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A generalization of projective covers

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Abstract

Let M be a left module over a ring R and I an ideal of R. We call (P, f) a *projective I-cover* of M if f is an epimorphism from P to M, P is projective, $\operatorname{Ker} f \subseteq IP$, and whenever $P = \operatorname{Ker} f + X$, then there exists a summand Y of P in $\operatorname{Ker} f$ such that $P = Y + X$. This definition generalizes projective covers and projective δ-covers. Similar to semiregular and semiperfect rings, we characterize I-semiregular and I-semiperfect rings which are defined by Yousif and Zhou using projective I-covers. In particular, we consider certain ideals such as $Z({}_RR)$, $\operatorname{Soc}({}_RR)$, $\delta({}_RR)$ and $Z_2({}_RR)$.

Keywords: Projective cover; δ-cover; *Soc*-cover; Semiregular; Semiperfect

1. Introduction

As is well known, projective covers play an important role in characterizing semiperfect and semiregular rings. Recently some authors have worked with various extensions of these rings (see for example [1,9,11,12,14,15]). Zhou calls a ring R *δ-semiperfect* (*δ-semiregular*) if for a (finitely generated) left ideal I of R, $I = Rf \oplus S$ where $f^2 = f \in R$ and $S \subseteq \delta({}_RR)$, and also by defining projective δ-cover, he characterizes these rings. After that, Yousif and Zhou extend semiperfect and semiregular rings to I-semiperfect and I-semiregular rings by taking an ideal I instead of

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$\delta({}_R R)$. Next, Nicholson and Zhou study I-semiperfect and I-semiregular rings by introducing strongly lifting ideals. A module theoretic version of these extensions is studied in [1,12]; also by defining projective *Soc*-covers, *Soc*-semiperfect modules are characterized in [12].

The purpose of this paper is to characterize I-semiregular and I-semiperfect rings by defining general projective covers.

In Section 2, we prove that any direct sum of I-semiregular modules is I-semiregular for an ideal I.

In Section 3, we introduce the notion of *DM* submodules and use this to define projective I-covers which are a generalization of some well-known projective covers. A submodule N of M is called *DM in* M if there is a summand S of M such that $S \leqslant N$ and $M = S + X$, whenever $N + X = M$ for a submodule X of M. A pair (P, f) is called a *projective* I*-cover* of M if P is projective and f is an epimorphism from P to M such that $\operatorname{Ker} f \subseteq IP$ and $\operatorname{Ker} f$ is *DM* in P. After investigating some properties of *DM* submodules, we prove that a module M has a projective $J(R)$-cover if and only if M has a projective cover (Proposition 3.6). M is called *DM for* I if any submodule of IM is *DM* in M. A ring R is called *a left DM ring for* I if for any finitely generated free left R-module is *DM* for I. We generalize Bass's Lemma (see [10, Lemma B.15]): for an ideal I and a projective module M, if M/N has a projective I-cover, then $M = Y \oplus X$ for some submodules Y and X with $Y \subseteq N$ and $X \cap N \subseteq IM$. The converse is true when M is *DM* for I (Lemma 3.10). Then we give some equivalent statements for I-semiregular rings when R is a left *DM* ring for I.

The last section is concerned with the characterization of I-semiperfect rings with a projective I-cover or a projective I-semicover. But first, we get a generalization of one of Azumaya's Theorems [3, Theorem 4] (Theorem 4.4). After that we prove that if R is a left *DM*-ring for I, then R is I-semiperfect if and only if every finitely generated left R-module has a projective I-cover. In addition, they are equivalent to the fact that every simple factor module of ${}_R R$ has a projective I-cover when R is *DM* for I with C_3 (Theorem 4.8). In the last two sections, we have satisfactory characterizations of I-semiregular and I-semiperfect rings when I is $\delta({}_R R)$, $\operatorname{Soc}({}_R R)$, $Z({}_R R)$ or $Z_2({}_R R)$ because some of the conditions can be omitted for these ideals.

Throughout this paper, R denotes an associative ring with unit and M denotes a unitary left R-module. We write $\operatorname{Rad}(M)$, $\operatorname{Soc}(M)$, $Z(M)$ and $Z_2(M)$ for the Jacobson radical, the socle, the singular submodule and the second singular submodule of M respectively. $J(R)$ is the Jacobson radical of R, and I will be any ideal in the paper. We write M^* for $\operatorname{Hom}(M, R)$.

2. I-semiregular modules

Nicholson [8] calls an element x of a module M *semiregular* if $Rx = A \oplus B$ where A is a projective summand of M and $B \subseteq \operatorname{Rad}(M)$. A module is called *semiregular* if each of its elements is semiregular. For a module M, this concept was extended in [1] to the notion of U-semiregular elements by taking any fully invariant submodule U of M instead of $\operatorname{Rad}(M)$.

In this section, we consider the fully invariant submodule IM for an ideal I. We use "I-semiregular" instead of "IM-semiregular" and we want to prove that if $M = \bigoplus_{i \in I} M_i$, then M is I-semiregular if and only if each M_i is I-semiregular.

Lemma 2.1. *(See [1, Proposition 2.2].) Let M be a module. The following are equivalent for any element m in M.*

(i) *There exists a decomposition* $M = P \oplus Q$ *where* P *is projective,* $P \subseteq Rm$ *and* $Rm \cap Q \subseteq IM$.
(ii) *There exists* $\lambda \in M^*$ *such that* $m\lambda = e = e^2$ *and* $(1-e)m \in IM$.
(iii) *There exits* $\gamma^2 = \gamma \in End({}_RM)$ *such that* $M\gamma$ *is projective and* $m - m\gamma \in IM$.

An element m of a module M is called *I-semiregular* if it satisfies the conditions in Lemma 2.1, and M is called an *I-semiregular module* if every element of M is I-semiregular. In [1], it is named by "IM-semiregular" but we use "I-semiregular" in this note for short. A ring R is called *I-semiregular* if ${}_RR$ is an I-semiregular module. Note that I-semiregular rings are left–right symmetric by [14].

For an ideal I of a ring R, IM is a fully invariant submodule of an R-module M. Hence another characterization of I-semiregular modules is given by the following theorem.

Theorem 2.2. *(See [1, Theorem 2.3].) The following are equivalent for a module* M.

(i) M *is I-semiregular.*
(ii) *If* $N \subseteq M$ *is finitely generated, there exists* $\gamma : M \to N$ *such that* $\gamma^2 = \gamma$ *and* $M\gamma$ *is projective and* $N(1-\gamma) \subseteq IM$.
(iii) *If* $N \subseteq M$ *is finitely generated, there exists a decomposition* $M = P \oplus Q$ *such that* P *is projective,* $P \subseteq N$ *and* $N \cap Q \subseteq IM$.

Now we give some lemmas to prove our result.

Lemma 2.3. *Let* $m \in M$. *If there exists* $\lambda \in M^*$ *such that* $m\lambda = e = e^2$ *and* $(1-e)m$ *is I-semiregular, then* m *is I-semiregular.*

Proof. Since $(1-e)m$ is I-semiregular choose $\beta \in M^*$ such that $f = ((1-e)m)\beta$ is an idempotent and $(1-f)(1-e)m \in IM$. Then $ef = 0$ so $g = e + f - fe$ is an idempotent in R and $(1-g)m \in IM$. Since M^* is a right R-module, define $\alpha \in M^*$ by $\alpha = \lambda + (\beta - \lambda \cdot m\beta)(1-e)$. Then $(1-g)m = (1-f)(1-e)m \in IM$, and

$$m\alpha = m\lambda + m(\beta - \lambda \cdot m\beta)(1-e) = e + [m\beta - e \cdot m\beta](1-e) = e + f(1-e) = g.$$

Thus m is I-semiregular by Lemma 2.1. □

Lemma 2.4. *Let* $M = N \oplus K$ *and* $m = n + k$ *where* $n \in IM$ *and* $k \in K$. *If* k *is I-semiregular in* K *then* m *is I-semiregular in* M.

Proof. Let $\lambda : K \to R$ satisfy $k\lambda = e = e^2$ and $(1-e)k \in IK$. Extend λ to $M \to R$ by defining $N\lambda = 0$. Then $m\lambda = k\lambda = e$ and $(1-e)m = (1-e)n + (1-e)k \in (1-e)IM + IK \subseteq IM$. □

Lemma 2.5. *Let* $M = N \oplus K$ *and let* $n \in N$. *Then* n *is I-semiregular in* N *if and only if* n *is I-semiregular in* M.

Proof. If n is I-semiregular in N, let $\lambda : N \to R$ satisfy $n\lambda = e = e^2$ and $(1-e)n \in IN$. Then define $\alpha : M \to R$ by $(n+k)\alpha = n\lambda$. Then $n\alpha = e$ and $(1-e)n \in IM$.

Conversely, let $\gamma : M \to R$ satisfy $n\gamma = e = e^2$ and $(1-e)n \in IM$. Define $\lambda = \gamma_{|N} : N \to R$. Then $n\lambda = e$ and $(1-e)n \in N \cap IM$, so it remains to show that $N \cap IM \subseteq IN$. Let $x \in N \cap IM$, say $x = \sum a_i m_i$, $a_i \in I, m_i \in M$. For each i, write $m_i = n_i + k_i$, $n_i \in N, k_i \in K$. Then $x = \sum a_i n_i + \sum a_i k_i \in N \oplus K$ so, since $x \in N$, $x = \sum a_i n_i \in IN$, as required. □

Now we may prove our result which is a generalization of [8, Theorem 1.10].

Theorem 2.6. *Let $M = \bigoplus_{i \in \Lambda} M_i$ be a left R-module for any index set Λ. Then M is I-semiregular if and only if each M_i is I-semiregular.*

Proof. The forward implication follows from Lemma 2.5. Conversely, assume that each M_i is I-semiregular. Since each element of M is in a finite sum of the M_i, we may assume by Lemma 2.5 that Λ is finite and so by induction that $M = N \oplus K$, where both N and K are I-semiregular. If $m \in M$ write $m = n + k, n \in N, k \in K$. Let $\alpha : N \to R$ satisfy $n\alpha = e = e^2$ and $(1-e)n \in IN$. Define $\lambda : M \to R$ by $(n+k)\lambda = n\alpha$. Then $(m)\lambda = e = e^2$ so, by Lemma 2.3, it suffices to show that $(1-e)m$ is I-semiregular in M. But $(1-e)m = (1-e)n + (1-e)k$ where $(1-e)k$ is I-semiregular in K by hypothesis, it is I-semiregular in M by Lemma 2.4. This is what we wanted. □

3. *I*-semiregular rings and projective *I*-covers

Zhou extends the notion of small submodules to δ-small submodules in [15]. A submodule K of an R-module M is called *δ-small* in M (notation $K \ll_\delta M$) if $K + L \neq M$ for any proper submodule L of M with M/L singular. Also by [15, Lemma 1.2], $K \ll_\delta M$ if and only if $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq K$ whenever $X + K = M$. Then he defines and characterizes δ-semiregular rings and δ-semiperfect rings.

Zhou also consider the following fully invariant submodule of a module M.

$$\delta(M) = \bigcap \{K \leqslant M\colon\ M/K \text{ is singular simple}\}.$$

Then $\delta(M)$ is the sum of all δ-small submodules of M. If M is projective, then $\operatorname{Soc}(M) \subseteq \delta(M)$ [15, Lemma 1.9].

Özcan and Alkan prove in [12, Proposition 2.12] that $\operatorname{Rad}(M/\operatorname{Soc}(M)) = \delta(M)/\operatorname{Soc}(M)$ for a projective module M. In particular, for a projective module M, $\delta(M) = M$ if and only if M is semisimple.

Now we extend the notion of δ-small submodules to study a generalization of δ-semiregular rings and δ-semiperfect rings.

Definition 3.1. Let I be an ideal of a ring R, and N be a submodule of an R-module M. We say that

(a) N *decomposes* M (briefly N is *DM in M*) if there is a summand S of M such that $S \leqslant N$ and $M = S + X$, whenever $N + X = M$ for a submodule X of M,
(b) N is *SDM in M* if there is a summand S of M such that $S \leqslant N$ and $M = S \oplus X$, whenever $N + X = M$ for a submodule X of M,
(c) M is *DM for I* if any submodule of IM is *DM* in M,
(d) R is *a left DM ring for I* if for any finitely generated free left R-module is *DM* for I.

Clearly any *SDM* submodule of a module M is *DM*. Any δ-small submodule of M is *SDM* in M, but there exists a module M such that $\mathrm{Soc}(M)$ is *SDM* but not δ-small (see Example 4.13). Any summand of M is *DM* in M. On the other hand, any module is *DM* for $\mathrm{Soc}({}_R R)$. Moreover, any semisimple module is *DM* for any ideal I, and any finitely generated module is *DM* for a δ-small ideal I and so does a ring R.

Nicholson and Zhou in [11] define a strongly lifting left ideal that is a generalization of an idempotent lifting left ideal. A left ideal I of a ring R is called *strongly lifting* if $a^2 - a \in I$, then there exists $e^2 = e \in Ra$ such that $e - a \in I$. They characterize I-semiregular and I-semiperfect rings by using strongly lifting ideals I. For an ideal I, R is I-semiregular if and only if R/I is regular and I is strongly lifting [11, Theorem 28]. R is I-semiperfect if and only if R/I is semisimple and I is strongly lifting [11, Theorem 36].

Now if I is strongly lifting left ideal, then I is a *DM* left ideal in R. For, let $R = I + X$ for some left ideal X. Then $1 = a + x$ where $a \in I$, $x \in X$. Since $x^2 - x \in I$, there exists $e^2 = e \in Rx$ such that $e - x \in I$. Hence we have that $R = R(1-e) + X$ where $R(1-e) \subseteq I$. But the converse is not true in general, because for example $J(R)$ is *DM* but not strongly lifting in general. Therefore, the concept of *DM* is a generalization of the notion of δ-small submodules and of strongly lifting ideals.

If every submodule of M is *DM* in M, then M is called refinable (see [5]). By using *DM* submodules, modules with the finite exchange property (see [6] for the definition) are characterized in [5]. Let M be a self-projective module. Then every submodule of M is *DM* in M if and only if M has the finite exchange property [5, 11.31].

Now we study some properties of submodules having *DM*.

Lemma 3.2. *Let N be a summand of a module M and A be a submodule of N. Then A is DM in N if and only if A is DM in M.*

Proof. Let $M = N \oplus K$ for a submodule K of M. Assume that A is *DM* in N. Let $M = A + L$ for any submodule L of M. Then $N = A + (L \cap N)$ and by assumption there is a summand S of N such that $N = S + (L \cap N)$ and $S \leqslant A$. Let $x \in M$ and write $x = a + l$ where $a \in A \leqslant N$ and $l \in L$. Since $a = s + k$ where $s \in S$ and $k \in L \cap N$, $k + l \in L$ and so $x = s + (k + l) \in S + L$. It follows that $M = S + L$ and S is a summand of M. Hence A is *DM* in M.

Conversely, assume that A is *DM* in M. Let $N = A + L$ for any submodule L of N. Then $M = A + (L + K)$ and so there is a summand S of M such that $M = S + (L + K)$ and $S \leqslant A$. It follows that $N = S + L$ and S is a summand of N. This completes the proof. □

Corollary 3.3. *Let M be an R-module. If M is DM for an ideal I of R, then any summand of M is DM for I.*

Proof. Let $M = N \oplus K$ and A be a submodule of IN. Then $A \leqslant IM$ and so A is *DM* in M. Since N is a summand of M, A is *DM* in N. □

By Corollary 3.3, we have that R is a left *DM* ring for an ideal I if and only if any finitely generated projective left R-module is *DM* for I.

Proposition 3.4. *Let $M = \bigoplus_{i \in \Lambda} M_i$ where Λ is any index set. If N_i is DM in M_i for all i in a finite subset $\mathcal{F}$ of Λ, then $\bigoplus_{i \in \mathcal{F}} N_i$ is DM in M.*

Proof. Let N_i be DM submodule in M_i for all $i = 1, \dots, n$. Let $M = \bigoplus_{i=1}^{n} N_i + L$ for any submodule L of M. Since by Lemma 3.2, N_1 is DM in M, there is a decomposition $M_1 = S_1 \oplus K_1$ for a submodule K_1 of M_1 such that $S_1 \leqslant N_1$ and

$$M = S_1 + (N_2 + \cdots + N_n + L) = N_2 + (S_1 + \cdots + N_n + L).$$

Then similarly, we get a decomposition $M_2 = S_2 \oplus K_2$ for a submodule K_2 of M_2 such that $M = S_2 + (S_1 + N_3 + \cdots + N_n + L)$ and $S_2 \leqslant N_2$. Hence

$$M = S_1 \oplus S_2 \oplus K_1 \oplus K_2 \oplus \left(\bigoplus_{i \in \Lambda \setminus \{1,2\}} M_i \right).$$

After finite steps, we find the summands S_i of M_i such that $M = (\bigoplus_{i=1}^{n} S_i) + L$. This completes the proof since $\bigoplus_{i=1}^{n} S_i \subseteq \bigoplus_{i=1}^{n} N_i$ and it is a summand of M. □

Now we recall some projective covers. A module M is said to have a *projective cover* (*δ-cover* [15], *Soc-cover* [12], respectively) if there exists an epimorphism $f : P \to M$ such that P is projective and $\operatorname{Ker} f \ll P$ ($\operatorname{Ker} f \ll_\delta P$, $\operatorname{Ker} f \subseteq \operatorname{Soc}(P)$, respectively). Here we consider some generalizations of these covers.

Definition 3.5. Let I be an ideal of a ring R and M be an R-module.

(a) A pair (P, f) is called a *projective I-semicover* of M if P is projective and f is an epimorphism from P to M such that $\operatorname{Ker} f \subseteq IP$.
(b) A pair (P, f) is called a *projective I-cover* of M if (P, f) is a projective I-semicover and $\operatorname{Ker} f$ is DM in P.

Hence *Soc*-covers will be called $\operatorname{Soc}({}_R R)$-semicovers from now on. But since $\operatorname{Ker} f$ is DM in P whenever $\operatorname{Ker} f \subseteq \operatorname{Soc}({}_R R)P = \operatorname{Soc}(P)$, we have that a projective $\operatorname{Soc}({}_R R)$-semicover is the same as a projective $\operatorname{Soc}({}_R R)$-cover. Also for $\delta({}_R R)$ and $J(R)$ we have the following result which shows that a projective I-cover is a generalization of a projective cover and a projective δ-cover.

Proposition 3.6. *A module M has a projective $\delta({}_R R)$-cover* (*projective $J(R)$-cover, respectively*) *if and only if M has a projective δ-cover* (*projective cover, respectively*).

Proof. It is enough to prove the necessity. Let P be a projective module and $f : P \to M$ an epimorphism such that $\operatorname{Ker} f \subseteq \delta(P)$ and $\operatorname{Ker} f$ is DM in P. We claim that $\operatorname{Ker} f \ll_\delta P$. Let X be a submodule of P such that $P = \operatorname{Ker} f + X$. Since $\operatorname{Ker} f$ is DM in P, there exists a summand S of P such that $S \subseteq \operatorname{Ker} f$ and $P = S + X$. Write $P = S \oplus S'$. Then $\delta(P) = \delta(S) \oplus \delta(S')$. Since $S \subseteq \delta(P)$, we have that $S = \delta(S)$. Since S is projective, S is semisimple by [12, Proposition 2.12]. Then there exists a summand Y of P such that $Y \subseteq S$ and $P = Y \oplus X$. Hence $\operatorname{Ker} f \ll_\delta P$ by [15, Lemma 1.2].

If $\operatorname{Ker} f \subseteq \operatorname{Rad}(P)$ and $\operatorname{Ker} f$ is DM in P, in the above proof we have that $S = \operatorname{Rad}(S)$. But since S is projective, $S = 0$. Then we have that $P = X$ and hence $\operatorname{Ker} f \ll P$. □

Clearly a module M has a projective 0-cover if and only if M is projective.

Note that the projective $J(R)$-semicover was studied in [3,13] under the name "generalized projective cover." Also the projective $\rho({}_RR)$-semicover was defined by Nakahara [7] and named by "projective ρ-semicover" for any precover ρ.

We extend some well-known theorems about projective modules (see [2]).

Proposition 3.7. *Let I be an ideal of a ring R, a module M has a projective I-semicover and $IM = M$. Then*

(i) *if I is δ-small in ${}_RR$, then M is semisimple and projective,*
(ii) *if I is small or singular in ${}_RR$, then $M = 0$.*

Proof. Let $IM = M$ and f be an epimorphism from P to M such that $\operatorname{Ker} f \subseteq IP$. Then $P = IP$.

(i) Assume that I is δ-small in ${}_RR$. Then $P = IP \subseteq \delta({}_RR)P = \delta(P)$. By [12, Proposition 2.13], P is semisimple and so do M. On the other hand, $P = \operatorname{Ker} f \oplus L$ for a submodule L of P and so $M \cong P/\operatorname{Ker} f \cong L$ is projective.

(ii) For any nonzero projective module P, $\operatorname{Rad}(P) \neq P$ and $Z(P) \neq P$. □

Now we note another result about I-semicovers without a proof.

Proposition 3.8. *Let N be a submodule of a projective module M. If M has a decomposition $M = P \oplus Q$ such that $P \subseteq N$ and $N \cap Q \subseteq IM$, then M/N has a projective I-semicover.*

Lemma 3.9. *Let $\{M_i\}_{i=1}^n$ be a finite collection of modules such that each M_i has a projective I-cover. Then $\bigoplus_{i=1}^n M_i$ has a projective I-cover.*

Proof. Let f_i be an epimorphism from a projective module P_i to M_i such that $\operatorname{Ker} f_i$ is *DM* in P_i and $\operatorname{Ker} f_i \subseteq IP_i$, $i = 1, \dots, n$. Then $\operatorname{Ker}(\bigoplus_{i=1}^n f_i) = \bigoplus_{i=1}^n \operatorname{Ker} f_i \subseteq \bigoplus_{i=1}^n (IP_i) = I(\bigoplus_{i=1}^n P_i)$ and also $\operatorname{Ker}(\bigoplus_{i=1}^n f_i)$ is *DM* in $\bigoplus_{i=1}^n P_i$ by Proposition 3.4. Hence $\bigoplus_{i=1}^n f_i$ is a projective I-cover of $\bigoplus_{i=1}^n M_i$. □

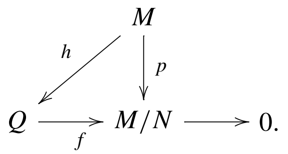
Clearly, in this lemma we may consider any direct sum of submodules for a projective I-semicover. The following lemma is a key result for our work.

Lemma 3.10. *Let I be an ideal of a ring R and M be a projective module and $N \leqslant M$. Consider the following conditions*:

(i) *M/N has a projective I-cover,*
(ii) *$M = Y \oplus X$ for some submodules Y and X with $Y \subseteq N$ and $X \cap N \subseteq IM$.*

Then (i) $\Rightarrow$ (ii)*, and* (ii) $\Rightarrow$ (i) *if M is DM for I.*

Proof. (i) $\Rightarrow$ (ii) Let N be a submodule of M. Assume f is an epimorphism from a projective module Q to M/N such that $\operatorname{Ker} f \subseteq IQ$ and $\operatorname{Ker} f$ is *DM* in Q. Let p be the projection map from M to M/N. Since M is projective, we have the following commutative diagram:



Since $(M)h + \operatorname{Ker} f = Q$ and $\operatorname{Ker} f$ is *DM* in Q, there is a summand K of Q such that $Q = (M)h + K$ and $K \subseteq \operatorname{Ker} f$. By [8, Lemma 1.16], we have $Q = A \oplus K$ for a submodule A of $(M)h$, and so $(M)h = A \oplus S$ where $S = K \cap (M)h \subseteq \operatorname{Ker} f$. On the other hand, $(M)h \cong M/\operatorname{Ker} h$ and so there is a decomposition $M/\operatorname{Ker} h = B/\operatorname{Ker} h \oplus Y/\operatorname{Ker} h$ such that $B/\operatorname{Ker} h \cong A$ and $Y/\operatorname{Ker} h \cong S$. Then there exists a homomorphism $g : A \to B$ such that $gh = 1_A$. Let $X = (A)g$. Then $B = X \oplus \operatorname{Ker} h$, and so $B + Y = X + Y = M$. Since $X \cap Y \subseteq B \cap Y = \operatorname{Ker} h$, we have that $M = X \oplus Y$. Let $y \in Y$. Since $(y)h \in S$, it follows that $(y)p = (y)hf = 0$ and so $y \in N$. Thus $Y \subseteq N$. Now let $x \in N \cap X$, and so $x = (t)g$ for some $t \in A$. Then $(t)f = (t)ghf = (x)hf = (x)p = 0$, and so $t \in \operatorname{Ker} f \subseteq IQ \cap A = IA$. Therefore, $x = (t)g \in IM$. Hence $N \cap X \subseteq IM$.

(ii) $\Rightarrow$ (i) Let $M = Y \oplus X$ for some Y and X with $Y \subseteq N$ and $X \cap N \subseteq IM$. Since M is assumed to be *DM* for I, it follows that $X \cap N$ is *DM* in X. Now define $f : X \to M/N$ be such that $f(x) = x + N$. Then f is an epimorphism with $\operatorname{Ker} f = X \cap N$. Hence f is a projective I-cover of M/N. $\square$

With Lemma 3.10, we can give the following characterization of I-semiregular rings related to projective I-covers.

Theorem 3.11. *Let I be an ideal of a ring R. Consider the following conditions*:

(i) *every finitely presented left R-module has a projective I-cover,*
(ii) *for every finitely generated left ideal K of R, R/K has a projective I-cover,*
(iii) *every cyclically presented left R-module has a projective I-cover,*
(iv) *R is I-semiregular.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv)*, and* (iv) $\Rightarrow$ (i) *if R is a left DM ring for I.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) They are clear.

(iii) $\Rightarrow$ (iv) Let K be a cyclic left ideal of R. Then R/K has a projective I-cover, and so by Lemma 3.10 R is I-semiregular.

(iv) $\Rightarrow$ (i) Let M be a finitely presented left R-module. Then $M \cong R^{(n)}/K$ where K is a finitely generated submodule of $R^{(n)}$ for some n. Since R is I-semiregular, $R^{(n)}$ is I-semiregular by Theorem 2.6. By hypothesis, ${}_R R^{(n)}$ is *DM* for I. Hence M has a projective I-cover by Lemma 3.10. $\square$

If $I \subseteq \delta({}_R R)$, then R is a left *DM* ring for I by [15, Lemma 1.5(4)], and so the conditions in Theorem 3.11 are equivalent. Hence if $I = \delta({}_R R)$, we have the characterization of

$\delta({}_RR)$-semiregular rings given in [15, Theorem 3.5]. Note that δ-semiregular rings of Zhou [15] are exactly the $\delta({}_RR)$-semiregular rings.

If $I = J(R)$, then Theorem 3.11 gives the characterization of semiregular rings (see [10]).

As for the singular ideal, it can be easily seen that $Z({}_RR) \subseteq J(R)$ if and only if $Z({}_RR)$ is *SDM* in R if and only if $Z({}_RR)$ is *DM* in R if and only if R is a left *DM* ring for $Z({}_RR)$. Also $Z({}_RR) \subseteq \delta({}_RR)$ if and only if $Z({}_RR) \subseteq J(R)$ by [1, Proposition 3.1]. Therefore, there is a ring R which is not *DM* for $Z({}_RR)$. But if R is $Z({}_RR)$-semiregular, then $Z({}_RR) = J(R) \subseteq \delta({}_RR)$ by [9, Theorem 2.4] or [1, Theorem 3.2]. Hence if R is $Z({}_RR)$-semiregular, then every finitely presented left R-module has a projective $Z({}_RR)$-cover by Theorem 3.11. Thus we have the following result.

Corollary 3.12. *The following are equivalent for a ring R.*

(i) *R is $Z({}_RR)$-semiregular.*
(ii) *Every cyclically presented left R-module has a projective $Z({}_RR)$-cover.*
(iii) *For every finitely generated left ideal K of R, R/K has a projective $Z({}_RR)$-cover.*
(iv) *Every finitely presented left R-module has a projective $Z({}_RR)$-cover.*

Similarly, if $I \subseteq \mathrm{Soc}({}_RR)$, then R is a left DM ring for I. Note that R is $\mathrm{Soc}({}_RR)$-semiregular if and only if $R/\mathrm{Soc}({}_RR)$ is regular (see [4]). Hence,

Corollary 3.13. *The following are equivalent for a ring R.*

(i) *R is* $\mathrm{Soc}({}_RR)$*-semiregular.*
(ii) *Every cyclically presented left R-module has a projective* $\mathrm{Soc}({}_RR)$*-cover.*
(iii) *For every finitely generated left ideal K of R, R/K has a projective* $\mathrm{Soc}({}_RR)$*-cover.*
(iv) *Every finitely presented left R-module has a projective* $\mathrm{Soc}({}_RR)$*-cover.*

4. *I*-semiperfect rings and projective *I*-covers

In this section, we study I-semiperfect rings related with projective I-semicovers and projective I-covers for an ideal I. First we generalize one of Azumaya's Theorems [3, Theorem 4] on projective $J(R)$-semicover. After that we give a characterization of I-semiperfect rings and consider certain ideals I such as $\delta({}_RR)$, $\mathrm{Soc}({}_RR)$, $Z({}_RR)$ and $Z_2({}_RR)$.

Proposition 4.1. *(See [12, Proposition 2.1].) Let I be an ideal of a ring R. The following are equivalent for a module M.*

(i) *For every submodule K of M, there is a decomposition $K = A \oplus B$ such that A is a projective summand of M and $B \subseteq IM$.*
(ii) *For every submodule K of M, there is a decomposition $M = A \oplus B$ such that A is projective, $A \subseteq K$ and $K \cap B \subseteq IM$.*

A module M is said to be *I-semiperfect* if it satisfies the conditions of Proposition 4.1. In [12], it is named by "τ-semiperfect" for the preradical τ where $IM = \tau(M) = \sum\{f(IM) \mid f : M \to M\}$ but we use "I-semiperfect" in this note for short.

By our definitions, any I-semiperfect module is I-semiregular for an ideal I. If M is a projective module with $\mathrm{Rad}(M) \ll M$, then M is $J(R)$-semiperfect if and only if M is semiperfect (i.e. every factor module of M has a projective cover).

First we generalize Azumaya's Theorem [3, Theorem 4] to projective I-semicovers where $I \subseteq \delta({}_R R)$. We need some lemmas.

Lemma 4.2. *Let I be an ideal of a ring R and S be a simple R-module having a projective I-semicover. Then S is M-projective for every R/I-module M.*

Proof. Let $f : P \to S$ be a projective I-semicover of S. If $IP = P$, then $S = f(P) = If(P) = IS$. Then since any homomorphism from S to an R/I-module M is zero, S is M-projective for every R/I-module M. If $IP \neq P$, then $\mathrm{Ker}\, f = IP$. Then $P/IP \cong S$ is a projective R/I-module by [2, p. 191]. Hence the proof is completed. □

Lemma 4.3. *Let I be an ideal of a ring R. If every proper submodule of a module M is contained in a maximal submodule and every simple factor module of M has a projective I-semicover, then M/IM is semisimple.*

Proof. Let $\bar{M} = M/IM$ and $C = \mathrm{Soc}(\bar{M})$. If $C \neq \bar{M}$, then there exists a maximal submodule D of $\bar{M}$ such that $C \subseteq D \subseteq \bar{M}$. Then $\bar{M}/D$ is a simple factor module of $\bar{M}$ whence of M, and so has a projective I-semicover and satisfies $I(\bar{M}/D) = 0$. Thus by Lemma 4.2, $\bar{M}/D$ is a projective R/I-module. This implies that D is a summand of $\bar{M}$. So $\bar{M} = D \oplus D'$ for some D'. This implies that $D' \subseteq C \subseteq D$, a contradiction. □

We can now prove Theorem 4.4, which restates Azumaya's Theorem.

Theorem 4.4. *Let I be an ideal of a ring R such that $I \subseteq \delta({}_R R)$. Then the following are equivalent for a module M.*

(i) *Every factor module of M has a projective I-semicover.*
(ii) *Every proper submodule of M is contained in a maximal submodule and every simple factor module of M has a projective I-semicover.*

Proof. (i) ⇒ (ii) Let U be a proper submodule of M and $f : P \to M/U$ be a projective I-semicover of M/U. If $\delta(P) \neq P$, then P has an essential maximal submodule V by [15, Lemma 1.9]. Then $\mathrm{Ker}\, f \subseteq \delta({}_R R)P = \delta(P) \subseteq V$. This implies that $(V)f$ is a maximal submodule of M/U. If $\delta(P) = P$, P and hence M/U is semisimple. It follows that M/U has a maximal submodule.

(ii) ⇒ (i) By Lemma 4.3, $\bar{M} = M/IM$ is semisimple. Then $M/IM = \bigoplus_{i\in\Lambda} S_i$ where each S_i is simple and Λ is any index set. Let $f_i : P_i \to S_i$ be a projective I-semicover of S_i. Then $f := \bigoplus_{i\in\Lambda} f_i : \bigoplus_{i\in\Lambda} P_i \to \bar{M}$ is an epimorphism and $P = \bigoplus_{i\in\Lambda} P_i$ is projective. Let $g : M \to \bar{M}$ be the canonical epimorphism. Then there exists a homomorphism $h : P \to M$ such that $hg = f$. Then we have that $M = (P)h + IM$. Since $IM \ll_\delta M$ by [15, Lemma 1.5], there exists a semisimple projective submodule X of M such that $M = (P)h \oplus X$. Since $h : P \to (P)h$ is a projective I-semicover, we have that M has a projective I-semicover. The hypotheses of the theorem are also satisfied for any factor module of M. Hence every factor module of M has a projective I-semicover. □

Corollary 4.5. *(See [3, Theorem 4].) Let M be an R-module. Then the following are equivalent.*

(i) *Every factor module of M has a projective $J(R)$-semicover.*
(ii) *Every proper submodule of M is contained in a maximal submodule and every simple factor module of M has a projective $J(R)$-semicover.*

From now on we consider projective I-covers to characterize I-semiperfect rings. First we prove the following theorem which shows that the projectivity condition in [12, Theorem 2.10] is removable by a similar proof.

Theorem 4.6. *Let $M = M_1 \oplus M_2$ a direct sum of modules M_1, M_2 such that M_i is I-semiperfect for $i = 1, 2$. Then M is I-semiperfect.*

Proof. Let $L \subseteq M$. We show that there exists a decomposition $M = A \oplus B$ such that $A \subseteq L$ is projective and $L \cap B \subseteq IM$.

Case (1). If $M_1 \cap (L + M_2) = 0$, then $L \subseteq M_2$. Since M_2 is I-semiperfect, there exists a decomposition $M_2 = B_1 \oplus B_2$ such that $B_1 \subseteq L$ is projective and $L \cap B_2 \subseteq IM_2$. Hence $M = M_1 \oplus B_1 \oplus B_2$ and $L \cap (M_1 \oplus B_2) = L \cap B_2 \subseteq IM_2 \subseteq IM$.

Case (2). If $M_1 \cap (L + M_2) \neq 0$, then M_1 has a decomposition $M_1 = A_1 \oplus A_2$ such that A_1 is projective, $A_1 \subseteq M_1 \cap (L + M_2)$ and $M_1 \cap (L + M_2) \cap A_2 = A_2 \cap (L + M_2) \subseteq IM_1 \subseteq IM$. Then $M = A_1 \oplus A_2 \oplus M_2 = L + (M_2 \oplus A_2)$.

Assume $M_2 \cap (L + A_2) = 0$. Since $L \cap A_2 \subseteq A_2$ and A_2 is I-semiperfect, A_2 has a decomposition $A_2 = C_1 \oplus C_2$ such that $C_1 \subseteq L \cap A_2$ is projective and $L \cap A_2 \cap C_2 = L \cap C_2 \subseteq IM_1$. Then $M = (A_1 \oplus C_1) \oplus (C_2 \oplus M_2) = L + (C_2 + M_2)$. Since $A_1 \oplus C_1$ is projective, there exists a summand L' of M such that $L' \subseteq L$ and $M = L' \oplus (C_2 \oplus M_2)$ (see [6, Lemma 4.47]). Then L' is projective. Since $M_2 \cap (L + A_2) = 0$, we have $L \cap (C_2 \oplus M_2) = L \cap C_2 \leqslant IM_1$.

Assume $M_2 \cap (L + A_2) \neq 0$. Then M_2 has a decomposition $M_2 = B_1 \oplus B_2$ such that $B_1 \leqslant M_2 \cap (L + A_2)$ is projective and $B_2 \cap (L + A_2) \subseteq IM_2$. Then $M = L + (A_2 + B_2) = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2)$. Since $A_1 \oplus B_1$ is projective, there exists $L' \subseteq L$ such that $M = L' \oplus A_2 \oplus B_2$ and then L' is projective. To show that $L \cap (A_2 \oplus B_2) \subseteq IM$, take $0 \neq l = a + b \in L \cap (A_2 \oplus B_2)$ where $l \in L, a \in A_2, b \in B_2$. Then $l - b = a \in A_2 \cap (L + M_2) \leqslant IM$ and $l - a = b \in B_2 \cap (L + A_2) \subseteq IM$ and so $l \in IM$. Hence M is I-semiperfect. □

A module M is said to have C_3 if, whenever M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is also a summand of M [6].

Theorem 4.7. *Let I be an ideal of a ring R and M be a finitely generated projective R-module. If every simple factor module of M has a projective I-cover and either IM is SDM in M or M is DM for I with C_3, then M is I-semiperfect.*

Proof. Let N be a submodule of M. Since by Lemma 4.3, M/IM is semisimple, $M/(IM + N)$ is semisimple. Hence it is a finite direct sum of simple modules S_i, $i \in \mathcal{F}$ where $\mathcal{F}$ is finite. Let $f_i : P_i \to S_i$ be a projective I-cover of S_i ($i \in \mathcal{F}$). Then $f = \bigoplus_{i \in \mathcal{F}} f_i : \bigoplus_{i \in \mathcal{F}} P_i \to M/(IM + N)$ is a projective I-cover of $M/(IM + N)$ by Lemma 3.9. Hence by Lemma 3.10, there is a decomposition $M = A \oplus B$ such that $IM + N = A \oplus (B \cap (IM + N))$ and $B \cap (IM + N) \subseteq IM$.

If IM is *SDM* in M, then since $M = IM + N + B$, $M = C \oplus (N + B)$ for a submodule C of IM. On the other hand, since $N + B$ is projective and B is a summand of $N + B$, there is a submodule K of N such that $N + B = K \oplus B$ and so $M = C \oplus K \oplus B$. It follows that $N = K \oplus ((C + B) \cap N)$, and since $(C + N) \cap B \subseteq (IM + N) \cap B \subseteq IM$, we have that $(C + B) \cap N \subseteq IM$. Hence M is I-semiperfect.

Now assume that M is *DM* for I with C_3. Since $M = IA + N + B$ and IA is *DM* in M, there is a summand C of IA such that $M = C + B + N$. Since $C \cap B = 0$ and M has C_3 we have $C + B$ is a summand of M, and so there is a submodule K of N such that $M = C \oplus K \oplus B$. It follows that $N = K \oplus ((C + B) \cap N)$ and $(C + B) \cap N \subseteq (C + N) \cap B + (N + B) \cap C \subseteq IM$. The proof is completed. □

Now we state our main result of this section which shows the relationship between projective I-covers and I-semiperfect rings.

Theorem 4.8. *Let I be an ideal of a ring R. Consider the following conditions*:

(i) *every finitely generated left R-module has a projective I-cover,*
(ii) *every factor module of ${}_RR$ has a projective I-cover,*
(iii) *for every countably generated left ideal L of R, R/L has a projective I-cover,*
(iv) *R is I-semiperfect,*
(v) *every simple factor module of ${}_RR$ has a projective I-cover.*

Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) *and* (ii) ⇒ (v); (iv) ⇒ (i) *if R is a left DM ring for I*; *and* (v) ⇒ (iv) *if either I is SDM in ${}_RR$ or ${}_RR$ is DM for I with C_3.*

Proof. (i) ⇒ (ii) ⇒ (iii) It is obvious.

(iii) ⇒ (iv) By Theorem 3.11, R is I-semiregular. By the proof of [12, Theorem 2.19 (3 ⇒ 1)], R/I is Noetherian and hence semisimple. By [11, Theorems 28 and 36], R is I-semiperfect.

(ii) ⇒ (v) It is obvious.

(iv) ⇒ (i) Assume that R is a left *DM* ring for I and R is an I-semiperfect ring. Let M be a finitely generated left R-module. Then there exists an epimorphism $f : F \to M$ where F is a finitely generated free module. Since F is I-semiperfect by Theorem 4.6 or [12, Corollary 2.11], $F = F_1 \oplus F_2$ where $F_1 \subseteq \operatorname{Ker} f$ and $F_2 \cap \operatorname{Ker} f \subseteq IF$. Now $f|_{F_2} : F_2 \to M$ is an epimorphism and $\operatorname{Ker}(f|_{F_2}) = F_2 \cap \operatorname{Ker} f$ is *DM* in F_2 since R is a left *DM* ring for I. Hence M has a projective I-cover.

(v) ⇒ (iv) By Theorem 4.7. □

If $I \subseteq \delta({}_RR)$, then the conditions of Theorem 4.8 are equivalent because R is a left *DM* ring for I and I is *SDM* in ${}_RR$. Hence if $I = \delta({}_RR)$, we have the characterization of δ-semiperfect rings which is proven by Zhou [15, Theorem 3.6]. Note that the δ-semiperfect rings of Zhou are exactly the $\delta({}_RR)$-semiperfect rings.

If $I = J(R)$, then Theorem 4.8 gives the characterization of semiperfect rings (see [10]).

If $I = \operatorname{Soc}({}_RR)$, then we have [12, Corollary 2.24].

For the singular ideal, if R is $Z({}_RR)$-semiperfect, then $Z({}_RR) = J(R)$ by [9, Theorem 2.4] or [1, Theorem 3.2] and hence $Z({}_RR)$ is *SDM*. If every simple factor module of ${}_RR$ has a projective $Z({}_RR)$-cover, then by Lemma 3.10 and [12, Theorem 2.12] $Z({}_RR) = J(R)$. Then by the remark above of Theorem 3.12 we have the following corollary.

Corollary 4.9. *The following are equivalent for a ring R.*

(i) *Every finitely generated module M has a projective $Z({}_RR)$-cover.*
(ii) *Every factor module of ${}_RR$ has a projective $Z({}_RR)$-cover.*
(iii) *For every countably generated submodule L of ${}_RR$, R/L has a projective $Z({}_RR)$-cover.*
(iv) *R is $Z({}_RR)$-semiperfect.*
(v) *Every simple factor module of ${}_RR$ has a projective $Z({}_RR)$-cover.*

Since any strongly lifting ideal is *DM* as a left and right ideal in R, by [11, Theorem 36] and Lemma 4.3, we can characterize I-semiperfect rings by using projective I-semicovers as follows.

Corollary 4.10. *Let I be a strongly lifting ideal of a ring R. Then the following are equivalent.*

(i) *R is I-semiperfect.*
(ii) *R/I is semisimple.*
(iii) *Every finitely generated left (right) R-module has a projective I-semicover.*
(iv) *Every factor module of ${}_RR(R_R)$ has a projective I-semicover.*
(v) *Every simple factor module of ${}_RR(R_R)$ has a projective I-semicover.*

Although $Z_2({}_RR)$ is not strongly lifting in general (see [11, Example 52]), it is proved in [11, Theorem 49] that R is $Z_2({}_RR)$-semiperfect if and only if $R/Z_2({}_RR)$ is semisimple.

Theorem 4.11. *The following are equivalent for a ring R.*

(i) *R is $Z_2({}_RR)$-semiperfect.*
(ii) *$R/Z_2({}_RR)$ is semisimple.*
(iii) *Every finitely generated left R-module has a projective $Z_2({}_RR)$-semicover.*
(iv) *Every factor module of ${}_RR$ has a projective $Z_2({}_RR)$-semicover.*
(v) *Every simple factor module of ${}_RR$ has a projective $Z_2({}_RR)$-semicover.*

Proof. (i) $\Leftrightarrow$ (ii) By [11, Theorem 49].

(i) $\Rightarrow$ (iii) By a proof similar to Theorem 4.8 ((iv) $\Rightarrow$ (i)).

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) They are obvious.

(v) $\Rightarrow$ (ii) By Lemma 4.3, $R/Z_2({}_RR)$ is semisimple. □

If $Z_2({}_RR) \subseteq \delta({}_RR)$, then $Z_2({}_RR)$-semiperfect rings are semisimple rings as the following corollary shows.

Corollary 4.12. *Let R be a ring with $Z_2({}_RR) \subseteq \delta({}_RR)$. The following are equivalent.*

(i) *R is $Z_2({}_RR)$-semiperfect.*
(ii) *For every countably generated submodule L of ${}_RR$, R/L has a projective $Z_2({}_RR)$-semicover.*
(iii) *R is semisimple.*

Proof. (i) $\Leftrightarrow$ (ii) By Theorem 4.8.

(i) $\Rightarrow$ (iii) If R is $Z_2({}_RR)$-semiperfect, then ${}_RR = Z_2({}_RR) \oplus L$ for a semisimple left ideal L by [11, Theorem 49]. Since $Z_2({}_RR)$ is δ-small in R, it follows that it is semisimple and so is R.

(iii) $\Rightarrow$ (i) By [11, Theorem 49]. $\square$

Example 4.13. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ be the ring of upper triangular matrices over a field F. Then $N = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ is a projective left ideal, $L = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ is a maximal left ideal and $I = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ is an ideal of R. Consider the R-module $M = N \oplus R/L$. Then $\mathrm{Soc}({}_RM) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \oplus R/L$ is *SDM* but not δ-small because $0 \oplus R/L$ is not δ-small in M.

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