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GENERALIZED SEMICOMMUTATIVE RINGS AND THEIR EXTENSIONS

Muhittin Başer, Abdullah Harmanci, and Tai Keun Kwak

Abstract. For an endomorphism α of a ring R, the endomorphism α is called *semicommutative* if $ab = 0$ implies $aR\alpha(b) = 0$ for $a \in R$. A ring R is called α*-semicommutative* if there exists a semicommutative endomorphism α of R. In this paper, various results of semicommutative rings are extended to α-semicommutative rings. In addition, we introduce the notion of an α*-skew power series Armendariz* ring which is an extension of Armendariz property in a ring R by considering the polynomials in the skew power series ring $R[[x;\alpha]]$. We show that a number of interesting properties of a ring R transfer to its the skew power series ring $R[[x;\alpha]]$ and vice-versa such as the Baer property and the p.p.-property, when R is α-skew power series Armendariz. Several known results relating to α-rigid rings can be obtained as corollaries of our results.

1. Introduction

Throughout this paper R denotes an associative ring with identity and α denotes a nonzero non identity endomorphism of a given ring, unless specified otherwise.

Recall that a ring is *reduced* if it has no nonzero nilpotent elements. Lambek [13] called a ring R *symmetric* provided $abc = 0$ implies $acb = 0$ for $a, b, c \in R$, Habeb [5] called a ring R *zero commutative* if R satisfies the condition: $ab = 0$ implies $ba = 0$ for $a, b \in R$, while Cohn [4] used the term *reversible* for what is called zero commutative. A generalization of a reversible ring is a semicommutative ring. A ring R is *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Historically, some of the earliest results known to us about semicommutative rings (although not so called at the time) was due to Shin [15]. He proved that (i) R is semicommutative if and only if $r_R(a)$ is an ideal of R where $r_R(a) = \{b \in R \mid ab = 0\}$ [15, Lemma 1.2]; (ii) every reduced ring is symmetric [15, Lemma 1.1] (but the converse does not hold [1, Example II.5]); and (iii) any symmetric ring is semicommutative but the converse does not

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hold ([15, Proposition 1.4 and Example 5.4(a)]. Semicommutative rings were also studied under the name *zero insertive* by Habeb [5].

Another generalization of a reduced ring is an Armendariz ring. Rege and Chhawchharia [14] called a ring R *Armendariz* if whenever any polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i and j. The Armendariz property of a ring was extended to one of the skew polynomial ring in [7]. For an endomorphism α of a ring R, a *skew polynomial* ring (also called an *Ore extension of endomorphism type*) $R[x;\alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$, while $R[[x;\alpha]]$ is called a *skew power series ring*. A ring R is called *α-skew Armendariz* [7, Definition] if for $p = a_0 + a_1x + \cdots + a_mx^m$ and $q = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\alpha]$, $pq = 0$ implies $a_i\alpha^i(b_j) = 0$ for all $0 \le i \le m$ and $0 \le j \le n$. Recall that an endomorphism α of a ring R is called *rigid* [12] if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. If R is an α-rigid ring then for $p = \sum_{i=0}^{\infty} a_ix^i$ and $q = \sum_{j=0}^{\infty} b_jx^j$ in $R[[x;\alpha]]$, $pq = 0$ if and only if $a_ib_j = 0$ for all $0 \le i$, $0 \le j$ [6, Proposition 17]; and R is an α-rigid ring if and only if a skew power series ring $R[[x;\alpha]]$ of R is reduced and α is monomorphism [6, Corollary 18].

Motivated by the above, we introduce the notion of an α-semicommutative ring with the endomorphism α (see Definition 2.1 in Section 2), as both a generalization of α-rigid rings and an extension of semicommutative rings, and study characterizations of α-semicommutative rings and their related properties. And then for some condition with respect to α, say *α-sps Armendariz property*, in a skew power series ring $R[[x;\alpha]]$ of R which is an extension of the Armendariz property of a ring R, the relationship between R and $R[[x;\alpha]]$ is studied, and the existence of strong connections among such rings and their various properties are also investigated. Moreover, we show that a number of interesting properties of a ring R satisfying *α-sps* Armendariz property transfer to its the skew power series ring $R[[x;\alpha]]$ and vice-versa such as the Baer property and the p.p.-property. Several known results relating to α-rigid rings can be obtained as corollaries of our results.

2. α-semicommutative rings and related rings

Our focus in this section is to introduce the concept of an α-semicommutative ring and study its properties. Observe that the notion of α-semicommutative rings not only generalizes that of α-rigid rings, but also extends that of semicommutative rings. We also investigate connections to other related conditions. Examples to illustrate the concepts and results are included. We start with the following definition.

Definition 2.1. An endomorphism α of a ring R is called *semicommutative* if whenever $ab = 0$ for $a, b \in R$, $aR\alpha(b) = 0$. A ring R is called *α-semicommutative* if there exists a semicommutative endomorphism α of R.

It is clear that a ring R is semicommutative if R is I_R-semicommutative, where I_R is the identity endomorphism of R. It is easy to see that every subring S with $\alpha(S) \subseteq S$ of an α-semicommutative ring is also α-semicommutative.

Remark 2.2. Let R be an α-semicommutative ring with $ab = 0$ for $a, b \in R$. Then $aR\alpha(b) = 0$ and, in particular, $a\alpha(b) = 0$. Since R is α-semicommutative, we get $aR\alpha^2(b) = 0$. So, by induction hypothesis, we obtain $aR\alpha^k(b) = 0$ and $a\alpha^k(b) = 0$ for any positive integer k.

Notice that in general the reverse implication in the above definition does not hold by the following example which also shows that there exists an endomorphism α of a semicommutative ring R such that R is not α-semicommutative.

Example 2.3. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and consider a ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Then R is semicommutative, since R is commutative reduced. Now, let $\alpha : R \to R$ be defined by $\alpha((a, b)) = (b, a)$. Then α is an automorphism of R. For $a = (1, 0) = b \in R$, $aR\alpha(b) = 0$ but $ab = (1, 0) \neq 0$. Moreover, R is not α-semicommutative: In fact, for $(1, 0), (0, 1) \in R$, $(1, 0)(0, 1) = (0, 0)$ but $(0, 0) \neq (1, 0)(1, 1)\alpha((0, 1)) \in (1, 0)R\alpha((0, 1))$.

Theorem 2.4. *A ring R is α-rigid if and only if R is a reduced α-semicommutative ring and α is a monomorphism.*

Proof. Let R be an α-rigid ring. Then R is reduced and α is a monomorphism by [6, p.218]. Assume that $ab = 0$ for $a, b \in R$. Let r be an arbitrary element of R. Then $ba = 0$ and $ar\alpha(b)\alpha(ar\alpha(b)) = ar\alpha(ba)\alpha(r)\alpha^2(b) = 0$. Since R is α-rigid, $ar\alpha(b) = 0$ and so $aR\alpha(b) = 0$. Thus R is α-semicommutative.

Conversely, assume that $a\alpha(a) = 0$ for $a \in R$. Since R is reduced and α-semicommutative, $\alpha(a)a = 0$ and so $\alpha(a)R\alpha(a) = 0$. Hence $\alpha(a^2) = 0$ and so $a = 0$, since α is a monomorphism of a reduced ring R. Therefore R is α-rigid. □

The following example shows that the conditions "R is a reduced ring" and "α is a monomorphism" in Theorem 2.4 cannot be dropped respectively.

Example 2.5. (1) Let $\mathbb{Z}$ be the ring of integers. Consider a ring $R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in \mathbb{Z}\}$. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha\left(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -b \\ 0 & a \end{smallmatrix}\right)$. Note that α is an automorphism. Clearly, R is not reduced and hence R is not α-rigid. Let $AB = O$ for $A = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right), B = \left(\begin{smallmatrix} c & d \\ 0 & c \end{smallmatrix}\right) \in R$. Then $ac = 0$ and $ad + bc = 0$. For an arbitrary $\left(\begin{smallmatrix} h & k \\ 0 & h \end{smallmatrix}\right) \in R, \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} h & k \\ 0 & h \end{smallmatrix}\right)\alpha\left(\left(\begin{smallmatrix} c & d \\ 0 & c \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} ahc & -ahd+akc+bhc \\ 0 & ahc \end{smallmatrix}\right)$. Since $ac = 0$, $a = 0$ or $c = 0$. If $a = 0$ then $bc = 0$. So $AR\alpha(B) = O$. If $c = 0$ then $ad = 0$. Again $AR\alpha(B) = O$. Thus R is an α-semicommutative ring.

(2) Let F be a field and $R = F[x]$ the polynomial ring over F. Define $\alpha : R \to R$ by $\alpha(f(x)) = f(0)$ where $f(x) \in R$. Then R is a commutative domain (and so reduced) and α is not a monomorphism. If $f(x)g(x) = 0$ for

$f(x), g(x) \in R$ then $f(x) = 0$ or $g(x) = 0$, and so $f(x) = 0$ or $\alpha(g(x)) = 0$. Hence $f(x)R\,\alpha(g(x)) = 0$, and thus R is α-semicommutative. Note that R is not α-rigid, since $x\alpha(x) = 0$ for $0 \neq x \in R$.

Observe that if R is a domain then R is both semicommutative and α-semicommutative for any endomorphism α of R. Example 2.5(1) also shows that there exists an α-semicommutative ring R which is not a domain.

Proposition 2.6. *Let R be an α-semicommutative ring. Then*

(1) $\alpha(1) = 1$ *where* 1 *is the identity of* R *if and only if* $\alpha(e) = e$ *for any* $e^2 = e \in R$.

(2) *If* $\alpha(1) = 1$, *then* R *is abelian (i.e., all its idempotents are central).*

Proof. (1) Suppose that $\alpha(1) = 1$. If $e^2 = e \in R$, then we get $e(1-e) = 0$ and $(1-e)e = 0$. Then $eR\alpha(1-e) = 0$ and $(1-e)R\alpha(e) = 0$ because R is α-semicommutative. Hence $e\alpha(1-e) = 0$ implies $e(1-\alpha(e)) = 0$ and so $e\alpha(e) = e$. From $(1-e)\alpha(e) = 0$, we get $\alpha(e) = e\alpha(e)$ and so $\alpha(e) = e$, since $e\alpha(e) = e$. The converse is obvious.

(2) Assume that $\alpha(1) = 1$. Let e be an arbitrary idempotent in R. By the same method as in (1), we get $eR\alpha(1-e) = 0$ and $(1-e)R\alpha(e) = 0$, and so $eR(1-e) = 0$ and $(1-e)Re = 0$ by (1). Hence $er(1-e) = 0$ and $(1-e)re = 0$ for all $r \in R$. So $er = ere = re$. Therefore R is an abelian ring. □

The concepts of α-semicommutative rings and abelian rings are independent on each other by Example 2.3 and Example 2.7 which also shows that the condition "$\alpha(1) = 1$" in Proposition 2.6(2) is not superfluous.

Example 2.7. Let $\mathbb{Z}$ be the ring of integers. Consider a ring $R = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in \mathbb{Z}\}$. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha\left(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. For $A = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right), B = \left(\begin{smallmatrix} a' & b' \\ 0 & c' \end{smallmatrix}\right) \in R$, if $AB = O$ then we obtain $aa' = 0$, and so $a = 0$ or $a' = 0$. This implies $AR\alpha(B) = O$, and thus R is an α-semicommutative ring. Note that $\alpha(1) \neq 1$ and R is not abelian.

Corollary 2.8. *Semicommutative rings are abelian.*

Given a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

For an endomorphism α of a ring R and the trivial extension $T(R, R)$ of R, $\bar{\alpha} : T(R, R) \to T(R, R)$ defined by $\bar{\alpha}\left(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix}$ is an endomorphism of $T(R, R)$. Since $T(R, 0)$ is isomorphic to R, we can identify the restriction of $\bar{\alpha}$ by $T(R, 0)$ to α.

Notice that the trivial extension of a semicommutative ring is not semicommutative by [8, Example 11]. Now, we may ask whether the trivial extension

$T(R,R)$ is $\bar{\alpha}$-semicommutative if R is α-semicommutative. But the following example erases the possibility, in general.

Example 2.9. Consider an α-semicommutative ring $R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in \mathbb{Z}\}$ with an endomorphism α defined by $\alpha\left(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -b \\ 0 & a \end{smallmatrix}\right)$ in Example 2.5(1). For

$$A = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix}, \quad B = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} \in T(R,R),$$

we have $AB = O$. However, for $C = \begin{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \end{pmatrix} \in T(R,R)$, we obtain

$$O \neq \begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} = AC\bar{\alpha}(B) \in AT(R,R)\bar{\alpha}(B),$$

and therefore $T(R,R)$ is not $\bar{\alpha}$-semicommutative.

However we have the following:

Proposition 2.10. *Let R be a reduced ring. If R is an α-semicommutative ring, then $T(R,R)$ is an $\bar{\alpha}$-semicommutative ring.*

Proof. We freely use the condition that R is reduced α-semicommutative and the fact that reduced rings are semicommutative. Note that R is a reduced ring if and only if for any $a, b \in R$, $ab^2 = 0$ implies $ab = 0$. Let $AB = O$ for $A = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right), B = \left(\begin{smallmatrix} c & d \\ 0 & c \end{smallmatrix}\right) \in T(R,R)$. Then $ac = 0$ and $ad + bc = 0$. So $0 = ad + bc = (ad + bc)c = bc^2$ implies $bc = 0$, and so $ad = 0$. Then $aR\alpha(c) = 0, bR\alpha(c) = 0$ and $aR\alpha(d) = 0$. Thus for any $C = \left(\begin{smallmatrix} h & k \\ 0 & h \end{smallmatrix}\right) \in T(R,R)$, $AC\bar{\alpha}(B) = \left(\begin{smallmatrix} ah\alpha(c) & ah\alpha(d)+ak\alpha(c)+bh\alpha(c) \\ 0 & ah\alpha(c) \end{smallmatrix}\right) = O$. Hence $AT(R,R)\bar{\alpha}(B) = O$, and thus $T(R,R)$ is $\bar{\alpha}$-semicommutative. □

Corollary 2.11. *If R is an α-rigid ring, then $T(R,R)$ is an $\bar{\alpha}$-semicommutative ring.*

Proof. It follows from Theorem 2.4 and Proposition 2.10. □

The trivial extension $T(R,R)$ of a ring R is extended to

$$S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a, b, c, d \in R \right\}$$

and an endomorphism α of a ring R is also extended to the endomorphism $\bar{\alpha}$ of $S_3(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. There exists a reduced ring R such that $S_3(R)$ is not $\bar{\alpha}$-semicommutative by the following example.

Example 2.12. We consider the commutative reduced ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the automorphism α of R defined by $\alpha((a,b)) = (b,a)$, in Example 2.3. Then $S_3(R)$ is not $\bar{\alpha}$-semicommutative. For, let

$$A = \begin{pmatrix} (1,0) & (0,0) & (0,0) \\ (0,0) & (1,0) & (0,0) \\ (0,0) & (0,0) & (1,0) \end{pmatrix} \text{ and } B = \begin{pmatrix} (0,1) & (0,0) & (0,0) \\ (0,0) & (0,1) & (0,0) \\ (0,0) & (0,0) & (0,1) \end{pmatrix} \in S_3(R).$$

Then $AB = O$, but $AA\bar{\alpha}(B) = A \neq O$. Thus $AS_3(R)\bar{\alpha}(B) \neq O$, and therefore $S_3(R)$ is not $\bar{\alpha}$-semicommutative.

However, we obtain that $S_3(R)$ is $\bar{\alpha}$-semicommutative for a reduced α-semicommutative ring R by the similar method to the proof of Proposition 2.10 as follows:

Proposition 2.13. *Let R be a reduced ring. If R is an α-semicommutative ring, then*

$$S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a, b, c, d \in R \right\}$$

is an $\bar{\alpha}$-semicommutative ring.

Proof. Let $AB = O$ for $A = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}, B = \begin{pmatrix} a' & b' & c' \\ 0 & a' & d' \\ 0 & 0 & a' \end{pmatrix} \in S_3(R)$. Then we have the following equations:

(1) $$aa' = 0,$$

(2) $$ab' + ba' = 0,$$

(3) $$ac' + bd' + ca' = 0,$$

(4) $$ad' + da' = 0.$$

From Eq.(1), we get $aR\alpha(a') = 0$. In Eq.(2), $0 = (ab' + ba')a' = ba'^2$, and so $ba' = 0$ and $ab' = 0$. Similarly, from Eq.(4), we have $da' = 0$ and $ad' = 0$. Also, in Eq.(3), $0 = (ac' + bd' + ca')a' = ca'^2$ implies $ca' = 0$ and $ac' + bd' = 0$. Then $0 = a(ac' + bd') = a^2c'$, and so $ac' = 0$ and $bd' = 0$. Hence, these yield that $aR\alpha(a') = 0, aR\alpha(b') = 0, bR\alpha(a') = 0, aR\alpha(c') = 0, aR\alpha(d') = 0, bR\alpha(d') = 0, cR\alpha(a') = 0$, and $dR\alpha(a') = 0$. Thus $AS_3(R)\bar{\alpha}(B) = O$, and therefore $S_3(R)$ is an $\bar{\alpha}$-semicommutative ring. □

Corollary 2.14 ([11, Proposition 1.2]). *Let R be a reduced ring. Then $S_3(R)$ is a semicommutative ring.*

For an α-rigid ring R and $n \geq 2$, let

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \;\middle|\; a, a_{ij} \in R \right\}.$$

From Proposition 2.13, we may suspect that $S_n(R)$ may be $\bar{\alpha}$-semicommutative for $n \geq 4$. But the possibility is eliminated by the next example.

Example 2.15. Let R be an α-rigid ring and

$$S_4(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \middle| \, a, a_{ij} \in R \right\}.$$

Note that if R is an α-rigid ring, then $\alpha(e) = e$ for $e^2 = e \in R$ by [6, Proposition 5]. In particular $\alpha(1) = 1$. For

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in S_4(R),$$

we obtain $AB = O$. But we have

$$O \neq \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = AC\bar{\alpha}(B) \in S_4(R) \text{ for } C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in S_4(R).$$

Thus $AS_4(R)\bar{\alpha}(B) \neq O$ and so $S_4(R)$ is not $\bar{\alpha}$-semicommutative. Similarly, it can be proved that $S_n(R)$ is not $\bar{\alpha}$-semicommutative for $n \geq 5$.

Observe that let R_i be a ring and α_i an endomorphism of R_i for each $i \in \Gamma$. Then, for the product $\Pi_{i \in \Gamma} R_i$ of R_i and the endomorphism $\bar{\alpha} : \Pi_{i \in \Gamma} R_i \to \Pi_{i \in \Gamma} R_i$ defined by $\bar{\alpha}((a_i)) = (\alpha_i(a_i))$, $\Pi_{i \in \Gamma} R_i$ is $\bar{\alpha}$-semicommutative if and only if each R_i is α_i-semicommutative.

3. Related topics

Following [14], a ring R is called *Armendariz* if whenever two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$ we have $a_i b_j = 0$ for every i and j. We extend the Armendariz property of a ring R to the skew power series ring $R[[x;\alpha]]$ of R. In [6, Proposition 17 and Corollary 18], if R is an α-rigid ring, then for $p = \sum_{i=0}^{\infty} a_i x^i$ and $q = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x;\alpha]]$, $pq = 0$ if and only if $a_i b_j = 0$ for all $0 \leq i$, $0 \leq j$; and the skew power series ring $R[[x;\alpha]]$ is reduced.

Hence, we define the following:

Definition 3.1. Let α be an endomorphism of a ring R. A ring R is called a *skew power series Armendariz ring with the endomorphism* α (simply, an *α-sps Armendariz ring*) if whenever $pq = 0$ for $p = \sum_{i=0}^{\infty} a_i x^i$, $q = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$, then $a_i b_j = 0$ for all i and j.

It can be easily checked that if R is an α-rigid ring then R is α-*sps* Armendariz by [6, Proposition 17], and that every subring S with $\alpha(S) \subseteq S$ of an α-*sps* Armendariz ring is also α-*sps* Armendariz. We remark that in general the reverse implication in the above definition does not hold by the following example.

Example 3.2. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the ring of integers modulo 2, and $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$ as in Example 2.3. Then for $p = (1, 0)x$ and $q = (0, 1)$ in $R[[x; \alpha]]$, we have $(1, 0)(0, 1) = (0, 0) \in R$, but $pq \neq 0$. Moreover, R is not α-*sps* Armendariz: In fact, for $p = (1, 0) + (1, 0)x$ and $q = (0, 1) + (1, 0)x$ in $R[[x; \alpha]]$, $pq = 0$ but $(1, 0)(1, 0) \neq (0, 0) \in R$.

Theorem 3.3. *Let R be a ring. Then we have the following.*

(1) *R is an α-rigid ring if and only if R is a reduced α-sps Armendariz ring.*

(2) *If $R[[x; \alpha]]$ is a semicommutative ring, then R is an α-semicommutative ring.*

(3) *Let R be an α-sps Armendariz ring. Then*

(i) *if R is α-semicommutative, then $R[[x; \alpha]]$ is semicommutative;*

(ii) *if $ab = 0$ for $a, b \in R$, then $\alpha^n(a)b = 0$ for any positive integer n;*

(iii) *if $a\alpha^m(b) = 0$ for $a, b \in R$ and some positive integer m, then $ab = 0$.*

Proof. (1) It is enough to show that R is α-rigid when R is a reduced α-*sps* Armendariz ring. Assume $a\alpha(a) = 0$ for $a \in R$. Then for $p = ax$ and $q = a$ in $R[[x; \alpha]]$, $pq = axa = a\alpha(a)x = 0$. Since R is α-*sps* Armendariz, $a^2 = 0$. Thus $a = 0$ because R is reduced. Therefore R is an α-rigid ring.

(2) Assume that $R[[x; \alpha]]$ is a semicommutative ring. Let $ab = 0$ for $a, b \in R$. Then $aR[[x; \alpha]]b = 0$. Thus $arxb = 0$ for any $r \in R$. Hence $ar\alpha(b)x = 0$ and so $aR\alpha(b) = 0$. Therefore R is α-semicommutative.

(3) Let R be an α-*sps* Armendariz ring. (i) Assume that R is α-semicommutative. First we show that R is semicommutative. Let $ab = 0$, then $aR\alpha(b) = 0$ since R is α-semicommutative. Let $f = arx$ and $g = b \in R[[x; \alpha]]$ for any $r \in R$. Then $fg = arxb = ar\alpha(b)x = 0$ since $aR\alpha(b) = 0$, and so $arb = 0$ since R is α-*sps* Armendariz. Therefore $aRb = 0$, and thus R is semicommutative. Now, let $pq = 0$ for $p = \sum_{i=0}^{\infty} a_i x^i$, $q = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \alpha]]$. Then $a_i b_j = 0$ for all i, j, since R is α-*sps* Armendariz. Hence $a_i R\alpha(b_j) = 0$ and so $a_i R\alpha^k(b_j) = 0$ for all i, j and positive integer k because R is α-semicommutative. This implies that for $c_k x^k \in R[[x; \alpha]]$, $p(c_k x^k)q = (\sum_{i=0}^{\infty} a_i x^i)c_k x^k(\sum_{j=0}^{\infty} b_j x^j) = a_0 c_k \alpha^k(b_0)x^k + (a_0 c_k \alpha^k(b_1) + a_1\ \alpha(c_k)\alpha^{k+1}(b_0))x^{k+1} + \cdots = 0$, since $a_i R\alpha^k(b_j) = 0$ for all i, j and positive integer k. By this fact and R is semicommutative, we get $phq = 0$ for all $h \in R[[x; \alpha]]$. Therefore $pR[[x; \alpha]]q = 0$, and so $R[[x; \alpha]]$ is a semicommutative ring. (ii) Suppose that $ab = 0$ for $a, b \in R$. It is enough to show that $\alpha(a)b = 0$. Let $p = \alpha(a)x$ and $q = bx$ in $R[[x; \alpha]]$. Then $pq = \alpha(a)\alpha(b)x^2 = \alpha(ab)x^2 = 0$. Since R is α-*sps* Armendariz, $\alpha(a)b = 0$. (iii) Suppose that $a\alpha^m(b) = 0$ for some positive integer m. Let $p = ax^m$ and $q = bx$ in $R[[x; \alpha]]$. Then $pq = a\alpha^m(b)x^{m+1} = 0$, and thus $ab = 0$ since R is α-*sps* Armendariz. □

It can be easily checked that if R is an α-*sps* Armendariz ring, then α is always a monomorphism and $a\alpha(b) = 0$ for $a, b \in R$ implies $\alpha(a)b = 0$ by Theorem 3.3(3)(ii) and (iii). Moreover, we have the following.

Corollary 3.4. *If R is an α-sps Armendariz ring, then $\alpha(1) = 1$, where 1 is the identity of R. In this case, $\alpha(e) = e$ for any $e^2 = e \in R$.*

Proof. $(1 - \alpha(1))\alpha(1) = 0$ implies $\alpha(1 - \alpha(1))\alpha(1) = 0$ by Theorem 3.3(3)(ii). Hence we have $(\alpha(1) - \alpha(\alpha(1)))\alpha(1) = 0$, and so $\alpha(1) = \alpha(\alpha(1))$. Since α is a monomorphism, $\alpha(1) = 1$.

Now, let $e^2 = e \in R$. Then $e(1-e) = 0$ implies $\alpha(e)(1-e) = 0$ by Theorem 3.3(3)(ii). Hence $\alpha(e) = \alpha(e)e$. Similarly, $(1 - e)e = 0$ implies $e = \alpha(e)e$. Consequently, $\alpha(e) = e$. □

Lemma 3.5. *Let R be an α-sps Armendariz ring. Then the set of all idempotents in $R[[x;\alpha]]$ coincides with the set of all idempotents of R and $R[[x;\alpha]]$ is abelian.*

Proof. Let $e^2 = e \in R[[x;\alpha]]$, where $e = \sum_{i=0}^{\infty} e_i x^i$. Since $e(1-e) = 0 = (1-e)e$, we have $(e_0 + e_1x + \cdots + e_nx^n + \cdots)((1 - e_0) - e_1x - \cdots - e_nx^n - \cdots) = 0$ and $((1 - e_0) - e_1x - \cdots - e_nx^n - \cdots)(e_0 + e_1x + \cdots + e_nx^n + \cdots) = 0$. Since R is an α-*sps* Armendariz ring, $e_0(1 - e_0) = 0$, $e_0e_i = 0$ and $(1 - e_0)e_i = 0$ for $1 \leq i$. Thus $e_i = 0$ for $1 \leq i$, and so $e = e_0 = e_0^2$.

Now, we claim that R is abelian. Note that $\alpha(e) = e$ for any $e^2 = e \in R$ by Corollary 3.4. We adapt the method in the proof of [7, Proposition 20]. For idempotents e and $e' \in R$, $ee'R\bigcap(1 - e')(1 - e)\alpha(R) = 0$. Suppose that $0 \neq ee'(-t) = (1-e')(1-e)\alpha(s) \in ee'R\bigcap(1-e')(1-e)\alpha(R)$ for some $s, t \in R$. Then $((1 - e')x + e)(e'tx + (1 - e)s) = (1 - e')\alpha(e't)x^2 + (ee't + (1 - e')\alpha((1 - e)s)x + e(1 - e)s = (1-e')e'\alpha(t)x^2 + (ee't + (1-e')(1-e)\alpha(s))x + e(1-e)s = 0$, since $\alpha(e') = e'$ and $\alpha(1 - e) = 1 - e$. But $ee't \neq 0$; which is a contradiction since R is α-*sps* Armendariz. Furthermore, suppose that $e'e = 0$. Then $ee' = (1 - e')(1 - e)(-e') = (1 - e')(1 - e)(-\alpha(e')) \in ee'R\bigcap(1 - e')(1 - e)\alpha(R) = 0$. Thus, for any idempotent $e \in R$ and any $r \in R$, $e'' = e + er(1 - e)$ is an idempotent in R with $(1 - e)e'' = 0$. Hence $e''(1 - e) = 0$ and so $er(1 - e) = 0$, i.e., $er = ere$. Similarly, $e''' = (1 - e) + (1 - e)re$ is an idempotent in R with $ee''' = 0$. Thus $e'''e = 0$ and so $(1 - e)re = 0$, i.e., $re = ere$. Hence, $er = re$ for any $r \in R$ and so R is abelian. Therefore $R[[x;\alpha]]$ is abelian. □

Now we turn our attention to the relationship between the Baerness and p.p.-property of a ring R and these of the skew power series ring $R[[x;\alpha]]$ in case R is α-*sps* Armendariz.

In [10], Kaplansky introduced the concept of *Baer* rings as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [3], a ring R is called *quasi-Baer* if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. It is well-known that these two concepts are left-right symmetric.

A ring R is called a *right* (resp., *left*) *p.p.*-ring if the right (resp., left) annihilator of an element of R is generated by an idempotent. R is called a *p.p.*-ring if it is both a right and left p.p.-ring.

On the other hand, Birkenmeier, Kim and Park [2] called R a *right* (resp., *left*) *principally quasi-Baer* (or simply *right* (resp., *left*) *p.q.-Baer*) ring if the right (resp., left) annihilator of a principal right (resp., left) ideal of R is generated by an idempotent. R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer. The class of p.q.-Baer rings has been extensively investigated by them [2]. This class includes all biregular rings, all (quasi-) Baer rings and all abelian p.p.-rings. Various extensions of Baer, quasi-Baer, p.q.-Baer and p.p.-rings have been studied by many authors [2, 6, 7, 8].

For a nonempty subset A of a ring R, we write $r_R(A) = \{c \in R \mid dc = 0$ for any $d \in A\}$ which is called the right annihilator of X in R.

Theorem 3.6. *Let R be an α-sps Armendariz ring.*

(1) *R is a Baer* (resp., *quasi-Baer*) *ring if and only if $R[[x;\alpha]]$ is a Baer* (resp., *quasi-Baer*) *ring.*

(2) *If $R[[x;\alpha]]$ is a right p.p.-ring, then R is a right p.p.-ring.*

Proof. (1) Assume that R is Baer. Let A be a nonempty subset of $R[[x;\alpha]]$ and A^* be the set of all coefficients of elements of A. Then A^* is a nonempty subset of R and so $r_R(A^*) = eR$ for some idempotent $e \in R$. Since $e \in r_{R[[x;\alpha]]}(A)$ by Corollary 3.4, we get $eR[[x;\alpha]] \subseteq r_{R[[x;\alpha]]}(A)$. Now, we let $0 \neq q = b_0 + b_1x + \cdots + b_tx^t + \cdots \in r_{R[[x;\alpha]]}(A)$. Then $Aq = 0$ and hence $pq = 0$ for any $p \in A$. Since R is α-*sps* Armendariz, $b_0, b_1, \ldots, b_t, \ldots \in r_R(A^*) = eR$. Hence there exists $c_0, c_1, \ldots, c_t, \ldots \in R$ such that $q = ec_0 + ec_1x + \cdots + ec_tx^t + \cdots = e(c_0 + c_1x + \ldots + c_tx^t + \cdots) \in eR[[x;\alpha]]$. Consequently $eR[[x;\alpha]] = r_{R[[x;\alpha]]}(A)$, and therefore $R[[x;\alpha]]$ is Baer.

Conversely, assume that $R[[x;\alpha]]$ is Baer. Let B be a nonempty subset of R. Then $r_{R[[x;\alpha]]}(B) = eR[[x;\alpha]]$ for some idempotent $e \in R$ by Lemma 3.5. Thus $r_R(B) = r_{R[[x;\alpha]]}(B) \cap R = eR[[x;\alpha]] \cap R = eR$, and therefore R is Baer.

The proof for the case of the quasi-Baer property follows in a similar fashion: In fact, for any right ideal A of $R[[x;\alpha]]$, take A^* as the right ideal generated by all coefficients of elements of A.

(2) Assume that $R[[x;\alpha]]$ is a right p.p.-ring. Let $a \in R$, then there exists an idempotent $e \in R$ such that $r_{R[[x;\alpha]]}(a) = eR[[x;\alpha]]$ by Lemma 3.5. Hence $r_R(a) = eR$, and therefore R is a right p.p.-ring. □

As a consequence we obtain:

Corollary 3.7 ([6, Theorem 21]). *Let R be an α-rigid ring. Then R is a Baer ring if and only if $R[[x;\alpha]]$ is a Baer ring.*

There exists an α-*sps* Armendariz and right p.q.-Baer ring R such that $R[[x;\alpha]]$ is not right p.q.-Baer by the next example.

Example 3.8 ([6, p.225]). For a field F, let

$$R = \left\{ (a_n) \in \prod_{n=1}^{\infty} F_n \mid a_n \text{ is eventually constant} \right\},$$

which is the subring of $\prod_{n=1}^{\infty} F_n$, where $F_n = F$ for $n = 1, 2, 3 \ldots$. Then R is right p.q.-Baer and I_R-rigid (and so I_R-*sps* Armendariz), where I_R is the identity endomorphism of R. But $R[[x; I_R]]$ is not right p.q.-Baer. Furthermore, $R[[x; I_R]]$ is neither right p.p. nor left p.p..

Finally, we have the following result which can be compared with Lemma 3.5.

Proposition 3.9. *If R is an α-semicommutative ring with $\alpha(1) = 1$, then the set of all idempotents in $R[[x; \alpha]]$ coincides with the set of all idempotents of R and $R[[x; \alpha]]$ is abelian.*

Proof. Note that R is an abelian ring with $\alpha(e) = e$ for any $e^2 = e \in R$ by Proposition 2.6. Let $p^2 = p \in R[[x; \alpha]]$, where $p = e_0 + e_1 x + e_2 x^2 + \cdots$. Since $\alpha(e) = e$ for any $e^2 = e \in R$, $p^2 = p$ implies the following system of equations:

$$\begin{aligned} &e_0^2 = e_0 \ ; \\ &e_0 e_1 + e_1 \alpha(e_0) = e_1 \ ; \\ &e_0 e_2 + e_1 \alpha(e_1) + e_2 \alpha^2(e_0) = e_2 \ ; \\ &\qquad \vdots \\ &e_0 e_k + e_1 \alpha(e_{k-1}) + \ldots + e_{k-1} \alpha^{k-1}(e_1) + e_k \alpha^k(e_0) = e_k \ ; \\ &\qquad \vdots \end{aligned}$$

From $e_0^2 = e_0$, we see that e_0 is an idempotent of R, so e_0 is central. Then we get the following:

$$e_0 e_1 + e_1 e_0 = e_1 \tag{5}$$

$$e_0 e_2 + e_1 \alpha(e_1) + e_2 e_0 = e_2 \tag{6}$$

$$\vdots$$

$$e_0 e_k + e_1 \alpha(e_{k-1}) + \ldots + e_{k-1} \alpha^{k-1}(e_1) + e_k e_0 = e_k \tag{7}$$

$$\vdots$$

From Eq.(5)$\times(1 - e_0)$, we obtain $e_1(1 - e_0) = 0$, and so $e_1 = e_1 e_0 = 0$. Hence Eq.(6) becomes $2e_0 e_2 = e_2$. Similarly, $2e_0 e_2(1 - e_0) = e_2(1 - e_0)$ implies $e_2 = 0$. Continuing this procedure yields $e_i = 0$ for $i \geq 1$. Consequently $p = e_0 = e_0^2 \in R$ and also $R[[x; \alpha]]$ is abelian by Proposition 2.6. □

Corollary 3.10. *If a ring R has one of the following such that $R[[x; \alpha]]$ is a right p.p.-ring;*

(1) *R is an α-sps-Armendariz ring,*

(2) *R is an α-semicommutative ring with $\alpha(1) = 1$.*

Then $R[[x; \alpha]]$ is semicommutative.

Proof. If R has one of the conditions, then R is abelian by Lemma 3.5 and Proposition 3.9, respectively. Note that abelian right p.p.-rings are semicommutative by [2, Proposition 1.14]. □

Observe that the following example shows that the converse of Theorem 3.3(2) does not hold; and the condition "R is α-*sps* Armendariz" in Theorem 3.3(3) and the condition "$\alpha(1) = 1$" in Proposition 3.9 cannot be dropped, respectively.

Example 3.11. Consider the α-semicommutative ring $R = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in \mathbb{Z}\}$ in Example 2.7, where $\alpha\left(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then $\alpha(1) \neq 1$, and so R is not α-*sps* Armendariz by Corollary 3.4. Moreover, $R[[x;\alpha]]$ is not semicommutative, either: In fact, For $p = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)x$ and $q = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $R[[x;\alpha]]$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in R[[x;\alpha]]$, $pq = 0$ but $p\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)q \neq 0$ and so $pR[[x;\alpha]]q \neq 0$. Therefore $R[[x;\alpha]]$ is not semicommutative. Note that let $f = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)x \in R[[x;\alpha]]$, then $f^2 = f \in R[[x;\alpha]]$, but $f \notin R$. Finally, for $A = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in R$, we get $AB = O$ but $\alpha^n(A)B \neq O$ for any positive integer n and $BC \neq O$ even if $B\alpha(C) = O$.

References

[1] D. D. Anderson and V. Camillo, *Semigroups and rings whose zero products commute*, Comm. Algebra **27** (1999), no. 6, 2847–2852.

[2] G. F. Birkenmeier, J. Y. Kim, and J. K. Park, *Principally quasi-Baer rings*, Comm. Algebra **29** (2001), no. 2, 639–660.

[3] W. E. Clark, *Twisted matrix units semigroup algebras*, Duck Math. J. **34** (1967), 417–423.

[4] P. M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999), no. 6, 641–648.

[5] J. M. Habeb, *A note on zero commutative and duo rings*, Math. J. Okayama Univ. **32** (1990), 73–76.

[6] C. Y. Hong, N. K. Kim, and T. K. Kwak, *Ore extensions of Baer and p.p.-rings*, J. Pure Appl. Algebra **151** (2000), no. 3, 215–226.

[7] ———, *On skew Armendariz rings*, Comm. Algebra **31** (2003), no. 1, 103–122.

[8] ———, *Extensions of generalized reduced rings*, Algebra Colloq. **12** (2005), no. 2, 229–240.

[9] C. Y. Hong, T. K. Kwak, and S. T. Rizvi, *Rigid ideals and radicals of Ore extensions*, Algebra Colloq. **12** (2005), no. 3, 399–412.

[10] I. Kaplansky, *Rings of Operators*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[11] N. K. Kim and Y. Lee, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 207–223.

[12] J. Krempa, *Some examples of reduced rings*, Algebra Colloq. **3** (1996), no. 4, 289–300.

[13] J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), 359–368.

[14] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17.

[15] G. Y. Shin, *Prime ideals and sheaf representation of a pseudo symmetric ring*, Trans. Amer. Math. Soc. **184** (1973), 43–60.

MUHITTIN BAŞER
DEPARTMENT OF MATHEMATICS
KOCATEPE UNIVERSITY
AFYONKARAHISAR 03200, TURKEY
E-mail address: mbaser@aku.edu.tr

Abdullah Harmanci
Department of Mathematics
Hacettepe University
Ankara, Turkey
E-mail address: harmanci@hacettepe.edu.tr

Tai Keun Kwak
Department of Mathematics
Daejin University
Pocheon 487–711, Korea
E-mail address: tkkwak@daejin.ac.kr