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GLOBAL ATTRACTORS FOR STRONGLY DAMPED WAVE EQUATIONS WITH DISPLACEMENT DEPENDENT DAMPING AND NONLINEAR SOURCE TERM OF CRITICAL EXPONENT

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Abstract. In this paper the long time behaviour of the solutions of the 3-D strongly damped wave equation is studied. It is shown that the semigroup generated by this equation possesses a global attractor in $H_0^1(\Omega) \times L_2(\Omega)$ and then it is proved that this is also a global attractor in $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

1. **Introduction.** We consider the following initial-boundary value problem for the strongly damped wave equation:

$$w_{tt} - \Delta w_t + \sigma(w)w_t - \Delta w + f(w) = g(x) \qquad \text{in } (0,\infty) \times \Omega, \tag{1.1}$$

$$w = 0 \qquad \text{on } (0,\infty) \times \partial\Omega, \tag{1.2}$$

$$w(0,\cdot) = w_0 \ , \qquad w_t(0,\cdot) = w_1 \qquad \text{in } \Omega, \tag{1.3}$$

where $\Omega \subset R^3$ is a bounded domain with sufficiently smooth boundary and $g \in L_2(\Omega)$.

As shown in [6] and [13], equation (1.1) is related to the following reaction-diffusion equation with memory:

$$w_t(t,x) = \int_{-\infty}^{t} K(t,s)\Delta w(s,x)ds - f(w(t,x)) + g(x). \tag{1.4}$$

Namely, if $K(t,s) = \frac{1-\alpha}{\lambda} e^{-\frac{t-s}{\lambda}} + 2\alpha\delta(t-s)$ then (1.4) can be transformed into

$$\lambda w_{tt} - \alpha\lambda\Delta w_t + (1 + \lambda f'(w))w_t - \Delta w + f(w) = g,$$

where $\lambda > 0$, $\alpha \in [0,1)$ and δ is a Dirac delta function. This equation is interesting from a physical viewpoint as a model describing the flow of viscoelastic fluids (see [6] and [13] for details).

When $\sigma(\cdot) \equiv 0$ the equation (1.1) becomes

$$w_{tt} - \Delta w_t - \Delta w + f(w) = g. \tag{1.5}$$

The long time behaviour (in terms of attractors) of solutions in this case has been studied by many authors (see [2], [5], [7], [14], [15], [19], [22] and references therein). In [14] the existence of a global attractor for (1.5) with critical source term (i.e. in the case when the growth of f is of order 5) was proved. However, the regularity of

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the global attractor in that article was established only in the subcritical case. For the critical case, the regularity of the global attractor of (1.5) was proved in [15], under the assumptions

$$f \in C^1(R),\ |f'(s)| \le c(1+|s|^4),\ \forall s \in R \text{ and } \liminf_{|s|\to\infty} f'(s) > -\lambda_1 \tag{1.6}$$

or

$$f \in C^2(R),\ |f''(s)| \le c(1+|s|^3),\ \forall s \in R \text{ and } \liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{1.7}$$

where λ_1 is a first eigenvalue of $-\Delta$ with zero Dirichlet data. In that article the authors obtained a regular estimate for w_{tt} (when $w(t,x)$ is a weak solution of (1.5)) and then proved the asymptotic regularity of the solution of the non-autonomous equation

$$-\Delta w_t - \Delta w + f(w) = g - w_{tt}.$$

In [5] and [19], the regularity of the global attractor of (1.5) was proved under the following weaker condition on the source term:

$$f \in C(R),\ |f(u)-f(v)| \le c(1+|u|^4+|v|^4)|u-v|,\ \forall u,v \in R \text{ and } \liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1.$$

In [8], the authors investigated the weak attractor for the quasi-linear strongly damped equation

$$w_{tt} - \Delta w_t - \Delta w + f(w) = \nabla \cdot \varphi'(\nabla w) + g$$

under the following conditions on the nonlinear functions f and φ:

$$f \in C^1(R), \quad -C + a_1 |s|^q \le f'(s) \le C |s|^q,\ \forall s \in R,$$

$$\varphi \in C^2(R^3, R),\ \ a_2 |\eta|^{p-1} |\xi|^2 \le \sum_{i,j=1}^{3} \frac{\partial^2 \varphi(\eta)}{\partial \eta_i \partial \eta_j} \xi_i \xi_j \le a_3(1+|\eta|^{p-1}) |\xi|^2,\ \ \forall \xi, \eta \in R^3,$$

for some $a_i > 0$, $(i = 1,2,3)$, $C > 0$, $q > 0$ and $p \in [1,5)$. When $\frac{\partial^2 \varphi}{\partial \eta_i \partial \eta_j} = 0$, $(i,j = 1,2,3)$, the strong attractor has also been studied. Recently, in [3], the authors have studied the global attractor for the strongly damped abstract equation

$$w_{tt} + D(w, w_t) + Aw + F(w) = 0.$$

However, the approaches of the articles mentioned above, in general, do not seem to be applicable to (1.1). The difficulty is caused by the term $\sigma(w)w_t$, when the function $\sigma(\cdot)$ is not differentiable and the growth condition imposed on $\sigma(\cdot)$ is critical. In this paper we prove the existence of the global attractors for (1.1)-(1.3) in $H_0^1(\Omega) \times L_2(\Omega)$ and $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Then using the embedding $H^{\frac{3}{2}+\varepsilon}(\Omega) \subset C(\overline{\Omega})$ we show that these attractors coincide.

2. **Well-posedness and the statement of the main result.** We start with the conditions on nonlinear terms f and σ.

- $f \in C(R),\ \ |f(s) - f(t)| \le c(1+|s|^4+|t|^4)\,|s-t|\,,\ \forall s,t \in R,$ (2.1)

- $\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1,$ where $\lambda_1 = \inf_{\varphi \in H_0^1(\Omega), \varphi \ne 0} \frac{\|\nabla \varphi\|^2_{L_2(\Omega)}}{\|\varphi\|^2_{L_2(\Omega)}}$, (2.2)

- $\sigma \in C(R),\ \ \sigma(s) \ge 0,\ \ |\sigma(s)| \le c(1+|s|^4),\ \ \forall s \in R.$ (2.3)

By the standard Galerkin's method it is easy to prove the following existence theorem:

Theorem 2.1. *Let conditions (2.1)-(2.3) hold. Then for every $T > 0$ and every $(w_0, w_1) \in \mathcal{H} := H_0^1(\Omega) \times L_2(\Omega)$, the problem (1.1)-(1.3) admits a weak solution*

$$w \in C([0,T]; H_0^1(\Omega)), \ \ w_t \in C([0,T]; L_2(\Omega)) \cap L_2(0,T; H_0^1(\Omega)),$$

which satisfies the following energy equality

$$E(w(t)) + \int_s^t \|\nabla w_t(\tau)\|_{L_2(\Omega)}^2 d\tau + \int_s^t \langle \sigma(w(\tau))w_t(\tau), w_t(\tau)\rangle \, d\tau + \langle F(w(t)), 1\rangle -$$
$$- \langle g, w(t)\rangle = E(w(s)) + \langle F(w(s)), 1\rangle - \langle g, w(s)\rangle , \qquad 0 \le s \le t \le T, \qquad (2.4)$$

where $E(w(t)) = \frac{1}{2}(\|\nabla w(t)\|_{L_2(\Omega)}^2 + \|w_t(t)\|_{L_2(\Omega)}^2)$, $\langle u, v\rangle = \int\limits_\Omega u(x)v(x)dx$ and $F(w) = \int\limits_0^w f(u)du$.

Now using the method of [16, Proposition 2.2] let us prove the following uniqueness theorem:

Theorem 2.2. *Let conditions (2.1)-(2.3) hold. If $w(t,\cdot)$ and $\widehat{w}(t,\cdot)$ are the weak solutions of (1.1)-(1.3), determined by Theorem 2.1, with initial data (w_0, w_1) and $(\widehat{w}_0, \widehat{w}_1)$ respectively, then*

$$\|w(T) - \widehat{w}(T)\|_{H^1(\Omega)}^2 + \|w_t(T) - \widehat{w}_t(T)\|_{H^{-1}(\Omega)}^2 \le$$
$$\le c(T,R) \left(\|w_0 - \widehat{w}_0\|_{H^1(\Omega)} + \|w_1 - \widehat{w}_1\|_{H^{-1}(\Omega)} \right)$$

where $c : R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable and $R = \max\{\|(w_0, w_1)\|_{\mathcal{H}}, \|(\widehat{w}_0, \widehat{w}_1)\|_{\mathcal{H}}\}$.

Proof. By (2.1)-(2.4), it follows that

$$\|(w(t), w_t(t))\|_{\mathcal{H}} + \|(\widehat{w}(t), \widehat{w}_t(t))\|_{\mathcal{H}} \le c_1(R), \quad \forall t \ge 0.$$

Denote $u(t,\cdot) = w(t,\cdot) - \widehat{w}(t,\cdot)$ and $\widehat{u}(t,\cdot) = \int\limits_0^t u(\tau,\cdot)d\tau$. Integrating (1.1) for $w(t,\cdot)$ and $\widehat{w}(t,\cdot)$ on $[0,t]$ and taking the difference, we have

$$u_t - \Delta u + \Sigma(w) - \Sigma(\widehat{w}) - \Delta\widehat{u} + \int_0^t \left(f(w(\tau,)) - f(\widehat{w}(\tau,))\right) d\tau =$$
$$= \Sigma(w_0) - \Sigma(\widehat{w}_0) - \Delta(w_0 - \widehat{w}_0) + w_1 - \widehat{w}_1, \quad \forall t \ge 0, \qquad (2.5)$$

where $\Sigma(w) = \int\limits_0^w \sigma(s)ds$. Testing (2.5) by u and taking into account (2.1), (2.3), (2.4) and monotonicity of $\Sigma(\cdot)$, we find

$$\frac{d}{dt}E(\widehat{u}(t)) + \frac{1}{2}\|\nabla u(t)\|_{L_2(\Omega)}^2 \le$$
$$\le c_2(R) \left(\|\nabla(w_0 - \widehat{w}_0)\|_{L_2(\Omega)}^2 + \|w_1 - \widehat{w}_1\|_{H^{-1}(\Omega)}^2 \right) +$$
$$+ c_2(R)t \int_0^t \|\nabla u(\tau)\|_{L_2(\Omega)}^2 d\tau, \ \ \forall t \ge 0 \qquad (2.6)$$

and consequently

$$\frac{d}{dt}\widehat{E}(\widehat{u}(t)) \le c_2(R) \left(\|w_0 - \widehat{w}_0\|_{H^1(\Omega)}^2 + \|w_1 - \widehat{w}_1\|_{H^{-1}(\Omega)}^2 \right) + 2c_2(R)t\widehat{E}(\widehat{u}(t)),$$

where $\widehat{E}(\widehat{u}(t)) = E(\widehat{u}(t)) + \frac{1}{2}\int\limits_0^t \|\nabla u(\tau)\|^2_{L_2(\Omega)}\, d\tau$. Applying Gronwall's lemma to the last inequality, we get

$$\widehat{E}(\widehat{u}(t)) \le c_3(R)e^{c_2(R)t^2}\left(\|w_0 - \widehat{w}_0\|^2_{H^1(\Omega)} + \|w_1 - \widehat{w}_1\|^2_{H^{-1}(\Omega)}\right) \tag{2.7}$$

By (2.1), (2.3), (2.4) and (2.7), it follows that

$$\left|\frac{d}{dt}E(\widehat{u}(t))\right| \le |\langle u_t(t), u(t)\rangle| + |\langle \nabla\widehat{u}(t), \nabla u(t)\rangle| \le$$

$$\le c_4(R)\left(\|u(t)\|_{L_2(\Omega)} + \|\nabla\widehat{u}(t)\|_{L_2(\Omega)}\right) \le$$

$$\le c_5(R)e^{\frac{c_2(R)t^2}{2}}\left(\|w_0 - \widehat{w}_0\|_{H^1(\Omega)} + \|w_1 - \widehat{w}_1\|_{H^{-1}(\Omega)}\right), \quad \forall t \ge 0.$$

Taking into account (2.7) and the last inequality in (2.6), we obtain

$$\|\nabla u(t)\|^2_{L_2(\Omega)} \le c_6(R)(1+t)e^{c_2(R)t^2}\left(\|w_0 - \widehat{w}_0\|_{H^1(\Omega)} + \right.$$

$$\left. + \|w_1 - \widehat{w}_1\|_{H^{-1}(\Omega)}\right), \quad \forall t \ge 0.$$

Now, from (2.5), we have

$$\|u_t(t)\|_{H^{-1}(\Omega)} \le \|\nabla u(t)\|_{L_2(\Omega)} + \|\nabla\widehat{u}(t)\|_{L_2(\Omega)} + \|\Sigma(w(t)) - \Sigma(\widehat{w}(t))\|_{H^{-1}(\Omega)} +$$

$$+\int\limits_0^t \|f(w(\tau,)) - f(\widehat{w}(\tau,))\|_{H^{-1}(\Omega)}\, d\tau + \|\Sigma(w_0) - \Sigma(\widehat{w}_0)\|_{H^{-1}(\Omega)} +$$

$$+ \|\nabla(w_0 - \widehat{w}_0)\|_{L_2(\Omega)} + \|w_1 - \widehat{w}_1\|_{H^{-1}(\Omega)},$$

which due to the above inequalities gives

$$\|u_t(t)\|^2_{H^{-1}(\Omega)} \le c_7(R)(1+t)e^{c_2(R)t^2}\left(\|w_0 - \widehat{w}_0\|_{H^1(\Omega)} + \right.$$

$$\left. + \|w_1 - \widehat{w}_1\|_{H^{-1}(\Omega)}\right), \quad \forall t \ge 0.$$

□

Thus by Theorem 2.1 and Theorem 2.2, it follows that by the formula $S(t)(w_0, w_1) = (w(t), w_t(t))$, problem (1.1)-(1.3) generates a weakly continuous (in the sense, if $\varphi_n \to \varphi$ strongly then $S(t)\varphi_n \to S(t)\varphi$ weakly) semigroup $\{S(t)\}_{t\ge 0}$ in $\mathcal{H}$, where $w(t,\cdot)$ is a weak solution of (1.1)-(1.3), determined by Theorem 2.1, with initial data (w_0, w_1). To show the strong continuity of $\{S(t)\}_{t\ge 0}$ we firstly prove the following lemma:

Lemma 2.1. *Let $\varphi \in C(R)$ and $|\varphi(x)| \le c(1 + |x|^r)$ for every $x \in R$ and some $r \ge 1$. If $v_n \to v$ strongly in $L_q(\Omega)$ for $q \ge r$, then $\varphi(v_n) \to \varphi(v)$ strongly in $L_{\frac{q}{r}}(\Omega)$.*

Proof. By the assumption of the lemma, there exists a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ a.e. in Ω. Then by Egorov's theorem, for any $\varepsilon > 0$ there exists a subset $A_\varepsilon \subset \Omega$ such that $mes(A_\varepsilon) < \varepsilon$ and $v_{n_k} \to v$ uniformly in $\Omega \backslash A_\varepsilon$. Hence for large enough k

$$|v_{n_k}(x)| \le 1 + |v(x)| \quad \text{in } \Omega\backslash A_\varepsilon$$

and consequently

$$|\varphi(v_{n_k}(x))| \le c_1(1 + |v(x)|^r) \quad \text{in } \Omega\backslash A_\varepsilon.$$

Applying Lebesgue's theorem we get

$$\lim_{k\to\infty} \|\varphi(v_{n_k}) - \varphi(v)\|_{L_{\frac{q}{r}}(\Omega\setminus A_\varepsilon)} = 0. \tag{2.8}$$

On the other hand since we have

$$\lim_{k\to\infty} \|v_{n_k}\|_{Lq(A_\varepsilon)} = \|v\|_{Lq(A_\varepsilon)},$$

the inequality

$$\limsup_{k\to\infty} \|\varphi(v_{n_k})\|^{\frac{q}{r}}_{L_{\frac{q}{r}}(A_\varepsilon)} < c_3(\varepsilon + \|v\|^q_{Lq(A_\varepsilon)})$$

is satisfied. The last inequality together with (2.8) implies that

$$\limsup_{k\to\infty} \|\varphi(v_{n_k}) - \varphi(v)\|^{\frac{q}{r}}_{L_{\frac{q}{r}}(\Omega)} \le c_4 \lim_{\varepsilon\to 0}(\varepsilon + \|v\|^q_{Lq(A_\varepsilon)}) = 0.$$

□

Theorem 2.3. *Under conditions (2.1)-(2.3) the semigroup $\{S(t)\}_{t\ge 0}$ is strongly continuous in $\mathcal{H}$.*

Proof. Let $(w_{0n}, w_{1n}) \to (w_0, w_1)$ strongly in $\mathcal{H}$. Denoting $(w_n(t), w_{tn}(t)) = S(t)(w_{0n}, w_{1n})$, $(w(t), w_t(t)) = S(t)(w_0, w_1)$ and $u_n(t) = w_n(t) - w(t)$, by (1.1) we have

$$u_{ntt} - \Delta u_{nt} + \sigma(w_n)w_{nt} - \sigma(w)w_t - \Delta u_n + f(w_n(\tau)) - f(w(t)) = 0.$$

Since, by Theorem 2.1, every term of the above equation belongs to $L_2(0, T; H^{-1}(\Omega))$, testing it by u_{nt}, we obtain

$$E(u_n(t)) \le E(u_n(0)) + c\,\|\sigma(w_n) - \sigma(w)\|^2_{C([0,T];L_{\frac{3}{2}}(\Omega))} + c\int_0^t E(u_n(s))ds,\ \forall t \in [0, T].$$

Applying Gronwall's lemma we have

$$E(u_n(T)) \le \left(E(u_n(0)) + c\,\|\sigma(w_n) - \sigma(w)\|^2_{C([0,T];L_{\frac{3}{2}}(\Omega))} \right) e^{cT},\ \ \forall T \ge 0. \tag{2.9}$$

By Theorem 2.2, it follows that

$$\lim_{n\to\infty} \|w_n - w\|_{C([0,T];L_6(\Omega))} = 0.$$

Now applying Lemma 2.1 it is easy to see that

$$\lim_{n\to\infty} \|\sigma(w_n) - \sigma(w)\|_{C([0,T];L_{\frac{3}{2}}(\Omega))} = 0,$$

which together with (2.9) yields that $S(T)(w_{0n}, w_{1n}) \to S(T)(w_0, w_1)$ strongly in $\mathcal{H}$, for every $T \ge 0$. □

Now let us recall the definition of a global attractor.

Definition ([17])**.** Let $\{V(t)\}_{t\ge 0}$ be a semigroup on a metric space $(X,\ d)$. A compact set $\mathcal{A} \subset X$ is called a global attractor for the semigroup $\{V(t)\}_{t\ge 0}$ iff

- $\mathcal{A}$ is invariant, i.e. $V(t)\mathcal{A} = \mathcal{A}$, $\forall t \ge 0$;
- $\lim\limits_{t\to\infty} \sup\limits_{v\in B} \inf\limits_{u\in\mathcal{A}} d(V(t)v, u) = 0$ for each bounded set $B \subset X$.

Our main result is as follows:

Theorem 2.4. *Under the conditions (2.1)-(2.3), the semigroup* $\{S(t)\}_{t\geq 0}$ *generated by the problem (1.1)-(1.3) possesses a global attractor* $\mathcal{A}$ *in* $\mathcal{H}$*, which is also a global attractor in* $\mathcal{H}_1 := (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

Remark 2.1. We note that if the condition (2.3) is replaced by

$$\sigma \in C(R),\ \ \sigma(s) \geq 0,\ |\sigma(s)| \leq c(1 + |s|^p),\ 0 \leq p < 4,\ \forall s \in R,$$

then using the methods of [5] , [19] and [21] one can prove Theorem 2.4. If we assume

$$\sigma \in C^1(R),\ \ \sigma(s) \geq 0,\ |\sigma'(s)| \leq c(1 + |s|),\ \forall s \in R,$$

instead of (2.3), then the method of [15] can be applied to (1.1)-(1.3). In this case, as in [20], one can show that a global attractor $\mathcal{A}$ attracts every bounded subset of $\mathcal{H}$ in the topology of $H_0^1(\Omega) \times H_0^1(\Omega)$.

Remark 2.2. We also note that problem (1.1)-(1.3), in 3-D case, without the strong damping $-\Delta w_t$ was considered in [11] and [16]. In this case, when $\sigma(\cdot)$ is not globally bounded, the existence of a global attractor in the strong topology of $\mathcal{H}$ and the regularity of the weak attractor remain open (see [11] and [16] for details).

3. **Existence of the global attractor in** $\mathcal{H}$**.** We start with the following asymptotic compactness lemma:

Lemma 3.1. *Let conditions (2.1)-(2.3) hold and* B *be a bounded subset of* $\mathcal{H}$*. Then every sequence of the form* $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$*,* $\{\varphi_n\}_{n=1}^{\infty} \subset B$*,* $t_n \to \infty$*, has a convergent subsequence in* $\mathcal{H}$*.*

Proof. By (2.4), we have

$$\begin{cases} \sup\limits_{t\geq 0}\sup\limits_{\varphi\in B} \|S(t)\varphi\|_{\mathcal{H}} < \infty, \\ \sup\limits_{\varphi\in B}\int\limits_0^\infty \|PS(t)\varphi\|^2_{H_0^1(\Omega)}\, dt < \infty, \end{cases} \tag{3.1}$$

where $P : \mathcal{H} \to L_2(\Omega)$ is a projection map, i.e. $P\varphi = \varphi_2$, for every $\varphi = (\varphi_1, \varphi_2) \in \mathcal{H}$. So for any $T_0 \geq 1$ there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $t_{n_k} \geq T_0$ and

$$\begin{cases} w_k \to w \ \text{ weakly star in } L_\infty(0, \infty; H_0^1(\Omega)), \\ w_{kt} \to w_t \quad\ \text{ weakly in } L_2(0, \infty; H_0^1(\Omega)), \end{cases} \tag{3.2}$$

for some $w \in L_\infty(0, \infty; H_0^1(\Omega)) \cap W^{1,\infty}(0, \infty; L_2(\Omega)) \cap W^{1,2}_{loc}(0, \infty; H_0^1(\Omega))$, where $(w_k(t), w_{kt}(t)) = S(t + t_{n_k} - T_0)\varphi_{n_k}$. Now multiplying the equality

$$(w_k - w_m)_{tt} - \Delta(w_{kt} - w_{mt}) + \sigma(w_k)w_{kt} - \sigma(w_m)w_{mt} - \Delta(w_k - w_m) +$$
$$+ f(w_k) - f(w_m) = 0$$

by $(w_{kt} - w_{mt} + \frac{\lambda_1}{2}(w_k - w_m))$ and integrating over $(s, T) \times \Omega$, we obtain

$$\frac{1}{2}E(w_k(T) - w_m(T)) + \lambda_1 \int\limits_s^T E(w_k(t) - w_m(t))dt +$$

$$+ \int\limits_s^T \langle \sigma(w_k(t))w_{kt}(t) - \sigma(w_m(t))w_{mt}(t), w_{kt}(t) - w_{mt}(t) \rangle\, dt +$$

$$+\frac{\lambda_1}{2}\left\langle \widehat{\Sigma}(w_k(T))+\widehat{\Sigma}(w_m(T)),1\right\rangle-\frac{\lambda_1}{2}\int\limits_s^T\langle\sigma(w_k(t))w_{kt}(t),w_m(t)\rangle\,dt$$

$$-\frac{\lambda_1}{2}\int\limits_s^T\langle\sigma(w_m(t))w_{mt}(t),w_k(t)\rangle\,dt+\langle F(w_k(T))+F(w_m(T)),1\rangle-$$

$$-\int\limits_s^T\langle f(w_k(t)),w_{mt}(t)\rangle\,dt-\int\limits_s^T\langle f(w_m(t)),w_{kt}(t)\rangle\,dt+$$

$$+\frac{\lambda_1}{2}\int\limits_s^T\langle f(w_k(t))-f(w_m(t)),w_k(t)-w_m(t)\rangle\,dt\le$$

$$\le(\frac{3}{2}+\lambda_1)E(w_k(s)-w_m(s))+\frac{\lambda_1}{2}\left\langle \widehat{\Sigma}(w_k(s))+\widehat{\Sigma}(w_m(s)),1\right\rangle+$$
$$+\langle F(w_k(s))+F(w_m(s)),1\rangle\,,\quad 0\le s\le T,$$

where $\widehat{\Sigma}(w)=\int\limits_0^w s\sigma(s)ds$. Integrating the last inequality with respect to s from 0 to T we find

$$\frac{T}{2}E(w_k(T)-w_m(T))+\lambda_1\int\limits_0^T sE(w_k(s)-w_m(s))ds+$$

$$+\int\limits_0^T s\,\langle\sigma(w_k(s))w_{kt}(s)-\sigma(w_m(s))w_{mt}(s),w_{kt}(s)-w_{mt}(s)\rangle\,ds+$$

$$+\frac{\lambda_1}{2}T\left\langle \widehat{\Sigma}(w_k(T))+\widehat{\Sigma}(w_m(T)),1\right\rangle-\frac{\lambda_1}{2}\int\limits_0^T s\,\langle\sigma(w_k(s))w_{kt}(s),w_m(s)\rangle\,ds$$

$$-\frac{\lambda_1}{2}\int\limits_0^T s\,\langle\sigma(w_m(s))w_{mt}(s),w_k(s)\rangle\,ds+T\,\langle F(w_k(T))+F(w_m(T)),1\rangle-$$

$$-\int\limits_0^T s\,\langle f(w_k(s)),w_{mt}(s)\rangle\,ds-\int\limits_0^T s\,\langle f(w_m(s)),w_{kt}(s)\rangle\,ds+$$

$$+\frac{\lambda_1}{2}\int\limits_0^T s\,\langle f(w_k(s))-f(w_m(s)),w_k(s)-w_m(s)\rangle\,dt\le$$

$$\le(\frac{3}{2}+\lambda_1)\int\limits_0^T E(w_k(s)-w_m(s))ds+\int\limits_0^T\left\langle F(w_k(s))+\frac{\lambda_1}{2}\widehat{\Sigma}(w_k(s)),1\right\rangle ds+$$

$$+\int\limits_0^T\left\langle F(w_m(s))+\frac{\lambda_1}{2}\widehat{\Sigma}(w_m(s)),1\right\rangle ds,\qquad\forall T\ge 0.\tag{3.3}$$

By $(3.1)_1$, it follows that

$$(\frac{3}{2}+\lambda_1)\int\limits_0^T E(w_k(s)-w_m(s))ds \le c_1+$$

$$+\frac{\lambda_1}{2}\int\limits_{\frac{3+2\lambda_1}{\lambda_1}}^T sE(w_k(s)-w_m(s))ds, \quad \forall T \ge \frac{3+2\lambda_1}{\lambda_1}. \tag{3.4}$$

Since for every $\varepsilon > 0$ the embedding $H^1(\Omega) \subset H^{1-\varepsilon}(\Omega)$ is compact (see for example [12, Theorem 16.1]), applying [18, Corollary 1] to (3.2), we have

$$w_k \to w \text{ strongly in } C([0,T]; H^{1-\varepsilon}(\Omega)).$$

Applying Lemma 2.1 it yields that

$$\begin{cases} \sigma(w_k) \to \sigma(w) \text{ strongly in } C([0,T]; L_{\frac{3}{2}-\varepsilon}(\Omega)), \\ \sigma^{\frac{1}{2}}(w_k) \to \sigma^{\frac{1}{2}}(w) \text{ strongly in } C([0,T]; L_{3-\varepsilon}(\Omega)), \end{cases}$$

for small enough $\varepsilon > 0$. The last approximation together with (2.3) and $(3.2)_2$ implies that

$$\begin{cases} \sigma(w_k)w_{kt} \to \sigma(w)w_t \text{ weakly in } L_2([0,T]; L_{\frac{6}{5}}(\Omega)), \\ \sigma^{\frac{1}{2}}(w_k)w_{kt} \to \sigma^{\frac{1}{2}}(w)w_t \text{ weakly in } L_2([0,T]; L_2(\Omega)), \end{cases}$$

by which we obtain

$$\liminf_{m\to\infty}\liminf_{k\to\infty}\int\limits_0^T s\left\langle \sigma(w_k(s))w_{kt}(s)-\sigma(w_m(s))w_{mt}(s), w_{kt}(s)-w_{mt}(s)\right\rangle ds =$$

$$=\liminf_{k\to\infty}\int\limits_0^T s\left\|\sigma^{\frac{1}{2}}(w_k(s))w_{kt}(s)\right\|^2_{L_2(\Omega)} ds + \liminf_{m\to\infty}\int\limits_0^T s\left\|\sigma^{\frac{1}{2}}(w_m(s))w_{mt}(s)\right\|^2_{L_2(\Omega)} ds -$$

$$-2\int\limits_0^T s\left\|\sigma^{\frac{1}{2}}(w(s))w_t(s)\right\|^2_{L_2(\Omega)} ds \ge 0, \tag{3.5}$$

$$\lim_{m\to\infty}\lim_{k\to\infty}\int\limits_0^T s\left\langle \sigma(w_k(s))w_{kt}(s), w_m(s)\right\rangle ds = \int\limits_0^T s\left\langle \sigma(w(s))w_t(s), w(s)\right\rangle ds =$$

$$= T\int\limits_0^T \left\langle \widehat{\Sigma}(w(s)), 1\right\rangle ds - \int\limits_0^T \left\langle \widehat{\Sigma}(w(s)), 1\right\rangle ds \tag{3.6}$$

and

$$\lim_{m\to\infty}\lim_{k\to\infty}\int\limits_0^T s\left\langle \sigma(w_m(s))w_{mt}(s), w_k(s)\right\rangle ds = \int\limits_0^T s\left\langle \sigma(w(s))w_t(s), w(s)\right\rangle ds =$$

$$= T\int\limits_0^T \left\langle \widehat{\Sigma}(w(s)), 1\right\rangle ds - \int\limits_0^T \left\langle \widehat{\Sigma}(w(s)), 1\right\rangle ds \tag{3.7}$$

Also applying Fatou's lemma and using (2.1), (2.2), (2.3), (3.2), we have

$$\begin{cases} \liminf\limits_{k\to\infty} \left\langle \widehat{\Sigma}(w_k(T)), 1 \right\rangle \geq \left\langle \widehat{\Sigma}(w(T)), 1 \right\rangle, \\ \liminf\limits_{k\to\infty} \left\langle F(w_k(T)), 1 \right\rangle \geq \left\langle F(w(T)), 1 \right\rangle, \\ \liminf\limits_{k\to\infty} \int_0^T s \left\langle f(w_k(s)), w_k(s) \right\rangle ds \geq \int_0^T s \left\langle f(w(s)), w(s) \right\rangle ds. \end{cases} \tag{3.8}$$

Taking into account (3.4)-(3.8) in (3.3), we obtain

$$\frac{T}{2} \liminf_{m\to\infty} \liminf_{k\to\infty} E(w_k(T) - w_m(T)) + \frac{\lambda_1}{2} \liminf_{m\to\infty} \liminf_{k\to\infty} \int_0^T s E(w_k(s) - w_m(s)) ds \leq c_1 +$$

$$+ 2 \liminf_{k\to\infty} \int_0^T \left\langle F(w_k(s)) + \frac{\lambda_1}{2} \widehat{\Sigma}(w_k(s)) - F(w(s)) - \frac{\lambda_1}{2} \widehat{\Sigma}(w(s)), 1 \right\rangle ds, \tag{3.9}$$

for $T \geq \frac{3+2\lambda_1}{\lambda_1}$. Now let us estimate the right hand side of (3.9). By (2.1), $(3.1)_1$ and (3.2), we find that

$$\int_0^T \left| \left\langle F(w_m(s)) - F(w(s)), 1 \right\rangle \right| ds \leq c_2 \int_0^T \| w_m(s) - w(s) \|_{H_0^1(\Omega)} ds \leq c_3 + c_4(\varepsilon) \log(T) +$$

$$+ \varepsilon \int_1^T s \| w_m(s) - w(s) \|^2_{H_0^1(\Omega)} ds \leq c_3 + c_4(\varepsilon) \log(T) +$$

$$+ \varepsilon \liminf_{k\to\infty} \int_0^T s \| w_m(s) - w_k(s) \|^2_{H_0^1(\Omega)} ds, \quad \forall T \geq 1, \ \ \forall \varepsilon > 0. \tag{3.10}$$

By the same way, we have

$$\int_0^T \left| \left\langle \widehat{\Sigma}(w_m(s)) - \widehat{\Sigma}(w(s)), 1 \right\rangle \right| ds \leq c_5 + c_6(\varepsilon) \log(T) +$$

$$+ \varepsilon \liminf_{k\to\infty} \int_0^T s \| w_m(s) - w_k(s) \|^2_{H_0^1(\Omega)} ds, \quad \forall T \geq 1, \ \ \forall \varepsilon > 0. \tag{3.11}$$

Now, choosing ε small enough, by (3.9)-(3.11), we obtain

$$\liminf_{m\to\infty} \liminf_{k\to\infty} E(w_k(T) - w_m(T)) \leq \frac{c_7(1 + \log(T))}{T}, \quad \forall T \geq \max\left\{1, \frac{3 + 2\lambda_1}{\lambda_1}\right\}.$$

Choosing $T = T_0$ in the last inequality we find

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \| S(t_n)\varphi_n - S(t_m)\varphi_m \|_{\mathcal{H}} \leq c_8 \sqrt{\frac{(1 + \log(T_0))}{T_0}},$$

and passing to the limit as $T_0 \to \infty$ we have

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \| S(t_n)\varphi_n - S(t_m)\varphi_m \|_{\mathcal{H}} = 0.$$

Similarly one can show that

$$\liminf_{k\to\infty} \liminf_{m\to\infty} \| S(t_{n_k})\varphi_{n_k} - S(t_{n_m})\varphi_{n_m} \|_{\mathcal{H}} = 0, \tag{3.12}$$

for every subsequence $\{n_k\}_{k=1}^{\infty}$. Now if the sequence $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$ has no convergent subsequence in $\mathcal{H}$, then there exist $\varepsilon_0 > 0$ and a subsequence $\{n_k\}_{k=1}^{\infty}$, such that

$$\|S(t_{n_k})\varphi_{n_k} - S(t_{n_m})\varphi_{n_m}\|_{\mathcal{H}} \geq \varepsilon_0, \quad k \neq m.$$

The last inequality contradicts (3.12). □

Now since, by (2.4), the problem (1.1)-(1.3) has a strict Lyapunov function $L(w(t)) := E(w(t)) + \langle F(w(t)), 1\rangle - \langle g, w(t)\rangle$, according to [4, Corollary 2.29] we have the following theorem:

Theorem 3.1. *Under conditions (2.1)-(2.3), the semigroup* $\{S(t)\}_{t\geq 0}$ *possesses a global attractor* $\mathcal{A}_{\mathcal{H}}$ *in* $\mathcal{H}$.

4. **Existence of the global attractor in $\mathcal{H}_1$.** To prove the existence of a global attractor in $\mathcal{H}_1$ we need the following lemmas:

Lemma 4.1. *Let conditions (2.1)-(2.3) hold and B be a bounded subset of $\mathcal{H}_1$. Then*

$$\sup_{t\geq 0}\sup_{\varphi\in B} \|S(t)\varphi\|_{\mathcal{H}_1} < \infty. \tag{4.1}$$

Proof. We use the formal estimates which can be justified by Galerkin's approximations. Multiplying both sides of (1.1) by $-\Delta w_t$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|\nabla w_t(t)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\Delta w(t)\|^2_{L_2(\Omega)} + \langle g, \Delta w(t)\rangle\right) + \\ +\frac{1}{2}\|\Delta w_t(t)\|^2_{L_2(\Omega)} \leq \|f(w(t))\|^2_{L_2(\Omega)} + \\ + \|\sigma(w(t))w_t(t)\|^2_{L_2(\Omega)}, \quad \forall t \geq 0.\end{aligned} \tag{4.2}$$

By (2.1) and (2.3), we have

$$\begin{aligned}\|f(w(t))\|^2_{L_2(\Omega)} + \|\sigma(w(t))w_t(t)\|^2_{L_2(\Omega)} \leq c_1\left(1 + \|w(t)\|^{10}_{L_{10}(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)}\right) + \\ + c_2\|w(t)\|^8_{L_{10}(\Omega)}\|w_t(t)\|^2_{L_{10}(\Omega)}, \ \forall t \geq 0.\end{aligned} \tag{4.3}$$

On the other hand, by the embedding and interpolation theorems, we find

$$\|\varphi\|_{L_{10}(\Omega)} \leq c_2\|\varphi\|_{H^{\frac{6}{5}}(\Omega)} \leq c_3\|\varphi\|^{\frac{1}{5}}_{H^2(\Omega)}\|\varphi\|^{\frac{4}{5}}_{H^1(\Omega)}, \quad \forall \varphi \in H^2(\Omega). \tag{4.4}$$

Taking into account (2.4), (4.3) and (4.4) in (4.2) and applying Gronwall's lemma, we obtain

$$\|(w(t), w_t(t))\|_{\mathcal{H}_1} \leq C(t, r)(1 + \|(w_0, w_1)\|_{\mathcal{H}_1}), \quad \forall t \geq 0, \tag{4.5}$$

where $C : R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable and $r = \sup_{\varphi\in B}\|\varphi\|_{\mathcal{H}}$. Since the embedding $\mathcal{H}_1 \subset \mathcal{H}$ is compact, by (4.5), it follows that the set $\underset{0\leq t\leq T}{\cup} S(t)B$ is a relatively compact subset of $\mathcal{H}$, for every $T > 0$. This together with Lemma 3.1 implies the relative compactness of $\underset{t\geq 0}{\cup} S(t)B$ in $\mathcal{H}$. Now using this fact let us estimate $\|w(t)\|_{L_{10}(\Omega)}$:

$$\|w(t)\|^{10}_{L_{10}(\Omega)} \leq m^{10} mes(\Omega) + \int\limits_{\{x: x\in\Omega,\ |w(t,x)|>m\}} |w(t,x)|^{10}\,dx \leq$$

$$\leq m^{10} mes(\Omega) + \left(\int\limits_{\{x: x\in\Omega,\ |w(t,x)|>m\}} |w(t,x)|^6 \, dx \right)^{\frac{1}{3}} \|w(t)\|^8_{L_{12}(\Omega)} \leq$$

$$\leq m^{10} mes(\Omega) + c_4 \left(\int\limits_{\{x: x\in\Omega,\ |w(t,x)|>m\}} |w(t,x)|^6 \, dx \right)^{\frac{1}{3}} \|w(t)\|^2_{H^2(\Omega)} \|w(t)\|^6_{H^1(\Omega)} .$$

So for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\|w(t)\|_{L_{10}(\Omega)} \leq \varepsilon \|\Delta w(t)\|^{\frac{1}{5}}_{L_2(\Omega)} + c_\varepsilon, \ \forall t \geq 0,$$

which together with (4.2)-(4.4) yields

$$\frac{d}{dt}\left(\frac{1}{2} \|\nabla w_t(t)\|^2_{L_2(\Omega)} + \frac{1}{2} \|\Delta w(t)\|^2_{L_2(\Omega)} + \langle g, \Delta w(t)\rangle \right) + \frac{1}{4} \|\Delta w_t(t)\|^2_{L_2(\Omega)} \leq$$

$$\leq c_5 \|\nabla w_t(t)\|^2_{L_2(\Omega)} \|\Delta w(t)\|^2_{L_2(\Omega)} + \varepsilon \|\Delta w(t)\|^2_{L_2(\Omega)} + \widetilde{c}_\varepsilon + c_5, \quad \forall t \geq 0.$$

Now multiplying both sides of (1.1) by $-\mu\Delta w$ ($\mu \in (0,1)$) and integrating over Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\mu \|\Delta w(t)\|^2_{L_2(\Omega)} + \mu \langle \nabla w_t(t), \nabla w(t)\rangle \right) + \mu \|\Delta w(t)\|^2_{L_2(\Omega)} \leq$$

$$\leq \mu \|g\|_{L_2(\Omega)} \|\Delta w(t)\|_{L_2(\Omega)} + \mu \|\nabla w_t(t)\|^2_{L_2(\Omega)} + \mu \|\sigma(w(t)) w_t(t)\|_{L_2(\Omega)} \|\Delta w(t)\|_{L_2(\Omega)}$$
$$+\mu \|f(w(t))\|_{L_2(\Omega)} \|\Delta w(t)\|_{L_2(\Omega)}, \quad \forall t \geq 0.$$

Taking into account the relative compactness of $\underset{t\geq 0}{\cup} S(t)B$, similar to the argument done above, we can say that for any $\varepsilon > 0$ there exists $\widehat{c}_\varepsilon > 0$ such that

$$\|f(w(t))\|^2_{L_2(\Omega)} + \|\sigma(w(t)) w_t(t)\|^2_{L_2(\Omega)} \leq \varepsilon \left(\|\Delta w(t)\|^2_{L_2(\Omega)} + \|\Delta w_t(t)\|^2_{L_2(\Omega)} \right) +$$

$$+\widehat{c}_\varepsilon \|\Delta w(t)\|^2_{L_2(\Omega)} \|\nabla w_t(t)\|^2_{L_2(\Omega)} + \widehat{c}_\varepsilon, \ \forall t \geq 0.$$

By the last three inequalities we have

$$\frac{d}{dt}\left(\frac{1}{2} \|\nabla w_t(t)\|^2_{L_2(\Omega)} + \frac{1}{2}(1+\mu) \|\Delta w(t)\|^2_{L_2(\Omega)} + \mu \langle \nabla w_t(t), \nabla w(t)\rangle + \langle g, \Delta w(t)\rangle \right)$$

$$+(\frac{1}{4} - \mu c_6 - \varepsilon) \|\Delta w_t(t)\|^2_{L_2(\Omega)} + (\frac{1}{4}\mu - 2\varepsilon) \|\Delta w(t)\|^2_{L_2(\Omega)} \leq$$

$$\leq (c_5 + \widehat{c}_\varepsilon) \|\Delta w(t)\|^2_{L_2(\Omega)} \|\nabla w_t(t)\|^2_{L_2(\Omega)} + c_6 + \widehat{c}_\varepsilon + \widetilde{c}_\varepsilon, \ \forall t \geq 0.$$

Choosing μ small enough and $\varepsilon \in (0, \frac{1}{8}\mu)$, we obtain

$$\frac{d}{dt}\Phi(t) + c_7 \Phi(t) \leq c_8 \|\nabla w_t(t)\|^2_{L_2(\Omega)} \Phi(t) + c_8(1 + \|\nabla w_t(t)\|^2_{L_2(\Omega)}), \ \forall t \geq 0,$$

where $\Phi(t) = \frac{1}{2} \|\nabla w_t(t)\|^2_{L_2(\Omega)} + \frac{1}{2}(1+\mu) \|\Delta w(t)\|^2_{L_2(\Omega)} + \mu \langle \nabla w_t(t), \nabla w(t)\rangle +$ $+ \langle g, \Delta w(t)\rangle$. Multiplying both sides of the last inequality by $e^{\int\limits_0^t (c_7 - c_8 \|\nabla w_t(\tau)\|^2_{L_2(\Omega)}) d\tau}$, integrating over $[0,T]$ and multiplying both sides of obtained inequality by $e^{-\int\limits_0^T \left[c_7 - c_8 \|\nabla w_t(t)\|^2_{L_2(\Omega)}\right] dt}$, we find

$$\Phi(T) \leq \Phi(0) e^{-\int\limits_0^T (c_7 - c_8 \|\nabla w_t(t)\|^2_{L_2(\Omega)}) dt} +$$

$$+c_8\int_0^T (1+\|\nabla w_t(t)\|^2_{L_2(\Omega)})e^{-\int_t^T (c_7-c_8\|\nabla w_t(\tau)\|^2_{L_2(\Omega)})d\tau}dt,\ \forall T\geq 0,$$

which together with (2.4) yields (4.1). □

Lemma 4.2. *Let conditions (2.1)-(2.3) hold and B be a bounded subset of $\mathcal{H}_1$. Then every sequence of the form $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$, $\{\varphi_n\}_{n=1}^{\infty}\subset B$, $t_n\to\infty$, has a convergent subsequence in $\mathcal{H}_1$.*

Proof. Let us decompose $\{S(t)\}_{t\geq 0}$ as $S(t)=U(t)+C(t)$, where $U(t)$ is a linear semigroup generated by the problem

$$\begin{cases} u_{tt}-\Delta u_t-\Delta u=0, & \text{in } (0,\infty)\times\Omega, \\ u=0, & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,\cdot)=w_0\ ,\quad u_t(0,\cdot)=w_1, & \text{in } \Omega, \end{cases} \tag{4.6}$$

$C(t)$ is a solution operator of

$$\begin{cases} v_{tt}-\Delta v_t-\Delta v=g(x)-f(w)-\sigma(w)w_t, & \text{in } (0,\infty)\times\Omega, \\ v=0, & \text{on } (0,\infty)\times\partial\Omega, \\ v_k(0,\cdot)=0\ ,\qquad v_t(0,\cdot)=0, & \text{in } \Omega \end{cases} \tag{4.7}$$

(i.e. $(u(t),u_t(t))=U(t)(w_0,w_1)$ and $(v(t),v_t(t))=C(t)(w_0,w_1)$) and $(w(t),w_t(t))=S(t)(w_0,w_1)$. Multiplying $(4.6)_1$ by $(u_t-\frac{1}{2}\Delta u-\mu\Delta u_t-\nu t\Delta u_t)$ and integrating over Ω, we obtain

$$\frac{d}{dt}\left(E(u(t))+\frac{1}{4}\|\Delta u(t)\|^2_{L_2(\Omega)}-\frac{1}{2}\langle u_t,\Delta u\rangle+\frac{1}{2}(\mu+\nu t)\|\nabla u_t(t)\|^2_{L_2(\Omega)}+\right.$$

$$\left.+\frac{1}{2}(\mu+\nu t)\|\Delta u(t)\|^2_{L_2(\Omega)}\right)+\frac{1}{2}(1-\nu)\|\nabla u_t(t)\|^2_{L_2(\Omega)}+\frac{1}{2}(1-\nu)\|\Delta u(t)\|^2_{L_2(\Omega)}+$$

$$+(\mu+\nu t)\|\Delta u_t(t)\|^2_{L_2(\Omega)}=0,\qquad \forall t\geq 0.$$

Choosing $(\mu,\nu)=(1,0)$ and $(\mu,\nu)=(0,1)$ in the last equality, we find

$$\|U(t)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_1)}\leq Me^{-\omega t},\qquad \forall t\geq 0, \tag{4.8}$$

and

$$\|U(t)\|_{\mathcal{L}((H^2(\Omega)\cap H^1_0(\Omega))\times L_2(\Omega),\mathcal{H}_1)}\leq\frac{M}{\sqrt{t}},\quad \forall t>0, \tag{4.9}$$

respectively, where $M>0$ and $\omega>0$. Also applying Duhamel's principle to (4.7), we have

$$C(t)(w_0,w_1)=\int_0^t U(t-s)(0,\Phi_{(w_0,w_1)}(s))ds, \tag{4.10}$$

where $\Phi_{(w_0,w_1)}(s)=g-f(w(s))-\sigma(w(s))w_t(s)$. By Lemma 4.1 and equation (1.1), it follows that the set of functions $\left\{\Phi_{(w_0,w_1)}(s):(w_0,w_1)\in B\ \right\}$ is precompact in $C([0,t];L_2(\Omega))$. So, from (4.9) and (4.10) we obtain that the operator $C(t):\mathcal{H}_1\to\mathcal{H}_1$, $t\geq 0$, is compact. Since

$$S(t_n)\varphi_n=U(T)S(t_n-T)\varphi_n+C(T)S(t_n-T)\varphi_n$$

for $t_n\geq T$, by (4.1), (4.8) and the compactness of $C(t)$, we obtain that the sequence $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$ has a finite ε-net in $\mathcal{H}$, for every $\varepsilon>0$. This completes the proof. □

Now by Lemma 4.2, similar to Theorem 3.1, we obtain the following theorem:

Theorem 4.1. *Under conditions (2.1)-(2.3), the semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor $\mathcal{A}_{\mathcal{H}_1}$ in $\mathcal{H}_1$.*

5. **Regularity of the $\mathcal{A}_{\mathcal{H}}$.** To prove the regularity of $\mathcal{A}_{\mathcal{H}}$ we will use the method used in [9] and [10]. Since $\mathcal{A}_{\mathcal{H}}$ is invariant, by [1, p. 159], for every $(w_0, w_1) \in \mathcal{A}_{\mathcal{H}}$ there exists an invariant trajectory $\gamma = \{W(t) = (w(t), w_t(t)),\ t \in R\} \subset \mathcal{A}_{\mathcal{H}}$ such that $W(0) = (w_0, w_1)$. By an invariant trajectory we mean a curve $\gamma = \{W(t),\ t \in R\}$ such that $S(t)W(\tau) = W(t+\tau)$ for $t \geq 0$ and $\tau \in R$ (see [1, p. 157]). Let us decompose $w(t)$ as $w(t) = u_k(t,s) + v_k(t,s)$, where

$$\begin{cases} v_{ktt} - \Delta v_{kt} + \sigma_k(w) v_{kt} - \Delta v_k + f_k(w) = g(x), & \text{in} \quad (s,\infty)\times\Omega, \\ v_k = 0, & \text{on} \ \ (s,\infty)\times\partial\Omega, \\ v_k(s,s,\cdot) = 0\ , \qquad v_{kt}(s,s,\cdot) = 0, & \text{in} \ \ \Omega \end{cases}, \tag{5.1}$$

$$\begin{cases} u_{ktt} - \Delta u_{kt} + \sigma(w) w_t - \sigma_k(w) v_{kt} - \Delta u_k = \\ = f_k(w) - f(w), & \text{in} \quad (s,\infty)\times\Omega, \\ u_k = 0, & \text{on} \ \ (s,\infty)\times\partial\Omega, \\ u_k(s,s,\cdot) = w(s,\cdot)\ , \qquad u_{kt}(s,s,\cdot) = w_t(s,\cdot), & \text{in} \ \ \Omega \end{cases}, \tag{5.2}$$

$$f_k(s) = \begin{cases} f(k), & s > k, \\ f(s), & |s| \leq k, \\ f(-k), & s < -k \end{cases}, \quad \sigma_k(s) = \begin{cases} \sigma(k), & s > k, \\ \sigma(s), & |s| \leq k, \\ \sigma(-k), & s < -k \end{cases} \quad \text{and } k \in \mathbb{N}.$$

Now let us prove the following lemmas:

Lemma 5.1. *Assume that conditions (2.1)-(2.3) are satisfied. Then $(v_k(t,s), v_{kt}(t,s)) \in \mathcal{H}_1$ and for any $k \in \mathbb{N}$ there exists $T_k < 0$ such that*

$$\|v_{kt}(t,s)\|_{H^1(\Omega)} + \|v_k(t,s)\|_{H^2(\Omega)} \leq r_0 k^{\frac{128}{65}},\ \forall s \leq t \leq T_k, \tag{5.3}$$

where the positive constant r_0 is independent of k and (w_0, w_1).

Proof. Multiplying both sides of $(5.1)_1$ by $v_{kt} + \mu v_k$ $(\mu \in (0,1))$ and integrating over Ω, we obtain

$$\frac{d}{dt}\Big(E(v_k(t,s)) + \frac{\mu}{2}\|\nabla v_k(t,s)\|^2_{L_2(\Omega)} + \mu\,\langle v_{kt}(t,s), v_k(t,s)\rangle\Big) +$$

$$+\frac{1}{2}\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)} - \mu\|v_{kt}(t,s)\|^2_{L_2(\Omega)} + (\mu - c_1\mu^2)\|\nabla v_k(t,s)\|^2_{L_2(\Omega)} \leq c_2,\ \forall t \geq s.$$

Choosing μ small enough in the last inequality, we find

$$\|v_{kt}(t,s)\|_{L_2(\Omega)} + \|v_k(t,s)\|_{H^1_0(\Omega)} \leq c_3, \quad \forall t \geq s. \tag{5.4}$$

Multiplying both sides of $(5.1)_1$ by v_{kt}, integrating over $(\tau_1, \tau_2)\times\Omega$ and taking into account (5.4), we have

$$\int_{\tau_1}^{\tau_2} \|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}\,dt \leq c_4 + \int_{\tau_1}^{\tau_2} |\langle f_k'(w(t))w_t(t), v_k(t,s)\rangle|\,dt \leq c_4 +$$

$$+\,c_5\int_{\tau_1}^{\tau_2} \|\nabla w_t(t)\|_{L_2(\Omega)}\,dt, \quad \forall \tau_2 \geq \tau_1 \geq s. \tag{5.5}$$

On the other hand, by (2.4), we have

$$\int_{-\infty}^{\infty} \|\nabla w_t(t)\|^2_{L_2(\Omega)}\,dt < \infty, \tag{5.6}$$

which together with (5.5) yields

$$\int_{\tau_1}^{\tau_2} \|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}\,dt \le c_6(1+(\tau_2-\tau_1)^{\frac{1}{2}}), \quad \forall \tau_2 \ge \tau_1 \ge s. \tag{5.7}$$

Multiplying both sides of $(5.1)_1$ by $-\Delta v_{kt} - \mu \Delta v_k$ $(\mu \in (0,1))$, integrating over Ω and taking into account (5.4), we have

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\Delta v_k(t,s)\|^2_{L_2(\Omega)} + \mu\langle \nabla v_{kt}(t,s), \nabla v_k(t,s)\rangle\right) +$$
$$+(\frac{1}{2} - c_7\mu)\|\Delta v_{kt}(t,s)\|^2_{L_2(\Omega)} + (\mu-\mu^2)\|\Delta v_k(t,s)\|^2_{L_2(\Omega)} \le c_7 +$$
$$+ c_7\|\sigma_k(w(t))v_{kt}(t,s)\|^2_{L_2(\Omega)} + c_7\|f_k(w(t))\|^2_{L_2(\Omega)}, \ \forall t \ge s. \tag{5.8}$$

Now let us estimate the last two terms on the right side of (5.8). By (4.4) and (5.4), we find

$$\|\sigma_k(w(t))v_{kt}(t,s)\|^2_{L_2(\Omega)} \le \|\sigma_k(w(t))\|^2_{L_{\frac{5}{2}}(\Omega)}\|v_{kt}(t,s)\|^2_{L_{10}(\Omega)} \le$$
$$\le c_8\|\sigma_k(w(t))\|^2_{L_{\frac{5}{2}}(\Omega)}\|v_{kt}(t,s)\|^{\frac{2}{5}}_{H^2(\Omega)}\|v_{kt}(t,s)\|^{\frac{8}{5}}_{H^1(\Omega)} \le$$
$$\le c_9\|\sigma_k(w(t))\|^4_{L_{\frac{5}{2}}(\Omega)} + c_9\|\Delta v_{kt}(t,s)\|^2_{L_2(\Omega)}\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)} +$$
$$+\frac{1}{3c_7}\|\Delta v_{kt}(t,s)\|^2_{L_2(\Omega)}, \ \forall t \ge s. \tag{5.9}$$

Also by the definitions of $\sigma_k(\cdot)$ and $f_k(\cdot)$, we have

$$\|\sigma_k(w(t))\|^{\frac{5}{2}}_{L_{\frac{5}{2}}(\Omega)} = \int_\Omega |\sigma_k(w(t,x))|^{\frac{5}{2}}\,dx \le$$
$$\le \int_{\{x:x\in\Omega,\ |w(t,x)|\le 2m\}} |\sigma_k(w(t,x))|^{\frac{5}{2}}\,dx + \int_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |\sigma_k(w(t,x))|^{\frac{5}{2}}\,dx \le$$
$$\le c_{10}m^4 \int_{\{x:x\in\Omega,\ |w(t,x)|\le 2m\}} (1+|w(t,x)|^6)dx +$$
$$+c_{10}k^4 \int_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx \le c_{11}m^4 +$$
$$+c_{10}k^4 \int_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx, \quad \forall k \in \mathbb{N},\ \forall m \ge 1 \text{ and } \forall t \in R. \tag{5.10}$$

$$\|f_k(w(t))\|^2_{L_2(\Omega)} = \int_\Omega |f_k(w(t,x))|^2\,dx \le$$
$$\le c_{12}m^4 \int_{\{x:x\in\Omega,\ |w(t,x)|\le 2m\}} (1+|w(t,x)|^6)dx +$$
$$+c_{12}k^4 \int_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx \le c_{13}m^4 +$$

$$c_{12}k^4 \int\limits_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx, \quad \forall k\in\mathbb{N},\ \forall m\geq 1 \text{ and } \forall t\in R. \tag{5.11}$$

Now denote $w^{(m)}(t,x)=\begin{cases} w(t,x)-m, & w(t,x)>m \\ 0, & |w(t,x)|\leq m \\ w(t,x)+m, & w(t,x)<-m \end{cases}$. Since,

$$|w(t,x)|<2\left|w^{(m)}(t,x)\right|, \qquad \forall (t,x)\in\{(t,x)\in R\times\Omega,\ |w(t,x)|>2m\},$$

we have

$$\int\limits_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w(t,x)|^6\,dx \leq 2^6 \int\limits_{\{x:x\in\Omega,\ |w(t,x)|>2m\}} |w^m(t,x)|^6\,dx \leq$$

$$\leq 2^6\int\limits_{\Omega} |w^m(t,x)|^6\,dx \leq c_{14}\left\|\nabla w^{(m)}(t)\right\|^2_{L_2(\Omega)}, \quad \forall t\in R. \tag{5.12}$$

So, by (5.8)-(5.12), it follows that

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}+\frac{1}{2}\|\Delta v_k(t,s)\|^2_{L_2(\Omega)}+\mu\langle\nabla v_{kt}(t,s),\nabla v_k(t,s)\rangle\right)+$$

$$+(\frac{1}{6}-c_7\mu)\|\Delta v_{kt}(t,s)\|^2_{L_2(\Omega)}+(\mu-\mu^2)\|\Delta v_k(t,s)\|^2_{L_2(\Omega)}\leq c_{15}m^{\frac{32}{5}}+$$

$$+c_{15}\|\Delta v_{kt}(t,s)\|^2_{L_2(\Omega)}\|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}+$$

$$+c_{15}k^{\frac{32}{5}}\left\|\nabla w^{(m)}(t)\right\|^2_{L_2(\Omega)}, \quad \forall k\in\mathbb{N},\ \forall m\geq 1 \text{ and } \forall t\geq s. \tag{5.13}$$

On the other hand, testing (1.1) by $w^{(m)}$, we obtain

$$\frac{d}{dt}\left\langle w_t(t),w^{(m)}(t)\right\rangle+\left\|\nabla w^{(m)}(t)\right\|^2_{L_2(\Omega)}-\left\|w_t^{(m)}(t)\right\|^2_{L_2(\Omega)}+\left\langle\nabla w_t(t),\nabla w^{(m)}(t)\right\rangle=$$

$$=\left\langle g,w^{(m)}(t)\right\rangle-\left\langle\sigma(w(t))w_t(t),w^{(m)}(t)\right\rangle-\left\langle f(w(t)),w^{(m)}(t)\right\rangle,\ \forall t\in R. \tag{5.14}$$

Let us estimate each term on the right hand side of (5.14). By the definition of $w^{(m)}$, we have

$$\left\langle g,w^{(m)}(t)\right\rangle\leq\left(\int\limits_{\{x:x\in\Omega,\ |w(t,x)|>m\}}|g(x)|^{\frac{6}{5}}\,dx\right)^{\frac{5}{6}}\left\|w^{(m)}(t)\right\|_{L_6(\Omega)}\leq$$

$$\leq\frac{c_{16}}{m^2}\left\|\nabla w^{(m)}(t)\right\|_{L_2(\Omega)}, \quad \forall t\in R.$$

By (2.3), it follows that

$$\left|\left\langle\sigma(w(t))w_t(t),w^{(m)}(t)\right\rangle\right|\leq c_{17}\|\nabla w_t(t)\|_{L_2(\Omega)}\left\|\nabla w^{(m)}(t)\right\|_{L_2(\Omega)}, \quad \forall t\in R.$$

Also by (2.3), we obtain

$$\left\langle f(w(t)),w^{(m)}(t)\right\rangle>-\lambda_1\left\langle w(t),w^{(m)}(t)\right\rangle\geq$$

$$\geq-\lambda_1\left(\int\limits_{\{x:x\in\Omega,\ |w(t,x)|>m\}}|w(t,x)|^{\frac{6}{5}}\,dx\right)^{\frac{5}{6}}\left\|w^{(m)}(t)\right\|_{L_6(\Omega)}\geq$$

$$\geq -\frac{c_{18}}{m^4}\left\|\nabla w^{(m)}(t)\right\|_{L_2(\Omega)}, \quad \forall t \in R,$$

for large enough m. Taking into account the last three inequalities in (5.14), we have

$$\frac{d}{dt}\left\langle w_t(t), w^{(m)}(t)\right\rangle + c_{19}\left\|\nabla w^{(m)}(t)\right\|^2_{L_2(\Omega)} \leq$$
$$\leq c_{20}\left\|\nabla w_t(t)\right\|^2_{L_2(\Omega)} + \frac{c_{20}}{m^4}, \quad \forall t \in R. \tag{5.15}$$

for large enough m. Now multiplying (5.15) by $\frac{c_{15}}{c_{19}}k^{\frac{32}{5}}$, adding to (5.13) and then choosing $m = k^{\frac{8}{13}}$, we get

$$\frac{d}{dt}\Lambda_{k,s}(t) + \widehat{c}_1\Lambda_{k,s}(t) \leq \widehat{c}_2\Lambda_{k,s}(t)\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)} +$$
$$+\widehat{c}_2 k^{\frac{256}{65}} + \widehat{c}_2 k^{\frac{32}{5}}\left\|\nabla w_t(t)\right\|^2_{L_2(\Omega)} + \widehat{c}_2 k^{\frac{64}{5}}\left|\left\langle w_t(t), w^{(k^{\frac{13}{8}})}(t)\right\rangle\right|^2, \quad \forall t \geq s,$$

for large enough k and small enough μ, where $\widehat{c}_1$ and $\widehat{c}_2$ are positive constants and $\Lambda_{k,s}(t) := \frac{1}{2}\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)} + \frac{1}{2}\left\|\Delta v_k(t,s)\right\|^2_{L_2(\Omega)} + \mu\left\langle \nabla v_{kt}(t,s), \nabla v_k(t,s)\right\rangle + \frac{c_{15}}{c_{16}}k^{\frac{32}{5}}\left\langle w_t(t), w^{(k^{\frac{8}{13}})}(t)\right\rangle$. Since

$$\left|\left\langle w_t(t), w^{(k^{\frac{8}{13}})}(t)\right\rangle\right| \leq \left\|w_t(t)\right\|_{L_6(\Omega)}\left(\int\limits_{\left\{x: x\in\Omega,\ |w(t,x)|>k^{\frac{8}{13}}\right\}} |w(t,x)|^{\frac{6}{5}}\,dx\right)^{\frac{5}{6}} \leq$$
$$\leq \frac{\widehat{c}_3}{k^{\frac{32}{13}}}\left\|\nabla w_t(t)\right\|_{L_2(\Omega)}, \quad \forall t \in R,$$

by the last differential inequality, we obtain

$$\frac{d}{dt}\Lambda_{k,s}(t) + \widehat{c}_1\Lambda_{k,s}(t) \leq \widehat{c}_2\Lambda_{k,s}(t)\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)} +$$
$$+\widehat{c}_4 k^{\frac{256}{65}} + \widehat{c}_4 k^8\left\|\nabla w_t(t)\right\|^2_{L_2(\Omega)}, \quad \forall t \geq s.$$

Multiplying both sides of the above inequality by $e^{\int_s^t\left[\widehat{c}_1 - \widehat{c}_2\left\|\nabla v_{kt}(\tau,s)\right\|^2_{L_2(\Omega)}\right]d\tau}$, integrating over $[s,T]$, multiplying both sides of the obtained inequality by $e^{-\int_s^T\left[\widehat{c}_1 - \widehat{c}_2\left\|\nabla v_{kt}(t,s)\right\|^2_{L_2(\Omega)}\right]dt}$ and taking into account (5.7), we find

$$\Lambda_{k,s}(T) \leq \widehat{c}_5 k^{\frac{32}{5}}\left|\left\langle w_t(s), w^{(m)}(s)\right\rangle\right| + \widehat{c}_5 k^{\frac{256}{65}} +$$
$$+ \widehat{c}_5 k^8\int\limits_s^T \left\|\nabla w_t(t)\right\|^2_{L_2(\Omega)}dt, \quad \forall T \geq s, \tag{5.16}$$

for large enough k and small enough μ. On the other hand, since $\mathcal{A}_{\mathcal{H}}$ is compact subset of $\mathcal{H}$ and problem (1.1)-(1.3) admits a strict Lyapunov function, we have

$$w_t(t) \to 0 \text{ strongly in } L_2(\Omega) \text{ as } t \to -\infty \tag{5.17}$$

Thus, by (5.6) and (5.17), for any $k \in \mathbb{N}$ there exists $T_k = T_k(\gamma) < 0$ such that

$$\widehat{c}_5 k^{\frac{32}{5}}\left|\left\langle w_t(T), w^{(m)}(T)\right\rangle\right| + \widehat{c}_5 k^8\int\limits_{-\infty}^T \left\|\nabla w_t(t)\right\|^2_{L_2(\Omega)}dt \leq 1, \quad \forall T \leq T_k,$$

which together with (5.16) yields (5.3). □

Lemma 5.2. *Assume that conditions (2.1)-(2.3) are satisfied. Then there exists* $k_0 \in \mathbb{N}$ *such that*

$$\lim_{s\to-\infty}\Big(\|u_{k_0t}(t,s)\|_{L_2(\Omega)} + \|u_{k_0}(t,s)\|_{H^1(\Omega)}\Big) = 0, \quad \forall t \le T_{k_0} \tag{5.18}$$

Proof. Multiplying both sides of $(5.2)_1$ by $u_{kt} + \mu u_k$ ($\mu \in (0,1)$) and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(E(u_k(t,s)) + \frac{\mu}{2}\|\nabla u_k(t,s)\|^2_{L_2(\Omega)} + \mu\langle u_{kt}(t,s), u_k(t,s)\rangle\Big) + \\
&+\|\nabla u_{kt}(t,s)\|^2_{L_2(\Omega)} + \mu\|\nabla u_k(t,s)\|^2_{L_2(\Omega)} - \mu\|u_{kt}(t,s)\|^2_{L_2(\Omega)} \le \\
&\le \|\sigma(w(t)) - \sigma_k(w(t))\|_{L_{\frac{3}{2}}(\Omega)}\|v_{kt}(t,s)\|_{L_6(\Omega)}\|u_{kt}(t,s)\|_{L_6(\Omega)} + \\
&+\mu\|\sigma(w(t))\|_{L_{\frac{3}{2}}(\Omega)}\|u_{kt}(t,s)\|_{L_6(\Omega)}\|u_k(t,s)\|_{L_6(\Omega)} + \\
&+\mu\|\sigma(w(t)) - \sigma_k(w(t))\|_{L_{\frac{3}{2}}(\Omega)}\|v_{kt}(t,s)\|_{L_6(\Omega)}\|u_k(t,s)\|_{L_6(\Omega)} + \\
&+\|f(w(t)) - f_k(w(t))\|_{L_{\frac{6}{5}}(\Omega)}\|u_{kt}(t,s)\|_{L_6(\Omega)} + \\
&+\mu\|f(w(t)) - f_k(w(t))\|_{L_{\frac{6}{5}}(\Omega)}\|u_k(t,s)\|_{L_6(\Omega)}, \quad \forall t \ge s.
\end{aligned} \tag{5.19}$$

Taking into account (2.4) in (5.19) and choosing μ small enough, we find

$$\begin{aligned}
&\frac{d}{dt}\Big(E(u_k(t,s)) + \frac{\mu}{2}\|\nabla u_k(t,s)\|^2_{L_2(\Omega)} + \mu\langle u_{kt}(t,s), u_k(t,s)\rangle\Big) + \\
&+c_1\Big(E(u_k(t,s)) + \frac{\mu}{2}\|\nabla u_k(t,s)\|^2_{L_2(\Omega)} + \mu\langle u_{kt}(t,s), u_k(t,s)\rangle\Big) \le \\
&\le c_2\|\sigma(w(t)) - \sigma_k(w(t))\|^2_{L_{\frac{3}{2}}(\Omega)}\|v_{kt}(t,s)\|^2_{L_6(\Omega)} + \\
&+c_2\|f(w(t)) - f_k(w(t))\|^2_{L_{\frac{6}{5}}(\Omega)}, \quad s \le t \le T_k,
\end{aligned} \tag{5.20}$$

where c_1 and c_2 are positive constants. Now let us estimate the terms on the right side of (5.20). Since $H^{\frac{3}{2}+\varepsilon}(\Omega) \subset C(\overline{\Omega})$ and

$$\|\varphi\|_{H^{\frac{3}{2}+\varepsilon}(\Omega)} \le c_3(\varepsilon)\|\varphi\|^{\frac{1}{2}-\varepsilon}_{H^1(\Omega)}\|\varphi\|^{\frac{1}{2}+\varepsilon}_{H^2(\Omega)}, \ \forall\varphi \in H^2(\Omega), \ \forall\varepsilon \in (0,\frac{1}{2}],$$

from (5.3) and (5.4) it follows that

$$\|v_k(t,s)\|_{C(\overline{\Omega})} \le \frac{1}{2}k, \quad s \le t \le T_k,$$

for large enough k. The last inequality together with (2.1)-(2.4) yields that

$$\begin{aligned}
\|\sigma(w(t)) - \sigma_k(w(t))\|^{\frac{3}{2}}_{L_{\frac{3}{2}}(\Omega)} &\le c_4\int\limits_{\{x:x\in\Omega,|w(t,x)|>k\}}|w(t,x)|^6\,dx \le \\
&\le c_5\Bigg(\int\limits_{\{x:x\in\Omega,|w(t,x)|>k\}}|w(t,x)|^6\,dx\Bigg)^{\frac{1}{4}} \le \\
&\le c_5\Bigg(\int\limits_{\{x:x\in\Omega,|u_k(t,s,x)|>|v_k(t,s,x)|\}}|w(t,x)|^6\,dx\Bigg)^{\frac{1}{4}} \le
\end{aligned}$$

$$\leq c_6 \left(\int\limits_{\{x:x\in\Omega,|u_k(t,s,x)|>|v_k(t,s,x)|\}} |u_k(t,s,x)|^6 \, dx \right)^{\frac{1}{4}} \leq$$

$$\leq c_6 \|\nabla u_k(t,s)\|^{\frac{3}{2}}_{L_2(\Omega)}, \quad s \leq t \leq T_k, \tag{5.21}$$

and

$$\|f(w(t)) - f_k(w(t))\|^{\frac{6}{5}}_{L_{\frac{6}{5}}(\Omega)} \leq c_7 \int\limits_{\{x:x\in\Omega,|w(t,x)|>k\}} |w(t,x)|^6 \, dx \leq$$

$$\leq c_8 \left(\int\limits_{\{x:x\in\Omega,|w(t,x)|>k\}} |w(t,x)|^6 \, dx \right)^{\frac{4}{5}} \times$$

$$\times \left(\int\limits_{\{x:x\in\Omega,|u_k(t,s,x)|>|v_k(t,s,x)|\}} |w(t,x)|^6 \, dx \right)^{\frac{1}{5}} \leq$$

$$\leq c_9 \left(\int\limits_{\{x:x\in\Omega,|w(t,x)|>k\}} |w(t,x)|^6 \, dx \right)^{\frac{4}{5}} \times$$

$$\times \left(\int\limits_{\{x:x\in\Omega,|u_k(t,s,x)|>|v_k(t,s,x)|\}} |u_k(t,s,x)|^6 \, dx \right)^{\frac{1}{5}} \leq$$

$$\leq c_{10} \left(\int\limits_{\{x:x\in\Omega,|w(t,x)|>k\}} |w(t,x)|^6 \, dx \right)^{\frac{4}{5}} \|\nabla u_k(t,s)\|^{\frac{6}{5}}_{L_2(\Omega)}, \quad s \leq t \leq T_k, \tag{5.22}$$

for large enough k. On the other hand, since $\mathcal{A}_{\mathcal{H}}$ is compact subset of $\mathcal{H}$ and $(w(t), w_t(t)) \in \mathcal{A}_{\mathcal{H}}$, we have

$$\sup_{t\in R} \int\limits_{\{x:x\in\Omega,|w(t,x)|>k\}} |w(t,x)|^6 \, dx \to 0 \text{ as } k \to \infty \tag{5.23}$$

Thus choosing μ small enough, k large enough and taking into account (5.21)-(5.23) in (5.20), we obtain

$$\frac{d}{dt}\widetilde{\Lambda}_{k,s}(t) + \widehat{c}_1\widetilde{\Lambda}_{k,s}(t) \leq \widehat{c}_2 \|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)} \widetilde{\Lambda}_{k,s}(t), \ s \leq t \leq T_k,$$

where $\widehat{c}_1$ and $\widehat{c}_2$ are positive constants and $\widetilde{\Lambda}_{k,s}(t) = E(u_k(t,s)) + \frac{\mu}{2} \|\nabla u_k(t,s)\|^2_{L_2(\Omega)}$ $+ \mu \langle u_{kt}(t,s), u_k(t,s)\rangle$. Now multiplying both sides of the last inequality by $e^{\int_s^t \left[\widehat{c}_1 - \widehat{c}_2 \|\nabla v_{kt}(\tau,s)\|^2_{L_2(\Omega)}\right] d\tau}$, integrating over $[s, T_k]$ and multiplying both sides of the obtained inequality by $e^{-\int_s^{T_k} \left[\widehat{c}_1 - \widehat{c}_2 \|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}\right] dt}$, we find

$$\widetilde{\Lambda}_{k,s}(T) \leq \widetilde{\Lambda}_{k,s}(s) e^{-\int_s^{T_k} \left[\widehat{c}_1 - \widehat{c}_2 \|\nabla v_{kt}(t,s)\|^2_{L_2(\Omega)}\right] dt}, \ s \leq t \leq T_k,$$

which together with (5.7) yields (5.18). $\square$

By Lemma 5.1 and Lemma 5.2, we have $(w(T_{k_0}), w_t(T_{k_0})) \in \mathcal{H}_1$ and

$$\|w_t(T_{k_0})\|_{H^1(\Omega)} + \|w(T_{k_0})\|_{H^2(\Omega)} \leq \widehat{r}_0,$$

where $\widehat{r}_0$ is independent of (w_0, w_1). Now since $w(t,x)$ satisfies (1.1)-(1.3) on $(T_{k_0}, \infty) \times \Omega$, with initial data $(w(T_{k_0}),\ w_t(T_{k_0}))$, applying Lemma 4.1 and taking into account the last inequality, we find $(w_0, w_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and

$$\|(w_0, w_1)\|_{H^2(\Omega)\times H^1(\Omega)} \leq R_0,$$

where the positive constant R_0 is independent of (w_0, w_1). So $\mathcal{A}_{\mathcal{H}}$ is a bounded subset of $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and that is why it coincides with $\mathcal{A}_{\mathcal{H}_1}$.

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