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Global attractor for the one dimensional wave equation with displacement dependent damping

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ABSTRACT

We study the long-time behavior of solutions of the one dimensional wave equation with nonlinear damping coefficient. We prove that if the damping coefficient function is strictly positive near the origin then this equation possesses a global attractor.

1. Introduction

In this paper, we consider the following Cauchy problem:

$$u_{tt} + \sigma(u)u_t - u_{xx} + \lambda u + f(u) = g(x), \quad (t,x) \in (0,\infty) \times R, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad x \in R, \tag{1.2}$$

where λ is a positive constant, $g \in L_1(R) + L_2(R)$ and nonlinear functions $f(\cdot)$ and $\sigma(\cdot)$ satisfy the following conditions:

$$f \in C^1(R), \qquad f(u)u \geq 0, \quad \forall u \in R, \tag{1.3}$$

$$\sigma \in C(R), \qquad \sigma(0) > 0, \qquad \sigma(u) \geq 0, \quad \forall u \in R. \tag{1.4}$$

As was mentioned in [1], Eq. (1.1) describes a model for a vibrating string in a viscous medium. In particular, u represents the displacement from equilibrium, u_t is the velocity, and $\sigma(u)u_t$ is a resistance force. Applying Galerkin's method and using techniques of [2, Proposition 2.2], it is easy to prove the following existence and uniqueness theorem:

Theorem 1.1. *Assume that conditions (1.3) and (1.4) hold. Then for any $T > 0$ and $(u_0, u_1) \in \mathcal{H} := H^1(R) \times L_2(R)$ problem (1.1) and (1.2) has a unique weak solution $u \in C([0,T]; H^1(R)) \cap C^1([0,T]; L_2(R)) \cap C^2([0,T]; H^{-1}(R))$ on $[0,T] \times R$ such that*

$$\|(u(t), u_t(t))\|_{\mathcal{H}} \leq c(\|(u_0, u_1)\|_{\mathcal{H}}), \quad \forall t \geq 0,$$

where $c : R_+ \to R_+$ is a nondecreasing function. Moreover if $v \in C([0,T]; H^1(R)) \cap C^1([0,T]; L_2(R)) \cap C^2([0,T]; H^{-1}(R))$ is also a weak solution to (1.1) and (1.2) with initial data $(v_0, v_1) \in \mathcal{H}$, then

$$\|u(t) - v(t)\|_{L_2(R)} + \|u_t(t) - v_t(t)\|_{H^{-1}(R)} \leq \widetilde{c}(T, \widetilde{R})(\|u_0 - v_0\|_{L_2(R)} + \|u_1 - v_1\|_{H^{-1}(R)}), \quad \forall t \in [0,T],$$

where $\widetilde{c} : R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable and $\widetilde{R} = \max\{\|(u_0, u_1)\|_{\mathcal{H}}, \|(v_0, v_1)\|_{\mathcal{H}}\}$.

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Thus, by Theorem 1.1, under conditions (1.3) and (1.4) the solution operator $S(t)(u_0, u_1) = (u(t), u_t(t))$ of problem (1.1) and (1.2) generates a weakly continuous (in the sense that if $\varphi_n \to \varphi$ strongly then $S(t)\varphi_n \to S(t)\varphi$ weakly) semigroup in $\mathcal{H}$.

The attractors for Eq. (1.1) in the finite interval were studied in [1], assuming the positivity of $\sigma(\cdot)$. For the two dimensional case, the attractors for the wave equation with displacement dependent damping were investigated in [3] under conditions

$$\sigma \in C^1(R), \quad 0 < \sigma_0 \le \sigma(u) \le c(1 + |u|^q), \quad \forall u \in R,\ 0 \le q < \infty,$$

and

$$|\sigma'(u)| \le c[\sigma(u)]^{1-\varepsilon}, \quad \forall u \in R,\ 0 < \varepsilon < 1, \tag{1.5}$$

on the damping coefficient. Recently, in [4], condition (1.5) has been improved as

$$|\sigma'(u)| \le c\sigma(u), \quad \forall u \in R.$$

For the three dimensional bounded domain case, the existence of a global attractor for the wave equation with displacement dependent damping was proved in [2] when $\sigma(\cdot)$ is a strictly positive and globally bounded function. When $\sigma(\cdot)$ is not globally bounded, but equal to a positive constant in a large enough interval, the existence of a global attractor has been established in [5].

In the articles mentioned above, the existence of global attractors was proved under the positivity or strict positivity condition on the damping coefficient function $\sigma(\cdot)$. In this paper, we study a global attractor for problem (1.1) and (1.2) under weaker conditions on $\sigma(\cdot)$ and prove the following theorem:

Theorem 1.2. *Under conditions (1.3) and (1.4) a semigroup $\{S(t)\}_{t\ge 0}$ generated by problem (1.1) and (1.2) possesses a global attractor in $\mathcal{H}$.*

2. Proof of Theorem 1.2

To prove this theorem we need the following lemma:

Lemma 2.1. *Let conditions (1.3) and (1.4) hold and let B be a bounded subset of $\mathcal{H}$. Then for any $\varepsilon > 0$ there exist $T_\varepsilon = T_\varepsilon(B) > 0$ and $r_\varepsilon = r_\varepsilon(B) > 0$ such that*

$$\|S(t)\varphi\|_{H^1(R\setminus(-r_\varepsilon, r_\varepsilon))\times L_2(R\setminus(-r_\varepsilon, r_\varepsilon))} < \varepsilon, \quad \forall t \ge T_\varepsilon,\ \forall \varphi \in B. \tag{2.1}$$

Proof. Let $(u_0, u_1) \in B$ and $S(t)(u_0, u_1) = (u(t), u_t(t))$. Multiplying (1.1) by u_t and integrating over $(0, t) \times R$, we obtain

$$\|u_t(t)\|^2_{L_2(R)} + \|u(t)\|^2_{H^1(R)} + \int_0^t \int_R \sigma(u(\tau, x))u_t^2(\tau, x)\,dx\,d\tau \le c_1, \quad \forall t \ge 0. \tag{2.2}$$

Let $\eta \in C^1(R)$, $0 \le \eta(x) \le 1$, $\eta(x) = \begin{cases} 0, & |x| \le 1 \\ 1, & |x| \ge 2 \end{cases}$, $\eta_r(x) = \eta\left(\frac{x}{r}\right)$ and $\Sigma(u) = \int_0^u \sigma(s)\,ds$. Multiplying (1.1) by $\eta_r^2\Sigma(u)$, integrating over $(0, t) \times R$ and taking into account (2.2), we have

$$\begin{aligned}&\int_0^t \int_R \eta_r^2(x)\sigma(u(\tau, x))u_x^2(\tau, x)\,dx\,d\tau + \lambda \int_0^t \int_R \eta_r^2(x)\Sigma(u(\tau, x))u(\tau, x)\,dx\,d\tau \\ &\quad \le c_2\left(1 + \sqrt{t} + \frac{t}{r} + t\,\|g\|_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))}\right), \quad \forall t \ge 0, \forall r > 0.\end{aligned} \tag{2.3}$$

By (1.4), there exists $l > 0$ such that

$$\frac{\sigma(0)}{2} \le \sigma(s) \le 2\sigma(0), \quad \forall s \in [-l, l]. \tag{2.4}$$

Using the embedding $H^{\frac{1}{2}+\varepsilon}(R) \subset L_\infty(R)$ and taking into account (2.2) and (2.4), we find

$$\begin{aligned}\int_0^t \int_R \eta_r^2(x)u^2(\tau, x)\,dx\,d\tau &\le \frac{2}{\sigma(0)} \int_0^t \int_{\{x:|u(\tau,x)|\le l\}} \eta_r^2(x)\Sigma(u(\tau, x))u(\tau, x)\,dx\,d\tau \\ &\quad + c_3 \int_0^t \int_{\{x:|u(\tau,x)|>l\}} \eta_r^2(x)\,|u(\tau, x)|\,dx\,d\tau \\ &\le \frac{2}{\sigma(0)} \int_0^t \int_{\{x:|u(\tau,x)|\le l\}} \eta_r^2(x)\Sigma(u(\tau, x))u(\tau, x)\,dx\,d\tau \\ &\quad + \frac{2c_3}{\sigma(0)l} \int_0^t \int_{\{x:|u(\tau,x)|>l\}} \eta_r^2(x)\Sigma(u(\tau, x))u(\tau, x)\,dx\,d\tau\end{aligned}$$

and consequently

$$\int_0^t \|\eta_r u(\tau)\|^5_{L_\infty(R)}\,d\tau \le c_4 \int_0^t \|\eta_r u(\tau)\|^2_{L_2(R)}\,d\tau \le c_5 \int_0^t \int_R \eta_r^2(x)\Sigma(u(\tau,x))u(\tau,x)\,dx\,d\tau, \tag{2.5}$$

for $r \ge 1$. So by (2.2), (2.3) and (2.5), we get

$$\int_0^t \Big[\|\eta_{2r}\sigma^{\frac{1}{2}}(u(\tau))u_t(\tau)\|^2_{L_2(R)} + \|\eta_{2r}\sigma^{\frac{1}{2}}(u(\tau))u_x(\tau)\|^2_{L_2(R)} + \lambda\|\eta_{2r}\sigma^{\frac{1}{2}}(u(\tau))u(\tau)\|^2_{L_2(R)} + \|\eta_r u(\tau)\|^5_{L_\infty(R)}\Big]d\tau \le c_6\Bigg(1+\sqrt{t}+\frac{t}{r} + t\,\|g\|_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))}\Bigg), \quad \forall t \ge 0,\ \forall r \ge 1. \tag{2.6}$$

Now define

$$\Phi_r(u(t)) := \frac{1}{2}\|\eta_r u_t(t)\|^2_{L_2(R)} + \frac{1}{2}\|\eta_r u_x(t)\|^2_{L_2(R)} + \mu\,\langle \eta_r u_t(t), \eta_r u(t)\rangle + \frac{\lambda}{2}\|\eta_r u(t)\|^2_{L_2(R)} + \langle \eta_r F(u(t)),\ \eta_r\rangle + \langle \eta_r g,\ \eta_r u(t)\rangle,$$

where $\mu = \min\Big\{\sqrt{\frac{\lambda}{2}}, \frac{\sigma(0)}{5}, \frac{\lambda}{4\sigma(0)}\Big\}$, $\langle u,\ v\rangle = \int_R u(x)v(x)\,dx$ and $F(u) = \int_0^u f(s)\,ds$. By (2.4) and (2.6), it follows that for any $\delta > 0$ there exist $\widetilde{T}_\delta = \widetilde{T}_\delta(B) > 0$, $r_{1,\delta} = r_{1,\delta}(B) > 1$ and $t^*_\delta \in [0, \widetilde{T}_\delta]$ such that

$$\Phi_r(u(t^*_\delta)) < \delta, \quad \forall r \ge r_{1,\delta}. \tag{2.7}$$

Again by (2.2), we have

$$\|\eta_r u(t)\|_{L_2(R)} \le \big\|\eta_r u(t^*_\delta)\big\|_{L_2(R)} + \int_{t^*_\delta}^t \|\eta_r u_t(s)\|_{L_2(R)}\ ds \le \big\|\eta_r u(t^*_\delta)\big\|_{L_2(R)} + c_7(t-t^*_\delta)$$

and consequently

$$\big\|\eta_r u(t)\big\|^3_{L_\infty(R)} \le \bar{c}_8\,\big\|\eta_r u(t)\big\|_{L_2(R)} \le c_9\Big(\big(\Phi_r(u(t^*_\delta)) + \bar{c}_8\,\|g\|_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))}\big)^{\frac{1}{2}} + t - t^*_\delta\Big) < c_9\Big(\delta^{\frac{1}{2}} + \|g\|^{\frac{1}{2}}_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))} + t - t^*_\delta\Big), \quad \forall t \ge t^*_\delta,\ \forall r \ge r_{1,\delta}.$$

Denoting $T^*_\delta = t^*_\delta + \frac{l^3}{3c_9}$ and choosing $\delta \in (0, \frac{l^6}{9c_9^2})$, by the last inequality, we can say that there exists $r_{2,\delta} \ge 2r_{1,\delta}$ such that

$$\|u(t)\|_{L_\infty(R\setminus(-r_{2,\delta},r_{2,\delta}))} < l, \quad \forall t \in [t^*_\delta, T^*_\delta]. \tag{2.8}$$

Now, multiplying (1.1) by $\eta_r^2(u_t+\mu u)$, integrating over R and taking into account (2.4) and (2.8), we obtain

$$\frac{d}{dt}\Phi_r(u(t)) + c_{10}\Phi_r(u(t)) \le c_{11}\left(\frac{1}{r} + \|g\|_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))}\right), \quad \forall t \in [t^*_\delta, T^*_\delta],$$

and consequently

$$\Phi_r(u(t)) \le \Phi_r(u(t^*_\delta))\,e^{-c_{10}(t-t^*_\delta)} + c_{11}\left(\frac{1}{r} + \|g\|_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))}\right)\frac{1-\ e^{-c_{10}(t-t^*_\delta)}}{c_{10}} \quad \forall r \ge r_{2,\delta}. \tag{2.9}$$

By (2.7) and (2.9), there exists $r_{3,\delta} \ge r_{2,\delta}$ such that

$$\Phi_r(u(t)) < \delta \quad \forall r \ge r_{3,\delta}, \forall t \in [t^*_\delta, T^*_\delta].$$

Hence denoting by n_δ the smallest integer number which is not less than $\frac{3c_9\widetilde{T}_\delta}{l^3}$ and applying above procedure at most n_δ times, we find

$$\Phi_r(u(\widetilde{T}_\delta)) < \delta, \quad \forall r \ge r_{4,\delta},$$

for some $r_{4,\delta} \ge 2^{n_\delta} r_{1,\delta}$. From the last inequality it follows that for any $\varepsilon > 0$ there exist $\widehat{T}_\varepsilon = \widehat{T}_\varepsilon(B) > 0$ and $\widehat{r}_\varepsilon = \widehat{r}_\varepsilon(B) > 0$ such that

$$\big\|S(\widehat{T}_\varepsilon)\varphi\big\|_{H^1(R\setminus(-\widehat{r}_\varepsilon,\widehat{r}_\varepsilon))\times L_2(R\setminus(-\widehat{r}_\varepsilon,\widehat{r}_\varepsilon))} < \varepsilon, \quad \forall \varphi \in B.$$

Since, by (2.2), $B_0 = \cup_{t\geq 0} S(t)B$ is a bounded subset of $\mathcal{H}$, for any $\varepsilon > 0$ there exist $T_\varepsilon = T_\varepsilon(B) > 0$ and $r_\varepsilon = r_\varepsilon(B) > 0$ such that

$$\|S(T_\varepsilon)\varphi\|_{H^1(R\setminus(-r_\varepsilon, r_\varepsilon))\times L_2(R\setminus(-r_\varepsilon, r_\varepsilon))} < \varepsilon, \quad \forall \varphi \in B_0.$$

Taking into account the positive invariance of B_0, from the last inequality we obtain (2.1). □

By (2.1) and (2.4), for any bounded subset B of $\mathcal{H}$ there exist $\widehat{T}_B > 0$ and $\widehat{r}_B > 0$ such that

$$\sigma(u(t,x)) \geq \frac{\sigma(0)}{2}, \quad \forall t \geq \widehat{T}_B, \forall |x| \geq \widehat{r}_B, \tag{2.10}$$

where $u(t,x)$ is the weak solution of problem (1.1) and (1.2) with initial data from B. Hence using techniques of [6, Lemma 3.3] one can prove the asymptotic compactness of the semigroup $\{S(t)\}_{t\geq 0}$, which is included in the following lemma:

Lemma 2.2. *Assume that conditions (1.3) and (1.4) hold and B is a bounded subset of $\mathcal{H}$. Then every sequence of the form $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$, $\{\varphi_n\}_{n=1}^{\infty} \subset B$, $t_n \to \infty$, has a convergent subsequence in $\mathcal{H}$.*

By (2.10) and the unique continuation result of [7], it is easy to see that problem (1.1) and (1.2) has a strict Lyapunov function (see [8] for definition). Thus according to [8, Corollary 2.29] the semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor.

Remark 1. We note that, for the problem considered in [1], from compact embedding $H_0^1(0,\pi) \subset C[0,\pi]$, it immediately follows that $\sigma(u(t,x)) \geq \frac{\sigma(0)}{2}$, $\forall t \geq 0$, $\forall x \in [0,\varepsilon] \cup [\pi - \varepsilon, \pi]$, for some $\varepsilon \in \left(0, \frac{\pi}{2}\right)$. So a global attractor still exists if one replaces the positivity condition on $\sigma(\cdot)$ by the (1.4).

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