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Interpolating functions

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ABSTRACT

Let X, Y be sets with quasiproximities $\lhd_X$ and $\lhd_Y$ (where $A \lhd B$ is interpreted as "B is a neighborhood of A"). Let $f, g : X \to Y$ be a pair of functions such that whenever $C \lhd_Y D$, then $f^{-1}[C] \lhd_X g^{-1}[D]$. We show that there is then a function $h : X \to Y$ such that whenever $C \lhd_Y D$, then $f^{-1}[C] \lhd_X h^{-1}[D]$, $h^{-1}[C] \lhd_X h^{-1}[D]$ and $h^{-1}[C] \lhd_X g^{-1}[D]$. Since any function h that satisfies $h^{-1}[C] \lhd_X h^{-1}[D]$ whenever $C \lhd_Y D$, is continuous, many classical "sandwich" or "insertion" theorems are corollaries of this result. The paper is written to emphasize the strong similarities between several concepts

- the posets with auxiliary relations studied in domain theory;
- quasiproximities and their simplification, Urysohn relations; and
- the axioms assumed by Katětov and by Lane to originally show some of these results.

Interpolation results are obtained for continuous posets and Scott domains. We also show that (bi-)topological notions such as normality are captured by these order theoretical ideas.

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1. Introduction

Results concerning the possibility of finding, given a pair of real-valued functions g, h on a space X, such that $g \leqslant h$, a continuous function f such that $g \leqslant f \leqslant h$, form part of the classical theory of general topology. For example, recall that a real-valued g is *upper semicontinuous* (abbreviated USC below) if the sets $g^{-1}((-\infty, r))$ are open in X for each r in $\mathbb{R}$, and is *lower semicontinuous* (abbreviated LSC) if the sets $g^{-1}((r, \infty))$ are open in X for each r in $\mathbb{R}$. As early as 1917 Hahn [10] proved that if X is metrizable, g is USC, and h is LSC, then such an $f : X \to \mathbb{R}$ exists.

Dieudonné [2] later extended Hahn's [10] result to paracompact spaces, and also showed that for the property:

($*$) for each $g, h : X \to \mathbb{R}$, g USC, h LSC, and $g < h$ (at each point), there is a continuous $f : X \to \mathbb{R}$ such that $g < f < h$,

any paracompact space X with ($*$) is normal and countably paracompact. In fact, these so called insertion results characterize natural and important topological properties, as the following result from [12], Theorem 1, and [21] shows:

Theorem 1 *(Katětov, Tong). A space X is normal if and only if for each $g, h : X \to \mathbb{R}$, g USC, h LSC, such that $g \leqslant h$ (at each point) there is a continuous $f : X \to \mathbb{R}$ so that $g \leqslant f \leqslant h$.*

Many other similar results have been obtained, and are discussed in Section 8. Notice that the above results seem bitopological, in that they involve two topologies on $\mathbb{R}$, the lower, $\omega = \{(-\infty, a){:}\ -\infty \leqslant a \leqslant \infty\}$ and the upper, $\sigma = \{(a, \infty){:}\ -\infty \leqslant a \leqslant \infty\}$.

In fact, a function $g : X \to \mathbb{R}$ is USC if and only if it is continuous into $(\mathbb{R}, \omega)$ and is LSC if and only if it is continuous into $(\mathbb{R}, \sigma)$ and Theorem 1 is a special case of the corresponding bitopological result Theorem 28(b) below. It turns out, however, that all of these results are actually consequences of a general principle that holds for quasiproximities, and more generally for posets with auxiliary relations (defined in Section 3), a basic concept of domain theory. Indeed as Theorem 27 shows, the notion of an auxiliary relation encapsulates both the topology of the space and the idea of Katětov's proof in [12].

As a special case we obtain:

Suppose $(P, \leqslant)$ is a Scott domain (see [5]) and $g : (P, \omega) \to (\mathbb{R}, \omega)$, $h : (P, \sigma) \to (\mathbb{R}, \sigma)$ are continuous and such that $g \leqslant h$ (at each point). Then there is an $f : P \to \mathbb{R}$ which is continuous from $(P, \omega) \to (\mathbb{R}, \omega)$ and from $(P, \sigma) \to (\mathbb{R}, \sigma)$ such that $g \leqslant f \leqslant h$ (see Corollary 29(b)).

2. Binary relations and associated orders

In this section we introduce the order-theoretic concepts that we use to formulate our theory. We use the conventions that for a binary relation $\prec$ on a set P and any $A, B \subseteq P$, $c \in P$, $A \prec B$ means $a \prec b$ for each $a \in A$ and $b \in B$, $c \prec A$ means $\{c\} \prec A$ and $A \prec c$ means $A \prec \{c\}$. But note that $\prec$ can also denote a relation on 2^P below; hopefully the use of $A \prec B$ in that context will not cause difficulty.

Definition 2. Let $\prec$ be a binary relation on a set P. Define

$$\uparrow_\prec p = \{q{:}\ p \prec q\}, \qquad \downarrow_\prec p = \{q{:}\ q \prec p\}.$$

The *associated order*, $\leqslant_\prec$, on P is defined by $p \leqslant_\prec q$ if and only if $\downarrow_\prec p \subseteq \downarrow_\prec q$ and $\uparrow_\prec p \supseteq \uparrow_\prec q$.

Definition 3. A binary relation $\prec$ on a poset $(P, \leqslant)$, is *approximating* if and only if $p = \bigvee \downarrow_\prec p$ for all $p \in P$ and *dually approximating* if and only if $p = \bigwedge \uparrow_\prec p$.

Recall that a *preorder* on a set is a reflexive, transitive order.

Lemma 4. *Let $\prec$ be a binary relation on P and $\leqslant_\prec$ be its associated order.*

(1) *$\leqslant_\prec$ is a preorder.*
(2) *$\leqslant_\prec$ is a partial order if and only if for all $p, q \in P$, $p = q$ whenever both $\uparrow_\prec p = \uparrow_\prec q$ and $\downarrow_\prec p = \downarrow_\prec q$ (that is, $p = q$ if and only if, $p \prec a \Leftrightarrow q \prec a$ and $b \prec p \Leftrightarrow b \prec q$).*
(3) *$\prec$ is transitive if and only if $\prec \subseteq \leqslant_\prec$.*
(4) *$\prec$ is reflexive if and only if $\leqslant_\prec \subseteq \prec$.*
(5) *If $p \leqslant_\prec q \prec r \leqslant_\prec s$ then $p \prec s$.*

Assume also that $\leqslant$ is a partial order on P:

(6) *$\leqslant \subseteq \leqslant_\prec$ holds if and only if, for each $p, q, r, s \in P$, $p \leqslant q \prec r \leqslant s \Rightarrow p \prec s$.*
(7) *If $\prec$ is approximating, then $\leqslant_\prec \subseteq \leqslant$.*

Proof. Clearly (1) and (2) follow from the corresponding properties of $\subseteq$.

For (3), assume first that $\prec$ is transitive. If $q \prec r$ and p is any element of $\downarrow_\prec q$, then $p \prec q \prec r$, so $p \in \downarrow_\prec r$. Hence $\downarrow_\prec q \subseteq \downarrow_\prec r$. Similarly $\uparrow_\prec q \supseteq \uparrow_\prec r$, thus $q \leqslant_\prec r$. Conversely, suppose that $\prec \subseteq \leqslant_\prec$. If $p \prec q$ and $q \prec r$, then $q \leqslant_\prec r$, so $p \in \downarrow_\prec q \subseteq \downarrow_\prec r$ thus $p \prec r$.

To see (4), suppose that $\prec$ is reflexive. If $p \leqslant_\prec q$, then $p \in \downarrow_\prec p \subseteq \downarrow_\prec q$. Hence $p \prec q$ and $\prec \supseteq \leqslant_\prec$. Conversely, if $\prec \supseteq \leqslant_\prec$, then reflexivity of $\leqslant_\prec$ implies the reflexivity of $\prec$.

For (5), if $p \leqslant_\prec q \prec r \leqslant_\prec s$ then $r \in \uparrow_\prec q \subseteq \uparrow_\prec p$ so that $p \prec r$, which implies that $p \in \downarrow_\prec r \subseteq \downarrow_\prec s$. Hence $p \prec s$.

For (6), assume $p \leqslant q \prec r \leqslant s \Rightarrow p \prec s$. If $r \leqslant s$ and $q \in \downarrow_\prec r$ then $q \leqslant q \prec r \leqslant s$ so $q \prec s$, thus $q \in \downarrow_\prec s$, showing $\downarrow_\prec r \subseteq \downarrow_\prec s$; similarly if $t \in \uparrow_\prec s$ then $r \leqslant s \prec t \leqslant t$ so $t \in \uparrow_\prec r$, showing $\uparrow_\prec s \subseteq \uparrow_\prec r$. These two together show $r \leqslant_\prec s$. Conversely, if $\leqslant \subseteq \leqslant_\prec$ and $p \leqslant q \prec r \leqslant s$ then $p \leqslant_\prec q \prec r \leqslant_\prec s$ hence $p \prec s$.

Finally for (7) assume $\prec$ is approximating and let $a \leqslant_\prec b$. By definition, $\downarrow_\prec a \subseteq \downarrow_\prec b$, thus since $\prec$ is approximating, $a = \bigvee \downarrow_\prec a \leqslant \bigvee \downarrow_\prec b = b$. □

3. Auxiliary relations and the Katětov–Lane Axioms

In this section we compare the order-theoretic notions of Urysohn relation and auxiliary relation with the properties that Katětov [12] and Lane [17] isolate in considering insertion theorems.

Definition 5 *(The Auxiliary Relation Axioms).* Let $\leqslant$ be a partial order on the set P and let $\lhd$ be a binary relation on P. Then $\lhd$ is a *Urysohn relation* on $(P, \leqslant)$ provided:

(AR_{str}) $\lhd$ is *stricter than* $\leqslant$: $\lhd \subseteq \leqslant$;
(AR_{trn}) $\lhd$ is *transitive through* $\leqslant$: $c \lhd d$ whenever $c \leqslant a \lhd b \leqslant d$;
(AR_{in11}) $\lhd$ *interpolates between singletons*: if $a \lhd b$ then there is some c such that $a \lhd c \lhd b$.

The Urysohn relation $\lhd$ is said to be an *auxiliary relation* on $(P, \leqslant)$ if, in addition:

(AR_{in21}) $\lhd$ *interpolates between a pair and a singleton*: if $a, b \lhd c$, then $a, b \lhd d \lhd c$ for some $d \in P$.

We say that the auxiliary relation $\lhd$ is *dualizable* if it also satisfies

(AR_{in12}) $\lhd$ *interpolates between a singleton and a pair*, i.e. if $a \lhd b, c$, then $a \lhd d \lhd b, c$ for some $d \in P$.

The following lemma collects together a number of basic facts about the Auxiliary Relation Axioms. Recall that a set $R \subseteq P$ is directed by the relation $\lhd$ if, for each $a, b \in R$ there is some $c \in R$ such that $a, b \lhd c$.

Lemma 6.

(1) (AR_{in21}) *implies* (AR_{in11}) *and* (AR_{in12}) *implies* (AR_{in11}).
(2) *If a binary relation $\lhd$ satisfies both* (AR_{str}) *and* (AR_{trn})*, then it is transitive.*
(3) *An auxiliary relation $\lhd$ is dualizable if and only if the reverse order $\lhd^{-1}$ (also denoted at times by $\rhd$) is an auxiliary relation on $(P, \geqslant)$.*
(4) *If $\lhd$ is an auxiliary relation on P, then $\downarrow_\lhd a$ is directed by $\lhd$ for all $a \in P$. If $\lhd$ is dualizable, then $\uparrow_\lhd a$ is directed by $\lhd^{-1}$ for all $a \in P$.*
(5) *$\lhd$ satisfies* (AR_{trn}) *if and only if $\leqslant \subseteq \leqslant_\lhd$.*
(6) *If $\lhd$ is an approximating Urysohn relation, then $\leqslant_\lhd = \leqslant$.*

Proof. (1) and (3) are obvious. (2) holds since if $a \lhd b \lhd c$, then $a \leqslant a \lhd b \leqslant c$ by (AR_{str}) and so $a \lhd c$ by (AR_{trn}). (4) is immediate from (AR_{in21}) and (AR_{in12}). (5) follows directly from Lemma 4(6), and then (6) comes from (5) and Lemma 4(7). □

The auxiliary relations that we are interested in here are not always approximating:

Example 7. If X is a normal topological space and P is the power set 2^X of X ordered by $\subseteq$, then $A \lhd_\mathcal{N} B$ if and only if $\mathrm{cl}(A) \subseteq \mathrm{int}(B)$ defines an auxiliary relation. But $\lhd_\mathcal{N}$ need not be approximating; for example, if $X = \mathbb{R}$ and $a \lhd_\mathcal{N} b = (0,1) \cup \{2\}$, then $a \subseteq (0,1)$ and so $b \neq \bigvee \downarrow_{\lhd_\mathcal{N}} b$.

A common assumption is that in $(P, \leqslant)$, if $\{a, b\}$ is bounded above, then it has a join, $a \vee b$; a straightforward induction then shows that each finite set that is bounded above has a join. In this case we say that $(P, \leqslant)$ has *suprema for bounded pairs*, and *infima for bounded pairs* is similarly defined. If $(P, \leqslant)$ has such suprema and infima, and also has a largest and a smallest element, then we call $(P, \leqslant)$ a *bounded lattice*; in bounded lattices each finite set has a supremum and an infimum.

Lemma 8.

(1) *If $(P, \leqslant)$ has suprema for bounded pairs, then each Urysohn relation $\lhd$ on $(P, \leqslant)$ is contained in a smallest auxiliary relation.*
(2) *If $(P, \leqslant)$, has suprema and infima for bounded pairs, then each Urysohn relation $\lhd$ on $(P, \leqslant)$ is contained in a smallest dualizable auxiliary relation.*
(3) *Every Urysohn relation on a lattice (for example, on $(2^X, \subseteq)$) is contained in a smallest dualizable auxiliary relation.*

Proof. (3) follows from (2). For (1), set $\lhd_0 = \lhd$ and, for each n, $\lhd_{n+1} = \{(a,b)\colon (\exists c,d)(c,d \lhd_n b \ \&\ a \leqslant c \vee d)\}$. It is easily seen by induction that if $a \lhd_n b$ then $a \leqslant b$: it holds for $\lhd_0$ by (AR_{str}), and if it holds for $\lhd_n$ and $a \lhd_{n+1} b$ then for some c, d, $c, d \lhd_n b \ \&\ a \leqslant c \vee d$, so by induction, $c, d \leqslant b$ thus $c \vee d$ exists and $c \vee d \leqslant b$; since $a \leqslant c \vee d$, $a \leqslant b$ as required. Then set $A\lhd = \bigcup_{n=0}^{\infty} \lhd_n$. It is easy to check that each $\lhd_n$ is a Urysohn relation and $A\lhd$ is this smallest auxiliary relation.

Similarly, to see (2), set $\lhd^0 = \lhd$, and for each n, $\lhd^{n+1} = \{(a,b)\colon (\exists c,d)(c,d \lhd^n b \ \&\ a \leqslant c \vee d)\} \cup \{(a,b)\colon (\exists c,d)(b \lhd^n c,d \ \&\ c \wedge d \leqslant a)\}$, and $D\lhd = \bigcup_{n=0}^{\infty} \lhd^n$. It is easily seen that each $\lhd^n$ is a Urysohn relation and $D\lhd$ is this smallest dualizable auxiliary relation. □

Essentially Katětov [12] and Lane [17] isolate the following properties in their proof of insertion theorems.

Definition 9 *(The Katětov–Lane Axioms).* Let $(P, \leqslant)$ be a poset and $\lhd$ a binary relation on P. Let us call the following conditions on P the Katětov–Lane Axioms:

(KL_{str}) $\lhd \subseteq \leqslant$.
(KL_{trn}) $\leqslant \subseteq \leqslant_\lhd$.
$(KL_{inf,f})$ If $A, B \subseteq P$ are finite and $A \lhd B$, then there is some $c \in P$ such that $A \lhd c \lhd B$.
(KL_{bd}) For any finite $A \subseteq P$ there are $a, b \in P$ such that:
(a) $b \leqslant_\lhd A \leqslant_\lhd a$,
(b) $a \lhd c$, whenever $A \lhd c$, and
(c) $c \lhd b$, whenever $c \lhd A$.
$(KL_{in\omega,\omega})$ If $a, b \in P$, and A and B are countable subsets of P, such that $A \leqslant_\lhd a \lhd B$ and $A \lhd b \leqslant_\lhd B$, then there is $c \in P$ such that $A \lhd c \lhd B$.
(KL_{top}) If $A \lhd B$, then $\mathrm{cl}(A) \subseteq B$ and $A \subseteq \mathrm{int}(B)$ (in the case that P is the power set of a topological space and $\leqslant = \subseteq$).

We say that $\lhd$ is a *KL-relation on* P if and only if it satisfies (KL_{str}), (KL_{trn}) and $(KL_{inf,f})$.

(In Katětov [13], (KL_{bd}) is denoted by property (L) and $(KL_{in\omega,\omega})$ is denoted by property (I).)

Theorem 10. *Let $(P, \leqslant)$ be a poset and $\lhd$ be a binary relation on P.*

(1) *If $\lhd$ is a KL-relation on $(P, \leqslant)$, then it is a dualizable auxiliary relation on $(P, \leqslant)$.*
(2) *Let $(P, \leqslant)$ have suprema for pairs or have infima for pairs. If $\lhd$ is a dualizable auxiliary relation on $(P, \leqslant)$ then $\lhd$ is a KL-relation on $(P, \leqslant)$.*

Proof. (1) Note first that (KL_{str}) and (KL_{trn}) imply that $\lhd \subseteq \leqslant_\lhd$, so that $\lhd$ is transitive by Lemma 4. Clearly both (AR_{in21}) and (AR_{in12}) are special cases of $(KL_{inf,f})$, and (AR_{in11}) is a special case of (AR_{in21}). That (AR_{trn}) follows from (KL_{trn}) and Lemma 4(6), for if $c \leqslant a \lhd b \leqslant d$, then $c \leqslant_\lhd a \lhd b \leqslant_\lhd d$ so that $c \lhd d$.

(2) $(KL_{str}) = (AR_{str})$, and (KL_{trn}) follows by Lemma 4(6). To see $(KL_{inf,f})$, let A and B be finite subsets of P such that $a \lhd b$ for each $a \in A$ and $b \in B$; then assume that a' is a $\leqslant$-sup of A. By Lemma 6(4), for $b \in B$, $\downarrow_\lhd b$ is directed by $\lhd$ and since $A \subseteq \downarrow_\lhd b$, there is some $d_b \lhd b$ such that $A \lhd d_b$. By (KL_{str}), $A \leqslant d_b$ so that $a' \leqslant d_b \lhd b \leqslant_\lhd b$. (AR_{trn}) then implies that $a' \lhd b$. Since $\uparrow_\lhd a'$ is directed by $\lhd^{-1}$ and $B \subseteq \uparrow_\lhd a'$, there is some c so that $a' \lhd c \lhd B$, but $A \leqslant a' \lhd c \leqslant c$ so we have $A \lhd c \lhd B$. If A has no sup then (2) is shown by a similar proof using an inf of B. □

The next result modifies part of Katětov's [12] to fit the current setting.

Theorem 11 *(Katětov). If $\lhd$ is a dualizable auxiliary relation on a bounded lattice $(P, \leqslant)$, then $(P, \leqslant, \lhd)$ satisfies (KL_{bd}).*
If also every countable subset of P has a $\leqslant$-supremum and infimum, then $(P, \leqslant, \lhd)$ satisfies $(KL_{in\omega,\omega})$.

Proof. By Theorem 10(2), $\lhd$ is a *KL*-relation on $(P, \leqslant)$. For the property (KL_{bd}), suppose A is a finite subset of P. Let a be a $\leqslant$-supremum of A and b be a $\leqslant$-infimum of A. Then $b \leqslant A \leqslant a$ so, by (KL_{trn}) $b \leqslant_\lhd A \leqslant_\lhd a$. If $A \lhd c$, then by $(KL_{inf,f})$ there is some d such that $A \lhd d \lhd c$. Since $A \leqslant d$, $A \leqslant a \leqslant d \lhd c$, so that $a \lhd c$. Similarly, if $c \lhd A$, then $c \lhd b$.

For second part of the theorem, suppose every countable subset of P has a $\leqslant$-supremum and infimum and that $a, b \in P$ and $A = \{a_n\colon n \in \mathbb{N}\}$, $B = \{b_n\colon n \in \mathbb{N}\}$ are subsets of P such that $A \leqslant_\lhd a \lhd B$ and $A \lhd b \leqslant_\lhd B$.

We want $c \in P$ such that $a_n \lhd c \lhd b_n$ for all $n \in \mathbb{N}$. We first define inductively $\{c_n\colon n \in \mathbb{N}\}$ and $\{d_n\colon n \in \mathbb{N}\}$ such that $a_i \lhd c_i \lhd b$, $a \lhd d_j \lhd b_j$ and $c_i \lhd d_j$ for all i, j. For this, inductively assume that we have such c_i, d_j for $i, j < n$. Since $a_n \leqslant_\lhd a \lhd d_i$, for each $i < n$, Lemma 4(5) implies that $a_n \lhd d_i$. Hence $a_n \lhd \{b\} \cup \{d_i\colon i < n\}$, so by $(KL_{inf,f})$ there is a c_n such that $a_n \lhd c_n \lhd \{b\} \cup \{d_j\colon j < n\}$. Similarly, since $c_n \lhd b \leqslant_\lhd b_n$, $c_n \lhd b_n$. Also $c_i \lhd b_n$ for $i < n$, and $a \lhd b_n$. Hence $\{a\} \cup \{c_i\colon i \leqslant n\} \lhd b_n$, so there is some d_n such that $\{a\} \cup \{c_i \mid i \leqslant n\} \lhd d_n \lhd b_n$.

Let $c = \sup_{n\in\mathbb{N}} c_n$. Then $a_n \lhd c_n \leqslant c$ for each n, so $A \lhd c$. Moreover $c_i \lhd d_j$ for each $i, j \in \mathbb{N}$, so $c_k \leqslant d_j$. Thus $c_k \leqslant c \leqslant d_j \lhd b_j$ for each $j \in \mathbb{N}$, from which it follows that $c \lhd B$. □

4. Topologies, auxiliary relations and (KL_{top})

An auxiliary relation on the power set of a set X, ordered by inclusion, naturally gives rise to two topologies on X. It turns out, in fact, that when considering the insertion of a continuous real-valued function between two semicontinuous functions, both the topology on the space and the continuity of the functions are inherent in the natural auxiliary relation on the power set of X. In this section we show that when our order theoretic notions are applied to the poset $(2^X, \subseteq)$, they correspond naturally to normal or completely regular (bi-)topologies on the set X.

Definition 12. Let X be a set and $\lhd$ be a binary relation on the power set 2^X. The *topology arising from* $\lhd$, $\tau_\lhd$ is the collection of subsets U of X such that for each $x \in U$ there is some finite subset F of 2^X such that $\bigcap F \subseteq U$ and $\{x\} \lhd B$ for each $B \in F$.

We say that a Urysohn relation *satisfies* AR_{in1s2} or *has* (AR_{in12}) *for singletons* if for each $x \in X$, $B, C \subseteq X$, $\{x\} \lhd B$ & $\{x\} \lhd C \Rightarrow \{x\} \lhd D$ for some $D \subseteq B, C$.

Lemma 13. *If $\lhd$ is a Urysohn relation on $(2^X, \subseteq)$, then $\tau_\lhd$ is a topology on X. Moreover, if $\lhd$ has (AR_{in12}) for singletons, then $T \in \tau_\lhd$ if and only if $\{x\} \lhd T$ for all $x \in T$.*

Proof. To show that $\tau_\lhd$ is a topology, first let $\mathcal{S} \subseteq \tau_\lhd$ and $x \in \bigcup \mathcal{S}$. Then for some $T \in \mathcal{S}$, $x \in T$, so for some finite set F of subsets of X, $\{x\} \lhd B$ for each $B \in F$, and $\bigcap F \subseteq T \subseteq \bigcup \mathcal{S}$; this shows $\bigcup \mathcal{S} \in \tau_\lhd$ (as a special case, $\emptyset \in \tau_\lhd$).

Also, if $T, U \in \tau_\lhd$ and $x \in T \cap U$, then for some finite sets F, G of subsets of X, $\{x\} \lhd B$ for each $B \in F$ and $\bigcap F \subseteq T$, and $\{x\} \lhd B$ for each $B \in G$ and $\bigcap G \subseteq U$. Thus $\{x\} \lhd B$ for each $B \in F \cup G$, and $\bigcap (F \cup G) = (\bigcap F) \cap (\bigcap G) \subseteq T \cap U$, thus intersections of pairs of open sets are open. Finally, to see that $X \in \tau_\lhd$, for each $x \in X$ let $F = \emptyset$; then $\{x\} \lhd B$ for each $B \in F$ and $\bigcap F \subseteq X$.

Now suppose further that $\lhd$ satisfies AR_{in1s2}. If $x \in T \in \tau_\lhd$, then for some finite set F of subsets of X, $\{x\} \lhd B$ for each $B \in F$ and $\bigcap F \subseteq T$. Thus by induction on axiom AR_{in1s2}, there is a D such that $\{x\} \lhd D$ and $D \subseteq B$ for each $B \in F$. But then $\{x\} \lhd D \subseteq \bigcap F \subseteq T$, so by (AR_{trn}), $\{x\} \lhd T$. For the reverse implication (in an arbitrary Urysohn relation), suppose $x \in T \Rightarrow \{x\} \lhd T$; then $F = \{T\}$ is a finite collection of sets such that $\{x\} \lhd B$ for each $B \in F$ and $\bigcap F \subseteq T$. Thus $T \in \tau_\lhd$. □

Since Katětov's original result, Theorem 1 (from [12]), involves two topologies on the reals, it is not surprising that our setting naturally gives rise to two topologies on the domain set as well.

Definition 14. Given a Urysohn relation $\lhd$ on $(2^X, \subseteq)$, the *Urysohn dual* of $\lhd$ is denoted by $\lhd^*$ and defined by $A \lhd^* B$ if and only if $(X - B) \lhd (X - A)$.

It is simple to see that $\lhd^*$ is a Urysohn relation when $\lhd$ is one, and an auxiliary relation when $\lhd$ is a dualizable auxiliary relation. Also, clearly $(\lhd^*)^* = \lhd$.

It turns out that the axiom (KL_{top}) is inherently incorporated into the topology $\tau_\lhd$ arising from a Urysohn relation $\lhd$ as the following proposition shows.

Proposition 15. *Let $\lhd$ be a Urysohn relation on $(2^X, \subseteq)$ and let $A \subseteq X$. Then $x \in \operatorname{int}_{\tau_\lhd} A$ if and only if for some finite set F of subsets of X, $\{x\} \lhd B$ for each $B \in F$, and $\bigcap F \subseteq A$.*

Moreover, if $A \lhd B$ then $A \subseteq \operatorname{int}_{\tau_\lhd} B$ and $\operatorname{cl}_{\tau_{\lhd^*}} A \subseteq B$.

Proof. Let

$$A^o = \left\{x\colon \text{for some finite } F \subseteq 2^X,\ \bigcap F \subseteq A \text{ and, for all } B \in F,\ \{x\} \lhd B\right\}.$$

Certainly $A^o \subseteq A$, and if $x \in U \subseteq \operatorname{int}_{\tau_\lhd} A$, for some $U \in \tau_\lhd$, then, by the definition of $\tau_\lhd$, $x \in A^o$. Therefore $\operatorname{int}_{\tau_\lhd} A \subseteq A^o \subseteq A$. To show $\operatorname{int}_{\tau_\lhd} A = A^o$, it suffices to show that the latter is open. But if $x \in A^o$, then there is a finite F as above; for each $B \in F$, there is thus a C_B such that $\{x\} \lhd C_B \lhd B$; now let $G = \{C_B \mid B \in F\}$; G is finite, and if $y \in \bigcap G$ then for each $B \in F$, $\{y\} \subseteq C_B \lhd B$, so $\{y\} \lhd B$, and of course, $\bigcap F \subseteq A$. But this asserts that if $y \in \bigcap G$ then $y \in A^o$; as a result, for arbitrary $x \in A^o$ we have found a finite collection G of sets such that for each $C \in G$, $\{x\} \lhd C$, and $\bigcap G \subseteq A^o$; thus $A^o \in \tau_\lhd$ and so $A^o = \operatorname{int}_{\tau_\lhd} A$.

Now suppose $A \lhd B$; then for each $x \in A$, $\{x\} \subseteq A \lhd B$ so $\{x\} \lhd B$, whence $x \in B^o$; this shows $A \subseteq B^o = \mathrm{int}_{\tau_\lhd} B$. Further, $X - B \lhd^* X - A$, thus by the previous sentence applied to $\lhd^*$, $X - B \subseteq \mathrm{int}_{\tau_{\lhd^*}}(X - A) = X - \mathrm{cl}_{\tau_{\lhd^*}} A$, so $\mathrm{cl}_{\tau_{\lhd^*}} A \subseteq B$, as required. $\square$

In fact, Theorems 17 and 18 will show that for a Urysohn relation $\lhd$, we can say a good deal more about the topology $\tau_\lhd$ when we consider the bitopological setting. We start by recalling some key definitions from [14] and (in our notation) [15]:

Definition 16. For a topological space (X, τ), its (*Alexandroff*) *specialization order* is defined by $x \leqslant_\tau y$ if $x \in \mathrm{cl}_\tau\{y\}$.

A *bitopological space* is a triple (X, τ, τ^*) such that X is a set and τ, τ^* are topologies on X. Given bitopological spaces (X, τ_X, τ_X^*) and (Y, τ_Y, τ_Y^*) a *pairwise continuous* map from (X, τ_X, τ_X^*) to (Y, τ_Y, τ_Y^*) is a function $f : X \to Y$ such that f is continuous both from (X, τ_X) to (Y, τ_Y) and from (X, τ_X^*) to (Y, τ_Y^*).

A bitopological space (X, τ, τ^*) is *weakly symmetric* if $x \notin \mathrm{cl}_\tau\{y\} \Rightarrow y \notin \mathrm{cl}_{\tau^*}\{x\}$; it is T_1 if weakly symmetric and $\tau \vee \tau^*$ is T_0.

A bitopological space (X, τ, τ^*) is *pseudoHausdorff* (*pH*) if whenever $x \notin \mathrm{cl}_\tau\{y\}$ then for some $T \in \tau$, $U \in \tau^*$, $x \in T$, $y \in U$, and $T \cap U = \emptyset$.

For any property Q of bitopological spaces, (X, τ, τ^*) is said to be *pairwise* Q if both (X, τ, τ^*) and its *bitopological dual*, (X, τ^*, τ) is Q.

A bitopological space (X, τ, τ^*) is *joincompact* it is pairwise pH, and $\tau \vee \tau^*$ is compact and T_0.

A bitopological space (X, τ, τ^*) is *completely regular* if whenever $x \in U \in \tau$, then there is a pairwise continuous f from (X, τ, τ^*) to $(\mathbb{I}, \sigma, \omega)$ such that $f(x) = 1$ and $f(y) = 0$ whenever $y \notin U$.

A bitopological space (X, τ, τ^*) is *normal* if whenever $C \subseteq U$, C is τ^*-closed and U τ-open, then there is a τ^*-closed D and a τ-open V such that $C \subseteq V \subseteq D \subseteq U$. It is T_4 if normal and T_1.

Theorem 17. *The following are equivalent*:

(1) *The bitopological space (X, τ, τ^*) is pairwise completely regular.*
(2) *There is a Urysohn relation $\lhd$ on 2^X such that $\tau = \tau_\lhd$ and $\tau^* = \tau_{\lhd^*}$.*
(3) *There is a dualizable auxiliary relation $\lhd$ on 2^X such that $\tau = \tau_\lhd$ and $\tau^* = \tau_{\lhd^*}$.*

Summary of proof. (1) $\Leftrightarrow$ (2) The definition of Urysohn relation is designed so that for each set of functions F from a set X into $[0, 1]$, the relation

$$A \lhd_F B \quad \Leftrightarrow \quad \bigl(\exists r, s \in [0,1],\ f \in F\bigr)\bigl(r < s \;\&\; A \subseteq f^{-1}[[s,1]] \;\&\; f^{-1}[(r,1]] \subseteq B\bigr)$$

is a Urysohn relation (the reader can easily check this, or see [15]), and to support the classic proof of the Urysohn Lemma (also easily checked, or see [15], Lemma 2.8). If each function in F is pairwise continuous from (X, τ, τ^*) to $([0, 1], \sigma, \omega)$, then $\tau_{\lhd_F} \subseteq \tau$ and $\tau_{\lhd_F^*} \subseteq \tau^*$.

Thus if there is a Urysohn relation $\lhd$ on 2^X such that $\tau = \tau_\lhd$ and $\tau^* = \tau_{\lhd^*}$, then by Urysohn's Lemma, for each $x \in T \in \tau$ there is a pairwise continuous $f : (X, \tau, \tau^*) \to ([0, 1], \sigma, \omega)$, and similar reasoning applies to the dual, (X, τ^*, τ), so (X, τ, τ^*) is pairwise completely regular. Conversely, if (X, τ, τ^*) is pairwise completely regular, and $F = \{f\colon\ f$ is pairwise continuous from (X, τ, τ^*) to $(\mathbb{I}, \sigma, \omega)\}$, then by the previous paragraph, $\lhd_F$ is a Urysohn relation for which $\tau_{\lhd_F} \subseteq \tau$ and $\tau_{\lhd_F^*} \subseteq \tau^*$. But in fact if $x \in T \in \tau$ there is an $f \in F$ such that $f(x) = 1$ and $f^{-1}[(0, 1]] \subseteq T$, so $T \in \tau_{\lhd_F}$. This shows $\tau = \tau_{\lhd_F}$ and similarly $\tau^* = \tau_{\lhd_F^*}$.

Clearly (3) $\Rightarrow$ (2), for the converse, if there is a Urysohn relation $\lhd$ on 2^X such that $\tau = \tau_\lhd$ and $\tau^* = \tau_{\lhd^*}$, then construct $D\lhd$ as in Lemma 8, and note that for each n, $\tau_{\lhd_n} = \tau_\lhd$ and $\tau_{\lhd_n^*} = \tau_{\lhd^*}$ and then that $\tau_{D\lhd} = \tau_\lhd$ and $\tau_{D\lhd^*} = \tau_{\lhd^*}$. Thus $D\lhd$ is a dualizable auxiliary relation that gives rise to the same bitopology as does $\lhd$. $\square$

In [15], the following is proved:

Theorem 18. *The following are equivalent*:

(1) *The bitopological space (X, τ, τ^*) is normal.*
(2) *The binary relation $\lhd_{\mathcal{N}}$ on $(2^X, \subseteq)$ is a dualizable auxiliary relation, where $A \lhd_{\mathcal{N}} B$ if and only if $\mathrm{cl}_{\tau^*} A \subseteq \mathrm{int}_\tau B$.*

Further, if (X, τ, τ^*) and (X, τ^*, τ) are weakly symmetric, then $\tau = \tau_{\lhd_{\mathcal{N}}}$ and $\tau^* = \tau_{\lhd_{\mathcal{N}}^*}$.

Remark. Theorem 17 easily yields the fact that a topological space (X, τ) is completely regular if and only if there is a Urysohn relation $\lhd$ on 2^X such that $\lhd$ is self-dual (that is, $\lhd = \lhd^*$). Thus of course, $\tau = \tau_\lhd = \tau_{\lhd^*}$:

Simply note that (X, τ) is completely regular if and only if (X, τ, τ) is pairwise completely regular, and f is pairwise continuous from (X, τ, τ) to $(\mathbb{I}, \sigma, \omega)$ if and only if, f is continuous from (X, τ) to $(\mathbb{I}, us)$, where us is the usual topology on the unit interval.

Then consider $\lhd_F$ as defined in the proof of Theorem 17. Note that $(\lhd_F)_d$ is a proximity in this situation, giving the usual characterization of complete regularity.

Also, by Theorem 18, a topological space (X, τ) is T_4 if and only if $\lhd_{\mathcal{N}}$ is a self-dual Urysohn relation on 2^X and $\tau = \tau_{\lhd_{\mathcal{N}}}$.

5. Auxiliary relations in domain theory

We point out some topological uses of the idea of auxiliary relation in domain theory.

Definition 19. Suppose $(P, \leqslant, \lhd)$ is a poset with Urysohn relation. For $A, B \subseteq P$, define $A \prec_\lhd B$ to mean that for some r, s in P, $A \subseteq \uparrow_\leqslant s \subseteq \uparrow_\leqslant r \subseteq B$ and $r \lhd s$.

Note that $A \prec_\lhd B \Leftrightarrow$ for some $r, s \in P$, $r \lhd s$, $A \subseteq \uparrow_\leqslant s$ and $\uparrow_\leqslant r \subseteq B \Leftrightarrow$ for some $u, s \in P$, $A \subseteq \uparrow_\leqslant s$ and $\uparrow_\lhd u \subseteq B$ (choose $r \lhd u \lhd s$).

It is easily seen that $\prec_\lhd$ is a Urysohn relation on $(2^P, \subseteq)$; indeed each of (AR_{str})–(AR_{in11}) for $\prec_\lhd$ arises from the corresponding axiom for $\lhd$. But note that for (AR_{in12}) to hold for $\prec_\lhd$ as defined, we need (AR_{in21}) for $\lhd$ and that P have suprema of pairs: Thus if $A \prec_\lhd B, C$ then there are $r_B, r_C, s_B, s_C \in P$ such that $A \subseteq \uparrow_\leqslant s_B, s_C$, $\uparrow_\lhd r_B \subseteq B$ and $\uparrow_\lhd r_C \subseteq C$. Thus if P has suprema for pairs, then $A \subseteq \uparrow_\leqslant (s_B \vee s_C)$. Also, $r_B \lhd s_B \leqslant s_B \vee s_C$, so $r_B \lhd s_B \vee s_C$; similarly, $r_C \lhd s_B \vee s_C$. So if $\lhd$ is an auxiliary relation, there is a $v \in P$ such that $r_B, r_C \lhd v \lhd s_B \vee s_C$, so $\uparrow_\lhd v \subseteq \uparrow_\lhd r_B \cap \uparrow_\lhd r_C \subseteq B \cap C$, $A \subseteq \uparrow_\leqslant (s_B \vee s_C)$ and $v \lhd s_B \vee s_C$, as required. Of course in general, $A \prec_\lhd$ is an auxiliary relation, and $D \prec_\lhd$ is a dualizable auxiliary relation:

Definition 20. For a poset with auxiliary relation $(P, \leqslant, \lhd)$, its *pseudoScott topology*, ρ, is the one whose open sets are generated by all sets of the form $\uparrow_\lhd p$ for $p \in P$, while its *lower topology*, ω, is the one whose closed sets are generated by all sets of the form $\uparrow_\leqslant p$ for $p \in P$.

Theorem 21. *For a poset with auxiliary relation $(P, \leqslant, \lhd)$, the pseudoScott topology is $\tau_{\prec_\lhd}$. If also $\lhd$ is approximating, the lower is $\tau_{\prec_{\lhd^*}}$, and further, $\leqslant_{\tau_{\prec_\lhd}}$ is $\leqslant$ and $\leqslant_{\tau_{\prec_{\lhd^*}}}$ is $\geqslant$.*

Proof. For the first assertion let $p \in P$. If $q \in \uparrow_\lhd p$, then $p \lhd q$ so $\{q\} \subseteq \uparrow_\leqslant q \subseteq \uparrow_\lhd p$, whence $\{q\} \prec_\lhd \uparrow_\lhd p$. This shows that $\uparrow_\lhd p$ is open in $\tau_{\prec_\lhd}$. If also $q \in T \in \tau_{\prec_\lhd}$, then by the last assertion of Lemma 13 $\{q\} \prec_\lhd T$, so for some $p, r \in P$, $\{q\} \subseteq \uparrow_\leqslant r \subseteq \uparrow_\lhd p \subseteq T$, so in particular, $q \in \uparrow_\lhd p \subseteq T$. Thus the $\uparrow_\lhd p$ form an open base for $\tau_{\prec_\lhd}$, showing that $\rho = \tau_{\prec_\lhd}$.

If $q \in T \in \tau_{\prec_{\lhd^*}}$ then for some $n, s_1, \ldots, s_n, r_1, \ldots, r_n \in P$, each $r_i \lhd s_i$ and $\{q\} \subseteq \bigcap_1^n (P \setminus \uparrow_\lhd r_i) \subseteq \bigcap_1^n (P \setminus \uparrow_\leqslant p_i) \subseteq T$. In particular $q \in \bigcap_1^n (P \setminus \uparrow_\leqslant p_i) \subseteq T$, showing that T is an ω neighborhood of q, and so T is an ω neighborhood of each of its elements q, so it is ω-open. This shows $\tau_{\prec_{\lhd^*}} \subseteq \omega$, even without the assumption that $\lhd$ is approximating.

To see that if $\lhd$ is approximating, then the lower is $\tau_{\prec_{\lhd^*}}$ let $q \in P \setminus \uparrow_\leqslant p$. Then $q \ngeqslant p$ so there is an $r \in P$ such that $q \ngeqslant r$ and $r \lhd p$. That is, $\{q\} \subseteq P \setminus \uparrow_\lhd r \subseteq P \setminus \uparrow_\leqslant p$; so each subbasic ω-open $P \setminus \uparrow_\leqslant p$ is a $\tau_{\prec_{\lhd^*}}$ neighborhood of each of its elements q, so it is $\tau_{\prec_{\lhd^*}}$-open. As a result, $\omega \subseteq \tau_{\prec_{\lhd^*}}$, so by the last paragraph, $\tau_{\prec_{\lhd^*}} = \omega$.

Note that by (AR_{str}) and (AR_{trn}), each basic $\uparrow_\lhd p$, thus each open set, is a $\leqslant$-upper set, so each closed set is a $\leqslant$-lower set, therefore $y \leqslant x \Rightarrow y \in \mathrm{cl}_\rho(\{x\})$, so $\leqslant \subseteq \leqslant_{\tau_{\prec_\lhd}}$. If $\lhd$ is approximating and $y \nleqslant x$ then for some $z \lhd y$, $z \ntriangleleft x$, so $\uparrow_\lhd z$ is a neighborhood of y not meeting $\{x\}$, thus $y \notin \mathrm{cl}_\rho(\{x\})$, and so $\leqslant \supseteq \leqslant_{\tau_{\prec_\lhd}}$. Also in this case (P, ρ, ω) is pairwise completely regular, thus $\leqslant_\omega = (\leqslant_\rho)^{-1} = \geqslant$. □

Definition 22. A *dcpo* is a poset in which directed (nonempty) subsets all have suprema, and a dcpo is *continuous* if each element is the directed supremum of those *way below* (*compactly below*) it:

The way below relationship is defined by declaring $p \ll q$ if and only if

$$\left(q \leqslant \bigvee D \Rightarrow (\exists r \in D)(p \leqslant r)\right)$$

for all directed sets D. Thus a dcpo is continuous if for each $p \in P$, $\downarrow_\ll p$ is directed and $p = \bigvee \downarrow_\ll p$.

A dcpo is *bounded complete* if each set which is bounded above has a supremum, and a *Scott domain* is a bounded complete continuous dcpo.

Note that $(\mathbb{I}, \leqslant, <)$ is a Scott domain; its upper topology is its Scott topology, a fact we have foreshadowed by using σ to denote it. Among the good references to domain theory we particularly recommend [5] and [1].

A useful example of a continuous dcpo is the collection of open proper subsets of a locally compact space (X, τ), $\mathcal{K} = (\tau \setminus \{X\}, \subseteq)$. Here $T \ll U \Leftrightarrow (\exists \text{ compact } K)(T \subseteq K \subseteq U)$. Verification is left to the reader, or can be found in [5].

Theorem 23.

(a) *For each continuous dcpo, $(P, \leqslant)$, $\ll$ is an approximating auxiliary relation on P, and for each Scott domain, (P, σ, ω) is join-compact.*

(b) *For each continuous dcpo,* $(P, \leqslant)$, $\sigma = \tau_{\prec_{\ll}}$ *and* $\omega = \tau_{\prec_{(\ll^*)}}$.
(c) *For each Scott domain* $(P, \leqslant)$, *the bitopological space* (P, σ, ω) *arises from the dualizable auxiliary relation* $\lhd_{\mathcal{N}}$.

Proof. Most assertions of (a) are well known (see for example [5]), but we show them here for the convenience of the reader. Certainly if $p \ll q$, since $\{q\}$ is directed, and $q \leqslant \bigvee\{q\}$, $p \leqslant q$, showing (AR_{str}); it is also clear that if $r \leqslant p \ll q \leqslant s$ and $s \leqslant \bigvee D$, D directed, then $q \leqslant \bigvee D$, so for some $d \in D$, $r \leqslant p \leqslant d$, showing (AR_{trn}). To see (AR_{in11}), suppose $p \ll q$ and consider $D = \downarrow_{\ll}(\downarrow_{\ll} q)$.

Note that D is directed, for if $s, t \in D$ then for some $s', t' \in \downarrow_{\ll} q$, $s \in \downarrow_{\ll} s'$ and $t \in \downarrow_{\ll} t'$. Since $\downarrow_{\ll} q$ is directed, there is a $u \in \downarrow_{\ll} q$ such that $s', t' \ll u$, and then since $\downarrow_{\ll} u$ is directed, there is a $v \in \downarrow_{\ll} u$ such that $s', t' \leqslant v$. Then $v \in D$, and $s \leqslant s' \ll v$, $t \leqslant t' \ll v$, so $s, t \leqslant v$.

Since the above $p \in \downarrow_{\ll} q$, we have $\downarrow_{\ll} p \subseteq \downarrow_{\ll}(\downarrow_{\ll} q) = D$, so $p = \bigvee \downarrow_{\ll} p \leqslant \bigvee D$, thus $p \leqslant t$ for some $t \in D$; that is, for some u, $p \leqslant t \ll u \ll q$, so $p \ll u \ll q$ showing (AR_{in11}). Since each $\downarrow_{\ll} q$ is directed, (AR_{in21}) holds as well; thus $\ll$ is an auxiliary relation; it is approximating also, since we have required that $p = \bigvee \downarrow_{\ll} p$ for all $p \in P$.

For (b), as a special case of Theorem 21, if $(P, \leqslant)$ is a continuous dcpo, then σ is $\tau_{\prec_{\ll}}$ and ω is $\tau_{\prec_{(\ll^*)}}$, so (P, σ, ω) is pairwise completely regular; also $\leqslant_{\sigma} = \leqslant$, so σ is T_0, thus so is $\sigma \vee \omega$, because it is stricter.

For (c), if $(P, \leqslant)$ is a Scott domain, then $\sigma \vee \omega$ is also compact (see [5]), so (P, σ, ω) is joincompact. Each joincompact bitopological space is T_4 by reasoning similar to the topological case (see [15], Theorem 3.6). So the theorem results from these observations as well as Theorems 17 and 18. □

6. Adjoints and interpolating relations on functions

Given two posets with auxiliary relations $(P, \leqslant_P, \lhd_P)$ and $(Q, \leqslant_Q, \lhd_Q)$, one can define a relation on order preserving functions from P to Q in terms of $\lhd_P$ and $\lhd_Q$. To do this, we consider adjoints.

Let P and Q be posets, and $f : P \to Q$, $g : Q \to P$ be order preserving maps. Then g is an *upper adjoint* for f if, for each $p \in P$ and $q \in Q$, $p \leqslant g(q) \Leftrightarrow f(p) \leqslant q$. In this case, f is a *lower adjoint* for g. A function from one poset to another has at most one upper adjoint. We denote this by $g = f^u$ and by $f = g^l$.

For the example most familiar to topologists, let $f : X \to Y$ be any function and, for $A \subseteq X$, $B \subseteq Y$, let $f^{\to}(A) = \{f(x):\ x \in A\}$ and $f^{\leftarrow}(B) = \{x:\ f(x) \in B\}$. Then $f^{\leftarrow}$ is an upper adjoint to $f^{\to}$ between the posets $(2^X, \subseteq)$ and $(2^Y, \subseteq)$, since $A \subseteq f^{\leftarrow}(B)$ if and only if $f^{\to}(A) \subseteq B$.

Many useful observations on adjoints are gathered in Section 0.3 of [5]: Each function with an upper adjoint preserves $\bigvee$; as a partial converse, if the domain is a complete lattice then each function that preserves $\bigvee$ has an upper adjoint. Results on adjoints are easily dualizable, since if g is an upper adjoint for f regarded as a map from $(P, \leqslant_P)$ to $(Q, \leqslant_Q)$ then f is an upper adjoint for g, seen as a map from $(Q, \leqslant_Q^{-1})$ to $(P, \leqslant_P^{-1})$.

Definition 24. Let $(P, \leqslant_P, \lhd_P)$ and $(Q, \leqslant_Q, \lhd_Q)$ be posets with Urysohn relations. Let $OR(P, Q)$ denote the poset of order preserving maps $f : P \to Q$, with the pointwise order on $OR(P, Q)$ (that is, $f \leqslant_{OR} g$ if, for each $p \in P$, $f(p) \leqslant_Q g(p)$), and let $UA(P, Q)$ denote the subset of $OR(P, Q)$, of maps with an upper adjoint; we denote that upper adjoint by f^u. Thus $a \leqslant f^u(b) \Leftrightarrow f(a) \leqslant b$.

Let $\lhd_{OR}$ be the relation on $OR(P, Q)$ defined by: $f \lhd_{OR} g$ if and only if for each $a \in P$, $c \in Q$, if $g(a) \lhd_Q c$ then for some $b \in P$, $a \lhd_P b$ and $f(b) \leqslant_Q c$.

Also, for $f, g \in UA(P, Q)$ let $\lhd_{UA}$ be defined by $f \lhd_{UA} g$ if and only if $f^u(q) \lhd_P g^u(r)$ whenever $q \lhd_Q r$.

Here are useful basic facts about the relationship between $\lhd_{OR}$ and $\lhd_{UA}$, and their connection with continuity:

Theorem 25. *Let* $(P, \leqslant_P, \lhd_P)$, $(Q, \leqslant_Q, \lhd_Q)$ *be posets with auxiliary relations and let* $f, g \in UA(P, Q)$.

(a) *If* $f \lhd_{UA} g \Rightarrow f \lhd_{OR} g$, *and if* $\lhd_Q$ *is dually approximating, then* $f \lhd_{OR} g \Rightarrow f \leqslant g$.
(b) *If* $f^{\to} \lhd_{UA} g^{\to}$ *then* $g^{\to} \lhd_{UA}^* f^{\to}$.
(c) *If* $f^{\to} \lhd_{OR} f^{\to}$ *then* f *is continuous from* $(X, \tau_{\lhd_X})$ *to* $(Y, \tau_{\lhd_Y})$. *Thus if* $f^{\to} \lhd_{UA} f^{\to}$ *then* f *is pairwise continuous from* $(X, \tau_{\lhd_X}, \tau_{\lhd_X^*})$ *to* $(Y, \tau_{\lhd_Y}, \tau_{\lhd_Y^*})$.

Proof. To see (a), suppose $f^{\to} \lhd_{UA} g^{\to}$ and let $f(a) \lhd_Q c$. Then for some $d \in Q$, $f(a) \lhd_Q d \lhd_Q c$, so $a \leqslant f^u(f(a)) \lhd_P g^u(d)$, so $a \lhd_P g^u(d)$. Thus there is some $b \in P$ so that $a \lhd_P b \lhd_P g^u(d)$; therefore $b \leqslant_P g^u(d)$, so $g(b) \leqslant_Q d \lhd_Q c$, showing $g(b) \lhd_Q c$. Thus $f \lhd_{OR} g$.

If $\lhd_Q$ is dually approximating, then $g(a) = \bigwedge\{c:\ g(a) \lhd_Q c\} \geqslant \{f(b):\ a \lhd_P b \text{ and } f(b) \leqslant c\} \geqslant f(a)$.

For (b), let $f^{\to} \lhd_{UA} g^{\to}$. If $A \lhd_Q^* B$, then $(Y - B) \lhd_Q (Y - A)$, so

$$X - f^{\leftarrow}[B] = f^{\leftarrow}[Y - B] \lhd_P g^{\leftarrow}[Y - A] = X - g^{\leftarrow}[A],$$

thus $g^{\leftarrow}[A] \lhd_P^* f^{\leftarrow}[B]$. Since $f^{\leftarrow}$ is an upper adjoint for $f^{\to}$, this shows that $g^{\to} \lhd_{UA}^* f^{\to}$.

For (c), if $f^{\rightarrow} \lhd_{OR} f^{\rightarrow}$ and $x \in f^{\leftarrow}[T]$, $T \in \tau_{\lhd_Y}$ then $f(x) \in T$, so for some finite set $\{B_1, \ldots, B_n\}$ of subsets of Y, $\{f(x)\} \lhd_Q B_i$ for each i and $\bigcap_{i=1}^n B_i \subseteq T$. Thus for each i there is a C_i such that $\{x\} \lhd C_i$ and $f^{\rightarrow}[C_i] \subseteq B_i$. But then $f^{\rightarrow}[\bigcap_{i=1}^n C_i] \subseteq \bigcap_{i=1}^n f^{\rightarrow}[C_i] \subseteq \bigcap_{i=1}^n B_i \subseteq T$. By the arbitrary nature of $x \in f^{\leftarrow}[T]$, this shows that $f^{\leftarrow}[T] \in \tau_{\lhd_X}$ and therefore f is continuous from $(X, \tau_{\lhd_X})$ to $(Y, \tau_{\lhd_Y})$.

Also, if $f^{\rightarrow} \lhd_{UA} f^{\rightarrow}$ then $f^{\rightarrow} \lhd^*_{UA} f^{\rightarrow}$ by (a), so $f^{\rightarrow} \lhd^*_{OR} f^{\rightarrow}$ by what we have just shown, so f is also continuous from $(X, \tau_{\lhd^*_X})$ to $(Y, \tau_{\lhd^*_Y})$. Thus f is pairwise continuous from $(X, \tau_{\lhd_X}, \tau_{\lhd^*_X})$ to $(Y, \tau_{\lhd_Y}, \tau_{\lhd^*_Y})$. □

7. Insertion of functions

The following extends a result of Lane [17]. Below let $\mathbb{I} = ([0,1], \leqslant, <)$.

Theorem 26 *(Lane). Let $(P, \leqslant, \lhd)$ and $(Q, \leqslant', \lhd')$ be posets with dualizable auxiliary relations such that $(P, \leqslant)$ is a bounded lattice, and Q is countable. Let $F, G : (Q, \leqslant') \to (P, \leqslant)$ be order-preserving functions. If $F(r) \lhd G(s)$ whenever $r \lhd' s$, then there is an order-preserving function $H' : (Q, \leqslant') \to (P, \leqslant)$ such that $F(r) \lhd H'(s) \lhd H'(t) \lhd G(u)$ whenever $r \lhd' s \lhd' t \lhd' u$.*

Moreover, if Q is a countable dense subset of $[0,1]$, and $\leqslant'$, $\lhd'$ are the restrictions of the usual $\leqslant$, $<$ on $\mathbb{I}$, then there is an order-preserving function $H : [0,1] \to P$ such that $F(r) \lhd H(s) \lhd H(t) \lhd G(u)$, whenever $r < s < t < u$, $r, u \in Q$ and $s, t \in [0,1]$. Moreover, $H(\bigwedge D) = \bigwedge H(D)$ for any $D \subseteq [0,1]$.

Proof. We define H' inductively. Index $Q = \{t_n\colon n \in \mathbb{N}\}$, and suppose that for some $n \in \mathbb{N}$, we have defined $H'(t_k)$ for each $k < n$, so that whenever $t \in Q$ and $j, k < n$: if $t \lhd' t_k$ then $F(t) \lhd H'(t_k)$, if $t_k \lhd' t$ then $H'(t_k) \lhd G(t)$ and if $t_k \lhd' t_j$ then $H'(t_k) \lhd H'(t_j)$. Now we define:

$$A = \{H'(t_k)\colon k < n, t_k \lhd' t_n\} \cup \{F(t)\colon t \lhd' t_n\}, \qquad A' = \{H'(t_k)\colon k < n, t_k \lhd' t_n\} \cup \{F(t_n)\},$$
$$B = \{H'(t_k)\colon k < n, t_n \lhd' t_k\} \cup \{G(t)\colon t_n \lhd' t\}, \qquad B' = \{H'(t_k)\colon k < n, t_n \lhd' t_k\} \cup \{G(t_n)\}.$$

Then A' and B' are finite nonempty sets, A' is bounded above by each element of B', and B' is bounded below by each element of A', so by (KL_{bd}), there are $a, b \in P$ so that $A' \leqslant a$, $b \leqslant B'$. Also for each $e \in Q$, $a \lhd e$ whenever $A' \lhd e$, and $e \lhd b$ whenever $e \lhd B'$. If $d \in A$, then either $d \in A'$ or $d = F(t) \leqslant F(t_n) \in A'$; in either case $d \leqslant a \lhd e$, whenever $a' \lhd e$. Thus $A \leqslant a \lhd B$; similarly $A \lhd b \leqslant B$. Hence by $(KL_{in\omega,\omega})$, there exists $c \in P$ such that $A \lhd c \lhd B$. Let $H'(t_n) = c$, completing the definition of H'.

Suppose that $t_i \lhd' t_j \lhd' t_k \lhd' t_l$, then $i, j, k, l < n$ for some n, and so we have $F(t_i) \lhd H'(t_j)$, $H'(t_j) \lhd H'(t_k)$, and $H'(t_k) \lhd G(t_l)$ as required.

If Q is a countable dense subset of $[0,1]$, with $\leqslant'$ its usual order and $\lhd' = <$, we define $H : [0,1] \to P$ by $H(r) = \bigwedge\{H'(q)\colon r < q \in Q\}$. Let $r < s < t < u$ where $r, u \in Q$ and $s, t \in [0,1]$, then there are $r', s', t', u' \in Q$ such that $r < r' < s < s' < t' < t < u' < u$; then

$$F(r) \lhd H'(r') \leqslant H(s) \leqslant H'(s') \lhd H'(t') \leqslant H(t) \leqslant H'(u') \lhd G(u),$$

from which it follows that $F(r) \lhd H(s) \lhd H(t) \lhd G(u)$.

Finally, for a subset $D \subseteq [0,1]$,

$$H\Big(\bigwedge D\Big) = \bigwedge\Big\{H'(q) \Bigm| \bigwedge D < q \in Q\Big\}.$$

Also, since $\{H'(q) \mid \bigwedge D < q \in Q\} = \bigcup_{d \in D}\{H'(q) \mid d < q \in Q\}$ and $\bigwedge(\bigcup_{d \in D}\{H'(q) \mid d < q \in Q\}) = \bigwedge H(D)$ we have $H(\bigwedge D) = \bigwedge H(D)$. □

Of course, by Theorems 10 and 11, the conclusions of Theorem 26 hold for KL-relations satisfying (KL_{bd}). Now we show that in this situation, $\lhd_{OR}$ is a Urysohn relation on $UA(P, \mathbb{I})$:

Theorem 27.

(a) *Let $(P, \leqslant, \lhd)$ be a bounded lattice with dualizable auxiliary relation. If $f, g \in UA(P, \mathbb{I})$ and $f \lhd_{UA} g$, then for some $h : P \to \mathbb{I}$, $f \lhd_{UA} h \lhd_{UA} h \lhd_{UA} g$.*
(b) *(AR_{trn}) holds for $\lhd_{OR}$ and if $\lhd_Q$ is dually approximating then (AR_{str}) holds for $\lhd_{OR}$ as well.*

Proof. (a) Given such $f, g : P \to \mathbb{I}$, let $Q = \mathbb{Q} \cap \mathbb{I}$ and define $F, G : (Q, \leqslant) \to (P, \leqslant)$ by $F = f^u \restriction_Q$ and $G = g^u \restriction_Q$. Then, by Theorem 26, there is an order preserving $H : (\mathbb{I}, \leqslant) \to (P, \leqslant)$ such that, whenever $p < u < v < q$, $p, q \in Q$ and $u, v \in \mathbb{I}$, then $F(p) \lhd H(u) \lhd H(v) \lhd G(q)$. In addition, for each $D \subseteq [0,1]$, $H(\bigwedge D) = \bigwedge H[D]$.

By the comments on adjoints, H thus has a lower adjoint, $h : (P, \leqslant) \to (\mathbb{I}, \leqslant)$, so H is the upper adjoint to h, which we denote $H = h^u$. Thus if $u < v$ then $h^u(u) = H(u) \lhd H(v) = h^u(v)$, so $h \lhd_{OR} h$.

Since $f^u, g^u : \mathbb{I} \to P$ preserve order, if $u < v$, $u, v \in \mathbb{I}$, there is some $p \in Q$ such that $u < p < v$ and $f^u(u) \leqslant f^u(p) = F(p) \lhd H(v) = h^u(v)$, hence $f \lhd_{OR} h$. Also, $h^u(u) = H(u) \lhd G(p) = g^u(p) \leqslant g^u(v)$, from which it follows that $h \lhd_{OR} g$. Thus (AR_{in11}) holds.

(b) To see (AR_{trn}): if $h \leqslant_{OR} f \lhd_{OR} g \leqslant_{OR} k$, then whenever $k(p) < b$, we have that $g(p) < b$, so for some c, $p \lhd c$ and $h(c) \leqslant f(c) \leqslant b$ so $h \lhd_{OR} k$. To see (AR_{str}), assume $f \lhd_{OR} g$; then if $g(p) \lhd_Q b$, there is a q so that $p \lhd_P q$ and $f(q) \leqslant b$. Thus $f(p) \leqslant \bigwedge\{b\colon\ g(p) \lhd_Q b\} = g(p)$, so $f \leqslant_{OR} g$. $\square$

Theorem 28.

(a) *Suppose $\lhd$ is a Urysohn relation on $(2^X, \subseteq)$, and let $f, g : X \to \mathbb{I}$. If $f^{\to} \lhd_{UA(X,\mathbb{I})} g^{\to}$ then there is a pairwise continuous h from $(X, \tau_{\lhd}, \tau^*_{\lhd})$ to $(\mathbb{I}, \sigma, \omega)$ such that $f \leqslant h \leqslant g$.*
(b) *Suppose (X, τ, τ^*) is a pairwise T_4 bitopological space, f is continuous from (X, τ) to $(\mathbb{I}, \sigma)$, g is continuous from (X, τ^*) to $(\mathbb{I}, \omega)$, and $f \leqslant g$. Then for some pairwise continuous h from (X, τ, τ^*) to $(\mathbb{I}, \sigma, \omega)$, $f \leqslant h \leqslant g$.*

Proof. (a) First by Lemma 8, $\lhd$ is contained in a smallest dualizable auxiliary relation, $D\lhd$. Since $f^{\to} \lhd_{UA(X,\mathbb{I})} g^{\to}$, whenever $r < s$ we have $f^{\leftarrow}[\uparrow_{\leqslant} s] \lhd g^{\leftarrow}[\uparrow_{\leqslant} r]$, so $f^{\leftarrow}[\uparrow_{\leqslant} s] D \lhd g^{\leftarrow}[\uparrow_{\leqslant} r]$, which says that $f^{\to} D \lhd_{UA(X,\mathbb{I})} g^{\to}$.

Since $D\lhd$ is a dualizable auxiliary relation and $(2^X, \subseteq)$ is a complete lattice, there is by Theorem 27, an $H \in \mathbb{I}^{2^X}$ so that $f^{\to} \lhd_{UA2^X,\mathbb{I}} H \lhd_{UA2^X,\mathbb{I}} H \lhd_{UA2^X,\mathbb{I}} g^{\to}$. Let h be the restriction of H to X (formally, $h(x) = H(\{x\})$ for each $x \in X$). Then $h \in \mathbb{I}^X$, and for each $A \subseteq X$, $A = \bigcup_{x \in A}\{x\} = \bigvee_{x \in A}\{x\}$, so $h^{\to}(A) = \bigvee_{x \in A} H(\{x\}) = H(A)$, so $h^{\to} = H$, and thus $f^{\to} \lhd_{UA(X,\mathbb{I})} h^{\to} \lhd_{UA(X,\mathbb{I})} h^{\to} \lhd_{UA(X,\mathbb{I})} g^{\to}$. Then by Theorem 27, $f \leqslant h \leqslant g$. Finally, h is pairwise continuous by Theorem 25.

For part (b) let $\lhd = \lhd_{\mathcal{N}}$; then $\tau = \tau_{\lhd}$ and $\tau^* = \tau_{\lhd^*}$. Note that $f \lhd_{\mathbb{I}^X} g$, for if $A \lhd_{\mathbb{I}} B$ then for some $r < s$, $A \subseteq \uparrow_{\leqslant} s$ and $\uparrow_{<} r \subseteq B$, so $\mathrm{cl}_{\omega} A \subseteq \mathrm{int}_{\sigma} B$ (if $r = 0$ then $\mathbb{I} = \uparrow_{<} r$). By continuity of f from (X, τ) to $(\mathbb{I}, \omega)$ and g from (X, τ^*) to $(\mathbb{I}, \sigma)$, we have that $\mathrm{cl}_{\tau^*} f^{\leftarrow}[A] \subseteq \mathrm{int}_{\tau} g^{\leftarrow}[B]$, which is to say that $f^{\leftarrow}[A] \lhd_{\mathcal{N}} g^{\leftarrow}[B]$. Thus $f \lhd_{OR} g$, so by Theorem 27 (with $P = 2^X$), there is an $h : P \to [0, 1]$ such that $f \lhd_{OR} h \lhd_{OR} h \lhd_{OR} g$. Then by Theorem 25, h is pairwise continuous from (X, τ, τ^*) to $(\mathbb{I}, \sigma, \omega)$. $\square$

We now have:

Corollary 29.

(a) *Let P be a continuous dcpo and $f, g : P \to \mathbb{I}$ be such that g is continuous from (P, σ) to $(\mathbb{I}, \sigma)$, f is continuous from (P, ω) to $(\mathbb{I}, \omega)$, and $f \prec_{\ll} g$. Then there is an $h : P \to \mathbb{I}$ such that $f \leqslant h \leqslant g$ and h is pairwise continuous from (P, σ, ω) to $(\mathbb{I}, \sigma, \omega)$. In particular each $f : P \to \mathbb{I}$ which is Scott continuous is the directed sup of the $h \prec_{\ll} f$ which are pairwise continuous from (P, σ, ω) to $(\mathbb{I}, \sigma, \omega)$.*
(b) *Let P be a Scott domain and $f, g : P \to \mathbb{I}$ be such that $f \leqslant g$, f is continuous from (P, ω) to $(\mathbb{I}, \omega)$, and g is continuous from (P, σ) to $(\mathbb{I}, \sigma)$. Then there is an $h : P \to \mathbb{I}$ such that $f \leqslant h \leqslant g$ and h is pairwise continuous from (P, σ, ω) to $(\mathbb{I}, \sigma, \omega)$. In particular, each $f : P \to \mathbb{I}$ which is Scott continuous is the directed sup of the $h \leqslant f$ which are pairwise continuous from (P, σ, ω) to $(\mathbb{I}, \sigma, \omega)$.*

Proof. Part (a) results from Theorems 23(a) and 28(a), while (b) comes from Theorems 23(b) and 28(b). $\square$

8. Classical examples

Several classical insertion theorems now follow from the above results. Notice that (by appropriately rescaling or considering functions with restricted range) we can equally well consider functions from a space X to either $\mathbb{R}$ or $[0, 1]$. We state our theorems using the more convenient range in each case.

The Katětov–Tong [12,21] Insertion Theorem is an immediate consequence of Theorem 28.

Theorem 30 *(Katětov–Tong). X is normal if and only if whenever $f : X \to \mathbb{R}$ is lower semicontinuous, $g : X \to \mathbb{R}$ is upper semicontinuous and $g \leqslant f$ then there is a continuous $h : X \to \mathbb{R}$ such that $g \leqslant h \leqslant f$.*

From this one can deduce a number of similar well-known results. For us, the existence of a continuous insertion $f \leqslant h \leqslant g$ in Theorem 30 follows from the fact that $g \lhd h \lhd h \lhd f$; we cannot directly deduce that $g(x) < h(x) < f(x)$ for any $x \in X$, so some of our proofs rely on topological facts.

Corollary 31.

(1) *The Tietze Extension Theorem: X is normal if and only if every continuous function $f : C \to [0, 1]$ on a closed set C can be extended to a continuous function $f' : X \to [0, 1]$.*

(2) *Dowker's Insertion Theorem* [3]: *X is normal and countably paracompact iff whenever $f : X \to \mathbb{R}$ is lower semicontinuous, $g : X \to \mathbb{R}$ is upper semicontinuous and $f < g$ then there is a continuous $h : X \to \mathbb{R}$ such that $f < h < g$.*
(3) *Michael's Insertion Theorem* [19]: *X is perfectly normal iff whenever $f : X \to \mathbb{R}$ is lower semicontinuous, $g : X \to \mathbb{R}$ is upper semicontinuous and $f \leqslant g$ then there is a continuous $h : X \to \mathbb{R}$ such that $f \leqslant h \leqslant g$ and $f(x) < h(x) < g(x)$ whenever $f(x) < g(x)$.*

Proof. In each case the converse is standard, so we only prove one direction.

For (1), if C is a closed subset of X and $f : C \to [0,1]$ is continuous, let $\varphi(x) = \psi(x) = f(x)$, for all $x \in C$, and define $\varphi(x) = 0$ and $\psi(x) = 1$, for $x \notin C$. Then $\varphi \leqslant \psi$, φ is usc and ψ is lsc. Theorem 30 provides us with a continuous f' which is equal to f on C.

Simple, geometric proofs of both (2) and (3) given Katětov's Theorem appear in [8], but here we give more 'functional' proofs. A normal space X is countably paracompact (see [3]) if and only for every decreasing sequence of closed sets (D_n) such that $\bigcap_{n\in\mathbb{N}} D_n = \varnothing$ there are open sets $U_n \supseteq D_n$ such that $\bigcap_{n\in\mathbb{N}} U_n = \varnothing$. A normal space X is perfect if and only if for every closed set D there are open sets $U_n \supseteq D$ such that $\bigcap_{n\in\mathbb{N}} U_n = D$. In fact it is easy to prove (see [4] for example) that X is perfect if and only if for every decreasing sequence of closed sets (D_n), there are open sets $U_n \supseteq D_n$ such that $\bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} D_n$.

For (2), suppose that X is both normal and countably paracompact and that $g < f$, where g is usc and f is lsc. Let $D_n = \{x\colon\ f(x) - g(x) \leqslant 1/3^{n+1}\}$; D_n is then closed and $\bigcap D_n = \varnothing$. By countable paracompactness, for each $n \in \mathbb{N}$, there is an open $U_n \supseteq D_n$ such that $\bigcap_{n\in\mathbb{N}} U_n = \varnothing$. By (1) we can extend the continuous function taking the value 0 on D_n and 1 on $X - U_n$ to a continuous function $\varphi_n : X \to [0,1]$. Let $\varphi = \sum \varphi_n/3^n$ so that $\varphi : X \to [0,1/2]$ is continuous and $\varphi(x) \leqslant \frac{1}{3^n 2}$ for $x \in D_n$. Every $x \in X$ is in $X - D_1$ or in $D_n - D_{n+1}$, for some n, so that $2\varphi(x) < f(x) - g(x)$ for all $x \in X$. We can now apply the Katětov–Tong Theorem to the functions $g' = g + \varphi \leqslant f' = h - \varphi$. The argument for (3) is similar: if $g \leqslant f$, where g is usc and f is lsc, then defining D_n as above we have $\bigcap_{n\in\mathbb{N}} D_n = D = \{x\colon\ f(x) = g(x)\}$, so that $\varphi(x) = 0$ for all $x \in D$. The rest of the argument is identical. □

A space is monotonically normal [11] if and only if there is an operator H assigning an open set $H(C,D)$ to each pair of disjoint closed sets such that

(1) $C \subseteq H(C,D) \subseteq \overline{H(C,D)} \subseteq X - D$, and
(2) $H(C,D) \subseteq H(C',D')$, whenever $C \subseteq C'$ and $D' \subseteq D$.

For more on the significance of monotonically normal spaces see [9]. It turns out that there is a natural monotone version of the Katětov–Tong Insertion Theorem due to Kubiak [16] (see also [18]). It is convenient to introduce some notation. Let $C(X)$ denote the set of all continuous $\mathbb{R}$-valued functions on X and let $UL(X) = \{(g,f)\colon\ g \leqslant f,\ \ f : X \to \mathbb{R} \text{ lsc},\ \ g : X \to \mathbb{R} \text{ usc}\}$, ordered by $(g,f) \leqslant (g',f')$ iff $g \leqslant g'$ and $f \leqslant f'$.

Theorem 32 *(Kubiak). X is monotonically normal iff there is an order preserving map $\Phi : UL(X) \to C(X)$ such that $g \leqslant \Phi(g,f) \leqslant f$.*

Proof. Order the power set of X, $\mathcal{P}(X)$ by inclusion. Let $P = \{\varphi : UL(X) \to \mathcal{P}(X)\colon\ \varphi \text{ is order reversing}\}$. Let $\leqslant$ be the partial order on P defined by $\varphi \leqslant \varphi'$ iff $\varphi(g,f) \subseteq \varphi'(g,f)$ for all $(g,f) \in UL(X)$. Define $\varphi \lhd \varphi'$ iff $\overline{\varphi(g,f)} \subseteq \varphi(g,f)^\circ$.

Clearly $(P,\leqslant)$ has finite sups and infs, for example define $(\bigvee_{\varphi\in R} \varphi)(g,f) = \bigcup_{\varphi\in R}(\varphi(g,f))$ for any $R \subseteq P$. And so $(P,\leqslant,\lhd)$ satisfies (AR_{str})–(AR_{in21}). To see (AR_{in21}) (hence (AR_{in11})), suppose that $\varphi, \varphi' \lhd \psi$. Let H be a monotone normality operator. Define

$$\chi(g,f) = H\big(\overline{\varphi(g,f)}, \psi(g,f)^\circ\big) \cup H\big(\overline{\varphi'(g,f)}, \psi(g,f)^\circ\big).$$

Then

$$\overline{\varphi(g,f)} \cup \overline{\varphi'(g,f)} \subseteq \chi(g,f)^\circ = \chi(g,f) \subseteq \overline{\chi(g,f)} \subseteq \psi(g,f)^\circ.$$

Also χ is order reversing since φ and φ' are and H is monotone.

Now we can apply Theorem 26 to the functions $F, G : \mathbb{Q} \to P$ defined by $F(r)(g,f) = \{x\colon\ f(x) \leqslant r\}$ and $G(r)(g,f) = \{x\colon\ g(x) < r\}$ so that $F \lhd G$ to get $H : \mathbb{Q} \to P$ such that $F \lhd H \lhd H \lhd G$. Defining $\Phi(g,f)(x) = \inf\{r\colon\ x \in H(r)(g,f)\}$ completes the proof. □

There are natural monotone versions of the Dowker and Michael Insertion Theorems, though both versions turn out to be equivalent to stratifiability. A space is stratifiable if and only there is an operator U assigning an open set $U(n,D)$ to every closed set D and $n \in \mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}} \overline{U(n,D)} = D$ and $U(n,D) \subseteq U(n,D')$ whenever $D \subseteq D'$. The following two results appear in [7] (see also [6]) and [20] respectively. One can also prove these results from Kubiak's in exactly the same way as Dowker's and Michael's follow from the Katětov–Tong Theorem so we omit the proofs here.

Corollary 33.

(1) *X is stratifiable iff there is an order preserving map Ψ assigning to each pair $(g, f) \in UL(X)$, with $g < f$, a continuous function $\Psi(g, f)$ such that $g < \Psi(g, f) < f$.*
(2) *X is stratifiable iff there is an order preserving map $\Theta : UL(X) \to C(X)$ such that $g \leqslant \Theta(g, f) \leqslant f$ and $g(x) < \Theta(g, f)(x) < f(x)$, whenever $g(x) < f(X)$.*

Finally the results of Lane [17] are clearly incorporated in our development. Given a function f, let $f_*(x) = \sup_{x\in U \text{ open}} \inf_{y\in U\cap X} f(y)$ and $f^*(x) = \inf_{x\in U \text{ open}} \sup_{y\in U\cap X} f(y)$. A function is *normal lsc* if $f = (f^*)_*$ and is *normal usc* if $f = (f_*)^*$.

Theorem 34 *(Lane).*

(1) *Suppose disjoint regular closed sets are separated by disjoint open sets. If $g \leqslant f$, g normal usc, f normal lsc then there is continuous h such that $g \leqslant h \leqslant f$.*
(2) *Suppose disjoint closed sets, at least one of which is regular closed, are separated by disjoint open sets. If $g \leqslant f$, and either g usc, f normal lsc or g normal usc, f lsc, then there is continuous h such that $g \leqslant h \leqslant f$.*
(3) *Suppose X is extremally disconnected. If $g \leqslant f$, g lsc and f usc, then there is a continuous h so that $g \leqslant h \leqslant f$.*

Proof. (1) follows from Theorem 26 defining $\lhd$ on the power set of X by $A \lhd B$ iff $\overline{A} \subseteq F \subseteq G \subseteq B^\circ$ where F is regular closed and G is regular open. Then, if f is normal lsc and g is normal usc, $\overline{\{x\colon\ f(x) < r\}}$ is regular closed and $\{x\colon\ g(x) \leqslant r\}^\circ$ is regular open, so $\{x\colon\ f(x) < r\} \lhd \{x\colon\ g(x) \leqslant r\}$.

(2) and (3) follow from Theorem 26 defining $A \lhd B$ iff there is some open G such that $\overline{A} \subseteq G \subseteq \overline{G} \subseteq B^\circ$. □

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