Sigma, \{\mathbf{h}_p\}_0^2)$.

Before writing the Reidemeister torsion formula for representations, we will give some definitions: Kronecker pairing, cup product and intersection forms. Recall that let Σ be a compact hyperbolic surface, $\phi : \pi_1(\Sigma) \to G$ be a homomorphism, and let K be a cell decomposition of Σ. We associated the twisted chains $C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi})$ and cochains $C^*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}) = \mathrm{Hom}_{\mathbb{Z}[\pi_1(\Sigma)]}(C_*(\widetilde{K};\mathbb{Z}), \mathfrak{g})$. Here, $\widetilde{K}$ is the lift of K to the universal covering $\widetilde{\Sigma}$ of Σ.

Definition 2.3.2 The *Kronecker pairing*

$$\langle \cdot, \cdot \rangle : C^i\left(K;\mathfrak{g}_{\mathrm{Ad}_\phi}\right) \times C_i\left(K;\mathfrak{g}_{\mathrm{Ad}_\phi}\right) \longrightarrow \mathbb{F}$$

is defined by

$$\langle \theta, \sigma \otimes_\phi t \rangle = B\left(t, \theta\left(\sigma\right)\right),$$

where B denotes the Cartan-Killing form.

Clearly, the pairing can be extended to

$$\langle \cdot, \cdot \rangle : H^i\left(\Sigma; \mathfrak{g}_{\mathrm{Ad}_\phi}\right) \times H_i\left(\Sigma; \mathfrak{g}_{\mathrm{Ad}_\phi}\right) \longrightarrow \mathbb{F}.$$

Definition 2.3.3 The *cup product*

$$\smile_B : C^i(K; \mathfrak{g}_{\mathrm{Ad}_\phi}) \times C^j(K; \mathfrak{g}_{\mathrm{Ad}_\phi}) \to C^{i+j}(\widetilde{\Sigma}; \mathbb{F})$$

defined by $(\theta_i \smile_B \theta_j)(\sigma_{i+j}) = B(\theta_i((\sigma_{i+j})_{\mathrm{front}})), \theta_j((\sigma_{i+j})_{\mathrm{back}})$.

Note that $\smile_B$ can be extended

$$\smile_B : H^i(\Sigma; \mathfrak{g}_{\mathrm{Ad}_\phi}) \times H^j(\Sigma; \mathfrak{g}_{\mathrm{Ad}_\phi}) \to H^{i+j}(\Sigma; \mathbb{F}).$$

Assume that K' is the dual cell-decomposition of Σ associated to the cell-decomposition K. Assume also that cells $\sigma \in K$, $\sigma' \in K'$ meet at most once, this assumption is not loss of generality because of the invariance of R-torsion under subdivision. Let us denote by c'_p the basis of $C_p(\widetilde{K'}; \mathbb{Z})$ associated to the basis c_p of $C_p(\widetilde{K}; \mathbb{Z})$, and also by $\mathbf{c}'_p = c'_p \otimes_\phi \mathcal{A}$ the basis for $C_p(K'; \mathfrak{g}_{\mathrm{Ad}_\phi})$, where $\mathcal{A}$ is a B-orthonormal basis of the Lie algebra $\mathfrak{g}$ of G.

Definition 2.3.4 The *intersection form*

$$(\cdot, \cdot)_{i,2-i} : C_i(K; \mathfrak{g}_{\mathrm{Ad}_\phi}) \times C_{2-i}(K'; \mathfrak{g}_{\mathrm{Ad}_\phi}) \to \mathbb{F}$$

defined by

$$(\sigma_1 \otimes t_1, \sigma_2 \otimes t_2)_{i,2-i} = \sum_{\gamma \in \pi_1(\Sigma)} \sigma_1.(\gamma \cdot \sigma_2)\ B(t_1, \gamma \cdot t_2).$$

Here, "." denotes the intersection number pairing. Clearly, "." is compatible with the usual boundary operator and thus $(\cdot, \cdot)_{i,2-i}$ are $\partial-$compatible. It is also anti-symmetric, because of the fact that intersection number form "." is anti-symmetric and B is invariant under adjoint action.

We can naturally extend the intersection form to twisted homologies. From the fact that twisted homologies are independent of the cell-decomposition, we get the following non-degenerate form

$$(\cdot, \cdot)_{i,2-i} : H_i(\Sigma; \mathfrak{g}_{\mathrm{Ad}_\phi}) \times H_{2-i}(\Sigma; \mathfrak{g}_{\mathrm{Ad}_\phi}) \to \mathbb{F}. \tag{2.8}$$

The isomorphisms induced by the Kronecker pairing and the intersection form yield the Poincaré duality isomorphisms

$$\mathrm{PD}: H^{2-i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) \overset{\text{Kronecker pairing}}{\cong} H_{2-i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi})^* \overset{\text{Intersection form}}{\cong} H_i(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}).$$

For $i=0,1,2$, there is the following commutative diagram

$$\begin{array}{ccccc} H^{2-i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) & \times & H^{i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) & \stackrel{\smile_B}{\longrightarrow} & H^2(\Sigma;\mathbb{F}) \\ \downarrow \mathrm{PD} & & \downarrow \mathrm{PD} & \circlearrowleft & \downarrow \int_\Sigma \\ H_{i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) & \times & H_{2-i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) & \stackrel{(,)_{i,2-i}}{\longrightarrow} & \mathbb{F}. \end{array}$$

Therefore, for $i=0,1,2$,

$$\omega_{2-i,i}: H^{2-i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) \times H^{i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) \stackrel{\smile_B}{\longrightarrow} H^2(\Sigma;\mathbb{F}) \stackrel{\int_\Sigma}{\longrightarrow} \mathbb{F} \tag{2.9}$$

is a dual pairing.

In the case of ϕ is irreducible we have $H_0(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi})$, $H_2(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi})$, $H^0(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi})$, and $H^2(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi})$ are all zero. Hence,

$$\begin{array}{ccccc} H^{1}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) & \times & H^{1}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) & \stackrel{\smile_B}{\longrightarrow} & H^2(\Sigma;\mathbb{F}) \\ \downarrow \mathrm{PD} & & \downarrow \mathrm{PD} & \circlearrowleft & \downarrow \int_\Sigma \\ H_{1}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) & \times & H_{1}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) & \stackrel{(,)_{1,1}}{\longrightarrow} & \mathbb{F}. \end{array} \tag{2.10}$$

Recall that $\omega_G: H^1(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) \times H^1(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) \stackrel{\smile_B}{\longrightarrow} H^2(\Sigma;\mathbb{F}) \stackrel{\int_\Sigma}{\longrightarrow} \mathbb{F}$ is called the Atiyah-Bott-Goldman symplectic form for the Lie group G.

Theorem 2.3.5 *([18]) If D_* denotes $C_*(K;\mathfrak{g}_{\mathrm{Ad}_\phi}) \oplus C_*(K';\mathfrak{g}_{\mathrm{Ad}_\phi})$, then it is a symplectic chain complex with ω-compatible bases, which are obtained from the geometric bases.*

Since the product of the determinant of the matrix associted to

$$(\cdot,\cdot)_{i,2-i}: H_i(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) \times H_{2-i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) \to \mathbb{F} \tag{2.11}$$

for basis $\mathbf{h}_i, \mathbf{h}_{2-i}$ and the determinant of the matrix associated to

$$\omega_{2-i,i}: H^{2-i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) \times H^{i}(\Sigma;\mathfrak{g}_{\mathrm{Ad}_\phi}) \stackrel{\smile_B}{\longrightarrow} H^2(\Sigma;\mathbb{F}) \stackrel{\int_\Sigma}{\longrightarrow} \mathbb{F}. \tag{2.12}$$

for basis $\mathbf{h}^i, \mathbf{h}^{2-i}$ is equal 1, one can easily get rid of the coefficient in [32, Theorem 4.4]. For more detail about this see [35, 36]. Using this fact the formula in [32] turns into following theorem:

Theorem 2.3.6 *Let Σ be a closed oriented Riemann surface of genus at least 2 and $\phi : \pi_1(\Sigma) \to G$ be an irreducible, purely loxodromic representation. Let K be a cell decomposition of Σ, $\mathbf{c}_p$ the geometric basis of $C_p(K; \mathfrak{g}_{\mathrm{Ad}_\phi})$. Then the following formulas hold:*

- *If the Lie group G is* $\mathrm{PSL}(3, \mathbb{R})$*, then*

$$\mathrm{Tor}(\Sigma, \{0, \mathbf{h}^1, 0\}) = \sqrt{\det \begin{bmatrix} \omega_G \\ \mathbf{h}^1 \end{bmatrix}}.$$

- *If the Lie group G is* $\mathrm{PSL}(2, \mathbb{C})$*, then*

$$\left|\mathrm{Tor}(\Sigma, \{0, \mathbf{h}^1, 0\})\right| = \sqrt{\left|\det \begin{bmatrix} \omega_G \\ \mathbf{h}^1 \end{bmatrix}\right|}.$$

3 REAL PROJECTIVE STRUCTURES

The real projective plane, denoted by $\mathbb{RP}^2$, is a very well-known object for many reasons. It can be the simplest example of a closed non-orientable surface. If we remove a disc from the real projective plane, then we get another familiar non-orientable surface the Möbius band. It is also the unique non-orientable surface with Euler characteristic equal to 1. The real projective plane is one of the first examples of a non-Euclidean geometry. Therefore, it is an elementary example in topology or algebraic geometry.

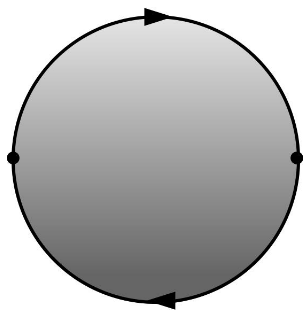


Figure 3.1: The real projective plane

One can define the real projective plane in two different point of view. The first one is topologically. It can be described as the quotient space of the closed disc by identifying opposite points on the boundary. The other one is geometrically. It can be described as the space of lines through the origin in 3-space.

Let us consider an open subset Ω of the real projective plane $\mathbb{RP}^2$. A map $\Phi : \Omega \to \mathbb{RP}^2$ is called *locally projective* if for each component $W \subset \Omega$, there is a projective transformation $g \in \mathrm{PGL}(3, \mathbb{R})$ such that the restriction $\Phi|_W$ equals the restriction $g|W$. Distinctly a locally projective map is a local diffeomorphism.

Let Σ be a connected smooth surface. An $\mathbb{RP}^2$-*atlas* on Σ is a collection of coordinate charts $\{\phi_i : U_i \to \mathbb{RP}^2\}_{i \in I}$ satisfying the following:

- $\{U_i\}$ is an open covering of Σ,
- Each ϕ_i is a diffeomorphism $U_i \to \phi_i(U_i)$,

- For each U_i and U_j, the coordinate change function

$$\phi_j\phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

  is locally projective.

A maximal $\mathbb{RP}^2$-atlas on Σ is called a *real projective structure* (or $\mathbb{RP}^2$-structure) on Σ and a manifold with an $\mathbb{RP}^2$-structure is called an $\mathbb{RP}^2$-*manifold*.

Let $f : M \to N$ be a smooth map, where M and N are $\mathbb{RP}^2$-manifolds. If for each coordinate chart (U_i, ϕ_i) on M and each (U_j, ϕ_j) on N, the composition

$$\phi_j \circ f \circ \phi_i^{-1} : \phi_i(U_i \cap f^{-1}(U_j)) \to \phi_j(U_j \cap f(U_i))$$

is a locally projective map, then f is called a *projective map*(or $\mathbb{RP}^2$-map).

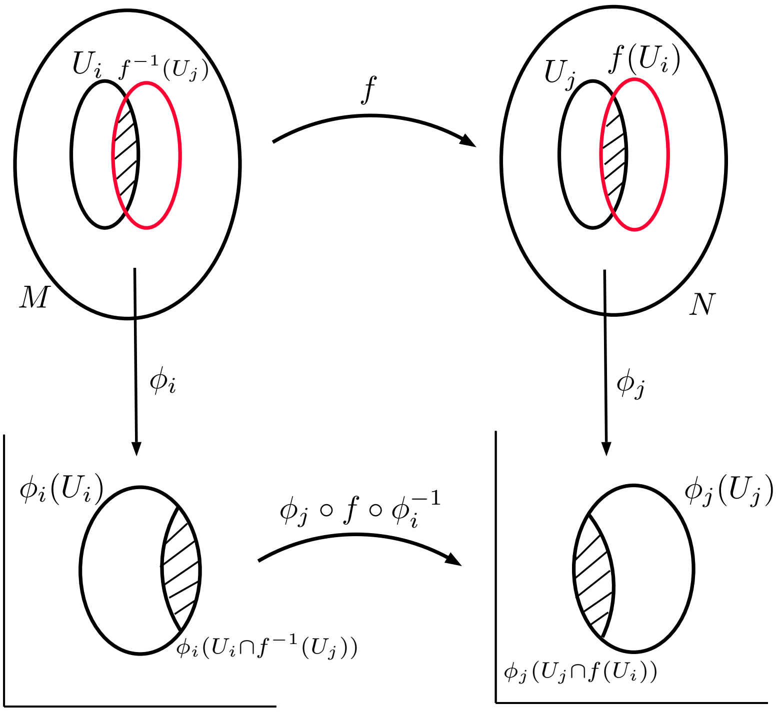


Figure 3.2: A projective map

An $\mathbb{RP}^2$-map between $\mathbb{RP}^2$-manifolds is necessarily a local diffeomorphism. Contrarily, if $f : M \to N$ is a smooth map which is a local diffeomorphism, and N is an $\mathbb{RP}^2$-manifold, there is a unique $\mathbb{RP}^2$-structure on M such that f is an $\mathbb{RP}^2$-map with respect to these structures.

Now, we will recall the following well-known basic theorem.

Theorem 3.0.1 *[37] Let $p : \widetilde{M} \to M$ be a universal covering map of an $\mathbb{RP}^2$-manifold M and π denote the corresponding group of covering transformations.*

1. *There exists a projective map* $\mathrm{dev} : \widetilde{M} \to \mathbb{RP}^2$ *and a homomorphism $h : \pi \to$* $\mathrm{SL}(3, \mathbb{R})$ *such that for each $\gamma \in \pi$ the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\mathrm{dev}} & \mathbb{RP}^2 \\ \gamma\downarrow & & \downarrow h(\gamma) \\ \widetilde{M} & \xrightarrow{\mathrm{dev}} & \mathbb{RP}^2 \end{array}$$

2. *Let* (dev', h') *be another pair satisfying above conditions. Then there exists a projective transformation $g \in$* $\mathrm{SL}(3, \mathbb{R})$ *such that* $\mathrm{dev}' = g \circ \mathrm{dev}$ *and $h' = \iota_g \circ h$ where ι_g :* $\mathrm{SL}(3, \mathbb{R}) \to \mathrm{SL}(3, \mathbb{R})$ *denotes the inner automorphism defined by g, namely $h'(\gamma) = (\iota_g \circ h)(\gamma) = g \circ h(\gamma) \circ g^{-1}$:*

$$\begin{array}{ccccc} \widetilde{M} & \xrightarrow{\mathrm{dev}} & \mathbb{RP}^2 & \xrightarrow{g} & \mathbb{RP}^2 \\ \gamma\Big\downarrow & & \Big\downarrow h(\gamma) & & \Big\downarrow h'(\gamma) \\ \widetilde{M} & \xrightarrow{\mathrm{dev}} & \mathbb{RP}^2 & \xrightarrow{g} & \mathbb{RP}^2 \end{array}$$

The projective map $\mathrm{dev} : \widetilde{M} \to \mathbb{RP}^2$ is called a *developing map* and $h : \pi \to \mathrm{SL}(3, \mathbb{R})$ is called the *holonomy homomorphism.* The image $\Gamma = h(\gamma)$ is called the *holonomy group.*

Definition 3.0.2 A domain Ω in $\mathbb{RP}^2$ is called *convex*, if the following two conditions are satisfied:

- There exists a projective line $l \subset \mathbb{RP}^2$ such that $\Omega \subset \mathbb{RP}^2 - l$,
- For all $x, y \in \Omega$, the line segment $\overline{xy}$ lies in Ω.

As is well-known that a discrete group Γ is a topological group in which the topology is discrete. Recall also that a discrete group Γ *acts properly* on Ω if for every compact sets $A, B \subset \Omega$, the following set

$$\Gamma(A, B) = \{g \in \Gamma | gA \cap B \neq \emptyset\}$$

is finite. Moreover, if a discrete group Γ has a trivial stabilizer subgroup namely $\Gamma_x = \{g \in \Gamma | gx = x\}$ is trivial then it is called *acting freely* on Ω.

3.1 Goldman Coordinates of Deformation Space $\mathcal{B}(\Sigma)$

In this section, we are going to first consider Goldman's very significant article [19] which gives the parametrization of the deformation space of convex $\mathbb{RP}^2$-structures on Σ and then the Hong Chan Kim's paper [20].

Let $\mathbb{RP}^2$ be the real projective plane and $\mathrm{PGL}(3, \mathbb{R})$ the group of projective transformations $\mathbb{RP}^2 \to \mathbb{RP}^2$. A convex $\mathbb{RP}^2$-manifold is a quotient $M = \Omega/\Gamma$ where Ω is a convex domain in $\mathbb{RP}^2$ and Γ is a discrete group of $\mathrm{PGL}(3, \mathbb{R})$ acting properly on Ω. Let us consider the two such quotients $M_1 = \Omega_1/\Gamma_1$ and $M_2 = \Omega_2/\Gamma_2$. They are *projectively equivalent*, if there exists a projective transformation $h \in \mathrm{PGL}(3, \mathbb{R})$ such that $h(\Omega_1) = \Omega_2$ and $h\Gamma_1 h^{-1} = \Gamma_2$.

Let Σ be a closed smooth surface and M be a convex $\mathbb{RP}^2$-manifold. A *convex* $\mathbb{RP}^2$-*structure* is an equivalence class $[(f, M)]$, where $f : \Sigma \to M$ is a diffeomorphism and two such pairs (f, M) and (f', M') are regarded as equivalent if there exists a projective equivalence $h : M \to M'$ such that $h \circ f$ isotopic to f'. For a convex $\mathbb{RP}^2$-structure on Σ, the action of the fundamental group $\pi_1(\Sigma)$ by deck transformations on the universal covering space of Σ determines a homomorphism $\phi : \pi_1(\Sigma) \to \mathrm{PGL}(3, \mathbb{R})$ which is well defined up to conjugacy in $\mathrm{PGL}(3, \mathbb{R})$. Up to above equivalence relation of convex $\mathbb{RP}^2$-structures on Σ, the equivalence classes has a natural topology which can be identified with an open subspace of the representations space $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbb{R}))/\mathrm{PGL}(3, \mathbb{R})$. This space is called *the deformation space of convex* $\mathbb{RP}^2$*-structures on* Σ and denoted by $\mathcal{B}(\Sigma)$. In [19], Goldman determines explicit coordinates on this space and proved the following theorem:

Theorem 3.1.1 *([19]) Let Σ be a closed orientable surface of genus $g > 1$. Then the deformation space $\mathcal{B}(\Sigma)$ of convex $\mathbb{RP}^2$-structures on Σ is diffeomorphic to an open cell of dimension $16(g-1)$.*

In this part we will give the necessary information for the parametrization of the deformation space $\mathcal{B}(\Sigma)$. For unexplained subjects and more details see [19].

One can consider the real projective plane as a space of all lines through the origin in $\mathbb{R}^3$. Therefore if $(x, y, z) \in \mathbb{R}^3 - \{0\}$ is a nonzero vector in $\mathbb{R}^3$ the corresponding point in $\mathbb{RP}^2$ will be denoted

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

in homogeneous coordinates.

Let A be an invertible element of the group of linear transformations of $\mathbb{R}^3$, namely $\mathrm{GL}(3, \mathbb{R})$. Then A preserves lines through the origin and induces a projective transformation of $\mathbb{RP}^2$. Recall that $\mathrm{PGL}(3, \mathbb{R})$ is the group of projective transformations of $\mathbb{RP}^2$. Clearly, we have the following exact sequence

$$\{1\} \to \mathbb{R}^* \to \mathrm{GL}(3, \mathbb{R}) \to \mathrm{PGL}(3, \mathbb{R}) \to \{1\},$$

where the scalar matrices $\mathbb{R}^*$ in $\mathrm{GL}(3, \mathbb{R})$ act trivially on $\mathbb{RP}^2$. The analytic homomorphism from $\mathrm{GL}(3, \mathbb{R})$ to $\mathrm{SL}(3, \mathbb{R})$ defined by

$$A \mapsto \frac{A}{(\det A)^{1/3}}$$

defines an isomorphism from $\mathrm{PGL}(3, \mathbb{R})$ to $\mathrm{SL}(3, \mathbb{R})$ as analytic groups. Thus, one can consider only the group $\mathrm{SL}(3, \mathbb{R})$.

Let us consider the three points

$$p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

corresponding to the coordinate axes in $\mathbb{R}^3$ and the three lines $l_1 = \overleftrightarrow{p_2 p_3}$, $l_2 = \overleftrightarrow{p_3 p_1}$, $l_3 = \overleftrightarrow{p_1 p_2}$, correspond to the coordinate planes and divide the real projective plane $\mathbb{RP}^2$ into four triangular regions:

$$\begin{aligned}
\Delta_0 &= \{[x, y, z] \in \mathbb{RP}^2 \mid x > 0, y > 0, z > 0\}, \\
\Delta_1 &= \{[x, y, z] \in \mathbb{RP}^2 \mid x < 0, y > 0, z > 0\}, \\
\Delta_2 &= \{[x, y, z] \in \mathbb{RP}^2 \mid x > 0, y < 0, z > 0\}, \\
\Delta_3 &= \{[x, y, z] \in \mathbb{RP}^2 \mid x > 0, y > 0, z < 0\}.
\end{aligned}$$

If a projective transformation $A \in \mathrm{SL}(3,\mathbb{R})$ fixes the points p_1, p_2, p_3 then, it is represented by a unique diagonal matrix in $\mathrm{SL}(3,\mathbb{R})$ and it leaves invariant one triangular region Δ_i if and only if it is represented by a diagonal matrix with positive eigenvalues. The full group of diagonal matrices in $\mathrm{SL}(3,\mathbb{R})$ is denoted by $\mathcal{A}$ and also the subgroup of diagonal matrices with positive eigenvalues by $\mathcal{A}_+$.

If an element of $\mathrm{SL}(3,\mathbb{R})$ has three distinct real eigenvalues then it is called *hyperbolic.* Moreover, if it is conjugate in $\mathrm{SL}(3,\mathbb{R})$ to a diagonal matrix with positive eigenvalues then, it is called *positive hyperbolic.* Let us denote this subset of $\mathrm{SL}(3,\mathbb{R})$ by Hyp_+.

Let us consider $A \in \mathrm{Hyp}_+$, then it is represented by the diagonal matrix

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \tag{3.1}$$

with the properties $\lambda\mu\nu = 1$ and $0 < \lambda < \mu < \nu$. The real eigenvalue of A having the smallest absolute value is denoted by $\lambda(A)$ and sum of the other two eigenvalues is denoted by $\tau(A)$. Namely, $\lambda(A) = \lambda$ and $\tau(A) = \mu + \nu$.

One can easily show that $A \in \mathrm{Hyp}_+$ is determined up to $\mathrm{SL}(3,\mathbb{R})$-conjugacy by the set of eigenvalues of A which are

$$\begin{aligned} \lambda &= \lambda(A), \\ \mu &= \frac{1}{2}\left[\tau(A) - \sqrt{\tau(A)^2 - \frac{4}{\lambda(A)}}\right], \\ \nu &= \frac{1}{2}\left[\tau(A) + \sqrt{\tau(A)^2 - \frac{4}{\lambda(A)}}\right]. \end{aligned}$$

Therefore, we get a complete invariant of the $\mathrm{SL}(3,\mathbb{R})$-conjugacy class of A, namely $(\lambda(A), \tau(A))$.

Proposition 3.1.2 *([19]) If we consider the action of* $\mathrm{SL}(3,\mathbb{R})$ *on* Hyp_+ *by conjugation, then the restriction of* $\mathrm{SL}(3,\mathbb{R}) \to \mathbb{R}^2$ *with* $A \mapsto (\lambda(A), \tau(A))$ *to* Hyp_+ *is a* $\mathrm{SL}(3,\mathbb{R})$*-invariant fibration with image the following region*

$$\mathfrak{R} = \left\{ (\lambda, \tau) \in \mathbb{R}^2 \,|\, 0 < \lambda < 1, \frac{2}{\sqrt{\lambda}} < \tau < \lambda + \frac{1}{\lambda^2} \right\}$$

and moreover, $\mathrm{Hyp}_+ = (\lambda, \tau)^{-1}(\mathfrak{R})$.

Let us now recall another pair of invariants (ℓ, m) of Hyp_+ which is more closely related to the geometry of convex $\mathbb{RP}^2$-manifolds. If $A \in \mathrm{Hyp}_+$ as in (3.1) with the properties $\lambda\mu\nu = 1$ and $0 < \lambda < \mu < \nu$, then $\ell(A), m(A)$ are defined as follows

$$\begin{aligned} \ell(A) &= \ln\left(\frac{\nu}{\lambda}\right) > 0, \\ m(A) &= 3\ln(\mu). \end{aligned} \tag{3.2}$$

Using the definitions, one can easily show that the conditions $\lambda\mu\nu = 1, 0 < \lambda < \mu < \nu$ are equivalent to the conditions

$$\begin{aligned} \ell(A) &> 0 \\ \ell(A) &> |m(A)|. \end{aligned}$$

And also the relation between these two invariant pairs can seen as follows

$$\begin{aligned} \lambda(A) &= \exp\left(-\frac{\ell(A)}{2} - \frac{m(A)}{6}\right), \\ \tau(A) &= \exp\left(\frac{\ell(A)}{2} - \frac{m(A)}{6}\right) + \exp\left(\frac{m(A)}{3}\right). \end{aligned} \tag{3.3}$$

The correspondence in (3.3) between $(\lambda(A), \tau(A))$ and $(\ell(A), m(A))$ defines a diffeomorphism

$$\mathfrak{R} \leftrightarrow \{(\ell, m) \in \mathbb{R}_+ \times \mathbb{R} \mid |m| < \ell\}$$

giving another set of parameters for conjugacy classes in Hyp_+.

Let us consider $A \in \mathrm{Hyp}_+$ which is represented by a diagonal matrix (3.1). Let us denote the fixed point corresponding to the eigenvector for λ as a repelling fixed point $\mathrm{Fix}_-(A)$, the fixed point corresponding to the eigenvector for ν as an attracting fixed point $\mathrm{Fix}_+(A)$, and the fixed point corresponding to the eigenvector for μ as a saddle point $\mathrm{Fix}_0(A)$. Let $\mathrm{Fix}(A)$ denotes the stationary set consisting of these three points.

Let $l(A) \subset \mathbb{RP}^2$ denote the *principle line* for A, namely the line joining the attracting and repelling fixed points of A. The *principal reflection* for A is the unique reflection $R \in \mathrm{SL}(3, \mathbb{R})$ with stationary set $\mathrm{Fix}(R) = l(A) \cup \mathrm{Fix}(A)$. Clearly, R commutes with A. Finally, the *principle segments* for A are two A-invariant segments which are the separation of $l(A)$ by two fixed points of A on $l(A)$.

Definition 3.1.3 The complement of a projective line l in $\mathbb{RP}^2$ is called as an *affine space* $\mathbb{A}$ in $\mathbb{RP}^2$ and the intersection of a projective line l' distinct from l with the affine space $\mathbb{A} = \mathbb{RP}^2 - l$ is called as *affine line* in $\mathbb{RP}^2$.

Definition 3.1.4 Let S be a subset of $\mathbb{RP}^2$. If there exists an affine space $\mathbb{A}$ in $\mathbb{RP}^2$ containing S such that S is convex in the usual sense (namely, if $x, y \in S$ then the line segment $\overline{xy}$ lies in S), then S is *convex.*

Lemma 3.1.5 *([19]) If $A \in \mathrm{Hyp}_+$ and $x \in \mathbb{RP}^2$ does not lie on an A-invariant line, then the closure of any convex set containing the $< A >$-orbit of x contains a principle segment for A.*

Now, let us consider the following matrix for $s \in \mathbb{R}$

$$A^s = \begin{bmatrix} \lambda^s & 0 & 0 \\ 0 & \mu^s & 0 \\ 0 & 0 & \nu^s \end{bmatrix}, \tag{3.4}$$

and a point in $\mathbb{RP}^2$ with homogeneous coordinates as

$$p_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix},$$

where x_0, y_0, z_0 are positive. The matrix in (3.1) lies on a unique one-parameter subgroup comprised of elements in (3.4). If we choose a projective line l_∞ which is not meeting the triangular region

$$\Delta_0 = \{[x, y, z] \in \mathbb{RP}^2 \mid x > 0, y > 0, z > 0\},$$

then the convex hull of the orbit $\{A^s(p_0) \mid s \in \mathbb{R}\}$ in $\mathbb{RP}^2 - l_\infty$ equals

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{RP}^2 \mid x, y, z > 0, \left(\frac{x}{x_0}\right)^{\ln(\nu/\mu)} \left(\frac{z}{z_0}\right)^{\ln(\mu/\lambda)} > \left(\frac{y}{y_0}\right)^{\ln(\nu/\lambda)} \right\}.$$

Here, the action of A^s to the point p_0 is matrix multiplication

$$A^s(p_0) = \begin{bmatrix} \lambda^s & 0 & 0 \\ 0 & \mu^s & 0 \\ 0 & 0 & \nu^s \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \lambda^s x_0 \\ \mu^s y_0 \\ \nu^s z_0 \end{bmatrix}.$$

In general, the families of $< A >$-invariant convex sets is defined as

$$W_\eta = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{RP}^2 \mid x, y, z \geq 0, x^{\ln(\nu/\mu)} z^{\ln(\mu/\lambda)} \geq \eta y^{\ln(\nu/\lambda)} \right\}$$

for each $\eta > 0$.

If we also consider the one-parameter subgroup of the diagonal matrices in $\mathrm{SL}(3, \mathbb{R})$ comprised of elements

$$B^s = \begin{bmatrix} e^{-s} & 0 & 0 \\ 0 & e^{2s} & 0 \\ 0 & 0 & e^{-s} \end{bmatrix} \tag{3.5}$$

for $s \in \mathbb{R}$, then B^s commutes with A. The line segments joining $\mathrm{Fix}_0(A)$ to the principal line of A are the orbits of the one-parameter subgroup $\{B^s \mid s \in \mathbb{R}\}$. Moreover, B^s maps the convex set W_η to $W_{\eta'}$ where

$$\eta' = (\nu/\lambda)^{-3s}\eta.$$

Finally, the invariant $\ell(A)$ in (3.2) can be interpreted geometrically. For a principal segment σ for A and an $x \in \sigma$, then the cross-ratio of the following four points

$$\mathrm{Fix}_- A, x, A(x), \mathrm{Fix}_+(A)$$

on the principal line l for A equals $e^{\ell(A)}$. If boundary of a convex domain Ω is a conic, then the Hilbert metric is the hyperbolic metric, and $\ell(A)$ equals the geodesic length displacement function. Let us now briefly explain the cross-ratio and Hilbert distance.

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex numbers and $\mathcal{D}_4(\widehat{\mathbb{C}}) \subset \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ denotes the set of all distinct four points. The mapping $X : \mathcal{D}_4(\widehat{\mathbb{C}}) \to \widehat{\mathbb{C}}$ defined by

$$X\{w_1, w_2; w_3, w_4\} = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_2)(w_3 - w_4)},$$

is called *cross-ratio* and it is invariant under the $\mathrm{GL}(2,\mathbb{C})$-action on $\mathcal{D}_4(\widehat{\mathbb{C}})$. If a_1, a_2, a_3, a_4 are collinear (i.e. they lie on a single straight line) distinct four points of $\mathbb{RP}^2$, then there is $B \in \mathrm{SL}(3,\mathbb{R})$ such that the second homogeneous coordinate of each $B(a_i)$ is zero for $i = 1,2,3,4$. If we consider the following identification

$$\begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{cases} x/z & \text{if} \quad z \neq 0, \\ \infty & \text{if} \quad z = 0, \end{cases}$$

we can think of $B(a_i) \in \mathbb{R} \cup \{\infty\}$ the extended real line and moreover, they are distinct. Let $\mathcal{CD}_4(\mathbb{RP}^2) \subset \mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$ consist of all collinear distinct four points. Then the cross-ratio $\mathrm{CR} : \mathcal{CD}_4(\mathbb{RP}^2) \to \mathbb{R}$ is defined by

$$\begin{aligned} \mathrm{CR}\{a_1, a_2; a_3, a_4\} &= X\{B(a_1), B(a_2), B(a_3), B(a_4)\} \\ &= X\left\{ \begin{bmatrix} a_1 \\ 0 \\ a_1' \end{bmatrix}, \begin{bmatrix} a_2 \\ 0 \\ a_2' \end{bmatrix}, \begin{bmatrix} a_3 \\ 0 \\ a_3' \end{bmatrix}, \begin{bmatrix} a_4 \\ 0 \\ a_4' \end{bmatrix} \right\} \\ &= X\left\{ \frac{a_1}{a_1'}, \frac{a_2}{a_2'}, \frac{a_3}{a_3'}, \frac{a_4}{a_4'} \right\} \\ &= \frac{\left(\frac{a_1}{a_1'} - \frac{a_3}{a_3'}\right)\left(\frac{a_2}{a_2'} - \frac{a_4}{a_4'}\right)}{\left(\frac{a_1}{a_1'} - \frac{a_2}{a_2'}\right)\left(\frac{a_3}{a_3'} - \frac{a_4}{a_4'}\right)}. \end{aligned}$$

Suppose that there is another such $B' \in \mathrm{SL}(3,\mathbb{R})$, then one can show that $B^{-1}B' \in \mathrm{SL}(2,\mathbb{R})$ via the identification

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \leftrightarrow \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}.$$

Therefore, the cross-ratio on $\mathcal{CD}_4(\mathbb{RP}^2)$ is independent of the choice of $B \in \mathrm{SL}(3,\mathbb{R})$.

The *Hilbert distance* $h : \mathbb{RP}^2 \times \mathbb{RP}^2 \to \mathbb{R}^+$ is defined by

$$h(a,b) = \inf_{\overrightarrow{xy}} (ln\, \mathrm{CR}\{x, a, b, y\}).$$

Here, a, b lie on the oriented segment $\overrightarrow{xy}$ with a is the first, b the second point and inf runs over all such $\overrightarrow{xy}$. Recall that if boundary of a convex domain Ω is a conic, then the Hilbert distance defines a metric and it is called the *Hilbert metric.* Furthermore, it is an hyperbolic metric.

Let $A \in \mathrm{Hyp}_+$. It can be uniquely decomposed as $A = HV$ product of two matrices up to $\mathrm{SL}(3, \mathbb{R})$-conjugation. Here,

$$H = \begin{bmatrix} \lambda\sqrt{\mu} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu\sqrt{\mu} \end{bmatrix}$$

is called *horizantal* factor and

$$V = \begin{bmatrix} \frac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \frac{1}{\sqrt{\mu}} \end{bmatrix}$$

is called *vertical* factor of the decomposition of a positive hyperbolic transformation.

Lemma 3.1.6 *If $a \in \mathbb{RP}^2$ and $A \in \mathrm{Hyp}_+$, then the Hilbert distance*

$$h(a, A(a)) = \ln \mathrm{CR}\{\mathrm{Fix}_-(A), a, A(a), \mathrm{Fix}_+(A)\}.$$

Now, let us consider the Hilbert distance between a and $H(a)$ for any $a = [1-s, 0, s]^t \in \sigma(A) = \sigma(H)$.

$$\begin{aligned} h(a, H(a)) &= \ln \mathrm{CR}\{\mathrm{Fix}_-(H), a, H(a), \mathrm{Fix}_+(H)\} \\ &= \ln \mathrm{CR}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1-s \\ 0 \\ s \end{bmatrix}, \begin{bmatrix} (1-s)\lambda\sqrt{\mu} \\ 0 \\ s\nu\sqrt{\mu} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &= \ln \mathrm{CR}\left\{ \infty, \frac{1-s}{s}, \frac{\lambda}{\nu}\frac{1-s}{s}, 0 \right\} \\ &= \ln \frac{\left(\infty - \frac{\lambda}{\nu}\frac{1-s}{s}\right)\left(\frac{1-s}{s} - 0\right)}{\left(\infty - \frac{1-s}{s}\right)\left(\frac{\lambda}{\nu}\frac{1-s}{s} - 0\right)} \\ &= \ln\left(\frac{\nu}{\lambda}\right) \\ &= \ell(A). \end{aligned}$$

So, we call $\ell(A) = h(a, H(a))$ the *horizontal translation length*, and it is the length of the boundary component represented by A.

Let us consider V and the stationary set which is the line joining $[1,0,0]^t$, $[0,0,1]^t$ and the fixed point $[0,1,0]^t$. We can assume $\mu > 1$ without loss of generality. So, for any $a = [1-s, y, s]^t$ in the line segment joining $[1-s,0,s]^t$ and $[0,1,0]^t$, the point $V(a)$ goes toward $[0,1,0]^t$ since $\mu > 1$. Thus, the Hilbert distance between a and $V(a)$ is

$$
\begin{aligned}
h(a, V(a)) &= \ln \mathrm{CR}\{\mathrm{Fix}_-(V), a, V(a), \mathrm{Fix}_+(V)\} \\
&= \ln \mathrm{CR}\left\{ \begin{bmatrix} 1-s \\ 0 \\ s \end{bmatrix}, \begin{bmatrix} 1-s \\ y \\ s \end{bmatrix}, \begin{bmatrix} \frac{(1-s)}{\sqrt{\mu}} \\ y\mu \\ \frac{(1-s)}{\sqrt{\mu}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\
&= \ln X \left\{ B \begin{bmatrix} 1-s \\ 0 \\ s \end{bmatrix}, B \begin{bmatrix} 1-s \\ y \\ s \end{bmatrix}, B \begin{bmatrix} \frac{(1-s)}{\sqrt{\mu}} \\ y\mu \\ \frac{(1-s)}{\sqrt{\mu}} \end{bmatrix}, B \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\
&= \ln X \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}, \begin{bmatrix} 1/\sqrt{\mu} \\ 0 \\ y\mu \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\
&= \ln X \left\{ \infty, \frac{1}{y}, \frac{1}{y\mu^{3/2}}, 0 \right\} \\
&= \ln \frac{\left(\infty - \frac{1}{y\mu^{3/2}}\right)\left(\frac{1}{y} - 0\right)}{\left(\infty - \frac{1}{y}\right)\left(\frac{1}{y\mu^{3/2}} - 0\right)} \\
&= \ln \left(\mu^{3/2}\right) \\
&= m(A).
\end{aligned}
$$

We call $m(A) = h(a, V(a))$ the *vertical translation length.*

Consequently, one can easily see the relations among H,V, and A as follows:

$$
\begin{aligned}
\ell(H) &= \ln \left(\frac{\nu\sqrt{\mu}}{\lambda\sqrt{\mu}} \right) = \ln \left(\frac{\nu}{\lambda} \right) = \ell(A), \\
m(H) &= \frac{3}{2} \ln(1) = 0, \\
\ell(V) &= \ln \left(\frac{1/\sqrt{\mu}}{1/\sqrt{\mu}} \right) = \ln(1) = 0, \\
m(V) &= \frac{3}{2} \ln(\mu) = m(A).
\end{aligned}
$$

Therefore, A and H have the same horizontal translation length; A and V have the same vertical translation length.

3.2 Convex $\mathbb{RP}^2$-structures on a Pair-of-pants

Let Σ be a pair-of-pants; that is, a compact oriented surface of genus zero with three boundary components A, B, C. The main result of Goldman is the following theorem.

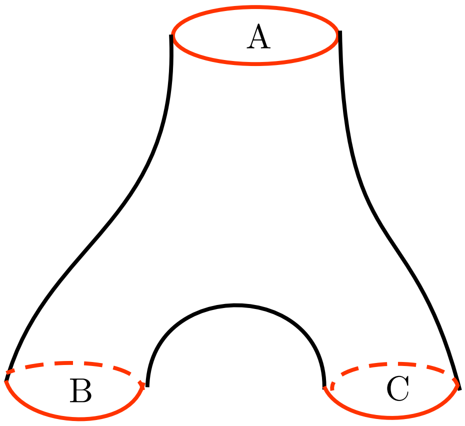


Figure 3.3: A pair-of-pants

Theorem 3.2.1 *([19]) The deformation space $\mathcal{B}(\Sigma)$ of convex $\mathbb{RP}^2$-structures on Σ is an open 8-dimensional cell and the map $\Theta_{\partial\Sigma} : \mathcal{B}(\Sigma) \to \mathcal{R}^3$ obtained by associating to a convex structure the boundary invariants*

$$((\lambda, \tau)_A, (\lambda, \tau)_B, (\lambda, \tau)_C)$$

is a fibration over an open 6-cell with fiber a 2-dimensional open cell.

Sketch of Proof. Let M be a convex $\mathbb{RP}^2$-structure representing a point in $\mathcal{B}(\Sigma)$ and (dev, h) be a development pair. Therefore, we have

$$\begin{aligned} h : \pi_1(\Sigma) &\to \mathrm{SL}(3, \mathbb{R}) \\ A &\mapsto h(A), \end{aligned}$$

and similarly we get $h(B)$ and $h(C)$ for the boundaries B, C, respectively. Let us also consider four triangular regions $\Delta_0, \Delta_a, \Delta_b, \Delta_c \subset \mathbb{RP}^2$ (see Figure 3.4) and three projective transformations $A, B, C \in \mathrm{SL}(3, \mathbb{R})$ (here, A, B, C denote the holonomy transformations of $h(A), h(B), h(C)$ of boundaries,respectively) which satisfy the following conditions:

- $\overline{\Delta}_a, \overline{\Delta}_b, \overline{\Delta}_c$ each intersect $\overline{\Delta}_0$ along each of the three edges of $\overline{\Delta}_0$,
- $\overline{\Delta}_0 \cup \overline{\Delta}_a \cup \overline{\Delta}_b \cup \overline{\Delta}_c$ is a convex hexagon,
- $CBA = I$ and $A(\Delta_b) = \Delta_c$, $B(\Delta_c) = \Delta_a$, $C(\Delta_a) = \Delta_b$,
- $A, B, C \in \mathrm{Hyp}_+$ and the vertices of Δ_0 are the repelling fixed points $\mathrm{Fix}_-(A)$, $\mathrm{Fix}_-(B)$,$\mathrm{Fix}_-(C)$ of A, B, C respectively and satisfy

$$\overline{\Delta}_a \cap \overline{\Delta}_b = \mathrm{Fix}_-(C), \quad \overline{\Delta}_b \cap \overline{\Delta}_c = \mathrm{Fix}_-(A), \quad \overline{\Delta}_c \cap \overline{\Delta}_a = \mathrm{Fix}_-(B).$$

The set of all $(\Delta_0, \Delta_a, \Delta_b, \Delta_c, A, B, C)$ satisfying above conditions is denoted by $\mathcal{O}'$ and the projective group $\mathrm{SL}(3, \mathbb{R})$ acts properly and freely on $\mathcal{O}'$, thus the quotient is denoted by $\mathcal{O}$.

The following lemma will conclude the proof of the Theorem 3.2.1.

Lemma 3.2.2 ([19]) *$\mathcal{O}$ is an open cell of dimension 8 and the map*

$$\begin{aligned} \mathcal{O} &\to \mathcal{R}^3 \\ (\Delta_0, \Delta_a, \Delta_b, \Delta_c, A, B, C) &\mapsto ((\lambda, \tau)_A, (\lambda, \tau)_B, (\lambda, \tau)_C) \end{aligned}$$

is a fibration with fiber an open 2-cell over the 6-cell $\mathcal{R}^3$. Moreover, there is an embedding $\mathrm{Teich}(\Sigma) \subset \mathcal{B}(\Sigma) \subset \mathcal{O}$, where $\mathrm{Teich}(\Sigma)$ is the deformation space of convex hyperbolic structures on Σ, namely the Teichmüller space.

Let us choose the coordinates in $\mathbb{RP}^2$ such that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are the homogeneous coordinates of the vertices of Δ_0. Here,

$$\Delta_0 = \{[x, y, z] \in \mathbb{RP}^2 \mid x > 0, y > 0, z > 0\},$$

and also $[1, 0, 0]$ is the repelling fixed point of A, $[0, 1, 0]$ is the repelling fixed point of B, and $[0, 0, 1]$ is the repelling fixed point of C. In the homogeneous coordinates, the remaining vertices of $\Delta_a, \Delta_b, \Delta_c$ are respectively $[-1, b_1, c_1], [a_2, -1, c_2], [a_3, b_3, -1]$. Note that the other triangular regions are given by

$$
\begin{aligned}
\Delta_a &= \{[x,y,z] \in \mathbb{RP}^2 \mid x < 0, 0 < y < -b_1 x, 0 < z < -c_1 x\}, \\
\Delta_b &= \{[x,y,z] \in \mathbb{RP}^2 \mid 0 < x < -a_2 y, y < 0, 0 < z < -c_2 z\}, \\
\Delta_c &= \{[x,y,z] \in \mathbb{RP}^2 \mid 0 < x < -a_3 z, 0 < y < -b_3 z, z < 0\},
\end{aligned}
$$

respectively.

Here, we will recall two lemmas about cross ratio. For more detail, see [38].

Lemma 3.2.3 *([38]) If a, b, c, d are four points in the projective plane $\mathbb{RP}^2$ and o is a point which is not on this line, then the cross-ratio can be calculated as*

$$(a, b; c, d) = \frac{[o, a, c][o, b, d]}{[o, a, d][o, b, c]}.$$

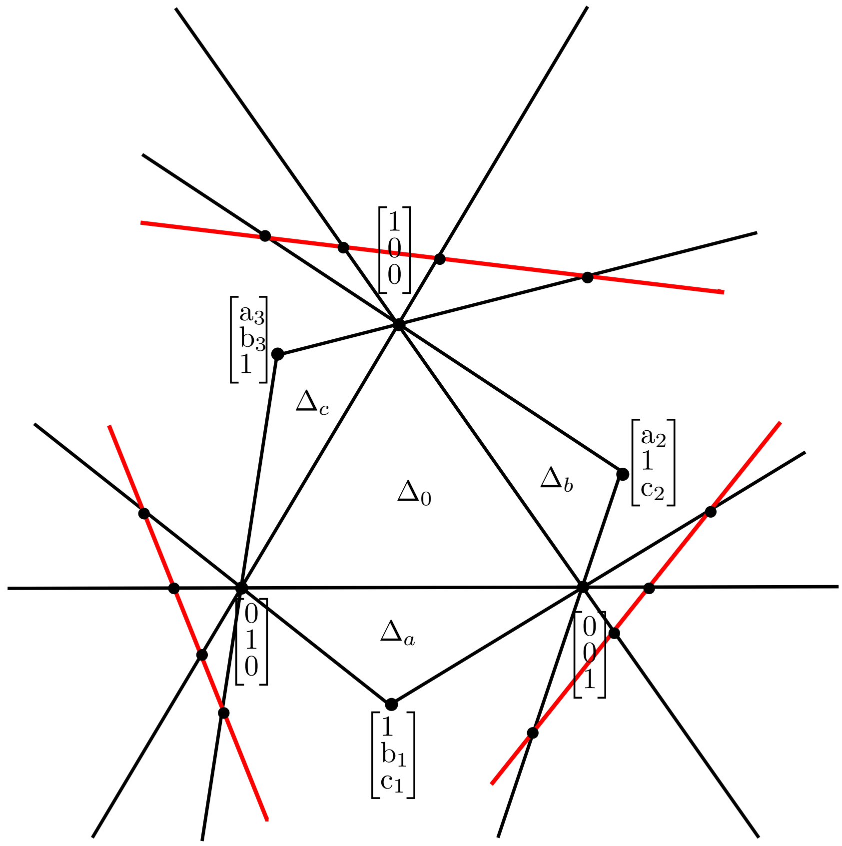


Figure 3.4: The cross-ratios of four lines

Thanks to above Lemma 3.2.3 and Figure 3.4, we can compute the cross-ratios of the four lines which contains edges of the incident triangles as follows:

$$\left(\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix};\begin{bmatrix}a_3\\b_3\\1\end{bmatrix},\begin{bmatrix}a_2\\1\\c_2\end{bmatrix}\right) = \frac{\begin{vmatrix}1&0&a_3\\0&0&b_3\\0&1&1\end{vmatrix}\begin{vmatrix}1&0&a_2\\0&1&1\\0&0&c_2\end{vmatrix}}{\begin{vmatrix}1&0&a_2\\0&0&1\\0&1&c_2\end{vmatrix}\begin{vmatrix}1&0&a_3\\0&1&b_3\\0&0&1\end{vmatrix}}$$

$$= \frac{(0-b_3)(c_2-0)}{(0-1)(1-0)} = b_3c_2 = \rho_1$$

$$\left(\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix};\begin{bmatrix}a_3\\b_3\\1\end{bmatrix},\begin{bmatrix}1\\b_1\\c_1\end{bmatrix}\right) = \frac{\begin{vmatrix}0&0&a_3\\1&0&b_3\\0&1&1\end{vmatrix}\begin{vmatrix}0&1&1\\1&0&b_1\\0&0&c_1\end{vmatrix}}{\begin{vmatrix}1&0&1\\1&0&b_1\\0&1&c_1\end{vmatrix}\begin{vmatrix}0&1&a_3\\1&0&b_3\\0&0&1\end{vmatrix}}$$

$$= \frac{(-1)(0-a_3)(-1)(c_1-0)}{(-1)(0-1)(-1)(1-0)} = a_3c_1 = \rho_2$$

$$\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix};\begin{bmatrix}a_2\\1\\c_2\end{bmatrix},\begin{bmatrix}1\\b_1\\c_1\end{bmatrix}\right) = \frac{\begin{vmatrix}0&0&a_2\\0&1&1\\1&0&c_2\end{vmatrix}\begin{vmatrix}0&1&1\\0&0&b_1\\1&0&c_1\end{vmatrix}}{\begin{vmatrix}0&0&1\\0&1&b_1\\1&0&c_1\end{vmatrix}\begin{vmatrix}0&1&a_2\\0&0&1\\1&0&c_2\end{vmatrix}}$$

$$= \frac{(0-a_2)(b_1-0)}{(0-1)(1-0)} = a_2b_1 = \rho_3$$

The hexagon $\Delta_0 \cup \Delta_a \cup \Delta_b \cup \Delta_c$ is convex if and only if all $b_1, c_1, a_2, c_2, a_3, b_3$ are positive and all the cross-ratios, namely ρ_1, ρ_2, ρ_3 are greater than 1.

There are also 2 internal parameters $s, t > 0$. From the details in [19], one can easily see that

$$t = \frac{a_2 b_3}{a_3}$$

and s is determined as the unique positive solution to any one of the three equations below:

$$\begin{aligned}
\rho_1 = b_3 c_2 &= 1 + \tau(A)\sqrt{\frac{\lambda(A)\lambda(C)}{\lambda(B)}}s + \frac{\lambda(C)}{\lambda(B)}s^2 \\
\rho_2 = a_3 c_1 &= 1 + \tau(B)\sqrt{\frac{\lambda(A)\lambda(B)}{\lambda(C)}}s + \frac{\lambda(A)}{\lambda(C)}s^2 \\
\rho_3 = a_3 b_1 &= 1 + \tau(C)\sqrt{\frac{\lambda(B)\lambda(C)}{\lambda(A)}}s + \frac{\lambda(B)}{\lambda(A)}s^2.
\end{aligned}$$

Finally, for the Σ which is a pair-of-pants, the parametrization is

$$((\lambda(A), \tau(A)), (\lambda(B), \tau(B)), (\lambda(C), \tau(C)), (s, t)) \in \mathbb{R}^8.$$

Thus, the fiber of the boundary invariant map $\mathcal{O} \to \mathcal{R}^3$ is parametrized by arbitrary pairs $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $\mathcal{O} \to \mathcal{R}^3$ is 2-cell fibration over the open 6-cell $\mathcal{R}^3$. This concludes the proof of Theorem 3.2.1.

Theorem 3.2.4 *([19]) Let Σ be a compact surface with negative Euler characteristic and having $n \geq 0$ boundary components. Then the map $\Theta_{\partial\Sigma} : \mathcal{B}(\Sigma) \to \mathcal{B}(\partial\Sigma)$ is a fibration over the $2n$-cell $\mathcal{B}(\partial\Sigma)$ with fiber an open cell of dimension $-8\chi(\Sigma) - 2n$, where χ denotes the Euler characteristic.*

Let Σ be a compact surface with boundary components $b_1, \ldots, b_n$, and cut Σ along disjoint one-sided simple closed curves $a_1, \ldots, a_m$. If we decompose the surface into pair-of-pants P_l for $l = 1, \ldots, -\chi(\Sigma)$ along simple closed curves $c_1, \ldots, c_p$. One can easily see that $n + m + 2p = -3\chi(\Sigma)$.

If, for example, Σ is a surface of genus 2 with 3 boundary components, then the Euler characteristic of this surface is

$$\chi(\Sigma) = 2 - 2g - n = 2 - 2 \cdot 2 - 3 = -5.$$

Moreover, if we decompose this surface into pair-of-pants then we get 5 pair-of-pants along 6 simple closed curves. So, we have

$$n + m + 2p = 3 + 0 + 2 \cdot 6 = 15$$

which equals $-3\chi(\Sigma) = (-3)(-5)$.

There is one more part of this parametrization. In [19], Goldman defines the $\mathbb{R}^2$ action Ψ on $\mathcal{B}(\Sigma)$ that generalizes the earthquake flow on the Teichmüller space. The action of an element $(u, v) \in \mathbb{R}^2$ is defined on a point in $\mathcal{B}(\Sigma)$. Therefore, a new convex $\mathbb{RP}^2$-manifold $\Psi_{(u,v)}(M)$ is constructed for $(u, v) \in \mathbb{R}^2$ which represents a point in $\mathcal{B}(\Sigma)$.

Let $p : \widetilde{M} \to M$ be a universal covering and (dev, h) be a development pair. For a simple closed geodesic C on M and a representative element $\gamma \in \pi_1(M)$ chosen such that $h(\gamma) \in \mathcal{A}$ is represented by the diagonal matrix (3.1) satisfying the properties $\lambda\mu\nu = 1$ and $0 < \lambda < \mu < \nu$. Obviously, the centralizer of $h(\gamma)$ in $\mathrm{SL}(3, \mathbb{R})$ equals $\mathcal{A}$. $\mathcal{A}_+$ is the identity component of $\mathcal{A}$ and it is the direct product of the two one-parameter groups

$$T^u = \begin{pmatrix} e^{-u} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^u \end{pmatrix}, \qquad U^v = \begin{pmatrix} e^{-v} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{-v} \end{pmatrix},$$

where $u, v \in \mathbb{R}$. The flows $\Psi_{(u,0)}$ and $\Psi_{(0,v)}$ on $\mathcal{B}(\Sigma)$ are special cases of the generalized twist flows on $\mathrm{Hom}(\pi, \mathrm{SL}(3, \mathbb{R}))$. These flows generates the vector fields $\frac{\partial}{\partial\theta_\gamma}, \frac{\partial}{\partial\beta_\gamma}$ and they are called the *generalized twisting vector fields* whose potential functions are $\ell(\gamma)$ and $m(\gamma)$, respectively.

Consequently, the following map

$$\mathcal{B}(\Sigma) \to \mathcal{R}^n \times \mathcal{R}^m \times (\mathcal{R} \times \mathbb{R}^2)^p \times (\mathbb{R}_+ \times \mathbb{R}_+)^{-\chi(\Sigma)}$$

defined by

$$\begin{aligned} M \quad \mapsto \quad & \{(\lambda(b_i), \tau(b_i))\}_{i=1}^n \times \{(\lambda(a_j), \tau(a_j))\}_{j=1}^m \\ & \times \{(\lambda(c_k), \tau(c_k), (u, v)(c_k))\}_{k=1}^p \times \{((s, t)(P_l))\}_{l=1}^{-\chi(\Sigma)} \end{aligned}$$

is a diffeomorphism of $\mathcal{B}(\Sigma)$ onto a $-8\chi(\Sigma)$-dimensional cell.

3.3 The Symplectic Structure on $\mathrm{Rep}(\pi, G)$

Let us recall that $\mathcal{B}(\Sigma)$ embeds into $\mathrm{Hom}(\pi, \mathrm{SL}(3,\mathbb{R}))/(\mathrm{SL}(3,\mathbb{R}))$. Therefore, we should recall the basic properties of $\mathrm{Hom}(\pi, G)$. Here, π denotes the fundamental group of a compact oriented smooth surface $M = \Sigma(g, n)$ with genus $g \geq 2$ and n boundary, and also G is a connected Lie group.

$\mathrm{Hom}(\pi, G)$ is not smooth, in general. Therefore, we look for a smooth part of it. First of all, let $\mathrm{Hom}(\pi, G)^-$ denote the set of nonsingular points of $\mathrm{Hom}(\pi, G)$. And then, $\mathrm{Hom}(\pi, G)^{--}$ be the subset of $\mathrm{Hom}(\pi, G)^-$ which consists of homomorphisms whose image does not lie in a parabolic subgroup of G. So, $\mathrm{Hom}(\pi, G)^{--}$ is a Zariski open subset of $\mathrm{Hom}(\pi, G)^-$, and $\mathrm{Hom}(\pi, G)^{--}/G$ is a Hausdorff smooth manifold of dimension $-\mathrm{dim}G \cdot \chi(M)$. Details and unexplained subjects can be found in [15].

In this thesis, we will consider the surfaces without boundary. So, we will study the symplectic form on the moduli space of a closed surface. And also, when we say $\mathrm{Rep}(\pi, G)$ we mean the smooth locus $\mathrm{Hom}(\pi, G)^{--}/G$.

First, let us recall Fox's calculus in [39]. By using this, one can define the explicit formula for the symplectic 2-form on $\mathrm{Rep}(\pi, G)$.

Let F be a free group with basis $\{x_1, \ldots, x_n\}$ and $\mathbb{Z}F$ be its integral group ring. The *Fox derivation* of $\mathbb{Z}F$ is a $\mathbb{Z}$-linear map $D : \mathbb{Z}F \to \mathbb{Z}F$ which satisfies

$$D(fg) = D(f)\xi(g) + fD(g).$$

Here, $f, g \in \mathbb{Z}F$ and $\xi : \mathbb{Z}F \to \mathbb{Z}$ is the *augmentation* homomorphism which defined by

$$\xi\left(\sum n_i\sigma_i\right) = \sum n_i.$$

$\mathbb{Z}F$ is an F-bimodule such that F acts on the right by trivial and on the left by left-multiplication. For an arbitrary $x, y \in F$, since $\xi(y) = 1$

$$\begin{aligned} D(xy) &= D(x)\xi(y) + xD(y) \\ &= D(x) + xD(y), \end{aligned}$$

the Fox derivation is a 1-cocycle on F with coefficients in $\mathbb{Z}F$.

Lemma 3.3.1 *([39]) Let $\{x_1, \ldots, x_n\}$ be the generators for group F and* $\mathrm{Der}(F)$ *be the set of all Fox derivations. If we define*

$$(D \circ f)(x) = D(x)\xi(f),$$

where $D \in \mathrm{Der}(F)$, $f \in \mathbb{Z}F$ and $x \in F$, then $\mathrm{Der}(F)$ is freely generated as a right $\mathbb{Z}F$-module by n elements $\partial_i = \partial/\partial x_i$, $i = 1, \ldots, n$ so that $(\partial/\partial x_i)(x_j) = \delta_{ij} I$. Here, I is the identity element of F.

Now, let us recall the group homology theory in [40]. Let F be a group and $\mathbb{Z}F$ denote its integral group ring. The freely generated $\mathbb{Z}$-module $F \times \cdots \times F$ is denoted by $C_n(F)$ and $C_0(F) = \mathbb{Z}$. The boundary operator $\partial_n : C_n(F) \to C_{n-1}(F)$ is defined as follows for $n \geq 2$

$$\begin{aligned} \partial_n(u_1, \ldots, u_n) &= \xi(u_1)(u_2, \ldots, u_n) + \sum_{i=1}^{n-1} (-1)^i (u_1, \ldots, u_i u_{i+1}, \ldots, u_n) \\ &\quad + (-1)^n (u_1, \ldots, u_{n-1})\xi(u_n), \end{aligned}$$

and $\partial_1(u) = 0$.

For example, let us write $\partial_2 : C_2(F) \to C_1(F)$:

$$\partial_2(u_1, u_2) = \xi(u_1)u_2 - u_1 u_2 + u_1 \xi(u_2),$$

and $\partial_3 : C_3(F) \to C_2(F)$:

$$\partial_3(u_1, u_2, u_3) = \xi(u_1)(u_2, u_3) - (u_1 u_2, u_3) + (u_1, u_2 u_3) - \xi(u_3)(u_1, u_2).$$

Now, let us check $\partial_2 \circ \partial_3$:

$$\begin{aligned}
\partial_2(\partial_3(u_1, u_2, u_3)) &= \xi(u_1)\partial_2(u_2, u_3) - \partial_2(u_1u_2, u_3) + \partial_2(u_1, u_2u_3) - \xi(u_3)\partial_2(u_1, u_2) \\
&= \xi(u_1)[\xi(u_2)u_3 - u_2u_3 + u_2\xi(u_3)] \\
&\quad -[\xi(u_1u_2)u_3 - u_1u_2u_3 + u_1u_2\xi(u_3)] \\
&\quad +[\xi(u_1)u_2u_3 - u_1u_2u_3 + u_1\xi(u_2u_3)] \\
&\quad -\xi(u_3)[\xi(u_1)u_2 - u_1u_2 + u_1\xi(u_2)] \\
&= 0.
\end{aligned}$$

One can easily see that $\partial_n \circ \partial_{n+1} = 0$. Therefore, the group homology is:

$$H_n(F) = \frac{Z_n(F)}{B_n(F)}.$$

Here, $Z_n(F)$ is the kernel of $\partial_n : C_n(F) \to C_{n-1}(F)$ and $B_n(F)$ is the image of $\partial_{n+1} : C_{n+1}(F) \to C_n(F)$.

For the rest of this subsection, we refer the reader [20]. Let us consider a closed surface group $\pi = F/R$ where π is a group generated by $2g$ generators $A_1, B_1, \ldots, A_g, B_g$ and with the relation

$$R = A_1B_1A_1^{-1}B_1^{-1} \cdots A_gB_gA_g^{-1}B_g^{-1}. \tag{3.6}$$

If we consider the 2-chain on π

$$\begin{aligned}
Z_R &= \sum_{i=1}^{g} \left((\frac{\partial R}{\partial A_i}, A_i) + (\frac{\partial R}{\partial B_i}, B_i) \right) \\
&= \sum_i n_i(x_i, y_i) \in \mathbb{Z}(\pi \times \pi),
\end{aligned}$$

then the boundary $\partial Z_R = 0$. Recall that Z_R is called the *fundamental cycle* of the fundamental group π.

Finally, we are ready to give the explicit formula of the symplectic form on $\mathrm{Rep}(\pi, G)$. Here, π is the fundamental group of a closed surface Σ, $G = \mathrm{SL}(3, \mathbb{R})$ is a connected algebraic Lie group and $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ is the Lie algebra of G. Let $B : \mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(3, \mathbb{R}) \to \mathbb{R}$ be an Ad-invariant nondegenerate symplectic bilinear form, e.g. it can be the trace form.

If $u, v \in Z^1(\pi; \mathfrak{sl}(3,\mathbb{R})_{Ad_\phi})$ then a $\mathbb{Z}$-linear map $B_*(u,v) : \mathbb{Z}(\pi \times \pi) \to \mathbb{R}$ is defined by

$$B_*(u,v)\left(\sum_{i=1}^{k} n_i(x_i, y_i)\right) = \sum_{i=1}^{k} n_i\{B(u(x_i), x_i \cdot v(y_i))\}.$$

Here, $x_i \cdot v(y_i) = Ad_{\phi(x_i)} v(y_i)$ for $(x_i, y_i) \in \pi \times \pi \subset C_2(\pi)$, therefore $B_*(u,v) \in Z^2(\pi; \mathbb{R})$.

Definition 3.3.2 The symplectic form ω_G on $\mathrm{Rep}(\pi, G)$, which is called *Atiyah-Bott-Goldman symplectic form*, is defined as follows:

$$\begin{aligned} \omega_G : H^1(\pi; \mathfrak{g}_{Ad_\phi}) &\times H^1(\pi; \mathfrak{g}_{Ad_\phi}) \to \mathbb{R} \\ \omega_G([u],[v]) &= B_*(u,v) Z_R. \end{aligned}$$

Here, Z_R is the fundamental cycle of π, G is a Lie goup and $\mathfrak{g}$ is the Lie algebra of G.

Let us consider the set of nontrivial homotopically distinct disjoint simply closed geodesics $\Gamma = \{\gamma_i\}_{i=1,\ldots,3g-3}$ on Σ so that Σ is decomposed as the disjoint union of $2g-2$ pair of pants by Γ. So, for each γ_i there are two length parameters ℓ_i, m_i and two twisting parameters θ_i, β_i, where $\ell_i \in \mathbb{R}_+$ and m_i, θ_i, β_i are real numbers. The coordinate fields $\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \beta_i}$ are the generalized twisting vector fields generated by the flows $\Psi_{\gamma_i(u,0)}, \Psi_{\gamma_i(0,v)}$ and their potential functions are ℓ_i, m_i. Thus, we have following

$$\omega_G\left(\frac{\partial}{\partial \theta_i}, \quad\right) = -d\ell_i, \quad \omega_G\left(\frac{\partial}{\partial \beta_i}, \quad\right) = -dm_i.$$

By using above duality formula, if

$$X \in \mathrm{Vect}(\mathcal{B}(\Sigma)) \backslash < \frac{\partial}{\partial \ell_k} > \qquad \text{and} \qquad Y \in \mathrm{Vect}(\mathcal{B}(\Sigma)) \backslash < \frac{\partial}{\partial m_k} >$$

then we can determine that

$$\begin{aligned} \omega_G\left(\frac{\partial}{\partial \theta_k}, X\right) &= -d\ell_k(X) = 0 \\ \omega_G\left(\frac{\partial}{\partial \beta_k}, Y\right) &= -dm_k(Y) = 0. \end{aligned}$$

Lemma 3.3.3 *([20]) Let Σ be a closed surface with genus g which is having an orientation reversing map ρ fixing the elements of a partition $\Gamma = \{\gamma_i\}$ and preserving the real projective structure on Σ. For each i and j,*

$$\omega_G\left(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right) = \omega_G\left(\frac{\partial}{\partial\ell_i}, \frac{\partial}{\partial\ell_j}\right) = \omega_G\left(\frac{\partial}{\partial\beta_i}, \frac{\partial}{\partial\beta_j}\right) = \omega_G\left(\frac{\partial}{\partial m_i}, \frac{\partial}{\partial m_j}\right) = 0,$$

and also for any j and k,

$$\omega_G\left(\frac{\partial}{\partial s_j}, \frac{\partial}{\partial s_k}\right) = \omega_G\left(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}\right) = 0.$$

Finally, the main theorem of [20] is following:

Theorem 3.3.4 *([20]) If Σ is a closed smooth surface with genus g, then the symplectic form on $\mathcal{B}(\Sigma)$ of convex real projective structures is*

$$\omega_G = \sum_{i=1}^{3g-3} d\ell_i \wedge d\theta_i + \sum_{i=1}^{3g-3} dm_i \wedge d\beta_i + \sum_{j=1}^{2g-2} dt_j \wedge ds_j. \tag{3.7}$$

Here,

$$\begin{aligned} \ell_i, m_i &\longrightarrow \quad \textit{length parameters,} \\ \theta_i, \beta_i &\longrightarrow \quad \textit{twisting parameters,} \\ s_j, t_j &\longrightarrow \quad \textit{internal parameters,} \end{aligned}$$

on $\mathcal{B}(\Sigma)$. Therefore, $\mathcal{B}(\Sigma)$ is symplectomorphic to $\mathbb{R}^{16g-16}$.

3.4 Reidemeister Torsion of Representations Associated to $\mathbb{RP}^2$ via the Goldman Parametrization

Thanks to Goldman parametrization of the deformation space of convex $\mathbb{RP}^2$-structures, we have the following diffeomorphism:

$$f : (\mathcal{B}(\Sigma), \omega_G) \hookrightarrow \left(\mathbb{R}^{16g-16}, \omega_{\text{nat}}\right), \tag{3.8}$$

where the image of f is an open cell of dimension $16g-16$, ω_G is the Atiyah-Bott-Goldman symplectic form, and ω_{nat} is the well-known natural symplectic form on $\mathbb{R}^{16g-16}$. This map defines a symplectomorphism, namely

$$f^*(\omega_{\text{nat}}) = \omega_G.$$

Here, f^* is the pullback of f.

Finally, we are ready to give the main result of this section of the thesis.

Theorem 3.4.1 *Let Σ be a closed orientable surface with genus at least 2 and ϕ : $\pi_1(\Sigma) \to \mathrm{SL}(3,\mathbb{R})$ be the element of the deformation space $\mathcal{B}(\Sigma)$. If $\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^{16g-16}$ is a basis of the open cell of dimension $16g-16$ then we have the following formula*

$$\mathrm{Tor}(\Sigma, \{\boldsymbol{\alpha}\}) = \sqrt{\det \begin{bmatrix} \omega_{\mathrm{nat}} \\ \boldsymbol{\alpha} \end{bmatrix}}.$$

Proof.

Let us consider the differential

$$f_* : H^1(\Sigma, \mathfrak{sl}(3,\mathbb{R})_{Ad_\phi}) \to \mathbb{R}^{16g-16}$$

of f in (3.8). Here, $H^1(\Sigma, \mathfrak{sl}(3,\mathbb{R})_{Ad_\phi})$ denotes the first cohomology group of the deformation space Σ with basis $\mathbf{h}^1 = f_*^{-1}(\boldsymbol{\alpha})$. Let σ_1, σ_2 be the elements of the $H^1(\Sigma, \mathfrak{sl}(3,\mathbb{R})_{Ad_\phi})$, so we have

$$\begin{aligned} \omega_G(\sigma_1, \sigma_2) &= f^*(\omega_{\mathrm{nat}})(\sigma_1, \sigma_2) \\ &= \omega_{\mathrm{nat}}(f_*(\sigma_1), f_*(\sigma_2)) \\ &= \omega_{\mathrm{nat}}(\alpha_1, \alpha_2), \end{aligned}$$

where, $\alpha_i \in \mathbb{R}^{16g-16}$ for $i = 1, 2$. Using the fact that $f^*(\omega_{\mathrm{nat}}) = \omega_G$, $f_*(\mathbf{h}^1) = \boldsymbol{\alpha}$ and applying Theorem 2.3.6 we conclude the proof.

$\square$

4 COMPLEX PROJECTIVE STRUCTURES

In this section, we will state well-known facts about Teichmüller space. For more information and unexplained subjects, we refer the reader to [27, 28].

Recall that a surface with a class of complex structures, in other words an element of Teich(Γ), is called a *marked* surface.

Let $p : \mathbb{H} \to \Sigma$ be the universal covering of Σ with covering transformation group Γ. Here, $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane and $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a strictly hyperbolic Fuchsian group acting on $\mathbb{H}$ by linear fractional transformations. Taking a base point $z \in \mathbb{H}$ lying over $x \in \Sigma$ establishes a canonical isomorphism between Γ and $\pi_1(\Sigma, x)$ and thus determines a system of generators $A_1, \ldots, A_g, B_1, \ldots, B_g$ of Γ that correspond to the elements $a_1, \ldots, a_g, b_1, \ldots, b_g$, so that

$$\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = 1.$$

This called a marking of Γ and also the group Γ with a marking is called a marked Fuchsian group. Throughout the following subsection, a marking of Σ and the corresponding marking of Γ will be fixed.

4.1 Bers Section

For more information about this subsection, we refer the reader to [21].

Definition 4.1.1 Let $q : \mathbb{H} \to \mathbb{C}$ be a holomorphic function. If q satisfies the following equality for all $\gamma \in \Gamma, z \in \mathbb{H}$

$$q(\gamma z)\gamma'(z)^2 = q(z)$$

then it is called a (holomorphic) *quadratic differential for* Γ. The space of all quadratic differentials denoted by $A_2(\mathbb{H}, \Gamma)$.

The Schwarzian derivative of a holomorphic function f of one complex variable z is defined by

$$(Sf)(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Note that, the Schwarzian derivative is a certain operator that is invariant under all Möbius transformation.

If we consider the differential equation

$$S(f)(z) = q(z)$$

for a quadratic differential $q \in A_2(\mathbb{H}, \Gamma)$, any solution of f of above equation turns out to be a locally biholomorphic mapping from $\mathbb{H}$ into the Riemann sphere $\widehat{\mathbb{C}}$. Therefore, it arises a homomorphism $\phi : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ such that

$$f(\gamma z) = \phi(\gamma) f(z)$$

for all $\gamma \in \Gamma, z \in \mathbb{H}$. This f determines a projective structure on Σ and we call ϕ the monodromy representation determined by f.

Definition 4.1.2 Let $\mu : \mathbb{H} \to \mathbb{C}$ be a C^∞- function. If the following equation

$$\mu(\gamma z) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z)$$

is satisfied for all $\gamma \in \Gamma$ and $z \in \mathbb{H}$, then μ is called a (smooth) *Beltrami differential for* Γ.

$B(\mathbb{H}, \Gamma)$ denotes the space of all Beltrami differentials for Γ and $B(\mathbb{H}, \Gamma)_1$ denotes the set

$$\{\mu \in B(\mathbb{H}, \Gamma)\,;\ \|\mu\|_\infty = \sup_{z \in \mathbb{H}} |\mu(z)| < 1\}.$$

If μ is an element of $B(\mathbb{H}, \Gamma)_1$ then there exists a unique quasiconformal mapping $w^\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which satisfies the Beltrami equation

$$\begin{cases} w^\mu_{\bar{z}} = \mu w^\mu_z & \text{in } \ \mathbb{H} \\ w^\mu_{\bar{z}} = 0 & \text{in } \ \mathbb{C} \setminus \mathbb{H} \end{cases}$$

and fixing the points 0,1 and ∞. Note that since μ is $\mathbb{C}^\infty$ on $\mathbb{H}$ the restriction $w^\mu|_{\mathbb{H}}$ is a diffeomorphism (see [27]).

Let us take two elements $\mu, \nu \in B(\mathbb{H}, \Gamma)_1$ and define an equivalence relation for them as

$$\mu \sim \nu \iff w^\mu \text{ and } w^\nu \text{ coincide on the lower half-plane } \mathbb{H}^*.$$

The equivalence class of an element μ is denoted by $[\mu]$.

The set of equivalence classes of these Beltrami differentials defines the Teichmüller space $\mathrm{Teich}(\Gamma)$.

$$\mathrm{Teich}(\Gamma) := B(\mathbb{H}, \Gamma)_1 \,/ \sim$$

If $[\mu]$ is an element of the Teichmüller space then one can consider the element $w^\mu|_{\mathbb{H}^*}$, which bring out a bijection between the Teichmüller space and a certain class of locally biholomorphic functions on $\mathbb{H}^*$ (see [27]).

Now, there is an important step like taking the Schwarzian derivative of the functions $w^\mu|_{\mathbb{H}^*}$. One can shows that

(i) $S(w^\mu|_{\mathbb{H}^*})$ is a (holomorphic) quadratic differential for Γ on $\mathbb{H}^*$, for any $\mu \in B(\mathbb{H}, \Gamma)_1$,

(ii) $S(w^\mu|_{\mathbb{H}^*}) = S(w^\nu|_{\mathbb{H}^*}) \Leftrightarrow [\mu] = [\nu]$ in $\mathrm{Teich}(\Gamma)$.

For more detail see [28, 27].

Definition 4.1.3 From above informations, there is a well-defined injective mapping

$$\begin{aligned} \beta : \mathrm{Teich}(\Gamma) &\longrightarrow A_2(\mathbb{H}^*, \Gamma) \\ [\mu] &\longmapsto S(w^\mu|_{\mathbb{H}^*}) \end{aligned} \tag{4.1}$$

which is called *Bers embedding*, where $A_2(\mathbb{H}^*, \Gamma)$ denotes the $(3g-3)$-dimensional complex vector space of (holomorphic) quadratic differentials on $\mathbb{H}^*$ for Γ.

If the following map (*the Bers projection*)

$$\begin{aligned} \Phi_\beta : B(\mathbb{H}, \Gamma)_1 &\longrightarrow A_2(\mathbb{H}^*, \Gamma) \\ \mu &\longmapsto S(w^\mu|_{\mathbb{H}^*}) \end{aligned} \tag{4.2}$$

is analyzed closely, one can show that the image of β, which is denoted by $\mathrm{Teich}_\beta(\Gamma)$, is an open (bounded) domain in $A_2(\mathbb{H}^*, \Gamma)$ (see [27]). One of the possible definitions of the complex structure of $\mathrm{Teich}(\Gamma)$ is the natural complex structure via the embedding β; thus it can be biholomorphically identified with the open domain $\mathrm{Teich}_\beta(\Gamma)$ in $A_2(\mathbb{H}^*, \Gamma)$.

Let $\mathbb{H}^\mu = w^\mu(\mathbb{H})$ be the quasidisk which depends only the Teichmüller class μ and that mapping w^μ conjugates Γ into a quasi-Fuchsian group $\Gamma^\mu = w^\mu\Gamma(w^\mu)^{-1}$ acting on $\mathbb{H}^\mu$.

Definition 4.1.4 Attaching over each $[\mu]$ in $\mathrm{Teich}(\Gamma)$ the marked Riemann surface $\Sigma^\mu = \mathbb{H}^\mu/\Gamma^\mu$, a representative of the Teichmüller point $[\mu]$ defines the *universal Teichmüller curve* $V(\Gamma)$.

Definition 4.1.5 If $q \in A_2(\mathbb{H}, \Gamma)$ and $z \in \mathbb{H}$, then one can define a Beltrami differential $\mu[q](z) \in B(\mathbb{H}, \Gamma)$ by

$$\mu[q](z) = \lambda_{\mathbb{H}}(z)^{-2}\,\overline{q(z)}.$$

$\mu[q]$ is called the *harmonic Beltrami differential* formed from q. $q \mapsto \mu[q]$ induces a complex antilinear isometric embedding of $A_2(\mathbb{H}, \Gamma)$ into $B(\mathbb{H}, \Gamma)$. The image space of this embedding is denoted by $HB(\mathbb{H}, \Gamma)$.

Let us consider the quasidisk $\mathbb{H}^\mu = w^\mu(\mathbb{H})$ and quasi-Fuchsian group $\Gamma^\mu = w^\mu\Gamma(w^\mu)^{-1}$. The Bers projection Φ_β in (4.2) can be expressed in terms of the generalized Bers projection

$$\Phi^\mu : B(\mathbb{H}^\mu, \Gamma^\mu)_1 \to A_2((\mathbb{H}^*)^\mu, \Gamma^\mu),$$

through the canonical biholomorphism $B(\mathbb{H}^\mu, \Gamma^\mu)_1 \to B(\mathbb{H}, \Gamma)_1$ and the isomorphism $A_2((\mathbb{H}^*)^\mu, \Gamma^\mu) \to A_2(\mathbb{H}^*, \Gamma)$. For more detail, see [27, 21].

Considering the differential of Φ_β at the point $0 \in B(\mathbb{H}, \Gamma)_1$ one has the following isomorphisms

$$\begin{aligned} \mathrm{T}_{[\mu]}\mathrm{Teich}(\Gamma) &\cong HB(\mathbb{H}^\mu, \Gamma^\mu) \\ \mathrm{T}^*_{[\mu]}\mathrm{Teich}(\Gamma) &\cong A_2(\mathbb{H}^\mu, \Gamma^\mu) \end{aligned}$$

[27]. Here, $\mathrm{T}_{[\mu]}\mathrm{Teich}(\Gamma)$ denotes the holomorphic tangent space and $\mathrm{T}^*_{[\mu]}\mathrm{Teich}(\Gamma)$ denotes the holomorphic cotangent space to $\mathrm{Teich}(\Gamma)$ at the base point $[\mu] \in \mathrm{Teich}(\Gamma)$.

Let Q be the total space of the holomorphic cotangent bundle $\mathrm{T}^*\mathrm{Teich}(\Gamma)$ of the Teicmüller space $\mathrm{Teich}(\Gamma)$. It is well-known that Q has a canonical symplectic structure ω_{nat}. Let us recall the natural symplectic form on the cotangent bundle.

Let M be an n-dimensional smooth manifold and its cotangent bundle is defined as follows

$$\mathrm{T}^*M := \{ \text{ linear maps } f : \mathrm{T}_q M \to \mathbb{R};\ q \in M\}.$$

If $q = (q_1, \ldots, q_n)$ is a choice of local coordinates on $U \subseteq M$, then for a fixed $q \in U$, a 1-form $\sum_{i=1}^n p_i dq_i$ on $\mathrm{T}_q M$ is determined by the coefficients $(p_1, \ldots, p_n)$. Therefore, local coordinates of an element $l \in \mathrm{T}^*M$ are $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$.

Let $X \in \mathrm{T}_l(\mathrm{T}^*M)$ be a vector tangent to the cotangent bundle at the point $l = (p, q) \in \mathrm{T}^*M$. Through the derivative

$$\pi_* : \mathrm{T}(\mathrm{T}^*M) \to \mathrm{T}M,$$

of the natural projection, the tangent vector $X \in \mathrm{T}_l(\mathrm{T}^*M)$ is mapped to the tangent vector $\pi_* X \in \mathrm{T}_q M$. This defines the 1-form υ on T^*M by the relation $\upsilon(X) = l(\pi_* X)$.

A symplectic form on T^*M is defined by the exterior derivative $\omega_{\mathrm{nat}} := d\upsilon$. Clearly, it is closed and non-degenerate. Note that in the local coordinates (p, q) described above the natural symplectic structure ω_{nat} is equal to $dp \wedge dq$ [42].

The space $\mathrm{Rep}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ also admits a natural symplectic structure namely, the Atiyah-Bott-Goldman symplectic form $\omega_{\mathrm{PSL}(2,\mathbb{C})}$. Let $\tau \in \mathrm{Teich}(\Gamma)$ and $q \in A_2(\mathbb{H}^\tau, \Gamma^\tau)$. One has a conjugacy class of representations $\Gamma^\tau \to \mathrm{PSL}(2, \mathbb{C})$ and hence one can obtain the monodromy mapping

$$F : Q \to \mathrm{Rep}(\Gamma, \mathrm{PSL}(2, \mathbb{C})) \tag{4.3}$$

via the canonical isomorphisms $\Gamma^\tau \cong \Gamma$.

It was proved in [21] by S. Kawai that the mapping $F : Q \to \mathrm{Rep}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ is a symplectomorphism, more precisely

$$F^* \omega_{\mathrm{PSL}(2,\mathbb{C})} = \frac{1}{\pi} \omega_{\mathrm{nat}}.$$

Using this fact, we prove:

Theorem 4.1.6 *Let Σ be a closed orientable surface with genus at least 2 and $\phi : \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{C})$ be an element of the space $\mathrm{Rep}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$. If $\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^{12g-12}$ is a basis of $\mathbb{R}^{12g-12}$, then*

$$|\mathrm{Tor}(\Sigma, \{\boldsymbol{\alpha}\})| = \pi^{6-6g}\sqrt{\det\begin{bmatrix}\omega_{\mathrm{nat}}\\ \boldsymbol{\alpha}\end{bmatrix}}.$$

Proof. We have the symplectomorphism (4.3)

$$F : Q \to \mathrm{Rep}(\Gamma, \mathrm{PSL}(2,\mathbb{C})),$$

and let $\phi \in \mathrm{Rep}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ be a representation. We can consider the differential of this map

$$F_* : \mathrm{T}_{F^{-1}(\phi)}Q \to \mathrm{T}_\phi\mathrm{Rep}(\Gamma, \mathrm{PSL}(2,\mathbb{C})).$$

Considering the isomorphisms $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{C})_{Ad_\phi}) \cong \mathrm{T}_\phi\mathrm{Rep}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ and $\mathrm{T}_{F^{-1}(\phi)}Q \cong \mathbb{R}^{12g-12}$, we get

$$F_*^{-1} : H^1(\Sigma, \mathfrak{sl}(2,\mathbb{C})_{Ad_\phi}) \to \mathbb{R}^{12g-12}.$$

Here, $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{C})_{Ad_\phi})$ is the first cohomology group of Σ with twisted coefficients. Considering the corresponding basis $\mathbf{h}^1 = F_*(\boldsymbol{\alpha})$ of $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{C})_{Ad_\phi})$ and Theorem 2.3.6 we have

$$\left|\mathrm{Tor}(\Sigma, \{\mathbf{h}^1\})\right| = \sqrt{\left|\det\begin{bmatrix}\omega_{\mathrm{PSL}(2,\mathbb{C})}\\ \mathbf{h}^1\end{bmatrix}\right|}. \tag{4.4}$$

Let σ_1, σ_2 be elements of $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{C})_{Ad_\phi})$. From the fact that $F^*(\omega_{\mathrm{PSL}(2,\mathbb{C})}) = \frac{1}{\pi}\omega_{\mathrm{nat}}$ it follows

$$\begin{aligned}\omega_{\mathrm{PSL}(2,\mathbb{C})}(\sigma_1, \sigma_2) &= \frac{1}{\pi}(F^*)^{-1}(\omega_{\mathrm{nat}})(\sigma_1, \sigma_2)\\ &= \frac{1}{\pi}\omega_{\mathrm{nat}}(F_*^{-1}(\sigma_1), F_*^{-1}(\sigma_2))\\ &= \frac{1}{\pi}\omega_{\mathrm{nat}}(\beta_1, \beta_2), \end{aligned} \tag{4.5}$$

where, $\beta_i \in \mathbb{R}^{12g-12}$ for $i = 1, 2$.

Combining the equations (4.4), (4.5) and using the fact $\dim_{\mathbb{R}}\left(H^1(\Sigma, \mathfrak{sl}(2,\mathbb{C})_{Ad_\phi})\right) = 12g - 12$ we obtain the formula

$$|\mathrm{Tor}(\Sigma, \{\boldsymbol{\alpha}\})| = \pi^{6-6g}\sqrt{\det\begin{bmatrix}\omega_{\mathrm{nat}}\\ \boldsymbol{\alpha}\end{bmatrix}}.$$

□

4.2 Schottky Uniformization

Let $\mathcal{CP}(\Sigma)$ be the deformation space of complex projective structures, namely the space of equivalence classes of projective structures associated to Σ and $\omega_{\mathcal{CP}}$ denote the holomorphic symplectic structure on this space obtained by pulling back, using the developing map. Let us also consider the natural symplectic structure ω_{nat} on the cotangent bundle $\mathrm{T}^*\mathrm{Teich}(\Sigma)$.

Definition 4.2.1 *([41])*Let Γ be a Fuchsian group acting on the unit disc Δ so that one can consider the closed Riemann surface Σ as Δ/Γ. The tuple $(\Delta, \Gamma, F : \Delta \to \Sigma)$ is a *Fuchsian uniformization* of Σ. Here $\Gamma : \langle A_1, \ldots, A_g, B_1, \ldots, B_g : \prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1} = 1\rangle$, $N = \langle\langle B_1, \ldots, B_g\rangle\rangle$ be the normal envelope of $B_1, \ldots, B_g$ inside Γ. The free group $G = \Gamma/N$ with rank g is a *Schottky group* and $\Omega = \Delta/N$ is the region of discontinuity. Now, the tuple $(\Omega, G, P : \Omega \to \Sigma)$ is a *Schottky uniformization.*

Considering the section $s : \mathrm{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$ obtained by Schottky uniformization, I. Biswas proved in [22] that

$$L_s{}^*\omega_{\mathcal{CP}} = \frac{1}{\pi}\omega_{\mathrm{nat}},$$

where $L_s : \mathrm{T}^*\mathrm{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$ is a smooth diffeomorphism.

4.3 Earle Uniformization

Recall that the deformation space of complex projective structures $\mathcal{CP}(\Sigma)$ is the space of equivalence classes of projective structures associated to Σ with the symplectic form $\omega_{\mathcal{CP}}$. Let us also recall that $\mathrm{T}^*\mathrm{Teich}(\Sigma)$ is the holomorphic cotangent bundle of the Teicmüller space $\mathrm{Teich}(\Sigma)$ with the symplectic form ω_{nat}.

The natural projection $p : \mathcal{CP}(\Sigma) \to \mathrm{Teich}(\Sigma)$ sends a projective structure on Σ to the underlying complex structure on Σ. For any smooth section $f : \mathrm{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$ of this projection, namely $p \circ f = Id_{\mathrm{Teich}(\Sigma)}$, there is a diffeomorphism $T_f : \mathrm{T}^*\mathrm{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$ which sends (t, w) to the projective structure $f(t)+w$. Here, w is any cotangent vector of Teichmüller space, over $t \in \mathrm{Teich}(\Sigma)$.

First of all, we will recall the Earle uniformization. For more detail about this uniformization, we refer to reader [43].

Let Γ be a quasi-Fuchsian group, namely the limit set $\Lambda(\Gamma)$ is a Jordan curve in the extended plane, that Γ maps each of the Jordan regions Ω^+ and Ω^- bounded by $\Lambda(\Gamma)$ into itself and that the quotient maps $\Omega^+ \to \Omega^+/\Gamma$ and $\Omega^- \to \Omega^-/\Gamma$ are unramified coverings of closed Riemann surfaces with genus g. If we lift a canonical dissection of the surface Ω^+/Γ to Ω^+, we can choose an ordered $2g$-tuple $\sigma = (A_1, B_1, A_2, B_2, \ldots, A_g, B_g)$ of Möbius transformations such that the A_j and B_j generate Γ and satisfy the relation

$$\prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1} = I.$$

Definition 4.3.1 The pair (σ, Γ) as in above, is called a *marked quasi-Fuchsian group.*

Theorem 4.3.2 *([43]) Let Σ be a closed Riemann surface of genus at least 2 with canonical homotopy basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ and let ϕ be an automorphism of $\pi_1(\Sigma)$ induced by an orientation reversing diffeomorphism of Σ. Then, there is a unique normalized marked quasi-Fuchsian group Γ such that:*

- *the map $\pi_1(\Sigma) \to \Gamma$ that sends a_j to A_j and b_j to B_j, $1 \le j \le g$, is induced by a conformal map $\Sigma \to \Omega^+/\Gamma$,*

- *there is a conformal map $F : \Omega^- \to \Omega^+$ such that*

$$F(\gamma z) = \phi(\gamma) F(z),$$

for all $\gamma \in \Gamma$ and $z \in \Omega^-$.

If ϕ is an involution then F is a Möbius transformation of order two. Moreover, F and Γ generate a Kleinian group whose deformation space is $\mathrm{Teich}(\Sigma)$.

In 2008, P. Ares-Gastesi and I. Biswas in [23] proved that for the holomorphic section

$$e : \mathrm{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$$

given by the Earle uniformization [43], following equality satisfies

$$L_e^* \omega_{\mathcal{CP}} = \pi \omega_{\mathrm{nat}}. \tag{4.6}$$

Here, $L_e : \mathrm{T}^*\mathrm{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$ is the biholomorphism given by the section e.

Note 4.3.3 Note that, P. Ares-Gastesi and I. Biswas in [23] also expressed that the equality in 4.6 remains true if e is replaced by a large class of sections f satisfying the following conditions:

- f is holomorphic, and
- The Kleinian reciprocity aplies to f.

Here, for the definition of the Kleinian reciprocity see [44]. Namely, for any section

$$f : \mathrm{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$$

satisfying the above conditions we can consider the biholomorphism L_f and then we get the equality

$$L_f^* \omega_{\mathcal{CP}} = \pi \omega_{\mathrm{nat}}. \tag{4.7}$$

Note that, Schottky section also satisfies these conditions.

By combining Note 4.3.3 and Theorem 4.1.6, we get the following result:

Corollary 4.3.4 *Let Σ be a closed orientable surface with genus at least 2 and $\phi : \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{C})$ be an element of the space $\mathrm{Rep}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$. Let $f : \mathrm{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$ be a section satisfying the above conditions. If $\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^{12g-12}$ is a basis of $\mathbb{R}^{12g-12}$, then*

$$|\mathrm{Tor}(\Sigma, \{\boldsymbol{\alpha}\})| = \pi^{6g-6} \sqrt{\det \begin{bmatrix} \omega_{\mathrm{nat}} \\ \boldsymbol{\alpha} \end{bmatrix}}.$$

Here, $\boldsymbol{\alpha} = f_*^{-1}(\mathbf{h}^1)$ is obtained by the corresponding basis $\mathbf{h}^1$ of $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{C})_{Ad_\phi})$.

5 APPLICATIONS

5.1 Complex Projective Structures on the Boundary of a Compact 3-manifold

For a marked complex structure X on Σ, there exists a representation $\phi : \pi_1(\Sigma) \to PSL(2, \mathbb{R})$ by uniformization theorem such that $X \approx \mathbb{H}/\Gamma$ as a Riemann surface. Here, Γ denotes $\phi(\pi_1(\Sigma))$. Since this quotient inherits a projective structure then it defines a section

$$\sigma_{\mathcal{F}} : \text{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$$

to the projection

$$p : \mathcal{CP}(\Sigma) \to \text{Teich}(\Sigma) \tag{5.1}$$

which is called the *Fuchsian section.* $\sigma_{\mathcal{F}}(\text{Teich}(\Sigma))$ is called as the deformation space of Fuchsian structures on Σ. It can be considered as an embedded copy of $\text{Teich}(\Sigma)$ in $\mathcal{CP}(\Sigma)$.

Let us consider two marked complex structures $(X^+, X^-) \in \text{Teich}(\Sigma) \times \text{Teich}(\overline{\Sigma})$. Here, $\overline{\Sigma}$ denotes the reversed oriented surface Σ. Then, there exists a unique representation $\phi : \pi_1(\Sigma) \to \Gamma \subset \text{PSL}(2, \mathbb{C})$ up to conjugation. Let Ω be the domain of discontinuity and it is the disjoint union of two simply connected domains Ω^+ and Ω^-. Therefore, the given two marked complex structures (X^+, X^-) can be considered as $X^+ \approx \Omega^+/\Gamma$ and $X^- \approx \Omega^-/\Gamma$. From Bers' Theorem

$$\begin{aligned} \beta = (\beta^+, \beta^-) : \text{Teich}(\Sigma) \times \text{Teich}(\overline{\Sigma}) &\to \mathcal{CP}(\Sigma) \times \mathcal{CP}(\overline{\Sigma}) \\ (X^+, X^-) &\mapsto (\beta^+(X^+, X^-), \beta^-(X^+, X^-)) \end{aligned}$$

is a holomorphic section of

$$p \times p : \mathcal{CP}(\Sigma) \times \mathcal{CP}(\overline{\Sigma}) \to \text{Teich}(\Sigma) \times \text{Teich}(\overline{\Sigma}).$$

Definition 5.1.1

- If $X^- \in \text{Teich}(\overline{\Sigma})$ is fixed, then the following map

$$\sigma_{X^-} := \beta^+(., X^-) : \text{Teich}(\Sigma) \to \mathcal{CP}(\Sigma)$$

  is a holomorphic section to p in (5.1), which is called a *Bers section* and its image $\sigma_{X^-}(\text{Teich}(\Sigma)) \subset \mathcal{CP}(\Sigma)$ is called a *Bers slice.*

- If $X^+ \in \text{Teich}(\Sigma)$ is fixed, then the following map

$$f_{X^+} := \beta^+(X^+, .) : \text{Teich}(\overline{\Sigma}) \to \mathcal{CP}(\Sigma)$$

is an embedding of $\text{Teich}(\overline{\Sigma})$ in the fiber $p^{-1}(X^+) \subset \mathcal{CP}(\Sigma)$. Thus, f_{X^+} is called a *Bers embedding*.

- Finally,

$$\mathcal{QF}(\Sigma) := \beta^+(\text{Teich}(\Sigma) \times \text{Teich}(\overline{\Sigma})) \subset \mathcal{CP}(\Sigma)$$

is called *the deformation space of standard quasi-Fuchsian structures* on Σ. This space is an open neighborhood of the deformation space of Fuchsian structures on Σ in $\mathcal{CP}(\Sigma)$.

Let $\widehat{M}$ be a smooth, connected,oriented, compact,irreducible, atoroidal 3-manifold with boundary and infinite fundamental group. Recall that irreducible means that every embedded 2-sphere bounds a ball and atoroidal means that it does not contain any embedded, non-boundary parallel, incompressible tori.

Let M denote the interior of the manifold $\widehat{M}$, namely $M = \widehat{M} - \partial\widehat{M}$. Assume that the boundary $\partial\widehat{M}$ is incompressible and contains no tori. Here, incompressible means that the map $\iota_* : \pi_1(\partial\widehat{M}) \to \pi_1(\widehat{M})$ induced by the inclusion map ι is injective. Therefore, $\partial\widehat{M}$ consists of a finite number of surfaces $\Sigma_1, \ldots, \Sigma_N$ of genera at least 2.

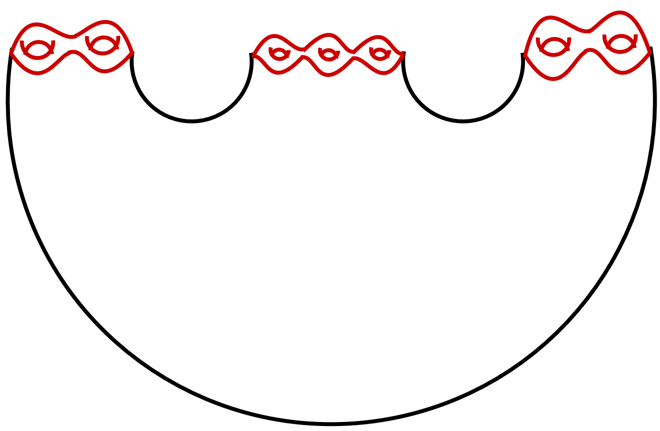


Figure 5.1: A 3-manifold with incompressible boundary

The Teichmüller space of the boundary $\partial\widehat{M}$ is considered as follows

$$\mathrm{Teich}(\partial\widehat{M}) = \mathrm{Teich}(\Sigma_1) \times \cdots \times \mathrm{Teich}(\Sigma_N),$$

and also the deformation space of complex projective structures on the boundary $\partial\widehat{M}$ is described

$$\mathcal{CP}(\partial\widehat{M}) = \mathcal{CP}(\Sigma_1) \times \cdots \times \mathcal{CP}(\Sigma_N),$$

and finally there is a holomorphic "forgetful" projection

$$p = p_1 \times \cdots \times p_N : \mathcal{CP}(\partial\widehat{M}) \to \mathrm{Teich}(\partial\widehat{M}).$$

Let

$$\mathrm{proj}_k : \mathcal{CP}(\partial\widehat{M}) \to \mathcal{CP}(\Sigma_k) \tag{5.2}$$

denotes the k-th projection map.

Definition 5.1.2 *The space of convex cocompact hyperbolic structures* $\mathcal{HC}(M)$ is the quotient of the set of convex cocompact hyperbolic metrics on M by the group of orientation-preserving diffeomorphisms of M that are homotopic to the identity.

Note that $\mathcal{HC}(M)$ is a connected component of the interior of the subset of discrete and faithful representations in the character variety $\mathrm{Hom}(M, \mathrm{PSL}(2,\mathbb{C}))$ (see [48, 49]).

Let $\varphi : \mathcal{HC}(M) \to \mathcal{CP}(\partial\widehat{M})$ be the map such that for any element of $\mathcal{HC}(M)$ there is a marked complex structure. Moreover, this map is holomorphic (see [50]). If we consider the induced conformal structure on $\partial\widehat{M}$, then in the following diagram

$$\begin{tikzcd}
\mathcal{HC}(M) \arrow[r, "\varphi"] \arrow[dr, dashed, "\psi"] & \mathcal{CP}(\partial\widehat{M}) \arrow[d, "p"] \\
 & \mathrm{Teich}(\partial\widehat{M})
\end{tikzcd}$$

the map $\psi = p \circ \varphi$ can be defined. The following theorem due to [51, 52, 53, 54, 48, 55, 49] as follows:

Theorem 5.1.3 *The map* $\psi : \mathcal{HC}(M) \to \mathrm{Teich}(\partial\widehat{M})$ *is bijective.*

The direct result of above theorem is that the map

$$\beta = \varphi \circ \psi^{-1} : \mathrm{Teich}(\partial\widehat{M}) \to \mathcal{CP}(\partial\widehat{M})$$

is a canonical holomorphic section to

$$p : \mathcal{CP}(\partial\widehat{M}) \to \mathrm{Teich}(\partial\widehat{M}),$$

and β is called *the generalized simultaneous uniformization section.* Thanks to this map "generalized Bers section" and "generalized Bers embeddings" can be defined as follows. Firstly, only one of the boundary components' conformal structure is varied then some other or the same boundary components' the resulting complex projective structure is checked. More clearly, let $X_i \in \mathrm{Teich}(\Sigma_i)$ be marked complex structures which are fixed for all $i \neq j$, where $j \in \{1, \ldots, N\}$. By using the canonical injection

$$\begin{aligned} \iota_{X_i} : \mathrm{Teich}(\Sigma_j) &\to \mathrm{Teich}(\partial\widehat{M}) \\ X &\mapsto (X_1, \ldots, X_{j-1}, X, X_{j+1}, \ldots, X_N) \end{aligned}$$

the map $f_{X_i,k} = \mathrm{proj}_k \circ \beta \circ \iota_{X_i}$ defines as follows:

$$\begin{array}{ccc} \mathrm{Teich}(\partial\widehat{M}) & \xrightarrow{\beta} & \mathcal{CP}(\partial\widehat{M}) \\ \iota_{X_i}\Big\uparrow & & \Big\downarrow \mathrm{proj}_k \\ \mathrm{Teich}(\Sigma_j) & \xrightarrow{f_{X_i,k}} & \mathcal{CP}(\Sigma_k). \end{array}$$

Here, if $j = k$ then $\sigma_{(X_i)} := f_{X_i,j}$ is called a *generalized Bers section* to $\mathrm{proj}_j : \mathcal{CP}(\Sigma_j) \to \mathrm{Teich}(\Sigma_j)$. Otherwise, if $j \neq k$ then $f_{X_i,k}$ is called a generalized Bers embedding which maps $\mathrm{Teich}(\Sigma_j)$ in the affine fiber $P(X_k) \subset \mathcal{CP}(\Sigma_k)$. Here, $P(X_k)$ denotes the set of marked projective structures on Σ_k whose underlying complex structure is a fixed point X_k in $\mathrm{Teich}(\Sigma_k)$. For more detail see [50].

Let $\widehat{M}$ be a smooth connected oriented compact irreducible atoroidal 3-manifold with boundary and infinite fundamental group. $\partial\widehat{M}$ denotes its incompressible boundary which consists of a finite number of surfaces $\Sigma_1, \ldots, \Sigma_N$ of genera at least 2 and contains no tori. Let $\mathrm{D}(\widehat{M})$ be the double of $\widehat{M}$ and $\phi : \pi_1(\widehat{M}) \to \mathrm{PSL}(2, \mathbb{C})$ be a representation so that the restriction $\phi \circ \iota_*$ of this representation to $\partial\widehat{M}$ belongs to $\mathrm{Teich}(\partial\widehat{M})$. Here,

ι_* is the induced map obtained by the inclusion $\partial\widehat{M} \hookrightarrow \widehat{M}$. Thus, note that the map $\phi \circ \iota_*$ is an element of $\mathrm{Rep}(\partial\widehat{M}, \mathrm{PSL}(2,\mathbb{C}))$. We can also consider the representation $\varrho : \pi_1(\mathrm{D}(\widehat{M})) \to \mathrm{PSL}(2,\mathbb{C})$ which is defined as

$$\pi_1(\mathrm{D}(\widehat{M})) \xrightarrow{\mathrm{proj}_{\pi_1(\widehat{M})}} \pi_1(\widehat{M}) \xrightarrow{\phi} \mathrm{PSL}(2,\mathbb{C}),$$

where $\mathrm{proj}_{\pi_1(\widehat{M})}$ is the projection map.

If we consider the short-exact sequence of chain complexes

$$0 \to C_*\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right) \to C_*\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \oplus C_*\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \to C_*\left(\mathrm{D}(\widehat{M}); \mathfrak{g}_{Ad_\varrho}\right) \to 0, \tag{5.3}$$

where $C_*\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right) = \oplus_{t=1}^N C_*\left(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}}\right)$ and $\iota_{t*} : \pi_1(\Sigma_t) \to \pi_1(\widehat{M})$, then we get the following Mayer-Vietoris long exact sequence:

$$\begin{aligned}
0 \to H_3\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right) &\to H_3\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \oplus H_3\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \to H_3\left(\mathrm{D}(\widehat{M}); \mathfrak{g}_{Ad_\varrho}\right) \\
\to H_2\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right) &\to H_2\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \oplus H_2\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \to H_2\left(\mathrm{D}(\widehat{M}); \mathfrak{g}_{Ad_\varrho}\right) \\
\to H_1\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right) &\to H_1\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \oplus H_1\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \to H_1\left(\mathrm{D}(\widehat{M}); \mathfrak{g}_{Ad_\varrho}\right) \\
\to H_0\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right) &\to H_0\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \oplus H_0\left(\widehat{M}; \mathfrak{g}_{Ad_\phi}\right) \to H_0\left(\mathrm{D}(\widehat{M}); \mathfrak{g}_{Ad_\varrho}\right) \to 0.
\end{aligned} \tag{5.4}$$

Here, $H_i\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right) = \oplus_{t=1}^N H_i\left(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}}\right)$ for $i = 0,1,2,3$.

Let $\mathbf{h}_i^{\widehat{M}}$ denote the basis of $H_i(\widehat{M}; \mathfrak{g}_{Ad_\phi})$ for $i = 0,1,2,3$. By using above sequences, there are bases $\mathbf{h}_j^{\mathrm{D}(\widehat{M})}$ and $\mathbf{h}_k^{\Sigma_t}$ for $j = 0,1,2,3$, $k = 0,1,2$ and $t = 1,\dots,N$ of $H_j(\mathrm{D}(\widehat{M}); \mathfrak{g}_{Ad_\phi})$, $H_k(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}})$, respectively such that Reidemeister torsion of the long exact sequence (5.4) in these bases equals to 1 [35, Theorem 6.2.2]. In the following result, we will denote the basis of $H_k\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right)$ with $\mathbf{h}_k^{\partial\widehat{M}}$ which is nothing but $\oplus_{t=1}^N \mathbf{h}_k^{\Sigma_t}$.

Theorem 5.1.4 *Let $\widehat{M}, \partial\widehat{M}, \mathrm{D}(\widehat{M}), \phi, \iota_*$ be as in above. Let $\mathbf{h}_i^{\widehat{M}}, \mathbf{h}_j^{\mathrm{D}(\widehat{M})}, \mathbf{h}_k^{\Sigma_t}$ denote the bases for $H_i(\widehat{M}; \mathfrak{g}_{Ad_\phi}), H_j(\mathrm{D}(\widehat{M}); \mathfrak{g}_{Ad_\phi}), H_k(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}})$, respectively, $i, j = 0, 1, 2, 3$, $k = 0, 1, 2$ and $t = 1, \ldots, N$. Then, we get the following formula:*

$$\left|\mathrm{Tor}\left(\widehat{M}, \left\{\mathbf{h}_i^{\widehat{M}}\right\}_0^3\right)\right| = \sqrt[4]{\left|\det\begin{bmatrix}\omega_G \\ \mathbf{h}^1\end{bmatrix}\right|}.$$

Here, $\mathbf{h}^1$ is the Poincare dual basis of $H^1(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}})$ corresponding to basis $\mathbf{h}_1^{\partial\widehat{M}}$ of $H_1(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}})$ and Atiyah-Bott-Goldman symplectic form ω_G is considered as in [50].

Proof. Let $\widehat{M}$ be the 3-manifold as in above with boundary which is incompressible and contains no tori (e.g. see Figure 5.1). Recall that the boundary is incompressible means that the map

$$\iota_* : \pi_1(\partial\widehat{M}) \to \pi_1(\widehat{M})$$

induced by the inclusion map

$$\iota : \partial\widehat{M} \hookrightarrow \widehat{M}$$

is injective. Let $\Sigma_1, \ldots, \Sigma_N$ be the boundary components of $\widehat{M}$. Let us denote by $g_t \geq 2$ the genus of Σ_t, for $t = 1, \ldots, N$. Recall that $\mathrm{D}(\widehat{M})$ denotes the double of the manifold $\widehat{M}$ which is a closed 3-manifold.

Let us consider a cell-decomposition K of $\partial\widehat{M}$ which is obtained by disjoint union of cell-decompositions K_t of Σ_t for $t = 1, \ldots, N$ and K' be the dual cell-decomposition of K. Considering the intersection number pairing in (5.1)

$$(\cdot, \cdot)_{i,2-i} : C_i(K; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}) \times C_{2-i}(K'; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}) \to \mathbb{C},$$

we conclude that the chain complex $C_*\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right)$ is a symplectic chain complex. We can extend the intersection form to twisted homologies as in subsection 2.3 and we get the following commutative diagram for each Σ_t

$$\begin{array}{ccccc}
H^1(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}}) & \times & H^1(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}}) & \xrightarrow{\smile_B} & H^2(\Sigma_t; \mathbb{C}) \\
\downarrow \mathrm{PD} & & \downarrow \mathrm{PD} & \circlearrowleft & \downarrow \int_{\Sigma_t} \\
H_1(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}}) & \times & H_1(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}}) & \xrightarrow{(,)_{1,1}} & \mathbb{C}.
\end{array}$$

Recall that $\omega_G^{(t)} : H^1(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}}) \times H^1(\Sigma_t; \mathfrak{g}_{Ad_{\phi\circ\iota_{t*}}}) \xrightarrow{\smile_B} H^2(\Sigma_t; \mathbb{C}) \xrightarrow{\int_{\Sigma_t}} \mathbb{C}$ is the Atiyah-Bott-Goldman symplectic form. By using these symplectic forms, we have the Atiyah-Bott-Goldman symplectic form

$$\omega_G : H^1(\partial\widehat{M}, \mathfrak{g}_{Ad_{\phi\circ\iota_*}}) \times H^1(\partial\widehat{M}, \mathfrak{g}_{Ad_{\phi\circ\iota_*}}) \to H^2(\partial\widehat{M}, \mathbb{C}) \xrightarrow{\int_{\partial\widehat{M}}} \mathbb{C},$$

which was proved in [50] that

$$\omega_G = \mathrm{proj}_1^*\omega_G^{(1)} + \ldots + \mathrm{proj}_N^*\omega_G^{(N)}. \tag{5.5}$$

Here, proj_t is the t-th projection map in (5.2).

Now, if we combine the fact that Reidemesiter torsion of the long exact sequence (5.4) corresponding to above bases equals to 1 and the Theorem 2.2.3 we get:

$$\left(\mathrm{Tor}\left(\widehat{M}, \left\{\mathbf{h}_i^{\widehat{M}}\right\}_0^3\right)\right)^2 = \mathrm{Tor}\left(\partial\widehat{M}, \left\{\mathbf{h}_k^{\partial\widehat{M}}\right\}_0^2\right) \mathrm{Tor}\left(\mathrm{D}(\widehat{M}), \left\{\mathbf{h}_j^{\mathrm{D}(\widehat{M})}\right\}_0^3\right).$$

On the other hand, from the fact that in [32], we have $\mathrm{Tor}\left(\mathrm{D}(\widehat{M}), \left\{\mathbf{h}_j^{\mathrm{D}(\widehat{M})}\right\}_0^3\right) = 1.$ Finally, if we use Theorem 2.3.6, then we can write the Reidemeister torsion formula of 3-manifold $\widehat{M}$ through the symplectic form ω_G as follows:

$$\left|\mathrm{Tor}\left(\widehat{M}, \left\{\mathbf{h}_i^{\widehat{M}}\right\}_0^3\right)\right| = \sqrt[4]{\left|\det\begin{bmatrix}\omega_G \\ \mathbf{h}^1\end{bmatrix}\right|}.$$

Clearly from the definition of ω_G through the symplectic forms $\omega_G^{(t)}$ it follows:

$$\det\begin{bmatrix}\omega_G \\ \mathbf{h}^1\end{bmatrix} = \det\left[\begin{array}{c|c|c} \begin{matrix}\omega_G^{(1)} \\ \mathbf{h}^1_{\Sigma_1}\end{matrix} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \begin{matrix}\omega_G^{(N)} \\ \mathbf{h}^1_{\Sigma_N}\end{matrix} \end{array}\right]. \tag{5.6}$$

□

We should note that by not using symplectic property of the chain complex $C_*\left(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}}\right)$ similar result was obtained in [35] . On the other hand, in this thesis thanks to the definition of Atiyah-Bott-Goldman symplectic form ω_G as in [50] we proved it by using the symplectic chain complex theory.

Let ω_G denote the complex symplectic structure on $\mathcal{CP}(\partial\widehat{M})$ as in equation (5.5) and $(\tau^\beta)^*\omega_{\mathrm{nat}}$ be the complex symplectic structure obtained in [50] by the identification

$\tau^{\beta} : \mathcal{CP}(\partial\widehat{M}) \xrightarrow{\sim} \mathrm{T}^*\mathrm{Teich}(\partial\widehat{M})$. B. Loustau proved in [50, Theorem 6.14] that

$$(\tau^{\beta})^*\omega_{\mathrm{nat}} = -i\omega_G. \tag{5.7}$$

Here, as in ω_G, ω_{nat} is also obtained by similarly as follows

$$\mathrm{proj}_1^*\omega_{\mathrm{nat}}^{(1)} + \ldots + \mathrm{proj}_N^*\omega_{\mathrm{nat}}^{(N)},$$

where $\omega_{\mathrm{nat}}^{(t)}$ denotes the complex symplectic structure on $\mathrm{T}^*\mathrm{Teich}(\Sigma_t)$ and proj_t is the t-th projection map from $\mathrm{T}^*\mathrm{Teich}(\partial\widehat{M})$ to $\mathrm{T}^*\mathrm{Teich}(\Sigma_t)$.

Combining Theorem 5.1.4, equation (5.7) and using the method as in Theorem 4.1.6, we get the following result:

Corollary 5.1.5 *Under the conditions in Theorem 5.1.4 and considering the equation (5.7), then we get*

$$\left|\mathrm{Tor}\left(\widehat{M}, \left\{\mathbf{h}_i^{\widehat{M}}\right\}_0^3\right)\right| = \sqrt[4]{\det\begin{bmatrix}\omega_{\mathrm{nat}} \\ \alpha\end{bmatrix}}.$$

Here, $\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^{\sum_{t=1}^N(6g_t-6)}$ *is the basis of* $\mathbb{R}^{\sum_{t=1}^N(6g_t-6)}$ *obtained by* $\boldsymbol{\alpha} = \tau_*^{\beta}(\mathbf{h}^1)$. *Recall that,* $\mathbf{h}^1$ *is the Poincare dual basis of* $H^1(\partial\widehat{M}; \mathfrak{g}_{Ad_{\phi\circ\iota_*}})$.

Note that, here ω_{nat} denotes the standard symplectic form on $\mathbb{R}^{\sum_{t=1}^N(6g_t-6)}$.

5.2 Future Work

Let Σ be a surface with a hyperbolic metric m (i.e. Riemannian metric with constant curvature -1). A *geodesic lamination* is a closed subset of Σ which can be decomposed as a disjoint union of simple complete m-geodesics, they are called its *leaves*. Here, a geodesic is *complete* if it cannot be extend to a longer geodesic and it is *simple* if it has no transverse self-intersection point. A geodesic lamination λ is *maximal* if it is not contained in any larger geodesic lamination which is equivalent to the property that the complement $\Sigma - \lambda$ consists of finitely many disjoint infinite triangles.

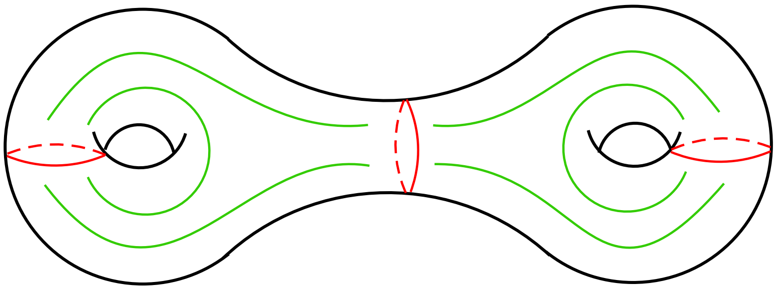


Figure 5.2: A maximal lamination

Let us fixed a geodesic lamination $\lambda \subset \Sigma$. An $\mathbb{R}$-*valued transverse cocycle* σ for λ is a real-valued function on the set of all arcs k transverse to the leaves of λ which satisfies the following two conditions:

- If k is transverse to λ which is decomposed into two arcs k_1, k_2 with disjoint interiors, we have the property

$$\sigma(k) = \sigma(k_1) + \sigma(k_2),$$

- If k and k' are homotopic through a family of arcs which are all transverse to λ, then we have

$$\sigma(k) = \sigma(k').$$

The $\mathbb{R}$-valued transverse cocycles for the geodesic lamination λ form a real vector space $\mathcal{Z}(\lambda; \mathbb{R})$.

A *train track* Φ on the surface Σ is a family of finitely many 'long' rectangles $e_1, \ldots, e_n$ which are foliated by arcs parallel to the 'short' sides and which meet only along arcs (possibly reduced to a point) contained in their short sides. And also, a train track must satisfy the following conditions:

- each point of the 'short' side of a rectangle also belongs to another rectangle, and each component of the union of the short sides of all rectangles is an arc, as opposed to a closed curve,
- the closure of the complement $\Sigma - \Phi$ has a certain number of 'spikes', namely the points where at least three rectangles meet and no component of this closure is a disc with $0, 1$ or 2 spikes or an annulus with no spike.

The rectangles e_i are the *edges* of the train track Φ and the leaves of the foliation of Φ are the *ties* of the train track. The finitely many ties where several edges meet are the *switches* of the train track. A tie which is not a switch is *generic*. The geodesic lamination λ is *carried* by the train track Φ if it is contained in the interior of Φ and if its leaves are transverse to the ties of Φ. Every geodesic lamination is carried by some train track. For more detail see, for instance [56, 57].

For a fixed train track Φ, let $\mathcal{W}(\Phi, \mathbb{R})$ denotes the vector space of all *edge weight systems* for Φ. More precisely, an edge weight system assigns $a(e) \in \mathbb{R}$ to each edge e of the train track such that for each switch s the following *switch relation* holds

$$a(e_i) = a(e_j) + a(e_k)$$

where s is adjacent to the edge e_i on one side and to the edges e_j and e_k on the other side.

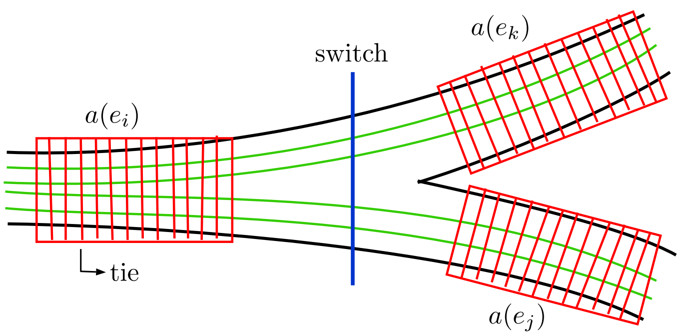


Figure 5.3: The switch relation

If the geodesic lamination λ is carried by the train track Φ, a transverse cocycle $\sigma \in \mathcal{Z}(\lambda; \mathbb{R})$ defines an edge weight system $a_\sigma \in \mathcal{W}(\Phi, \mathbb{R})$ by

$$a_\sigma(e) = \sigma(k_e),$$

where k_e is an arbitrary tie of the edge e. This map defines a linear isomorphism between the spaces $\mathcal{Z}(\lambda; \mathbb{R})$ and $\mathcal{W}(\Phi, \mathbb{R})$. In addition, if λ is a maximal geodesic lamination, then these two vector spaces are isomorphic to $\mathbb{R}^{3|\chi(\Sigma)|}$. See [58] for more detail.

So far, we consider the $\mathbb{R}$-valued transverse cocycle. This can be generalized to $\mathbb{R}^{n-1}$-valued transverse cocycles, straightforwardly. Here, the orientation is really important point, so for more detail see [59, 60].

For a geodesic lamination $\lambda \subset \Sigma$, a *twisted* $\mathbb{R}^{n-1}$*-valued transverse cocycle* for λ assigns a vector $\sigma(k) \in \mathbb{R}^{n-1}$ to each oriented arc $k \subset \Sigma$ that is transverse to λ such that the following conditions hold:

- σ is finitely additive, namely

  $$\sigma(k) = \sigma(k_1) + \sigma(k_2)$$

  where the oriented arc k transverse to λ is split into two oriented arcs k_1 and k_2 with disjoint interiors,

- σ is invariant under homotopy. More precisely, if k and k' are homotopic through a family of arcs which are all transverse to λ,

- For every oriented transverse arc k the following equality holds

  $$\sigma(\overline{k}) = \overline{\sigma(k)}.$$

  Here, $\overline{k}$ denotes the reverse orientated of k and $x \mapsto \overline{x}$ is the involution map of $\mathbb{R}^2$ associating $\overline{x} = (x_{n-1}, x_{n-2}, \ldots, x_1)$ to $x = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Let $\mathcal{Z}(\lambda; \mathbb{R}^{n-1})$ denote the vector space of all twisted $\mathbb{R}^{n-1}$-valued transverse cocycles for a maximal geodesic lamination λ and the dimension of this space equal to $6(g-1)(n-1) + \lfloor \frac{1}{2}(n-1) \rfloor$. Here, $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.

Let $\mathfrak{sl}(n,\mathbb{R})$ be the Lie algebra of $\mathrm{PSL}(n,\mathbb{R})$. Let ϕ be an element of $\mathrm{Hit}_n(\Sigma)$. Consider the adjoint representation defined by for every $\gamma \in \pi_1(\Sigma)$

$$\begin{array}{rcl} Ad_\phi(\gamma) : \mathfrak{sl}(n,\mathbb{R}) & \to & \mathfrak{sl}(n,\mathbb{R}) \\ u & \mapsto & \phi(\gamma)u\phi(\gamma)^{-1}. \end{array}$$

Let us consider the Cartan-Killing bilinear form $B : \mathfrak{sl}(n,\mathbb{R}) \times \mathfrak{sl}(n,\mathbb{R}) \to \mathbb{R}$ is defined as $B(u,v) = 2n\mathrm{Tr}(uv)$, where Tr denotes the trace. Clearly B is preserved by the adjoint representation. Therefore, this enables us to define a cup product

$$\smile_B : C^1(\Sigma, \mathfrak{sl}(n,\mathbb{R})_{Ad_\phi}) \times C^1(\Sigma, \mathfrak{sl}(n,\mathbb{R})_{Ad_\phi}) \to C^2(\Sigma,\mathbb{R})$$

which induces an antisymmetric bilinear form

$$\omega_{\mathrm{PSL}(n,\mathbb{R})} : H^1(\Sigma, \mathfrak{sl}(n,\mathbb{R})_{Ad_\phi}) \times H^1(\Sigma, \mathfrak{sl}(n,\mathbb{R})_{Ad_\phi}) \to H^2(\Sigma,\mathbb{R}) \xrightarrow{\int_\Sigma} \mathbb{R}.$$

Here, $H^2(\Sigma,\mathbb{R})$ isomorphic to $\mathbb{R}$ thanks to evaluation on the fundamental class of the oriented surface Σ.

$\omega_{\mathrm{PSL}(n,\mathbb{R})}$ is a symplectic form on $\mathrm{Hit}_n(\Sigma)$ called Atiyah-Bott-Goldman symplectic form (see [15]). Recall that $\mathrm{T}_\phi\mathrm{Hit}_n(\Sigma)$ is isomorphic to $H^1(\Sigma, \mathfrak{sl}(n,\mathbb{R})_{Ad_\phi})$ [61].

In [62] for general n, $\omega_{\mathrm{PSL}(n,\mathbb{R})}(u_{\sigma_1}, u_{\sigma_2})$ was computed for the vectors $u_{\sigma_1}, u_{\sigma_2} \in \mathrm{T}_\phi\mathrm{Hit}_n(\Sigma)$ associated to the infinitesimal shearing of $\phi \in \mathrm{Hit}_n(\Sigma)$ according to twisted transverse cocycles $\sigma_1, \sigma_2 \in \mathcal{Z}(\lambda; \mathbb{R}^{n-1})$ for the geodesic lamination λ. Let us recall this formula:

Theorem 5.2.1 *([62]) Let Σ be a closed oriented surface with genus at least 2 and let λ be a maximal geodesic lamination carried by a train track Φ in Σ. If the vectors $u_{\sigma_1}, u_{\sigma_2} \in \mathrm{T}_\phi\mathrm{Hit}_n(\Sigma)$ are tangent to the shearing deformations of $\phi \in \mathrm{Hit}_n(\Sigma)$ along λ respectively associated to the transverse cocycles $\sigma_1, \sigma_2 \in \mathcal{Z}(\lambda; \mathbb{R}^{n-1})$, then the Atiyah-Bott-Goldman symplectic form,*

$$\omega_{\mathrm{PSL}(n,\mathbb{R})}(u_{\sigma_1}, u_{\sigma_2}) = \frac{1}{2} \sum_{a,b=1}^{n-1} \sum_s C(a,b)(\sigma_1^{(a)}(e_s^{\mathrm{right}})\sigma_2^{(b)}(e_s^{\mathrm{left}}) - \sigma_1^{(a)}(e_s^{\mathrm{left}})\sigma_2^{(b)}(e_s^{\mathrm{right}}))$$

where s changes over all switches of the train track $\widehat{\Phi}$ *and* $e_s^{\text{right}}, e_s^{\text{left}}$ *denote the two edges of* $\widehat{\Phi}$ *outgoing from switch s on the right and on the left, respectively. Here,*

$$C(a,b) = \begin{cases} 2a(n-b) & \text{if } a \leq b \\ 2b(n-a) & \text{if } a \geq b. \end{cases}$$

Note that, the above formula is a generalization of the case for $n = 2$. For more detail about the $n = 2$ case see [17].

As an application, by using the above computation of the Atiyah-Bott-Goldman symplectic form, we can write the Reidemeister torsion formula of some special representations. However, we should be careful about the non-degeneracy of the Atiyah-Bott-Goldman symplectic form on the given submanifold $\mathcal{Z}(\lambda; \mathbb{R}^2)$. For example, if $n = 3$ this space has the odd dimension $12g - 11$. Obviously, the symplectic form can not be non-degenerate on this space. The reason of this is probably that the restriction of the symplectic form to this submanifold has a non-trivial kernel.

As a future work, we first consider to compute this kernel of the shearing transverse cocycle space $\mathcal{Z}(\lambda; \mathbb{R}^2)$ and then we can establish the Reidemeister torsion formula through the restriction of Atiyah-Bott-Goldman symplectic form to this space.

REFERENCES

[1] K. Reidemeister, Homotopieringe und linsenräume, Abh. Math. Semin. Univ. Hambg., 11 **(1935)** 102-109.

[2] W. Franz, Über die Torsion einer Überdeckung, J. Reine Angew. Math., 173 **(1935)** 245-254.

[3] G. de Rham, Reidemeister's torsion invariant and rotations of S^n, in: Differential Analysis (Mumbay,1964), Oxford Univ. Press,London **1964** 27 - 36.

[4] R.C. Kirby, L.C. Siebenmann, On triangulation of manifolds and Haupvermutung, Bull. Amer. Math. Soc., 75 **(1969)** 742-749.

[5] T.A. Chapman, Hilbert cube manifolds and the invariance of Whitehead torsion, Bull. Amer. Math. Soc., 79 **(1973)** 52-56.

[6] T.A. Chapman, Topological invariance of Whitehead torsion, Amer. J. Math., 96 **(1974)** 488-497.

[7] J. Milnor, Two complexes which are homeomorphic but combinatorially distinct, Ann. of Math., 74 (3) **(1961)** 575-590.

[8] J. Milnor, A duality theorem for Reidemeister torsion, Ann. of Math., 76 (1) **(1962)** 137-147.

[9] J. Milnor, Whitehead torsion, Bull. Amer. Math. Soc., 72 (3) **(1966)** 358-426.

[10] E. Witten, On quantum gauge theories in two dimension, Communications in Mathematical Physics, 141 **(1991)** 153 - 209.

[11] J. Dubois, Non abelian Reidemeister torsion and volume form on the SU(2)-presentation space of knot groups, Ann. Inst. Fourier, 55 **(2005)** 1685-1734.

[12] J. Dubois, A volume form on the SU(2)-representation space of knot groups, Algebr. Geom. Topol., 6 **(2006)** 373-404.

[13] N. Hitchin, Lie groups and Teichmüller spaces, Topology, 31 **(1992)** 449 - 473.

[14] A. Weil, Modules des surfaces de Riemann, Séminaire Bourbaki, 168 **(1958)** 413-419.

[15] W.M. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. in Math., (54) (**1984**) 200 - 255.

[16] F. Bonahon, Shearing hyperbolic surfaces, bending pleated surfaces, and Thurston's symplectic form, Ann. Fac. Sci. Toulouse, 5 (**1996**) 233 - 297.

[17] Y. Sozen, F. Bonahon, Weil-Petersson and Thurston symplectic forms, Duke Math. J., 108 (3) **(2001)** 581 - 597.

[18] Y. Sozen, On a volume element of Hitchin component, Fund. Math., 217 **(2012)** 249-264.

[19] W.M. Goldman, Convex real projective Structures on compact surfaces, J. Differential Geometry, (31) (**1990**) 791 - 845.

[20] H.C. Kim, The symplectic global coordinates on the moduli spaceof real projective structures, J. Differential Geometry, (53) **(1999)** 359-401.

[21] S. Kawai, The Symplectic Nature of the Space of Projective Connections on Riemann Surfaces, Math. Ann., (305) **(1966)** 161-182.

[22] I. Biswas, Schottky uniformization and the symplectic structure of the cotangent bundle of a Teichmüller space, Journal of Geometry and Physics, 35 (**2000**) 57-62.

[23] P. Ares-Gastesi, I. Biswas, On the symplectic form of the moduli space of projective structures, Journal of Symplectic Geometry, 6 (3) (**2008**) 239 - 246.

[24] M. Mirzakhani, Weil-Petersson volumes and Intersection theory on the moduli space of curves, Journal of The American Mathematical Society, 20 (1) (**2007**) 1 - 23.

[25] J.G. Ratcliffe, Foundations of Hyperbolic Manifolds, Graduate Texts in Math. 149. Springer, New York **1994**.

[26] R. Benedetti, C. Petronio, Lectures on Hyperbolic Geometry, Springer-Verlag Berlin Heidelberg New York **1992**.

[27] S. Nag, The complex analytic theory of Teichmüller spaces, New York, Wiley **1988**.

[28] Y. Imayoshi, M. Taniguchi, An introduction to Teichmüller spaces, Berlin Heidelberg New York, Springer **1992**.

[29] J.M. Lee, Introduction to Smooth Manifolds, 2nd edn. Graduate Texts in Mathematics New York, Springer **2013**.

[30] J. Porti, Torsion de Reidemeister pour les Varieties Hyperboliques, Memoirs of the Amer. Math. Soc., 128 (612), Amer. Math. Soc., Providence, RI, **1997**.

[31] Y. Sozen, On Reidemeister torsion of a symplectic complex, Osaka J. Math., 45 (1) **(2008)** 1-39.

[32] Y. Sozen, A note on Reidemeister torsion and pleated surfaces, J. Knot Theory Ramifications, 23 (3) **(2014)** 1-32.

[33] J. Milnor, Whitehead torsion, Bull. Amer. Math. Soc., 72 (3) **(1966)** 358-426.

[34] Y. Sozen, Symplectic chain complex and Reidemeister torsion of compact manifolds, Math. Scand., 111 (1) **(2012)** 65-91.

[35] F. Hezenci, Representation Varieties of Free or Surface Group and Reidemeister Torsion, PhD thesis, Hacettepe University **2019**.

[36] F. Hezenci, Y. Sözen, A note on Exceptional Groups and Reidemeister Torsion, Journal of Mathematical Physics, 59 **(2018)** 1 - 17.

[37] C. Ehresmann, Sur les espaces localement homogenes, Enseign. Math., 35 (**1937**) 317 -333.

[38] J. Richter-Gebert, Perspectives on Projective Geometry, Berlin Heidelberg , Springer-Verlag **2011**.

[39] R. Fox, The free differential calculus. I: Derivation in the free group ring, Ann. of Math., 57 (**1953**) 547 -560.

[40] K. Brown, Cohomology of groups, Graduate Texts in Math., 87, Springer, Berlin **1982**.

[41] R.A. Hidalgo, A theoretical algorithm to get a Schottky uniformization from a Fuchsian one, In: Analysis and Mathematica Physics. Trends in Mathematics, Birkhäuser, Basel, **2009** 193 - 204.

[42] B. Aesbischer, M. Borer, M. Kälin, Ch. Leuenberger, H.M. Reimann, Symplectic Geometry, (Progr. in Math.), vol. 124, Basel: Birkhäuser **1994**.

[43] C.J. Earle, Some intrinsic coordinates on Teichmüller space, Proc. Amer. Math. Soc., 83 (**1981**) 527 -531.

[44] C.T. McMullen, The moduli space of Riemann surfaces is Kähler hyperbolic, Ann. of Math., 151 (**2000**) 327 -357.

[45] C.H. Taubes, Minimal surfaces in germs of hyperbolic 3-manifolds, In: Proceedings of the Casson Fest, volme 7 of Geom. Topol. Monogr., pp. 69-100 (electronic). Geom. Topol. Publ., Coventry **2004**.

[46] B. Loustau, Minimal surfaces and symplectic structures of moduli spaces, Geom. Dedicata 175 (**2015**) 309 - 322.

[47] K.K. Uhlenbeck, Closed Minimal Surfaces in Hyperbolic 3-manifolds, In: Seminar on Minimal Submanifolds, Volume 103 of Annals of Mathematical Studies, Princeton University Press, Princeton, NJ, **1983**.

[48] A. Marden, The geometry of finitely generated kleinian groups, Ann. of Math. 2 (99) (**1974**) 383 - 462.

[49] D. Sullivan, Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups, Acta Math., 155 (3-4) **(1985)** 243 - 260.

[50] B. Loustau, The complex symplectic geometry of the deformation space of complex projective structures, Geometry and Topology, 19 **2015** 1737 - 1775.

[51] L. Ahlfors, L. Bers, Riemann's mapping theorem for variable metrics, Ann. of Math., 2 (72) (**1960**) 385 - 404.

[52] L. V.Ahlfors, Finitely generated Kleinian groups, Amer. J. Math., 86 (**1964**) 413 - 429.

[53] L. Bers, On Sullivan's proof of the finiteness theorem and the eventual periodicity theorem, Amer. J. Math., 109 (5) (**1987**) 833 - 852.

[54] I. Kra, On spaces of Kleinian groups, Comment. Math. Helv., 47 **(1972)** 53 - 69.

[55] B. Maskit, Self-maps on Kleinian groups, Amer. J. Math. 93 (**1971**) 840 - 856.

[56] W.P. Thurston, The geometry and topology of 3-manifolds, Princeton lecture notes, **(1978 - 1981)** http://library.msri.org/books/gt3m/ (Access date: April 20, 2020).

[57] R.D. Canary, D.B.A. Epstein, P. Green, Notes on notes of Thurston, Analytical and Geometric Aspects of Hyperbolic Space, L.M.S Lecture Notes Series, Cambridge Univ. Press, 111 (**1987**) 3 - 92.

[58] F. Bonahon, Transverse Hölder distributions for geodesic laminations, Topology, 36 (**1997**) 103 - 122.

[59] F. Bonahon, G. Dereyer, Parameterizing Hitchin components, Duke Math. J., 163 (15) (**2014**) 2935 - 2975.

[60] F. Bonahon, G. Dereyer, Hitchin characters and geodesic laminations, Acta Math., 218 (2) (**2017**) 201 - 295.

[61] A. Weil, Remarks on the cohomology of groups, Ann. of Math., 80 **(1964)** 149 - 157.

[62] H. Zeybek, Shearing deformations of Hitchin representations and the Atiyah-Bott-Goldman symplectic form, https://arxiv.org/pdf/1905.11118.pdf (Access date: April 20, 2020).