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LONG-TIME DYNAMICS OF THE PARABOLIC p-LAPLACIAN EQUATION

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Abstract. In this paper, we study the long-time behaviour of solutions of Cauchy problem for the parabolic p-Laplacian equation with variable coefficients. Under the mild conditions on the coefficient of the principal part and without upper growth restriction on the source function, we prove that this problem possesses a compact and invariant global attractor in $L^2(R^n)$.

1. Introduction.

The main goal of this paper is to discuss the long-time behaviour (in the terms of attractors) of the solutions for the following equation

$$u_t - div(\sigma(x)\,|\nabla u|^{p-2}\,\nabla u) + \beta(x)u + f(u) = g(x), \qquad (t,x) \in (0,\infty) \times R^n, \quad (1.1)$$

with the initial data

$$u(0,x) = u_0(x), \quad (1.2)$$

where $p \geq 2$, $g \in L^2(R^n)$, $u_0 \in L^2(R^n)$, $n \geq 2$. Here, the functions σ, β and f satisfy the following assumptions:

$$\sigma \in L^1_{loc}(R^n),\ \sigma(\cdot) \geq 0,\ \sigma^{-\frac{2n}{n(p-2)+2p}} \in L^1_{loc}(R^n), \quad (1.3)$$

$$\beta \in L^\infty(R^n),\ \beta(\cdot) \geq 0,\ \beta(x) \geq \beta_0 > 0 \text{ a.e. in } \{|x| \geq r_0\} \text{ for some } r_0 > 0, \quad (1.4)$$

$$f \in C^1(R),\ f(s)s \geq 0,\ \forall\ s \in R,\ f'(\cdot) > -c,\ c > 0. \quad (1.5)$$

The understanding of the long-time behaviour of dynamical systems is one of the most important problems of modern mathematics. One way of approaching to this problem is to analyse the existence of the global attractor. The existence of the global attractors for the parabolic equations has extensively been studied by many authors. We refer to [1-7] and the references therein for the reaction-diffusion equations and to [3, 8-15] for the evolution p-Laplacian equations. When $\sigma(x) \equiv 1$, $\beta(x) \equiv \lambda$, the existence of the global attractor for equation (1.1) was studied in [3, 8-11] for bounded domains and in [12-16] for unbounded domains.

In this paper we deal with the equation (1.1) which contains the variable coefficients $\sigma(\cdot)$ and $\beta(\cdot)$. This type of equations have recently taken an interest by several authors. In [17], for the case $\beta(x) \equiv 0$, the authors have shown the existence of the global attractor for equation (1.1) in a bounded domain. In that paper the diffusion coefficient $\sigma(\cdot)$ is assumed to be like $|x|^\alpha$ for $\alpha \in (0,p)$ and due to the studying in a bounded domain the authors prove the asymptotic compactness

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property of the solutions by using the compact embeddings of Sobolev spaces. The existence of the global attractor for equation (1.1), under the assumption

$$\sigma(x) \sim |x|^{\alpha} + |x|^{\gamma} \alpha \in (0, p), \gamma > p + \frac{n}{2}(p-2),$$

has been shown in [18]. Although the authors in [18] have studied the problem in an arbitrary domain, the compact embeddings could also be used to obtain the asymptotic compactness of solutions because of the conditions imposed on $\sigma(\cdot)$.

The main novelty in our paper is the following: (i) we weaken the conditions on the function $\sigma(\cdot)$ which are given in [17] and [18], so that the embedding of the space with the norm $\left(\|\nabla u\|_{L^p_\sigma(\Omega)} + \|u\|_{L^2(\Omega)}\right)$ into the space $L^2(\Omega)$ is not compact, for each subdomain $\Omega \subset R^n$; (ii) we remove the upper growth condition on the source term.

The absence of the upper growth condition on f and the lack of the compact embedding cause some difficulties for the existence of the solutions and the asymptotic compactness of the solution operator in $L^2(R^n)$. We prove the existence of the solutions by Galerkin's method and to overcome the difficulties related to the limit transition in the source term f, we apply the weak compactness theorem in the Orlicz spaces. To prove the asymptotic compactness of the solutions, we first establish the validity of the energy equalities by using the approximation of the weak solutions by the bounded functions and then apply the approach of [19] by using the weak compactness argument.

Our main result is as follows:

Theorem 1.1. *Let conditions (1.3)-(1.5) hold. Then problem (1.1)-(1.2) possesses a compact and invariant global attractor in* $L^2(R^n)$.

The present paper is organized as follows. In the next section, we give some definitions and lemmas which will be used in the following sections. In section 3, the well-posedness of problem (1.1)-(1.2) is proved. In section 4, we show the existence of the absorbing set and present the proof of the asymptotic compactness to establish our main result.

2. Preliminaries.

This section is devoted to give some definitions and lemmas which will be used in the next sections. In order to study problem (1.1)-(1.2), let us begin with the introduction of the spaces W and W_b.

Definition 2.1. Under conditions (1.3)-(1.4), we define the spaces W and W_b as the closure of $C_0^\infty(R^n)$ in the following norms respectively,

$$\begin{aligned}\|u\|_W &:= \|\nabla u\|_{L^p_\sigma(R^n)} + \|u\|_{L^2_\beta(R^n)} \\ &= \left(\int_{R^n} \sigma(x) |\nabla u(x)|^p \, dx\right)^{\frac{1}{p}} + \left(\int_{R^n} \beta(x) |u(x)|^2 \, dx\right)^{\frac{1}{2}}, \\ \|u\|_{W_b} &= \|u\|_W + \sup_{x \in R^n} |u(x)| .\end{aligned}$$

One can show that W is a separable, reflexive Banach space and W_b is a separable Banach space. Now, before giving the definition of the weak solution of problem (1.1)-(1.2), let us define the operator $A : W \to W^*$ as $A\varphi = -div(\sigma(x) |\nabla\varphi|^{p-2} \nabla\varphi) + \beta(x)\varphi$, where W^* is the dual of W. It is easy to show that the operator $A : W \to W^*$ is bounded, monotone and hemicontinuous.

Definition 2.2. The function $u \in C([0,T];L^2(R^n)) \cap L^p(0;T;W)$ $\cap W^{1,2}_{loc}(0,T;L^2(R^n))$, satisfying $\int_0^T \int_{R^n} f(u(t,x))u(t,x)dxdt < \infty,\ u(0,x) = u_0(x)$ and the equation

$$\langle u_t, v\rangle + \langle Au, v\rangle + \langle f(u), v\rangle = \langle g, v\rangle \quad \text{a.e. on } (0,T),$$

for all $v \in W \cap L^\infty(R^n)$, is called the weak solution to problem (1.1)-(1.2), where $\langle \cdot,\cdot\rangle$ is the dual form between W and W^*.

Remark 2.1. To give a meaning to the third term on the left hand side of the equality given in the definition, it is enough to see that $f(u) \in L^1(0,T;L^1(R^n)) + L^2(0,T;L^2(R^n))$. Let χ_{Ω_1} and χ_{Ω_2} be the characteristic functions of the sets

$$\Omega_1 = \{(t,x) \in (0,T) \times R^n : |u(t,x)| > 1\},$$
$$\Omega_2 = \{(t,x) \in (0,T) \times R^n : |u(t,x)| \le 1\}.$$

Since

$$\int_0^T \int_{R^n} |f(u(t,x))\chi_{\Omega_1}(t,x)|\, dxdt = \iint_{\Omega_1} |f(u(t,x))|\, dxdt$$
$$\le \iint_{\Omega_1} f(u(t,x))u(t,x)dxdt \le \int_0^T \int_{R^n} f(u(t,x))u(t,x)dxdt < \infty,$$

we get $f(u)\chi_{\Omega_1} \in L^1(0,T;L^1(R^n))$. On the other hand, since

$$\int_0^T \int_{R^n} |f(u(t,x))\chi_{\Omega_2}(t,x)|^2\, dxdt = \iint_{\Omega_2} |f(u(t,x))|^2\, dxdt$$
$$= \iint_{\Omega_2} |(f(u(t,x)) - f(0))|^2\, dxdt,$$

by Mean Value theorem we have

$$\int_0^T \int_{R^n} |f(u(t,x))\chi_{\Omega_2}(t,x)|^2\, dxdt \le C \iint_{\Omega_2} |u(t,x)|^2\, dxdt \le C \int_0^T \int_{R^n} |u(t,x)|^2\, dxdt.$$

Taking into account that $u \in L^\infty(0;T;L^2(R^n))$, we get $f(u)\chi_{\Omega_2} \in L^2(0,T;L^2(R^n))$. Since

$$f(u(t,x)) = f(u(t,x))\chi_{\Omega_1}(t,x) + f(u(t,x))\chi_{\Omega_2}(t,x).$$

we have $f(u) \in L^1(0,T;L^1(R^n)) + L^2(0,T;L^2(R^n))$.

Lemma 2.1. *The inequality*

$$\|\nabla u\|^2_{L^{\frac{2n}{n+2}}(B(0,2r))} + \|u\|^2_{L^2(R^n \setminus B(0,r))} \ge C\,\|u\|^2_{L^2(R^n)}$$

is satisfied for all $u \in C_0^\infty(R^n)$ *and* $r > 0$, *where* $B(0,r) = \{x : x \in R^n,\ \ |x| < r\}$ *and the positive constant* C *depends on* r *and* n.

Proof. Let $\varphi(\cdot) \in C_0^\infty(R^n)$ be such that $0 \le \varphi(x) \le 1$ and

$$\varphi(x) = \begin{cases} 1, & x \in B(0,1), \\ 0, & x \in R^n \setminus B(0,2), \end{cases}$$

furthermore define $\varphi_r(x) = \varphi(\frac{x}{r})$ and $\widetilde{u}(x) = (u\varphi_r)(x)$. By the Sobolev inequality we have

$$\|\widetilde{u}\|^2_{L^2(R^n)} \leq c_1 \|\nabla \widetilde{u}\|^2_{L^{\frac{2n}{n+2}}(R^n)}. \tag{2.1}$$

Now, since

$$\|u\|^2_{L^2(B(0,r))} = \|\widetilde{u}\|^2_{L^2(B(0,r))} \leq \|\widetilde{u}\|^2_{L^2(R^n)}, \tag{2.2}$$

and

$$\begin{aligned}
&\left(\int_{R^n} |\nabla \widetilde{u}(x)|^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{n}} = \left(\int_{R^n} |\nabla (u\varphi_r)(x)|^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{n}} \\
\leq & c_2 \left(\int_{R^n} \left|\nabla u(x)\right|^{\frac{2n}{n+2}} \left|\varphi_r(x)\right|^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{n}} + \frac{c_2}{r^2} \left(\int_{R^n} \left|u\right|^{\frac{2n}{n+2}} \left|(\nabla\varphi)(\frac{x}{r})\right|^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{n}} \\
\leq & c_3 \left(\|\nabla u\|^2_{L^{\frac{2n}{n+2}}(B(0,2r))} + \|u\|^2_{L^2(B(0,2r)\setminus B(0,r))}\right),
\end{aligned} \tag{2.3}$$

from (2.1)-(2.3), we obtain

$$\|u\|^2_{L^2(B(0,r))} \leq c_4 \left(\|\nabla u\|^2_{L^{\frac{2n}{n+2}}(B(0,2r))} + \|u\|^2_{L^2(B(0,2r)\setminus B(0,r))}\right).$$

By adding $\|u\|^2_{L^2(R^n \setminus B(0,r))}$ to the both sides of the above inequality we get the claim of the lemma. □

Lemma 2.2. *Assume that the conditions (1.3)-(1.4) are satisfied. Then for all $u \in C_0^\infty(R^n)$ the inequality*

$$\|u\|_{L^2(R^n)} \leq \overline{C} \|u\|_W,$$

which yields $W \subset L^2(R^n)$, is satisfied.

Proof. The Holder inequality yields

$$\begin{aligned}
&\int_{B(0,r)} |\nabla u(x)|^{\frac{2n}{n+2}} dx \\
\leq & \left(\int_{B(0,r)} \sigma(x) |\nabla u(x)|^p dx\right)^{\frac{2n}{p(n+2)}} \left(\int_{B(0,r)} \sigma^{-\frac{2n}{p(n+2)-2n}}(x) dx\right)^{\frac{p(n+2)-2n}{p(n+2)}},
\end{aligned}$$

for every $r > 0$. From the assumption (1.3) it follows that

$$\|\nabla u\|_{L^{\frac{2n}{n+2}}(B(0,r))} \leq c(r) \|u\|_W, \tag{2.4}$$

On the other hand by condition (1.4), we have

$$\|u\|_{L^2(R^n \setminus B(0,r_0))} \leq \beta_0^{-\frac{1}{2}} \|u\|_W. \tag{2.5}$$

Taking into account the previous lemma and (2.4) - (2.5), we obtain the result. □

Lemma 2.3. *Let the conditions (1.3)-(1.4) are satisfied. Then $B_k : W \to W$ is a continuous map and $\lim_{k\to\infty} \|u - B_k(u)\|_W = 0$, for every $u \in W$, where*

$$B_k(s) = \begin{cases} k, & s > k, \\ s, & |s| \le k, \\ -k, & s < -k, \end{cases} \quad \text{for } s \in R.$$

Proof. By the definition of W, for any $u \in W$ there exists a sequence $\{u_m\}_{m=1}^{\infty} \subset C_0^{\infty}(R^n)$ such that

$$\lim_{m\to\infty} \|u - u_m\|_W = 0, \tag{2.6}$$

which according to the Lemma 2.2 yields

$$\lim_{m\to\infty} \|u - u_m\|_{L^2(R^n)} = 0. \tag{2.7}$$

Now, let us show that

$$\lim_{m\to\infty} \|B_k(u) - B_k(u_m)\|_W = 0. \tag{2.8}$$

Since

$$|B_k(u) - B_k(v)| \le |u - v|, \quad \forall u, v \in R,$$

by (1.4) and (2.7), we have

$$\limsup_{m\to\infty} \int_{R^n} \beta(x) |B_k(u)(x) - B_k(u_m)(x)|^2 dx$$
$$\le \|\beta\|_{L^\infty(R^n)} \limsup_{m\to\infty} \|u - u_m\|^2_{L^2(R^n)} = 0. \tag{2.9}$$

By (2.6), it follows that

$$\lim_{m\to\infty} \int_E \sigma(x) |\nabla u_m(x)|^p dx = \int_E \sigma(x) |\nabla u(x)|^p dx, \tag{2.10}$$

for every measurable $E \subset R^n$. By the last equality, we find

$$\limsup_{r\to\infty} \limsup_{m\to\infty} \int_{R^n\setminus B(0,r)} \sigma(x) |\nabla B_k(u)(x) - \nabla B_k(u_m)(x)|^p dx$$
$$\le 2^p \limsup_{r\to\infty} \limsup_{m\to\infty} \int_{R^n\setminus B(0,r)} \sigma(x) |\nabla u_m|^p dx + 2^p \limsup_{r\to\infty} \limsup_{m\to\infty} \int_{R^n\setminus B(0,r)} \sigma(x) |\nabla u|^p dx$$
$$= 2^{p+1} \limsup_{r\to\infty} \int_{R^n\setminus B(0,r)} \sigma(x) |\nabla u|^p dx = 0. \tag{2.11}$$

By (2.4) and (2.6), there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ such that

$$\nabla u_{m_j}(x) \to \nabla u(x) \text{ a.e. on } B(0,r).$$

Then by Egorov's theorem for any $\delta > 0$ there exists a measurable $E_\delta \subset B(0,r)$ such that $mes(E_\delta) < \delta$ and

$$\nabla u_{m_j}(x) \to \nabla u(x) \text{ uniformly on } B(0,r)\setminus E_\delta. \tag{2.12}$$

By (2.10) and (2.12), we get

$$\limsup_{j\to\infty}\int\limits_{B(0,r)}\sigma(x)\left|\nabla B_k(u)(x)-\nabla B_k(u_{m_j})(x)\right|^p dx\le 2^{p+1}\limsup_{\delta\to 0}\int\limits_{E_\delta}\sigma(x)\left|\nabla u(x)\right|^p dx$$

$$+\limsup_{\delta\to 0}\limsup_{j\to\infty}\int\limits_{\{x:\ k-\delta\le|u(x)|\le k+\delta\}\cap(B(0,r)\setminus E_\delta)}\sigma(x)\left|\nabla B_k(u)(x)-\nabla B_k(u_{m_j})\right|^p dx$$

$$\le 2^{p+1}\limsup_{\delta\to 0}\int\limits_{\{x:\ k-\delta\le|u(x)|\le k+\delta\}\cap B(0,r)}\sigma(x)\left|\nabla u(x)\right|^p dx$$

$$=2^{p+1}\int\limits_{\{x:\ |u(x)|=k\}\cap B(0,r)}\sigma(x)\left|\nabla u(x)\right|^p dx$$

$$\le 2^{p+1}\int\limits_{\{x:\ u(x)=k\}}\sigma(x)\left|\nabla u(x)\right|^p dx+2^{p+1}\int\limits_{\{x:\ u(x)=-k\}}\sigma(x)\left|\nabla u(x)\right|^p dx=0.$$

By the same way, one can show that every subsequence of $\{u_m\}_{m=1}^{\infty}$ has a subsequence satisfying the above equality. So, we have

$$\lim_{m\to\infty}\int\limits_{B(0,r)}\sigma(x)\left|\nabla B_k(u)(x)-\nabla B_k(u_m)(x)\right|^p dx=0. \tag{2.13}$$

By (2.9), (2.11) and (2.13), we obtain (2.8).

Now, let us show that for any $u\in C_0^\infty(R^n)$ and $k\in\mathbb{N}$ there exists $\{v_m\}_{m=1}^{\infty}\subset C_0^\infty(R^n)$ such that

$$\lim_{m\to\infty}\|B_k(u)-v_m\|_W=0. \tag{2.14}$$

Denote $v_m(x)=(\rho_m * B_k(u))(x)$, where $\rho_m(x)=\begin{cases} Km^n e^{-\frac{1}{1-m^2|x|^2}}, & |x|<\frac{1}{m},\\ 0, & |x|\ge\frac{1}{m},\end{cases}$ $m\in\mathbb{N}$ and $K^{-1}=\int_{\{x:\ |x|<1\}}e^{-\frac{1}{1-|x|^2}}dx$. Since $u\in C_0^\infty(R^n)$, by the definition of the function $B_k(\cdot)$, we have $B_k(u)\in W_0^{1,\infty}(B(0,r))\cap C_0(B(0,r))$ and $v_m\in C_0^\infty(B(0,r+1))$, for some $r>0$. It is well known that

$$\lim_{m\to\infty}\|B_k(u)-v_m\|_{W^{1,2}(R^n)}=0. \tag{2.15}$$

By (1.4) and (2.15), we obtain

$$\limsup_{m\to\infty}\int\limits_{R^n}\beta(x)\left|v_m(x)-B_k(u)(x)\right|^2 dx=0. \tag{2.16}$$

Also by (2.15), we have

$$\nabla v_m\to\nabla B_k(u) \text{ in measure on } R^n.$$

Since

$$\|\nabla v_m\|_{L^\infty(R^n)}\le\|\nabla B_k(u)\|_{L^\infty(R^n)}\le c,\qquad \forall m,k\in\mathbb{N},$$

applying Lebesgue's convergence theorem, we get

$$\lim_{m\to\infty}\int\limits_{R^n}\sigma(x)\,|\nabla v_m(x)-\nabla B_k(u)(x)|^p\,dx$$
$$=\lim_{m\to\infty}\int\limits_{B(0,r+1)}\sigma(x)\,|\nabla v_m(x)-\nabla B_k(u)(x)|^p\,dx=0.$$

which together with (2.16) yields (2.14). By (2.8) and (2.14), for every $u\in W$ and $k\in\mathbb{N}$ there exists a sequence $\{u_m\}_{m=1}^{\infty}\subset C_0^{\infty}(R^n)$ converging to $B_k(u)$ in the norm of W. It means that $B_k(u)\in W$, for every $u\in W$.

Now, using the argument done in the proof of (2.8), one can prove that if $\{u_m\}_{m=1}^{\infty}$ converges to u in W as $m\to\infty$, then $\{B_k(u_m)\}_{m=1}^{\infty}$ converges to $B_k(u)$ in W as $m\to\infty$. Also, by the definition of the function $B_k(\cdot)$, it is easy to show that $\{B_k(u)\}_{k=1}^{\infty}$ converges to u in W as $k\to\infty$. □

Remark 2.2. By (2.8) and (2.14), for every $u\in W$ and $k\in\mathbb{N}$ there exists a sequence $\{v_m\}_{m=1}^{\infty}\subset C_0^{\infty}(R^n)$ such that

$$\lim_{m\to\infty}\|B_k(u)-v_m\|_W=0\quad\text{and}\quad\sup_m\|v_m\|_{L^{\infty}(R^n)}\le k.$$

On the other hand, by the definition of $B_k(\cdot)$, for every $u\in L^{\infty}(R^n)$ and $k\ge\|u\|_{L^{\infty}(R^n)}$, we have

$$B_k(u)=u\quad\text{a.e. on}\quad R^n.$$

Hence, for every $u\in W\cap L^{\infty}(R^n)$ there exists a sequence $\{w_m\}_{m=1}^{\infty}\subset C_0^{\infty}(R^n)$ such that

$$\lim_{m\to\infty}\|u-w_m\|_W=0\quad\text{and}\quad\sup_m\|w_m\|_{L^{\infty}(R^n)}\le\|u\|_{L^{\infty}(R^n)}+1.$$

3. Well-posedness.

We prove the existence of the weak solution to problem (1.1)-(1.2) by Galerkin's method.

Theorem 3.1. *Assume that the conditions (1.3)-(1.5) are satisfied. Then for any $u_0\in L^2(R^n)$ and $T>0$, there exists a weak solution to (1.1)-(1.2).*

Proof. Let us consider the approximate solutions $\{u_m(t)\}_{m=1}^{\infty}$ in the form

$$u_m(t)=\sum_{k=1}^{m}c_{mk}(t)e_k,$$

where $\{e_j\}_{j=1}^{\infty}\subset C_0^{\infty}(R^n)$ is a basis of the space W_b and the functions $\{c_{mk}(t)\}_{k=1}^{m}$ are the solutions of the following problem :

$$\begin{cases}\left\langle\sum\limits_{k=1}^{m}c'_{mk}(t)e_k,e_j\right\rangle+\left\langle A(\sum\limits_{k=1}^{m}c_{mk}(t)e_k),e_j\right\rangle+\left\langle f(\sum\limits_{k=1}^{m}c_{mk}(t)e_k),e_j\right\rangle\\ =\langle g,e_j\rangle\,,\quad t>0,\quad j=1,...,m,\\ \sum\limits_{k=1}^{m}c_{mk}(0)e_k\to u_0\text{ strongly in }L^2(R^n)\text{ as }m\to\infty.\end{cases}\tag{3.1}$$

By the boundedness, monotonicity and hemicontinuity of $A:W\to W^*$ it follows that this operator is demicontinuous (see [20, Lemma 2.1 and Lemma 2.2, p. 38]). So, since $det(\langle e_j,e_k\rangle)\neq 0$ and f is continuous, by the Peano existence theorem, there exists at least one local solution to (3.1) in the interval $[0,T_m)$. Multiplying the equation $(3.1)_j$, by the function $c_{mj}(t)$, for each j, summing these relations for $j = 1, ..., m$ and integrating over $(0, t)$, we have

$$\|u_m(t)\|^2_{L^2(R^n)} + 2\int_0^t \int_{R^n} \left(\sigma(x)\left|\nabla u_m(\tau,x)\right|^p + \beta(x)\left|u_m(\tau,x)\right|^2\right) dxd\tau$$
$$+ 2\int_0^t \int_{R^n} f(u_m(\tau,x))u_m(\tau,x)dxd\tau$$
$$=2\int_0^t \int_{R^n} g(x)u_m(\tau,x)dxd\tau + \|u_m(0)\|^2_{L^2(R^n)}, \qquad 0 \le t < T_m. \tag{3.2}$$

Since by the last equality

$$\|u_m\|_{L^\infty(0,T_m;L^2(R^n))} \le c_1, \tag{3.3}$$

we can extend the approximate solution to the interval $[0, T]$, for every $T > 0$. Taking into account (3.3) in (3.2), we get

$$\|u_m(t)\|^2_{L^2(R^n)} + \int_0^t \int_{R^n} \left(\sigma(x)\left|\nabla u_m(\tau,x)\right|^p + \beta(x)\left|u_m(\tau,x)\right|^2\right) dxd\tau$$
$$+ \int_0^t \int_{R^n} \widetilde{f}(u_m(\tau,x))u_m(\tau,x)dxd\tau \le c_2, \quad \forall t \in [0,T], \tag{3.4}$$

where $\widetilde{f}(u_m(t,x)) = f(u_m(t,x)) + cu_m(t,x)$ and c is the constant in condition (1.5). Now, multiplying equation $(3.1)_j$ by the function $c'_{mj}(t)$, for each j, summing these relations for $j = 1, ..., m$, integrating over (s, T) and taking into account (3.3), we have

$$\int_s^T \|u_{mt}(t)\|^2_{L^2(R^n)} dt + \int_{R^n} F(u_m(T,x))dx$$
$$\le \int_{R^n} \sigma(x)\left|\nabla u_m(s,x)\right|^p dx + \int_{R^n} \beta(x)\left|u_m(s,x)\right|^2 dx + \int_{R^n} F(u_m(s,x))dx + c_3,$$

where $F(u) = \int_0^u f(s)ds$. Integrating the last inequality over $(0, T)$ with respect to the variable s and taking into account (3.4), we get

$$\|u_{mt}\|_{L^2(\varepsilon,T;L^2(R^n))} \le c_4(\varepsilon), \tag{3.5}$$

for every $\varepsilon \in (0, T)$. By the estimates (3.3)-(3.5) and the boundedness of the operator $A : L^p(0, T; W) \to L^{\frac{p}{p-1}}(0, T; W^*)$, we obtain (up to a subsequence) that

$$\begin{cases} u_m \to u \ \text{ weakly star in } L^\infty(0,T;L^2(R^n)), \\ u_{mt} \to u_t \ \text{ weakly in } L^2(\varepsilon,T;L^2(R^n)), \quad \forall \varepsilon \in (0,T), \\ u_m \to u \ \text{ weakly in } L^p(0,T;W), \\ Au_m \to \chi \ \text{ weakly in } L^{\frac{p}{p-1}}(0,T;W^*), \end{cases} \tag{3.6}$$

as $m \to \infty$, for some $\chi \in L^{\frac{p}{p-1}}(0,T;W^*)$. So, by (3.6), we get

$$u \in C([\varepsilon,T];L^2(R^n)). \tag{3.7}$$

Since by (2.4) and (3.4), the sequence $\{u_m\}$ is bounded in $L^p(0,T;W^{1,\frac{2n}{n+2}}_{loc}(R^n))$, using (3.5) and Aubin type compact embedding theorem (see [21, Corollary 4]), we have the compactness of $\{u_m\}$ in $L^1(\varepsilon,T;L^1_{loc}(R^n))$ for every $\varepsilon \in (0,T)$. Hence there exists subsequences $\{u^{(k)}_{m_n}\}_{n=1}^{\infty} \subset \{u^{(k-1)}_{m_n}\}_{n=1}^{\infty} \subset ... \subset \{u_m\}_{m=1}^{\infty}$ and $\varepsilon_k \searrow 0$ such that

$$u^{(k)}_{m_n} \to u \quad \text{a.e. on} \quad (\varepsilon_k,T)\times B(0,k),$$

as $n \to \infty$. Now, applying the diagonalization procedure, we obtain (up to a subsequence $\{u^{(k)}_{m_k}\}$) that

$$u_m \to u \quad \text{a.e. on} \quad (0,T)\times R^n, \tag{3.8}$$

as $m \to \infty$. Now, because the sign of the function $\widetilde{f}(u)$ is the same as the sign of u, together with (3.4), it follows that

$$\begin{cases} \int\limits_0^t \int\limits_{R^n} \widetilde{f}(u_m^+(\tau,x))u_m^+(\tau,x)dxd\tau \le c_5, \\ \int\limits_0^t \int\limits_{R^n} \widetilde{f}(u_m^-(\tau,x))u_m^-(\tau,x)dxd\tau \le c_5 \end{cases}, \quad \forall t \in [0,T], \tag{3.9}$$

where $u_m^+ = \max\{u_m,0\}$ and $u_m^- = \min\{u_m,0\}$. Since the function $\widetilde{f}(\cdot)$ is continuous, increasing, positive for $x>0$ and $\widetilde{f}(0)=0$, we can define an N-function (see [22] for definition)

$$\widetilde{F}(x) = \int\limits_0^{|x|} \widetilde{f}(s)ds,$$

which has a complementary N-function $\widetilde{G}$ as follows:

$$\widetilde{G}(y) = \int\limits_0^{|y|} \widetilde{f}^{-1}(\tau)d\tau.$$

By definition of $\widetilde{G}(\cdot)$ and $(3.9)_1$, we get

$$\int\limits_0^T \int\limits_{R^n} \widetilde{G}(\widetilde{f}(u_m^+(\tau,x))dxd\tau \le \int\limits_0^T \int\limits_{R^n} \widetilde{f}(u_m^+(\tau,x))u_m^+(\tau,x)dxd\tau \le c_5,$$

and consequently we obtain

$$\left\| \widetilde{f}(u_m^+) \right\|_{L^*_{\widetilde{G}}((0,T)\times B(0,k)\ )} \le c_5 + 1, \tag{3.10}$$

for every $k \in \mathbb{N}$, where $L^*_{\widetilde{G}}((0,T)\times B(0,k))$ is the Orlicz space (see [22] for definition). On the other hand, defining $g(s) = -\widetilde{f}^{-1}(-s)$ for $s>0$, we can construct a new N-function Φ such as

$$\Phi(y) = \int\limits_0^{|y|} g(\xi)d\xi.$$

Choosing $y = -\widetilde{f}(u_m^-)$ and taking into account $(3.9)_2$, we get

$$\int_0^T \int_{R^n} \Phi(\widetilde{f}(u_m^-(\tau,x))dxd\tau \le \int_0^T \int_{R^n} \widetilde{f}(u_m^-(\tau,x))u_m^-(\tau,x)dxd\tau \le c_5,$$

and consequently

$$\left\| \widetilde{f}(u_m^-) \right\|_{L^*_\Phi((0,T)\times B(0,k)\ )} \le c_5 + 1, \tag{3.11}$$

for every $k \in \mathbb{N}$. By using (3.8), continuity of $\widetilde{f}(\cdot)$ and the functions $\max\{s,0\}$ and $\min\{s,0\}$, it can be inferred that

$$\begin{cases} \widetilde{f}(u_m^+) \to \widetilde{f}(u^+) & \text{in measure on } (0,T)\times B(0,k), \\ \widetilde{f}(u_m^-) \to \widetilde{f}(u^-) & \text{in measure on } (0,T)\times B(0,k). \end{cases} \tag{3.12}$$

Now, taking into account (3.10)-(3.12) and using the [22, Theorem 14.6, p. 132], we get

$$\int_0^T \int_{B(0,k)} \widetilde{f}(u_m^+(t,x))v(t,x)dxdt \to \int_0^T \int_{B(0,k)} \widetilde{f}(u^+(t,x))v(t,x)dxdt, \quad \forall\ v \in E_{\widetilde{F}},$$

$$\int_0^T \int_{B(0,k)} \widetilde{f}(u_m^-(t,x))w(t,x)dxdt \to \int_0^T \int_{B(0,k)} \widetilde{f}(u^-(t,x))w(t,x)dxdt, \quad \forall w \in E_\Psi,$$

for every $k \in \mathbb{N}$, where Ψ is the complementary N-function to Φ and $E_{\widetilde{F}}$, E_Ψ are the closures of the set of bounded functions in the spaces $L^*_{\widetilde{F}}((0,T)\times B(0,k))$ and $L^*_\Psi((0,T)\times B(0,k))$, respectively. The last two approximations together with $(3.6)_1$ yield that

$$\int_0^T \int_{B(0,k)} f(u_m^+(t,x))v(t,x)dxdt \to \int_0^T \int_{B(0,k)} f(u^+(t,x))v(t,x)dxdt,$$

$$\int_0^T \int_{B(0,k)} f(u_m^-(t,x))v(t,x)dxdt \to \int_0^T \int_{B(0,k)} f(u^-(t,x))v(t,x)dxdt,$$

for every $v \in L^\infty((0,T)\times R^n)$. Now, since

$$f(u_m(t,x)) = f(u_m^+(t,x)) + f(u_m^-(t,x)),$$

we obtain

$$\int_0^T \int_{B(0,k)} f(u_m(t,x))v(t,x)dxdt \to \int_0^T \int_{B(0,k)} f(u(t,x))v(t,x)dxdt, \tag{3.13}$$

for every $v \in L^\infty((0,T)\times R^n)$ and $k \in \mathbb{N}$. By (3.4)-(3.6), we have

$$\langle u_m, e_j \rangle \to \langle u, e_j \rangle \text{ weakly star in } L^\infty(0,T), \tag{3.14}$$

$$\langle u_{mt}, e_j \rangle \to \langle u_t, e_j \rangle \text{ weakly in } L^2(\varepsilon,T),\ \forall \varepsilon \in (0,T),$$

$$\langle Au_m, e_j \rangle \to \langle \chi, e_j \rangle \text{ weakly in } L^{\frac{p}{p-1}}(0,T), \tag{3.15}$$

$$\langle f(u_m), e_j \rangle \to \langle f(u), e_j \rangle \text{ weakly in } L^1(0,T). \tag{3.16}$$

As a result, we can write that

$$\langle u_t, e_j\rangle = -\langle \chi, e_j\rangle - \langle f(u), e_j\rangle + \langle g, e_j\rangle \text{ weakly in } L^1(\varepsilon, T). \tag{3.17}$$

On the other hand by (3.4), we have

$$\int_0^T \int_{R^n} f(u_m(t,x))u_m(t,x))dxdt \le c_2.$$

Taking into account (3.8) and applying Fatou's lemma to the last inequality, we obtain

$$\int_0^T \int_{R^n} f(u(t,x))u(t,x))dxdt \le c_2.$$

As it was mentioned in the Remark 2.1, the last inequality gives us that

$$f(u) \in L^1(0,T; L^1(R^n) + L^2(R^n)).$$

So, by (3.17) and the density of $\{e_j\}_{j=1}^{\infty}$ in W_b, we get

$$\langle u_t, v\rangle = -\langle \chi, v\rangle - \langle f(u), v\rangle + \langle g, v\rangle \text{ a.e. on } (0,T),$$

for every $v \in W_b$, which together with Lemma 2.2 and Remark 2.2 , gives

$$\langle u_t, v\rangle = -\langle \chi, v\rangle - \langle f(u), v\rangle + \langle g, v\rangle \text{ a.e. on } (0,T),$$

for every $v \in W \cap L^\infty(R^n)$. From the last equality it follows that
$u_t \in L^1(0;T; L^1(R^n) + W^*)$ and

$$u_t = -\chi - f(u) + g, \quad \text{in} \quad L^1(0;T; L^1(R^n) + W^*). \tag{3.18}$$

By $u \in L^\infty(0,T; L^2(R^n))$ and $u_t \in L^1(0;T; L^1(R^n) + W^*)$, we have

$$u \in C([0,T]; L^1(R^n) + W^*)$$

and consequently

$$u \in C_s(0,T; L^1(R^n) + W^*)$$

Since by [23, Lemma 8.1, p. 275],

$$L^\infty(0,T; L^2(R^n)) \cap C_s(0,T; L^1(R^n) + W^*) = C_s(0,T; L^2(R^n)),$$

we get

$$u \in C_s(0,T; L^2(R^n)). \tag{3.19}$$

Also, by (3.1), (3.15) and (3.16), we have

$$\langle u_{mt}, e_j\rangle \to \langle g - \chi - f(u), e_j\rangle \text{ weakly in } L^1(0,T), \quad j = 1, 2, ...,$$

which, together with (3.18), yields that

$$\langle u_{mt}, e_j\rangle \to \langle u_t, e_j\rangle \text{ weakly in } L^1(0,T), \quad j = 1, 2, ...$$

The last approximation and (3.14) give us

$$\langle u_m(0), e_j\rangle \to \langle u(0), e_j\rangle, \quad j = 1, 2, ... \ .$$

On the other hand, since

$$u_m(0) \to u_0 \text{ strongly in } L^2(R^n),$$

we have $u(0) = u_0$. Hence, taking into account (1.3)-(1.5), (3.6) and passing to the limit in (3.2) when $m \to \infty$, we get

$$\|u(t)\|^2_{L^2(R^n)} \le 2 \int_0^t \int_{R^n} g(x)u(\tau,x)dxd\tau + \|u(0)\|^2_{L^2(R^n)},$$

and consequently we have

$$\limsup_{t\to 0} \|u(t)\|^2_{L^2(R^n)} \le \|u(0)\|^2_{L^2(R^n)}.$$

By (3.7), (3.19) and the last inequality we obtain

$$u \in C([0,T]; L^2(R^n)).$$

Now, since the operator $A : L^p(0,T;W) \to L^{\frac{p}{p-1}}(0,T;W^*)$ is bounded, monotone and hemicontinuous, to prove that $\chi = Au$, in addition to $(3.6)_3$ and $(3.6)_4$ we need to show that $\limsup_{m\to\infty} \int_0^T \langle Au_m(t), u_m(t)\rangle\, dt \le \int_0^T \langle \chi(t), u(t)\rangle\, dt$ (see [20, Lemma 2.1, p. 38]). By (3.2), we have

$$\begin{aligned}\int_0^T \langle Au_m(t), u_m(t)\rangle\, dt &= \int_0^T \int_{R^n} \left(\sigma(x)\left|\nabla u_m(\tau,x)\right|^p + \beta(x)\left|u_m(\tau,x)\right|^2\right) dxd\tau \\ &= \int_0^T \int_{R^n} (g(x)u_m(\tau,x) - f(u_m(\tau,x))u_m(\tau,x))\, dxd\tau \\ &\quad + \frac{1}{2}\|u_m(0)\|^2_{L^2(R^n)} - \frac{1}{2}\|u_m(T)\|^2_{L^2(R^n)}.\end{aligned}$$

Since $u_m(0) \to u_0$ in $L^2(R^n)$, taking into account (3.6) and (3.8), and applying Fatou's lemma, we find

$$\begin{aligned}\limsup_{m\to\infty} \int_0^T \langle Au_m, u_m\rangle\, dt &\le \int_0^T \int_{R^n} g(x)u(\tau,x)dxd\tau \\ &- \int_0^T \int_{R^n} f(u(\tau,x))u(\tau,x)dxd\tau + \frac{1}{2}\|u(0)\|^2_{L^2(R^n)} - \frac{1}{2}\|u(T)\|^2_{L^2(R^n)}. \quad (3.20)\end{aligned}$$

On the other hand, by Remark 2.1 and Lemma 2.3, we can test (3.18) by $B_k(u)$ on $(\varepsilon, T) \times R^n$, which gives us

$$\begin{aligned}\int_\varepsilon^T \langle \chi(t), B_k(u)(t)\rangle\, dt &= \int_\varepsilon^T \int_{R^n} g(x)B_k(u)(t,x)dxdt - \int_\varepsilon^T \int_{R^n} f(u(x,t))B_k(u)(t,x)dxdt \\ &+ \int_{R^n} u(\varepsilon,x)B_k(u)(\varepsilon,x)dx - \int_{R^n} u(T,x)B_k(u)(T,x)dx \\ &+ \frac{1}{2}\|B_k(u)(T)\|^2_{L^2(R^n)} - \frac{1}{2}\|B_k(u)(\varepsilon)\|^2_{L^2(R^n)}.\end{aligned}$$

Taking into account Lemma 2.3 and passing to the limit as $k \to \infty$, in the last equality, we obtain

$$\int_{\varepsilon}^{T} \langle \chi(t), u(t) \rangle \, dt = \int_{\varepsilon}^{T} \int_{R^n} g(x) u(t,x) dx dt - \lim_{k \to \infty} \int_{\varepsilon}^{T} \int_{R^n} f(u(t,x)) B_k(u)(t,x) dx dt + \frac{1}{2} \|u(\varepsilon)\|^2_{L^2(R^n)} - \frac{1}{2} \|u(T)\|^2_{L^2(R^n)} . \tag{3.21}$$

Since, the sequence $\{f(u(t,x))B_k(u)(t,x)\}_{k=1}^{\infty}$ is non-decreasing and $B_k(u) \to u$ in $C([0,T]; L^2(R^n))$, by monotone convergence theorem, we have

$$\lim_{\mathbf{k} \to \infty} \int_{\varepsilon}^{T} \int_{R^n} f(u(t,x)) B_k(u)(t,x) dx dt = \int_{\varepsilon}^{T} \int_{R^n} f(u(t,x)) u(t,x) dx dt,$$

which together with (3.21) yields

$$\int_{\varepsilon}^{T} \langle \chi(t), u(t) \rangle \, dt = \int_{\varepsilon}^{T} \int_{R^n} g(x) u(t,x) dx dt - \int_{\varepsilon}^{T} \int_{R^n} f(u(t,x)) u(t,x) dx dt + \frac{1}{2} \|u(\varepsilon)\|^2_{L^2(R^n)} - \frac{1}{2} \|u(T)\|^2_{L^2(R^n)} .$$

Since $u \in C([0,T]; L^2(R^n))$, passing to the limit in the last equality as $\varepsilon \to 0$, we get

$$\int_{0}^{T} \langle \chi(t), u(t) \rangle \, dt = \int_{0}^{T} \int_{R^n} g(x) u(t,x) dx dt - \int_{0}^{T} \int_{R^n} f(u(t,x)) u(t,x) dx dt + \frac{1}{2} \|u(0)\|^2_{L^2(R^n)} - \frac{1}{2} \|u(T)\|^2_{L^2(R^n)} .$$

Taking into account the last equality in (3.20), we obtain $\chi = Au$, which completes the proof of the existence of the solution. □

Theorem 3.2. *Let u and v be weak solutions of problem (1.1)-(1.2), with initial data u_0 and v_0, respectively. Then*

$$\|u(T) - v(T)\|_{L^2(R^n)} \le e^{cT} \|u_0 - v_0\|_{L^2(R^n)} , \quad \forall T \ge 0, \tag{3.22}$$

where c is the same constant in (1.5).

Proof. Denoting $w = u - v$, we have

$$\begin{cases} w_t + (A(v+w) - Av) - cw + \widetilde{f}(v+w) - \widetilde{f}(v) = 0, \\ w(0) = u_0 - v_0. \end{cases} \tag{3.23}$$

Testing $(3.23)_1$ by $B_k(w)$ on $(\varepsilon, T)\times R^n$ and taking into account the monotonicity of the function $\widetilde{f}$, we get

$$\begin{aligned}
&\int_{R^n} w(T,x)B_k(w)(T,x)dx - \frac{1}{2}\left\|B_k(w)(T)\right\|^2_{L^2(R^n)} \\
&+\int_{\varepsilon}^{T}\int_{R^n} \sigma(x)\left|\nabla u(t,x)\right|^{p-2}\nabla u(t,x)\cdot\nabla B_k(w)(t,x)dxdt \\
&-\int_{\varepsilon}^{T}\int_{R^n} \sigma(x)\left|\nabla v(t,x)\right|^{p-2}\nabla v(t,x)\cdot\nabla B_k(w)(t,x)dxdt \\
\leq &\int_{R^n} w(x,\varepsilon)B_k(w)(\varepsilon,x)dx - \frac{1}{2}\left\|B_k(w)(\varepsilon)\right\|^2_{L^2(R^n)} + c\int_{\varepsilon}^{T}\int_{R^n} w(x,t)B_k(w)(t,x)dxdt.
\end{aligned}$$

By the definition of $B_k(\cdot)$ and monotonicity of the function s^{p-1} for $s \geq 0$, we have

$$\begin{aligned}
&\int_{\varepsilon}^{T}\int_{R^n} \sigma(x)\left|\nabla u(t,x)\right|^{p-2}\nabla u(t,x)\cdot\nabla B_k(w)(t,x)dxdt \\
&-\int_{\varepsilon}^{T}\int_{R^n} \sigma(x)\left|\nabla v(t,x)\right|^{p-2}\nabla v(t,x)\cdot\nabla B_k(w)(t,x)dxdt \\
=&\int_{\varepsilon}^{T}\int_{\{x\in R^n, |w(t,x)|\leq k\}} \sigma(x)\left|\nabla u(t,x)\right|^{p-2}\nabla u(t,x)\cdot\nabla(u(t,x)-v(t,x))dxdt \\
&-\int_{\varepsilon}^{T}\int_{\{x\in R^n, |w(t,x)|\leq k\}} \sigma(x)\left|\nabla v(t,x)\right|^{p-2}\nabla v(t,x)\cdot\nabla(u(t,x)-v(t,x))dxdt \\
&\geq\int_{\varepsilon}^{T}\int_{\{x\in R^n, |w(t,x)|\leq k\}} \sigma(x)(\left|\nabla u(t,x)\right|^{p} - \left|\nabla u(t,x)\right|^{p-1}\left|\nabla v(t,x)\right|)dxdt \\
&+\int_{\varepsilon}^{T}\int_{\{x\in R^n, |w(t,x)|\leq k\}} \sigma(x)(\left|\nabla v(t,x)\right|^{p} - \left|\nabla v(t,x)\right|^{p-1}\left|\nabla u(t,x)\right|)dxdt \\
=&\int_{\varepsilon}^{T}\int_{\{x\in R^n, |w(t,x)|\leq k\}} \sigma(x)(\left|\nabla u(t,x)\right|^{p-1} - \left|\nabla v(t,x)\right|^{p-1})\left|\nabla u(t,x)\right|dxdt \\
&-\int_{\varepsilon}^{T}\int_{\{x\in R^n, |w(t,x)|\leq k\}} \sigma(x)(\left|\nabla u(t,x)\right|^{p-1} - \left|\nabla v(t,x)\right|^{p-1})\left|\nabla v(t,x)\right|)dxdt \geq 0.
\end{aligned}$$

By the last two inequalities, we find

$$\int\limits_{R^n} w(x,T)B_k(w)(T,x)dx - \frac{1}{2}\left\|B_k(w)(T)\right\|^2_{L^2(R^n)} \le \int\limits_{R^n} w(\varepsilon,x)B_k(w)(\varepsilon,x)dx$$

$$-\frac{1}{2}\left\|B_k(w)(\varepsilon)\right\|^2_{L^2(R^n)} + c\int\limits_{\varepsilon}^{T}\int\limits_{R^n} w(x,t)B_k(w)(t,x)dxdt$$

Passing to the limit as $k\to\infty$ and $\varepsilon\to 0$ in the above inequality and taking into account $(3.23)_2$, we obtain

$$\frac{1}{2}\left\|w(T)\right\|^2_{L^2(R^n)} \le \frac{1}{2}\left\|u_0-v_0\right\|^2_{L^2(R^n)} + c\int\limits_0^T \left\|w(t)\right\|^2_{L^2(R^n)}dt, \quad \forall T\ge 0,$$

which by Gronwall's lemma yields (3.22). □

Thus, by Theorem 3.1 and Theorem 3.2, under the conditions (1.3)-(1.5) the solution operator $S(t)u_0 = u(t)$ of problem (1.1)-(1.2) generates a strongly continuous semigroup in $L^2(R^n)$.

4. **Existence of the global attractor.** We begin with the existence of the absorbing set for the semigroup $\{S(t)\}_{t\ge 0}$.

Theorem 4.1. *Assume that the conditions (1.3)-(1.5) are satisfied. Then the semigroup $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set in $L^2(R^n)$, that is, there is a bounded set B_0 in $L^2(R^n)$ such that for any bounded subset B of $L^2(R^n)$, there exists a $T_0 = T_0(B) > 0$ such that $S(t)B \subset B_0$ for every $t \ge T_0$.*

Proof. Multiplying the equation $(3.1)_j$ by the function $c_{mj}(t)$, for each j, summing these relations for $j = 1,...,m$, we get the following equality :

$$\frac{1}{2}\frac{d}{dt}\left\|u_m(t)\right\|^2_{L^2(R^n)} + \int\limits_{R^n}\sigma(x)\left|\nabla u_m(t,x)\right|^p dx + \int\limits_{R^n}\beta(x)\left|u_m(t,x)\right|^2 dx$$

$$+\int\limits_{R^n} f(u_m(t,x))u_m(t,x)dx = \int\limits_{R^n} g(x)u_m(t,x)dx, \qquad \forall t\ge 0. \tag{4.1}$$

Since

$$1+\int\limits_{R^n}\sigma(x)\left|\nabla u_m(t,x)\right|^p dx \ge \left(\int\limits_{R^n}\sigma(x)\left|\nabla u_m(t,x)\right|^p dx\right)^{\frac{2}{p}},$$

by taking into account Lemma 2.2 in (4.1), we obtain

$$\frac{d}{dt}\left\|u_m(t)\right\|^2_{L^2(R^n)} + c_1\left\|u_m(t)\right\|^2_{L^2(R^n)} \le c_2\left\|g\right\|^2_{L^2(R^n)} + 2,$$

and consequently

$$\left\|u_m(t)\right\|^2_{L^2(R^n)} \le e^{-c_1 t}\left\|u_m(0)\right\|^2_{L^2(R^n)} + \frac{1}{c_1}(c_2\left\|g\right\|^2_{L^2(R^n)}+2),$$

for some $c_1>0$ and $c_2>0$. By $(3.6)_1$ and $(3.6)_2$, we have

$$u_m \to u \text{ weakly in } C([\varepsilon,T];L^2(R^n)), \;\; \forall T>\varepsilon>0,$$

which yields

$$u_m(t)\to u(t) \text{ weakly in } L^2(R^n), \;\; \forall t>0.$$

On the other hand, since $u_m(0) \to u_0$ strongly in $L^2(R^n)$, passing to the limit in the last inequality we find that

$$B_0 = \left\{ u \in L^2(R^n) : \ \|u\|_{L^2(R^n)} \leq \frac{1}{c_1}(c_2 \|g\|^2_{L^2(R^n)} + 2) + 1 \right\}$$

is an absorbing set for $\{S(t)\}_{t\geq 0}$. □

Now, let's prove the asymptotic compactness property of the semigroup $\{S(t)\}_{t\geq 0}$.

Theorem 4.2. *Let conditions (1.3)-(1.5) hold and B be a bounded subset of $L^2(R^n)$. Then the set $\{S(t_m)\varphi_m\}_{m=1}^{\infty}$ is relatively compact in $L^2(R^n)$, where $t_m \to \infty$, $\{\varphi_m\}_{m=1}^{\infty} \subset B$.*

Proof. By Theorem 4.1, there exists $T_0 = T_0(B) > 0$ such that

$$\|S(t)\varphi\|_{L^2(R^n)} \leq c_1, \qquad \forall \ \ t \geq T_0, \ \ \forall \varphi \in B. \tag{4.2}$$

For any $T > 0$ and $\{t_{m_k}\} \subset \{t_m\}$ such that $t_{m_k} \geq T + T_0$ let us define

$$u_k(t) := S(t + t_{m_k} - T)\varphi_{m_k}. \tag{4.3}$$

So, u_k is the solution of (1.1) with the initial condition $u_k(0) = S(t_{m_k} - T)\varphi_{m_k}$. Putting u_k instead of u in (1.1), formally multiplying the obtained equation by u_k and tu_{kt}, integrating over $(0,T) \times R^n$ and then taking into account condition (1.3) -(1.5) and (4.2)-(4.3), we find the following estimates

$$\begin{cases} \|u_k\|_{L^p(0,T;W)} + \|u_k\|_{L^\infty(0,T;L^2(R^n))} \\ + \int\limits_0^T \int\limits_{R^n} f(u_k(t,x))u_k(t,x)dxdt \leq c_2, \\ \|u_k\|_{L^p(0,T;W^{1,\frac{2n}{n+2}}(B(0,r))} \\ + \|u_{kt}\|_{L^2(\varepsilon,T;L^2(R^n))} \leq c_{\varepsilon,r}, \quad \forall \varepsilon \in (0,T), \quad \forall r > 0. \end{cases}$$

These estimates can be justified by using Galerkin's approximation as it was done in the previous section. So, repeating the argument done in the proof of Theorem 3.1, for the subsequence of u_k, without changing the name of it, we have

$$\begin{cases} u_k \to w \ \text{ weakly in } \ L^p(0,T;W), \\ u_k \to w \ \text{ weakly star in } \ L^\infty(0,T;L^2(R^n)) \\ u_{kt} \to w_t \ \text{ weakly in } L^2(\varepsilon,T;L^2(R^n)), \\ Au_k \to \chi \text{ weakly in } \ L^{\frac{p}{p-1}}(0,T;W^*), \\ u_k \to w \ \text{ a.e. in } \ (0,T) \times R^n, \\ f(u_k) \to f(w) \text{ in } \ D'(0,T \times R^n), \end{cases} \tag{4.4}$$

where $\chi \in L^{\frac{p}{p-1}}(0,T;W^*)$, $w \in L^\infty(0,T;L^2(R^n)) \cap L^p(0,T;W) \cap W^{1,2}(\varepsilon,T;L^2(R^n))$ and $\int\limits_0^T \int\limits_{R^n} f(w(t,x))w(t,x)dxdt < \infty$. Now, putting u_k instead of u in (1.1) and passing to the limit, we find

$$w_t + \chi + f(w) = g.$$

Taking into account Remark 2.1 and Lemma 2.3, and testing the above equation by $B_k(w)$ on $(s,T)\times R^n$, we obtain

$$\int_s^T \langle \chi(t), B_k(w)(t)\rangle\, dt = \int_s^T\int_{R^n} g(x)B_k(w)(t,x)dxdt - \int_s^T\int_{R^n} f(w(t,x))B_k(w)(t,x)dxdt$$
$$+\int_{R^n} w(s,x)B_k(w)(s,x)dx - \int_{R^n} w(T,x)B_k(w)(T,x)dx$$
$$+\frac{1}{2}\left\|B_k(w)(T)\right\|^2_{L^2(R^n)} - \frac{1}{2}\left\|B_k(w)(s)\right\|^2_{L^2(R^n)}, \quad \forall T\geq s>0.$$

Again repeating the argument done in the proof of Theorem 3.1, passing to the limit as $k\to\infty$ and integrating the obtained equality from ε to T with respect to s, we get

$$\frac{1}{2}(T-\varepsilon)\left\|w(T)\right\|^2_{L^2(R^n)} + \int_\varepsilon^T\int_s^T \langle \chi(t), w(t)\rangle\, dtds + \int_\varepsilon^T\int_s^T\int_{R^n} f(w(x,t))w(x,t)dxdtds$$
$$=\frac{1}{2}\int_\varepsilon^T \left\|w(s)\right\|^2_{L^2(R^n)} ds + \int_\varepsilon^T\int_s^T\int_{R^n} g(x)w(x,t)dxdtds, \quad \forall T\geq \varepsilon>0. \tag{4.5}$$

Now, putting u_k instead of u in (1.1), testing this equation by $B_m(u_k)$ on $(s,T)\times R^n$, integrating the obtained equality from ε to T with respect to s and passing to the limit as $m\to\infty$, we obtain

$$\frac{1}{2}(T-\varepsilon)\left\|u_k(T)\right\|^2_{L^2(R^n)} + \int_\varepsilon^T\int_s^T \langle Au_k(t), u_k(t)\rangle\, dtds$$
$$+\int_\varepsilon^T\int_s^T\int_{R^n} f(u_k(t,x))u_k(t,x)dxdtds$$
$$=\frac{1}{2}\int_\varepsilon^T \left\|u_k(s)\right\|^2_{L^2(R^n)} ds + \int_\varepsilon^T\int_s^T\int_{R^n} g(x)u_k(t,x)dxdtds, \quad \forall T\geq \varepsilon>0. \tag{4.6}$$

By (4.2)-(4.4), we have

$$\liminf_{k\to\infty}\int_\varepsilon^T\int_s^T \langle Au_k(t), u_k(t)\rangle\, dtds - \int_\varepsilon^T\int_s^T \langle \chi(t), w(t)\rangle\, dtds$$
$$=\liminf_{k\to\infty}\int_\varepsilon^T\int_s^T \langle Au_k(t)-Aw(t), u_k(t)-w(t)\rangle\, dtds$$
$$\geq c_3\liminf_{k\to\infty}\int_\varepsilon^T\int_s^T \left\|u_k(t)-w(t)\right\|^p_W dtds, \tag{4.7}$$

and

$$\liminf_{k\to\infty}\int_{\varepsilon}^{T}\int_{s}^{T}\int_{R^n} f(u_k(t,x))u_k(t,x)dxdtds \geq \int_{\varepsilon}^{T}\int_{s}^{T}\int_{R^n} f(w(t,x))w(t,x)dxdtds. \quad (4.8)$$

So, by (4.4)-(4.8), we obtain

$$(T-\varepsilon)\liminf_{k\to\infty}\|u_k(T)-w(T)\|^2_{L^2(R^n)} + 2c_3\liminf_{k\to\infty}\int_{\varepsilon}^{T}(s-\varepsilon)\|u_k(s)-w(s)\|^p_W ds$$

$$\leq \liminf_{k\to\infty}\int_{\varepsilon}^{T}\|u_k(s)-w(s)\|^2_{L^2(R^n)}ds, \quad \forall T\geq\varepsilon,$$

and consequently

$$(T-\varepsilon)\liminf_{k\to\infty}\|u_k(T)-w(T)\|^2_{L^2(R^n)} + c_4\liminf_{k\to\infty}\int_{2\varepsilon}^{T}s\|u_k(s)-w(s)\|^p_{L^2(R^n)} ds$$

$$\leq \liminf_{k\to\infty}\int_{\varepsilon}^{T}\|u_k(s)-w(s)\|^2_{L^2(R^n)}ds, \quad \forall T\geq 2\varepsilon.$$

Taking into account (4.2), (4.3) and $(4.4)_2$ in the integral on the right hand side of the above inequality, we have

$$\int_{\varepsilon}^{T}\|u_k(s)-w(s)\|^2_{L^2(R^n)}ds \leq 2c_1^2\varepsilon + 2c_1\int_{2\varepsilon}^{T}\|u_k(s)-w(s)\|_{L^2(R^n)}ds, \quad \forall T\geq 2\varepsilon.$$

Applying Holder inequality, we get

$$\int_{2\varepsilon}^{T}\|u_k(s)-w(s)\|_{L^2(R^n)}ds$$

$$\leq \begin{cases} \log^{\frac{1}{2}}\left(\frac{T}{2\varepsilon}\right)\left(\int_{2\varepsilon}^{T}s\|u_k(s)-w(s)\|^p_{L^2(R^n)}ds\right)^{\frac{1}{2}}, & p=2, \\ \left(\frac{p-1}{p-2}\left(T^{\frac{p-2}{p-1}}-(2\varepsilon)^{\frac{p-2}{p-1}}\right)\right)^{\frac{p-1}{p}}\left(\int_{2\varepsilon}^{T}s\|u_k(s)-w(s)\|^p_{L^2(R^n)}ds\right)^{\frac{1}{p}}, & p>2, \end{cases}$$

for $T\geq 2\varepsilon$. The last three inequalities together with $(4.4)_2$-$(4.4)_3$ give us

$$\liminf_{k\to\infty}\liminf_{i\to\infty}\|u_k(T)-u_i(T)\|^2_{L^2(R^n)}$$

$$\leq \frac{2c_1^2\varepsilon}{T-\varepsilon}+\frac{c_5}{T-\varepsilon}\begin{cases}\log\left(\frac{T}{2\varepsilon}\right), & p=2, \\ \frac{p-1}{p-2}\left(T^{\frac{p-2}{p-1}}-(2\varepsilon)^{\frac{p-2}{p-1}}\right), & p>2,\end{cases} \quad \forall T\geq 2\varepsilon,$$

and consequently

$$\liminf_{m\to\infty}\liminf_{k\to\infty}\|S(t_m)\varphi_m - S(t_k)\varphi_k\|^2_{L^2(R^n)}$$

$$\leq \frac{2c_1^2\varepsilon}{T-\varepsilon}+\frac{c_5}{T-\varepsilon}\begin{cases}\log\left(\frac{T}{2\varepsilon}\right), & p=2, \\ \frac{p-1}{p-2}\left(T^{\frac{p-2}{p-1}}-(2\varepsilon)^{\frac{p-2}{p-1}}\right), & p>2,\end{cases} \quad \forall T\geq 2\varepsilon.$$

Passing to the limit in the above inequality as $T \to \infty$, we have

$$\liminf_{m\to\infty} \liminf_{k\to\infty} \|S(t_m)\varphi_m - S(t_k)\varphi_k\|_{L^2(R^n)} = 0.$$

By the same way, one can show that

$$\liminf_{k\to\infty} \liminf_{i\to\infty} \|S(t_{m_i})\varphi_{m_i} - S(t_{m_k})\varphi_{m_k}\|_{L^2(R^n)} = 0, \tag{4.9}$$

for every subsequence $\{m_k\}_{k=1}^{\infty}$. Now, using the argument done at the end of the proof of [19, Lemma 3.4], let us show that the sequence $\{S(t_m)\varphi_m\}_{m=1}^{\infty}$ is relatively compact in $L^2(R^n)$. If not, then there exists $\varepsilon_0 > 0$ such that the set $\{S(t_m)\varphi_m\}_{m=1}^{\infty}$ has no finite ε_0-net in $L^2(R^n)$. This means that there exists a subsequence $\{m_k\}_{k=1}^{\infty}$, such that

$$\|S(t_{m_i})\varphi_{m_i} - S(t_{m_k})\varphi_{m_k}\|_{L^2(R^n)} \geq \varepsilon_0, \quad i \neq k.$$

The last inequality contradicts (4.9). □

Thus, taking into account Theorem 4.1, Theorem 4.2 and applying [24, Theorem 3.1] we have Theorem 1.1.

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