Is homotopy perturbation method the traditional Taylor series expansion

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Abstract

The homotopy perturbation method is studied in the present paper. The question of whether the homotopy perturbation method is simply the conventional Taylor series expansion is examined. It is proven that under particular choices of the auxiliary parameters the homotopy perturbation method is indeed the Taylor series expansion of the sought solution of nonlinear equations.

Key words: Nonlinear equations, Analytic solution, Homotopy perturbation method

1. Introduction

The search for a better and easy to use tool for the solution of nonlinear equations illuminating the nonlinear phenomena of our life is continuing.

Various methods therefore were proposed to find approximate solutions. One of the most recent popular technique is the homotopy perturbation method, which is a combination of the classical perturbation technique and homotopy concept as used in topology. In the homotopy perturbation method, which requires neither a small parameter nor a linear term in a differential equation, a homotopy with an embedding parameter $p \in [0, 1]$ is constructed. In [1] a basic idea of homotopy perturbation method for solving nonlinear differential equations was presented. A numerous nonlinear problems were recently treated by the method.

We in the present paper investigate the homotopy perturbation technique from a mathematical point of view. The aim is to analyze the method and to show that under certain circumstances, by particular choice of auxiliary linear operator and initial approximation, the homotopy perturbation method simply collapses onto the classical Taylor series expansion. The given theorem is justified exemplifying it by basic examples from ordinary and partial differential equations from the literature.

In the rest of the paper, §2. lays the basis of homotopy perturbation method. A theory is given in §3. with illustrative examples in §4. followed by the conclusions in §5..

2. The Homotopy Perturbation Method

The Homotopy perturbation method was first proposed by the Chinese mathematician He [1, 2], although it was defined earlier by Liao in [3]. The essential idea of this method is to introduce a homotopy parameter, say p, which varies from 0 to 1. At $p = 0$, the system of equations usually has been reduced to a simplified form which normally admits a rather simple solution. As p gradually increases continuously toward 1, the system goes through a sequence of deformations, and the solution at each stage is close to that at the previous stage of the deformation. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of the deformation gives the desired solution. Consider the nonlinear initial value problem

$$N(u) = 0, \quad B(u, \frac{du}{dn}) = 0, \tag{2.1}$$

where u is the function to be solved under the boundary constraints in B. He's homotopy perturbation technique [1, 4] defines a homotopy $u(r, p) : R\times[0, 1] \to R$ so that

$$H(u, p) = (1 - p)[L(u) - L(u_0)] + pN(u), \tag{2.2}$$

where L is a suitable auxiliary linear operator, u_0 is an initial approximation of equation (2.1) satisfying exactly the boundary conditions. It is obvious from equation (2.2) that

$$H(u, 0) = L(u) - L(u_0), \quad H(u, 1) = N(u). \tag{2.3}$$

As p moves from 0 to 1, $u(t, p)$ moves from $u_0(t)$ to $u(t)$. In topology, this called a deformation and $L(u) - L(u_0)$ and $N(u)$ are said to be homotopic. Our basic assumption is that the solution of equation (2.2) when equated to zero can be expressed as a power series in p

$$u(t, p) = u_0(t) + pu_1(t) + p^2u_2(t) + \cdots = \sum_{k=0}^{\infty} u_k(t)p^k. \tag{2.4}$$

The approximate solution of equation (2.1), therefore, can be readily obtained as

$$u(t) = \lim_{p\to 1} u(t, p) = \sum_{k=0}^{\infty} u_k(t). \tag{2.5}$$

Whenever the series (2.4) is known to be convergent, then (2.5) represents the exact solution of (2.1) [4].

3. Homotopy perturbation and Taylor expansion

To answer the question raised in the title of the paper, let's take into account the first-order initial value problem version of (2.1)

$$u'(t) = F(u), \quad u(0) = \alpha, \tag{3.6}$$

where α is a constant. A straightforward Taylor series representation for the solution $u(t)$ at point $t = 0$ can be given in the form

$$u(t) \;=\; u(0) + u'(0)t + \frac{u''(0)}{2!}t^2 + \cdots = \sum_{k=0}^{\infty} a_k t^k, \tag{3.7}$$

where $a_n = \frac{u^{(n)}(0)}{n!}$ can be immediately found from differentiating (3.6) successively and substituting $t = 0$. A few of the coefficients follow

$$\begin{aligned} a_1 &= u'(0) = F(\alpha), \\ a_2 &= \frac{u''(0)}{2!} = \frac{1}{2!}F_u(\alpha)a_1, \\ a_3 &= \frac{u'''(0)}{3!} = \frac{1}{3!}[F_{uu}(\alpha)a_1^2 + 2F_u(\alpha)a_2], \\ a_4 &= \frac{u^{(4)}(0)}{4!} = \frac{1}{4!}[F_{uuu}(\alpha)a_1^3 + 6F_{uu}(\alpha)a_1a_2 + 6F_u(\alpha)a_3], \\ &\vdots \end{aligned} \tag{3.8}$$

Theorem. If the auxiliary linear operator L and the initial approximation $u_0(t)$ to the solution $u(t)$ of equation (3.6) is taken in the homotopy procedure (2.2) as

$$L \;=\; \frac{\partial}{\partial t}, \quad u_0(t) = \alpha, \tag{3.9}$$

then the homotopy series solution (2.4) converges to the Taylor series expansion (3.7) whose coefficients are evaluated in the order given by (3.8).

Proof. Expanding the homotopy solution $u(t, p)$ from (2.2) into Taylor series according to the parameter p at $p = 0$, it reads

$$u(t, p) \;=\; \sum_{k=0}^{\infty} u_k(t)p^k. \tag{3.10}$$

When (3.10) is substituted into the homotopy equations (2.2) or equivalently differentiating (2.2) successively with respect to p and replacing $p = 0$ at the end yields a system of linear ordinary differential equations for the coefficients $u_k(t)$ of (3.10)

$$\begin{aligned} L(u_k - \chi_k u_{k-1}) &= -u'_{k-1} + \frac{1}{(k-1)!}[\frac{\partial^{k-1}F}{\partial p^{k-1}}]|_{p=0}, \\ u_k(0,p) &= \alpha, \end{aligned} \tag{3.11}$$

where $\chi_k = 0$ for $k = 1$ and $\chi_k = 1$ for $k > 1$. Having solved the equations (3.11) iteratively, the followings result for $u_k(t)$

$$\begin{aligned} u_1(t) &= F(\alpha)t = a_1 t, \\ u_2(t) &= \frac{1}{2!}F_u(\alpha)a_1 t^2 = a_2 t^2, \\ u_3(t) &= \frac{1}{3!}[F_{uu}(\alpha)a_1^2 + 2F_u(\alpha)a_2]t^3 = a_3 t^3, \\ u_4(t) &= \frac{1}{4!}[F_{uuu}(\alpha)a_1^3 + 6F_{uu}(\alpha)a_1 a_2 + 6F_u(\alpha)a_3]t^4 = a_4 t^4, \\ &\vdots \end{aligned} \tag{3.12}$$

which generates the homotopy series

$$u(t,p) = \sum_{k=0}^{\infty} u_k(t)p^k = \sum_{k=0}^{\infty} a_k t^k p^k. \tag{3.13}$$

The convergence assumption of (3.13) at $p = 1$ yields the homotopy series solution (2.5) which turns out to be the Taylor series expansion (3.7-3.8) to the solution.

Remark 1. Since the homotopy series (3.13) at $p = 1$ is the traditional Taylor series, then the convergence issue of the homotopy series (3.10) is guaranteed for those values t, $|t| < R$ such that $R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|$.

Remark 2. If $u = u(t,r)$ with r denoting space variables and the initial-value problem consists of a partial differential equation of the form

$$\begin{aligned} u_t &= F(u, u_r), \\ u(t=0,r) &= f(r), \end{aligned} \tag{3.14}$$

then by a similar argument to Theorem 1, the homotopy solution to (3.14) will be again the traditional Taylor series expansion at $t = 0$, provided that the auxiliary linear operator L and the initial approximation $u_0(t,r)$ are selected as

$$L = \frac{\partial}{\partial t}, \quad u_0(t,r) = f(r).$$

Remark 3. If higher-order ordinary or partial differential initial-value problems (or systems) are considered, by a particular choice of linear differential operator and initial guess, it can be shown that the homotopy perturbation series solution and the Taylor series solution are the same.

4. Illustrative Examples

To illustrate the validity of the Theorem outlined, we take into account the following examples taken from the homotopy perturbation studies in literature.

Example 1. Consider the first-order nonlinear differential equation [5]

$$y' + y^2 \;=\; 1, \quad y(0) = 0, \tag{4.15}$$

that governs the steady free convection flow over a vertical semi-infinite flat plate which is embedded in a fluid saturated porous medium of ambient temperature [6] and also the steady-state boundary-layer flows over a permeable stretching sheet [7]. In accordance with the Theorem, choosing $u_0(t) = 0$ and $L = \frac{d}{dt}$, the homotopy (2.2) becomes

$$\frac{\partial u(t,p)}{\partial t} + p\, u(x,p)^2 - p \;=\; 0, \quad u(0,p) = 0. \tag{4.16}$$

A few approximate homotopy solutions via the homotopy perturbation (4.16) can be calculated as

$$u_1(t) = t,\ u_2(t) = 0,\ u_3(t) = -\frac{t^3}{3},\ u_4(t) = 0,\ u_5(t) = \frac{2}{15}t^5,$$

which are the same as those generated from the classical Taylor series expansion of (4.15) at $t = 0$, the validity region is determined to be $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

Example 2. Consider now the second-order nonlinear differential equation

$$2y'' + y - y^2 \;=\; 0, \quad y(0) = 0, \quad y'(0) = \alpha = 1/\sqrt{6}, \tag{4.17}$$

that governs the steady mixed convection flow past a plane of arbitrary shape under the boundary layer and Darcy-Boussinesq approximations [8]. The Taylor series expansion of (4.17) at point $t = 0$ yields

$$y(t) = t\alpha - \frac{t^3\alpha}{12} + \frac{t^4\alpha^2}{24} + \frac{t^5\alpha}{480} - \frac{t^6\alpha^2}{288} + \frac{t^7\alpha\,(-1+40\alpha^2)}{40320} + \frac{t^8\alpha^2}{7680} + \cdots. \tag{4.18}$$

which totally corresponds to the homotopy perturbation series solution provided that we choose the auxiliary parameters as $u_0(t) = \alpha t$ and $L = 2\frac{\partial^2}{\partial t^2}$.

Example 3. Consider now the nonlinear partial differential Burger's equation

$$u_t + uu_x \;=\; u_{xx}, \quad u(x,0) = 2x, \; (x,t) \in R \times [0, 1/2), \tag{4.19}$$

that has been found to describe various kind of phenomena, such as a mathematical model of turbulence and the approximate theory of the flow through a shock wave traveling in a viscous fluid [9]. Equation (4.19) admits an exact solution given by

$$u(x,t) \;=\; \frac{2x}{1+2t}. \tag{4.20}$$

To approximate the exact solution (4.20), we choose the auxiliary parameters as $u_0(x,t) = 2x$ and $L = \frac{\partial}{\partial t}$. Then, the homotopy (2.2) turns out to be

$$u_t(x,t,p) + p(u(x,t,p)u_x(x,p,t) - u_{xx}(x,t,p) = 0, \; u(x,0,p) = 2x. \tag{4.21}$$

Equation (4.21) produces the below homotopy series for the solution of (4.19)

$$u(x,t) \;=\; 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 + ... + (-1)^n 2^{n+1} xt^n + \cdots, \tag{4.22}$$

which is the same as the classical Taylor series expansion of (4.20) around $t = 0$. The interval of convergence is easy to identify as $0 \leq t < 1/2$.

Example 4. Consider now the well-known KdV-Burger's equation involving both dispersion and dissipation terms

$$u_t + 2(u^3)_x - u_{xxx} + u_{xx} \;=\; 0, \quad u(x,0) = \frac{1}{6}\left(1 + \tanh\left[\frac{x}{6}\right]\right), \tag{4.23}$$

whose exact travelling-wave solution is given by

$$u(x,t) \;=\; \frac{1}{6}\left(1 + \tanh\left[\frac{1}{6}\left(x - \frac{2}{9}t\right)\right]\right). \tag{4.24}$$

To approximate the exact solution (4.24), if we choose the auxiliary parameters $u_0(x,t) = \frac{1}{6}\left(1 + \tanh\left[\frac{x}{6}\right]\right)$ and $L = \frac{\partial}{\partial t}$, the homotopy (2.2) turns out to be

$$u_t(x,t,p) + p(2(u(x,t,p)^3)_x - u_{xxx}(x,p,t) + u_{xx}(x,t,p) \;=\; 0, \tag{4.25}$$
$$u(x,0,p) = \frac{1}{6}\left(1 + \tanh\left[\frac{x}{6}\right]\right).$$

It is no hard to deduce that the Taylor series and homotopy perturbation series completely coincide again for this specific problem.

Example 5. Consider now the following fourth-order parabolic partial differential equation arising in the study of the transverse vibrations of a uniform flexible beam [10]

$$u_{tt} + (\frac{y+z}{2\cos x} - 1)u_{xxxx} + (\frac{z+x}{2\cos y} - 1)u_{yyyy} + (\frac{x+y}{2\cos z} - 1)u_{zzzz} \;=\; 0, \tag{4.26}$$
$$u(x,y,z,0) = -u_t(x,y,z,0) = x + y + z - (\cos x + \cos y + \cos z),$$

whose exact solution is given by

$$u(x,t) = (x+y+z-\cos x-\cos y-\cos z)e^{-t}. \quad (4.27)$$

To approximate the exact solution (4.27), if we choose the auxiliary parameters respectively, $u_0(x,t) = (x+y+z-\cos x-\cos y-\cos z)(1-t)$ and $L = \frac{\partial^2}{\partial t^2}$, the homotopy (2.2) then generates the subsequent homotopy series

$$u(x,t) = (x+y+z-\cos x-\cos y-\cos z)(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots) \quad (4.28)$$

which matches exactly onto the Taylor series expansion of (4.27) around $t=0$

$$u(x,t) = \sum_{n=0}^{\infty}(x+y+z-\cos x-\cos y-\cos z)\frac{t^n}{n!}, \quad (4.29)$$

that is obviously convergent for all t.

Example 6. As a final example, let us consider the linear partial differential equation

$$u_t + u_x - 2u_{xxt} = 0, \quad u(x,0) = e^{-x}, \quad (4.30)$$

whose exact separable solution is given by

$$u(x,t) = e^{-x-t}. \quad (4.31)$$

To approximate the exact solution (4.32), if we choose the auxiliary parameters $u_0(x,t) = e^{-x}$ and $L = \frac{\partial}{\partial t}$, then the homotopy (2.2) turns out to be

$$u_t(x,t,p) + p(u_x(x,p,t) - u_{xxt}(x,t,p) = 0, \; u(x,0,p) = e^{-x}. \quad (4.32)$$

The homotopy series solution of (4.27) from (2.1) can be found as

$$u(x,t) = \frac{e^{-x}}{720}(720+45360t+46440t^2+13320t^3+1470t^4+66t^5+\cdots), \quad (4.33)$$

whose radius of convergence is zero, so that the homotopy series (4.33) is convergent only at the point $t=0$. On the other hand, the classical Taylor series expansion applied to (4.30) predicts the exact result (4.31). It should be remarked that this example does not contradict at all with the Theorem, since (4.30) involves mixed partial derivatives. The weakness of the homotopy perturbation method on this example may be overcame by a better choice of auxiliary parameters.

It can be concluded as an answer to the title of the paper that for specific choices of auxiliary homotopy parameters, the homotopy perturbation technique produces exactly the same series as the traditional Taylor series. If this is the case, then there seems no a scientific merit to publish papers regarding the homotopy perturbation technique.

5. Concluding remarks

In this paper, the homotopy perturbation method has been analyzed with an aim to investigate the conditions which result in the traditional Taylor series expansion of the solutions of the nonlinear ordinary and partial differential equations. The theorem outlined in the paper has proved that if specific values are assigned to the auxiliary parameters in the homotopy perturbation method, then the approximate homotopy results entirely collide with the traditional Taylor series expansions. Examples have been provided to verify the theory. An example has also been given to demonstrate the advantage of the Taylor series expansion over the homotopy perturbation method. It can be concluded that a great deal of the papers published under the topic of homotopy perturbation technique is simply the traditional Taylor series expansion, whose contributions to science are questionable.

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