Continuous Dependence of Solutions to Fourth-Order Nonlinear Wave Equation

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Abstract

This paper gives a priori estimates and continuous dependence of the solutions to fourth-order nonlinear wave equation.

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1 Introduction

We consider the following initial boundary value problem

$$u_{tt} - \alpha\Delta u - \beta\Delta u_t - \gamma\Delta u_{tt} = f(u) \tag{1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x),\ x \in \Omega \tag{2}$$

$$u = 0,\ x \in \partial\Omega, t > 0, \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ is bounded region with smooth boundary $\partial\Omega$; α,β and γ are positive constants. $f(u)$ is a given nonlinear function which satisfies

$$f \in C^1(R), |f'(u)| \le c\left(1 + |u|^{p-1}\right), p \ge 1, (n-2)p \le n \tag{4}$$

and

$$\limsup_{u \to \infty} \frac{f(u)}{u} < \alpha\lambda_1 \tag{5}$$

where λ_1 is the first eigenvalue of the Laplace operator with the homogeneous Dirichlet boundary condition.

Continuous dependence of solutions on coefficients of equations is a type of structural stability, which reflects the effect of small changes in coefficients of equations on the solutions. Many results of this type can be found in [1].

In [2], authors studied asymptotic behaviour of solution to initial value problem of fourth order wave equation with dispersive and dissipative terms by taking coefficients $\alpha = \beta = \gamma = 1$ in (1). They proved that the global strong solution of the problem decays to zero exponentially as $t \to \infty$. The authors Guo-wang Chen and Chang-Shun Hou, in article [3], studied the following initial value problem for a class of fourth order nonlinear wave equations,

$$v_{tt} - a_1 v_{xx} - a_2 v_{xxt} - a_3 v_{xxtt} = f(v_x)_x \quad, x \in R, t > 0$$

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ x \in R$$

where a_1, a_2, a_3 are positive constants. They gave also the blow up results for this problem.

In [4], Shang studied the initial boundary value problem

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u), x \in \Omega, t > 0 \tag{1'}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x),\ x \in \Omega \tag{2'}$$

$$u = 0,\ x \in \partial\Omega, t > 0, \tag{3'}$$

Under the assumptions that $n = 1,2,3$; $f \in C^1$, $f'(u)$ is bounded above and satisfies (i) $|f'(u)| \le A|u|^p + B, 0 < p < \infty$ if $n = 2$; $0 < p \le \frac{2}{n-2}$ if $n = 3$; $u_i(x) \in$

$H^2(\Omega) \cap H_0^1(\Omega)\,(i = 0,1)$, it was proven that problem (1')-(3') admits unique global strong solution u such that $\forall T > 0, u \in W^{2,\infty}\left(0,T; H^2(\Omega) \cap H_0^1(\Omega)\right)$.

In [5], problem (1')-(3') were studied again for all $n \geq 1$. By supposing that $f \in C^1$ and $f'(u)$ is bounded above satisfying (ii)$|f'(u)| \leq A\,|u|^p + B, 0 < p < \infty$ if $n = 2$; $0 < p \leq \frac{4}{n-2}$ if $n \geq 3, u_i(x) \in H^2(\Omega) \cap H_0^1(\Omega)\,(i = 0,1)$, it was proven that problem (1')-(3') admits unique global strong solution u such that for all $T > 0, u \in W^{2,\infty}\left(0,T; H^2(\Omega) \cap H_0^1(\Omega)\right)$.

In [6], authors studied the spatial behavior of a coupled system of wave-plate type . They got the alternative results of Phragmen-Lindelof type in terms of an area measure of the amplitude in question based on a first-order differential inequality. They also got the spatial decay estimates based on a second-order differential inequality.

The aim of this paper is to prove the continuous dependence of solutions to the problem (1)-(3) on coefficients α, β and γ.

Throughout this paper , we use the notation $\|.\|_p$ for the norm in $L^P(\Omega)$. We use $\|.\|$ instead of $\|.\|_2$.

2 A Priori Estimates

In this section, we obtain a priori estimates for the problem (1)-(3).

Theorem 1 *Assume that the conditions (4) and (5) hold. Then for $u_0, u_1 \in H_0^1(\Omega)$ the solution u of problem (1)-(3) satisfies the following estimates:*

$$\|\nabla u\|^2 + \|\nabla u_t\|^2 \leq D_1 \tag{6}$$

and

$$\int_0^t \|\nabla u_{ss}\|^2\, ds \leq D_2 t \tag{7}$$

for any $t > 0$. Here $D_1 > 0$ and $D_2 > 0$ depend on initial data and the parameters of (1).

Proof. First, by taking the inner product of (1) by u_t in $L^2(\Omega)$ and integrating by parts, we get

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{\alpha}{2}\|\nabla u\|^2+\frac{\gamma}{2}\|\nabla u_t\|^2-\int\limits_{\Omega}F(u)dx\right]+\beta\|\nabla u_t\|^2=0 \tag{8}$$

and

$$E(t)\le E(0) \tag{9}$$

where $F(u)=\int\limits_0^u f(s)\,ds$ and $E(t)=\frac{1}{2}\|u_t\|^2+\frac{\alpha}{2}\|\nabla u\|^2+\frac{\gamma}{2}\|\nabla u_t\|^2-\int\limits_{\Omega}F(u)dx$. From (5) and definition of limsup we obtain

$$F(u)\le c+\frac{\alpha\lambda_1}{2}u^2-\frac{\varepsilon}{2}u^2 \tag{10}$$

Using (10) and Poincare's inequality from (9) we find (6).
Next we multiply (1) by u_{tt} in $L^2(\Omega)$ to get

$$\frac{d}{dt}\frac{\beta}{2}\|\nabla u_t\|^2+\gamma\|\nabla u_{tt}\|^2+\|u_{tt}\|^2+\alpha\int\limits_{\Omega}\nabla u\nabla u_{tt}dx=\int\limits_{\Omega}f(u)u_{tt}dx \tag{11}$$

Using Cauchy-Schwarz inequality, ε-Cauchy inequality and from (4), we take,

$$(\gamma\ -\frac{\varepsilon}{2})\|\nabla u_{tt}\|^2+\frac{d}{dt}\frac{\beta}{2}\|\nabla u_t\|^2\le c_2+\frac{|\alpha|^2}{2\varepsilon}\|\nabla u\|^2+\frac{c_1^2}{2}\int\limits_{\Omega}|u|^{2p}\,dx \tag{12}$$

where c_1, c_2 are constants and ε is sufficiently small and positive. Using Sobolev inequality and (6) we have

$$\int\limits_{\Omega}|u|^{2p}\,dx=\|u\|_{2p}^{2p}\le c_3\|\nabla u\|^{2p}\le c_4 \tag{13}$$

where c_3 is a Sobolev constant and $c_4=c_4(\alpha,\gamma,u_0,u_1)$. From (12) and (13) we obtain

$$(\gamma\ -\frac{\varepsilon}{2})\|\nabla u_{tt}\|^2+\frac{d}{dt}\frac{\beta}{2}\|\nabla u_t\|^2\le c_5 \tag{14}$$

where c_5 depends on initial data and the parameters of (1). Now, we integrate (14) from (0,t), then we obtain

$$\int\limits_0^t\|\nabla u_{ss}\|^2\,ds\le c_6 t \tag{15}$$

where c_6 depends on initial data and the parameters of (1). Hence, (7) follows from (15).

3 Continuous Dependence on the Coefficients

In this section, we prove that the solution of the problem (1)-(3) depends continuously on the coefficients α, β and γ in $H^1(\Omega)$.

We consider the problem

$$u_{tt} - \alpha_1 \Delta u - \beta_1 \Delta u_t - \gamma_1 \Delta u_{tt} = f(u) \tag{16}$$

$$u(x,0) = 0, u_t(x,0) = 0 \tag{17}$$

$$u|_{\partial\Omega} = 0 \tag{18}$$

and

$$v_{tt} - \alpha_2 \Delta v - \beta_2 \Delta v_t - \gamma_2 \Delta v_{tt} = f(v) \tag{19}$$

$$v(x,0) = 0, v_t(x,0) = 0 \tag{20}$$

$$v|_{\partial\Omega} = 0 \tag{21}$$

Let us define the difference variables w, α, β and γ by w=u-v, $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $\gamma = \gamma_1 - \gamma_2$ then w satisfy the following the initial boundary value problem:

$$w_{tt} - \alpha_1 \Delta w - \alpha \Delta v - \beta_1 \Delta w_t - \beta \Delta v_t - \gamma_1 \Delta w_{tt} - \gamma \Delta v_{tt} = f(u) - f(v) \tag{22}$$

$$w(x,0) = 0, w_t(x,0) = 0 \tag{23}$$

$$w|_{\partial\Omega} = 0 \tag{24}$$

The main result of this section is the following theorem.

Theorem 2 *Let w be the solution of the problem (22)-(24). If*

$$|f(u) - f(v)| \leq c_7 \left(1 + |u|^{p-1} + |v|^{p-1}\right) |u - v| \tag{25}$$

holds, then w satisfies the estimate

$$\|w_t\|^2 + \|\nabla w\|^2 + \|\nabla w_t\|^2 \leq e^{Mt} K \left[(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2 \right] t$$

where M and K are positive constants depending on initial data and the parameters of (1).

Proof. Let us take the inner product of (22) with w_t in $L^2(\Omega)$; we have

$$\frac{d}{dt}\left[\frac{1}{2}\|w_t\|^2 + \frac{\alpha_1}{2}\|\nabla w\|^2 + \frac{\gamma_1}{2}\|\nabla w_t\|^2\right] + \beta_1 \|\nabla w_t\|^2 +$$

$$\alpha \int_\Omega \nabla v \nabla w_t dx + \beta \int_\Omega \nabla v_t \nabla w_t dx + \gamma \int_\Omega \nabla v_{tt} \nabla w_t dx = \int_\Omega |f(u) - f(v)|\, w_t dx \quad (26)$$

From (26) we obtain

$$\frac{d}{dt}E_1(t) + \beta_1 \|\nabla w_t\|^2 \leq |\alpha| \|\nabla w_t\| \|\nabla v\| + |\beta| \|\nabla w_t\| \|\nabla v_t\| +$$

$$|\gamma| \|\nabla w_t\| \|\nabla v_{tt}\| + \int_\Omega |f(u) - f(v)|\, w_t dx \quad (27)$$

where $E_1(t) = \frac{1}{2}\|w_t\|^2 + \frac{\alpha_1}{2}\|\nabla w\|^2 + \frac{\gamma_1}{2}\|\nabla w_t\|^2$.
Using the Holder, Sobolev, Cauchy-Schwarz inequalities and (25) we obtain the estimate

$$\int_\Omega |f(u) - f(v)|\, w_t dx \leq c_7 \int_\Omega \left(1 + |u|^{p-1} + |v|^{p-1}\right) |w|\, w_t dx$$

$$\leq c_8 \left(1 + \|\nabla u\|^{p-1} + \|\nabla v\|^{p-1}\right) \|w\|_{\frac{2n}{n-2}} \|w_t\|$$

$$\leq C\left(\|\nabla w\|^2 + \|w_t\|^2\right) \quad (28)$$

where c_7, c_8 are constants and $C = C(c_7, c_8)$.Using Cauchy-Schwarz inequality and (28), from (27), we get

$$\frac{d}{dt}E_1(t) + (\beta_1 - \varepsilon)\|\nabla w_t\|^2 \leq \frac{3}{4\varepsilon}|\alpha|^2 \|\nabla v\|^2 + \frac{3}{4\varepsilon}|\beta|^2 \|\nabla v_t\|^2 +$$

$$\frac{3}{4\varepsilon}|\gamma|^2 \|\nabla v_{tt}\|^2 + c_9\left(\|\nabla w\|^2 + \|w_t\|^2\right) \quad (29)$$

and from (29) we can write

$$\frac{d}{dt}E_1(t) \leq \frac{3}{4\varepsilon}\left(|\alpha|^2\|\nabla v\|^2 + |\beta|^2\|\nabla v_t\|^2 + |\gamma|^2\|\nabla v_{tt}\|^2\right) + ME_1(t) \qquad (30)$$

where $M = \frac{2C(1+\alpha_1)}{\alpha_1}$. Applying Gronwall's inequality with (6) and (7), we get

$$E_1(t) \leq e^{Mt}K\left(|\alpha|^2 + |\beta|^2 + |\gamma|^2\right)t \qquad (31)$$

Hence proof is completed.

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