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Categories of rough sets and textures

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ABSTRACT

It is known that the theories of rough sets and fuzzy sets have successful applications in computing. Textures, as a theoretical model, provide a new perspective for both rough sets and fuzzy sets. Indeed, recent papers have shown that there is a natural link between rough sets and textures while a texture is an alternative point-set based setting for fuzzy sets. Relations are representatives of information systems and induce approximation operators. Therefore, the first step for the categorical discussions on rough sets involves the category **REL** of sets and relations. In this context, we observe that power sets and pairs of rough set approximation operators form a category denoted by **R-APR**. In particular, we prove that **R-APR** is isomorphic to a full subcategory of the category **cdrTex** whose objects are complemented textures and morphisms are complemented direlations. Therefore, **cdrTex** may be regarded as a suitable abstract model of rough set theory. Here, we show that **R-APR** and **cdrTex** are new examples of dagger symmetric monoidal categories.

0. Introduction

Rough set theory was introduced by the Polish mathematician, Z. Pawlak in the early 1980s as a new mathematical approach to deal with imprecision vagueness, and uncertainty in data analysis [28]. The starting point of the theory is a data set which consists of objects and attributes obtained from measurements and human experts. Formally, a data set is an information system with a universe U of objects and a set A of attributes related to objects of the universe. Any subset B of A determines an equivalence relation r on U, called an indiscernibility relation defined by $(x, y) \in r$ if and only if $a(x) = a(y)$ for every $a \in B$ where $a(x)$ denotes the value of attribute a for object x. Then we can approximate every subset $X \subseteq U$ using only the information contained in B in the following manner: if $[x]$ denotes an equivalence class of r containing x, then we may define two operators $\underline{apr}, \overline{apr} : \mathcal{P}(U) \to \mathcal{P}(U)$ as

$$\underline{apr}(X) = \{x \in U \mid [x] \subseteq X\} \text{ and } \overline{apr}(X) = \{x \in U \mid [x] \cap X \neq \emptyset\}$$

for all $X \subseteq U$, respectively. The pair $(\underline{apr}(X), \overline{apr}(X))$ is called *a rough set*. In rough set theory, equivalence relations can be replaced by ordinary relations (see e.g., [32–35]). This leads to very successful applications in machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, knowledge discovery, decision analysis and expert systems (for more information see [29]). Approximation operators are fundamental tools in rough set theory. Here we consider rough set models on two universes for arbitrary relations, and we show that the pairs of approximation operators and power sets form a category denoted by **R-APR**. In fact, **R-APR** is isomorphic to the category **REL** of sets and relations. The categorical discussions on rough sets are rare and recent studies on category theoretical approaches to rough set theory can be found in [3,4,17–19,23,24].

Recall that *a texturing* $\mathcal{U}$ is a family of subsets of a given universe U satisfying certain conditions related to the basic properties of the power set $\mathcal{P}(U)$. The pair $(U, \mathcal{U})$ is called *a texture space* or *a texture*, in brief [6]. The basic motivation

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for textures is to give an alternative point-set based setting for fuzzy lattices, that is, complete, completely distributive lattices with an order reversing involution [7]. Coincidentally, a texture is a T_0-topological space with completely distributive lattice of open (or closed) sets [11]. However, this side of textures is not the motivation for the alternative setting for fuzzy lattices and the morphisms are not the same as the ordinary functions between topological spaces. Since duality is an important tool of texture spaces, we notice that the morphisms between textures have two parts which are dual to each other. Namely, a direlation is a pair (r, R) where r (relation) and R (corelation) are the elements of a textural product satisfying certain conditions [8]. Presections with respect to direlations are natural generalizations of rough sets in that if (r, R) is a complemented direlation on a complemented texture $(U, \mathcal{U}, c_U)$, then the system $(U, \mathcal{U}, c_U, R^{\leftarrow}, r^{\leftarrow})$ defines an approximation space where $R^{\leftarrow}$ and $r^{\leftarrow}$ are the inverse corelation and inverse relation, respectively (see e.g., [12,27,28,33]). Here, we report that the complemented textures and complemented direlations form a category which is denoted by **cdrTex** and prove that the category **R-APR** is a full subcategory of **cdrTex**.

On the other hand, the concept of monoidal category goes back to works on monoid in abstract algebra. It is well-known that Abelian groups, vector spaces, more generally R-modules, or R-algebras constitute symmetric monoidal categories by means of ordinary tensor product (see e.g., [20,21,25,31]). Dagger (involutive) symmetric monoidal categories are also used in linear logic and quantum mechanics [1,2]. Here, we prove that **cdrTex** is a new example to a dagger symmetric monoidal category.

This paper is an extended and revised version of the conference paper [13] and it contains full proofs, more detailed remarks, and several further results.

For the benefit of the reader we give the necessary concepts and results related to textures and textural rough sets in Sections 1–4. For details, we refer to [5–8,11–16,26].

1. Textures

Let U be a set. Then $\mathcal{U} \subseteq \mathcal{P}(U)$ is called a *texturing* of U, and $(U, \mathcal{U})$ is called a *texture space*, or simply a *texture*, if

1. $(\mathcal{U}, \subseteq)$ is a complete lattice containing U and $\emptyset$, such that arbitrary meets coincide with intersections, and finite joins coincide with unions,
2. $\mathcal{U}$ is completely distributive, i.e., for all index set I, and for all $i \in I$, if J_i is an index set and if $A_i^j \in \mathcal{U}$, then we have

$$\bigcap_{i\in I} \bigvee_{j\in J_i} A_i^j = \bigvee_{\gamma\in\prod_i J_i} \bigcap_{i\in I} A^i_{\gamma(i)},$$

3. $\mathcal{U}$ separates the points of U, i.e., given $u_1 \neq u_2$ in U there exists $A \in \mathcal{U}$ such that $u_1 \in A,\ u_2 \notin A$, or $u_2 \in A,\ u_1 \notin A$.

Note that for any family $\{A_i \mid i \in I\} \subseteq \mathcal{U}$, we have

$$\bigvee_{i\in I} A_i = \bigcap \left\{ A \mid A \in \mathcal{U} \text{ and } \bigcup_{i\in I} A_i \subseteq A \right\}.$$

A mapping $c_U : \mathcal{U} \to \mathcal{U}$ is called *a complementation* on $(U, \mathcal{U})$ if it satisfies the conditions $c_U^2(A) = A$ for all $A \in \mathcal{U}$ and $A \subseteq B$ in $\mathcal{U}$ implies $c_U(B) \subseteq c_U(A)$. Then the triple $(U, \mathcal{U}, c_U)$ is said to be *a complemented texture space*.

For $u \in U$, the p-sets and q-sets are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\} \text{ and } Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\}.$$

A nonempty set $A \in \mathcal{U}$ is *a molecule* if $\forall B, C \in \mathcal{U}, A \subseteq B \cup C \Rightarrow A \subseteq B$ or $A \subseteq C$. Clearly, p-sets are molecules of a texture space. A texture space $(U, \mathcal{U})$ is called *simple* if all molecules of the space are p-sets. The p-sets and q-sets are important tools in the theory of texture spaces since complete distributivity can be written in terms of p-sets and the q-sets:

Theorem 1.1 (*[11]*)**.** *Let $(\mathcal{U}, \subseteq)$ be a complete lattice. The following statements are equivalent.*

(i) *$(U, \mathcal{U})$ is completely distributive.*
(ii) *For $A, B \in \mathcal{U}$, if $A \not\subseteq B$ then there exists $u \in U$ with $A \not\subseteq Q_u$ and $P_u \not\subseteq B$.*

Example 1.2 (*[8]*)**.** (i) The pair $(U, \mathcal{P}(U))$ is a texture space where $\mathcal{P}(U)$ is the power set of U. It is called *a discrete texture*. Clearly, $(U, \mathcal{P}(U))$ is simple and for $u \in U$ we have

$$P_u = \{u\} \text{ and } Q_u = U \setminus \{u\}$$

and $c_U : \mathcal{P}(U) \to \mathcal{P}(U)$ is the ordinary complementation on $(U, \mathcal{P}(U))$ defined by $c_U(A) = U \setminus A$ for all $A \in \mathcal{P}(U)$.
(ii) Let $M = (0, 1]$. The family $\mathcal{M} = \{(0, r] \mid r \in [0, 1]\}$ is a texture on M which is called the Hutton texture. Clearly, $\mathcal{M}$ is closed under arbitrary intersections. Then it is easy to see that it is a complete lattice with respect to set inclusion. It is also completely distributive. To see this, take $(0, r], (0, s] \in \mathcal{M}$ where $(0, r] \not\subseteq (0, s]$. Then we have $s < r$. Choose a point $t \in [0, 1]$ where $s < t < r$. Since we have $P_t = Q_t = (0, t]$, we may conclude that $(0, r] \not\subseteq Q_t$ and $P_t \not\subseteq (0, s]$. Therefore, by Theorem 1.1. $\mathcal{M}$ is completely distributive. Further, $\mathcal{M}$ is simple and the complementation $c_M : \mathcal{M} \to \mathcal{M}$ is defined by

$$\forall r \in (0, 1],\ c_M((0, r]) = (0, 1 - r].$$

Here, join is not always equal to union. For example, for the collection $\{(0, 1-\frac{1}{n}] \mid n \in N\} \subseteq \mathcal{M}$ we have

$$\bigcup_{n\in N}\left(0, 1-\frac{1}{n}\right] = (0,1) \text{ and } \bigvee_{n\in N}\left(0, 1-\frac{1}{n}\right] = (0,1].$$

(iii) Using a similar argument as in (ii), we may show that the pair $(I, \mathcal{I})$ where

$$I = [0,1] \text{ and } \mathcal{I} = \{[0,r) \mid r \in I\} \cup \{[0,r] \mid r \in I\}$$

is also a texture (*unit texture*). For $r \in I$, we have $P_r = [0,r]$, $Q_r = [0,r)$. Since Q_r is also a molecule, the texture is not simple. Further, the mapping $c_I : \mathcal{I} \to \mathcal{I}$ defined by

$$\forall r \in I,\ c_I([0,r]) = [0, 1-r),\ c_I([0,r)) = [0, 1-r]$$

is a complementation on $(I, \mathcal{I})$.

(iv) Let $U = \{a, b, c\}$. Then $\mathcal{U} = \{\emptyset, \{a\}, \{a,b\}, U\}$ is a texture on U. Clearly,

$$P_a = \{a\},\ P_b = \{a,b\},\ P_c = U \text{ and } Q_a = \emptyset,\ Q_b = \{a\},\ Q_c = \{a,b\}.$$

The mapping $c_U : \mathcal{U} \to \mathcal{U}$ defined by

$$c_U(\emptyset) = U,\ c_U(U) = \emptyset,\ c_U(\{a\}) = \{a,b\},\ c_U(\{a,b\}) = \{a\}$$

is a complementation on $(U, \mathcal{U})$. It is clearly simple.

2. Products of textures

Here, we discuss the product of any two texture spaces $(U, \mathcal{U})$ and $(V, \mathcal{V})$. For more information about the products of arbitrary families of textures we refer to [7]. Consider the family $\mathcal{A} = \{A \times V \mid A \in \mathcal{U}\} \bigcup \{U \times B \mid B \in \mathcal{V}\}$ and define

$$\mathcal{B} = \left\{\bigcup_{j\in J} E_j \mid \{E_j\}_{j\in J} \subseteq \mathcal{A}\right\}.$$

The family of arbitrary intersections of the elements of $\mathcal{B}$, that is, the lattice

$$\mathcal{U} \otimes \mathcal{V} = \left\{\bigcap_{i\in I} D_i \mid \{D_i\}_{i\in I} \subseteq \mathcal{B}\right\}$$

is a texture on $U \times V$. Clearly, for all $A \in \mathcal{U}$ and for all $B \in \mathcal{V}$, we have $A \times B \in \mathcal{U} \otimes \mathcal{V}$. Further, the p-sets and q-sets may be easily determined as

$$P_{(u,v)} = P_u \times P_v \text{ and } Q_{(u,v)} = (U \times Q_v) \cup (Q_u \times V).$$

If c_U and c_V are complementations on the textures $(U, \mathcal{U})$ and $(V, \mathcal{V})$, respectively, then for the complementation $c_{U\times V}$ on the product, it is enough to check that

$$c_{U\times V}(U \times B) = U \times c_V(B) \text{ and } c_{U\times V}(A \times V) = c_U(A) \times V$$

for all $A \in \mathcal{U}$ and $B \in \mathcal{V}$. In particular, if $\mathcal{P}(U)$ is a discrete texture on U, then for the textures $(U, \mathcal{P}(U)),\ (V, \mathcal{V})$, the p-sets and q-sets will be

$$\overline{P}_{(u,v)} = \{u\} \times P_v \text{ and } \overline{Q}_{(u,v)} = ((U \setminus \{u\}) \times V) \cup (U \times Q_v)$$

for the product texture $(U \times V,\ \mathcal{P}(U) \otimes \mathcal{V})$.

Now take the texture $(M, \mathcal{M}, c_M)$ in Example 1.2(ii). We determine the product texture $\mathcal{P}(U) \otimes \mathcal{M}$ on $U \times (0,1]$. It is easy to see that the sets $A \times (0,r]$ are the elements of the product texture for all $A \subseteq U$ and $r \in [0,1]$. Note that for $\mathcal{P}(U)$, we have $P_u = \{u\}$ and $Q_u = U \setminus \{u\}$ where $u \in U$. Further, we have $P_r = (0,r] = Q_r$ in $\mathcal{M}$. Therefore, the p-sets and q-sets of the product texture $\mathcal{P}(U) \otimes \mathcal{M}$ are

$$P_{(u,r)} = P_u \times P_r = \{u\} \times (0,r]$$

and

$$Q_{(u,r)} = (Q_u \times (0,1]) \cup (U \times Q_r) = (U \setminus \{u\} \times (0,1]) \cup (U \times (0,r]),$$

respectively. On the other hand, the complementations on $\mathcal{M}$ and $\mathcal{P}(U)$ are given by

$$\forall r \in (0,1],\ c_{(0,1]}(0,r] = (0, 1-r] \text{ and } \forall A \subseteq U,\ c_U(A) = U \setminus A,$$

For the complementation $c_{U\times M}$ on the product texture $\mathcal{P}(U) \otimes \mathcal{M}$, we have

$$c_{U\times M}((A \times (0,1]) \cup (U \times (0,r])) = (U \setminus A) \times (0, 1-r]$$

for every subset $A \subseteq U$ and $r \in (0,1]$.

3. Hutton textures

The basic motivation of textures is the correspondence between fuzzy lattices and simple textures [7]. Let $(L, \leq, ')$ be a fuzzy lattice (Hutton algebra), that is, a complete, completely distributive lattice with an order reversing involution "$'$". Recall that $m \in L$ is *join-irreducible*, if

$$\forall a, b \in L,\ m \leq a \vee b \Rightarrow m \leq a \text{ or } m \leq b.$$

Consider the sets

$$M_L = \{m \mid m \text{ is join-irreducible in } L\},$$
$$\mathcal{M}_L = \{\widehat{a} \mid a \in L\}, \text{ and } \widehat{a} = \{m \mid m \in M_L \text{ and } m \leq a\} \text{ for all } a \in L.$$

Then the mapping $\widehat{\ }: L \to \mathcal{M}_L$ defined by $\forall a \in L,\ a \mapsto \widehat{a}$ is a lattice isomorphism and the triple $(M_L, \mathcal{M}_L, c_{M_L})$ is a complemented simple texture space which is called *a Hutton texture*. Here the complementation $c_{M_L} : \mathcal{M}_L \to \mathcal{M}_L$ is defined by

$$\forall a \in L,\ c_{M_L}(\widehat{a}) = \widehat{a'}.$$

Conversely, every complemented simple texture may be obtained in this way from a suitable Hutton algebra [7].

Example 3.1. (i) The unit interval [0, 1] is a Hutton algebra with the usual ordering $\leq$ and the order reversing involution $'$ where $u' = 1 - u$ for all $u \in [0, 1]$. The simple texture corresponding to the Hutton algebra [0, 1] is the Hutton texture $(M, \mathcal{M}, c_M)$ given in Example 1.2(ii) where

$$\mathcal{M} = \{(0, u] \mid u \in [0, 1]\} \text{ and } c_M(0, u] = (0, 1 - u], \forall u \in [0, 1].$$

Indeed, the set of all join-irreducible elements of [0, 1] is $M = (0, 1]$ and for every $u \in [0, 1]$, we have $\widehat{u} = (0, u]$. Then the mapping

$$\widehat{\ }: [0, 1] \longrightarrow \mathcal{M}$$
$$u \longrightarrow (0, u], \forall u \in [0, 1]$$

is a lattice isomorphism.
(ii) Recall that a fuzzy subset α of U is a membership function $\alpha : U \to [0, 1]$. We denote the set of all fuzzy subsets of U by $\mathcal{F}(U)$. It is well known that $\mathcal{F}(U)$ is also an Hutton algebra with the pointwise ordering

$$\forall u \in U,\ \alpha \leq \beta \Longleftrightarrow \alpha(u) \leq \beta(u)$$

and the order reversing involution $\alpha'(u) = 1 - \alpha(u)$. Here the join and the meet of fuzzy sets are considered as

$$(\alpha \wedge \beta)(u) = \alpha(u) \wedge \beta(u) \text{ and } (\alpha \vee \beta)(u) = \alpha(u) \vee \beta(u)$$

for all $\alpha, \beta \in \mathcal{F}(U)$.

Now consider the fuzzy points u_s and fuzzy copoints u^s of $\mathcal{F}(U)$ defined by

$$u_s(z) = \begin{cases} s, & \text{if } z = u \\ 0, & \text{if } z \neq u \end{cases} \quad \text{and} \quad u^s(z) = \begin{cases} s, & \text{if } z = u \\ 1, & \text{if } z \neq u \end{cases}$$

Let us take the sets:

$$\widehat{\alpha} = \{u_s \mid u_s \leq \alpha\},$$
$$\mathcal{M}_{\mathcal{F}(U)} = \{\widehat{\alpha} \mid \alpha \in \mathcal{F}(U)\}, \text{ and}$$
$$M_{\mathcal{F}(U)} = \{u_s \mid u_s \text{ is a fuzzy point in } \mathcal{F}(U)\}.$$

Then under the lattice isomorphism $\widehat{\ } : \mathcal{F}(U) \to \mathcal{M}_{\mathcal{F}(U)}$, the corresponding texture space will be $(M_{\mathcal{F}(U)}, \mathcal{M}_{\mathcal{F}(U)})$. Every fuzzy point u_s can be regarded as an ordered pair $(u, s) \in U \times (0, 1]$ and then we may obtain that $\widehat{\alpha} = \{(u, s) \mid s \leq \alpha(u)\}$. Therefore, it can be shown that the texture $(M_{\mathcal{F}(U)}, \mathcal{M}_{\mathcal{F}(U)})$ is isomorphic to the product texture

$$(U \times M,\ \mathcal{P}(U) \otimes \mathcal{M},\ c_{U \times M})$$

while the complementation mapping is defined by $c_{U\times M}(\widehat{\alpha}) = \widehat{1 - \alpha}$ for all $\alpha \in \mathcal{F}(U)$ [7]. Further, for the p-sets and q-sets in this product we immediately have that

$$\widehat{u_s} = \{u\} \times (0, s] = P_{(u,s)} \quad \text{and} \quad \widehat{u^s} = (U \setminus \{u\} \times [0, 1]) \cup (U \times (0, s]) = Q_{(u,s)}.$$

4. Direlations

Direlations play a central role in the theory of texture spaces [8]. A direlation has two parts which are dual to each other. Now let $(U, \mathcal{U})$, $(V, \mathcal{V})$ be texture spaces and let us consider the product texture $\mathcal{P}(U) \otimes \mathcal{V}$ of the texture spaces $(U, \mathcal{P}(U))$ and $(V, \mathcal{V})$ and denote the p-sets and the q-sets by $\overline{P}_{(u,v)}$ and $\overline{Q}_{(u,v)}$ respectively. Then

(i) $r \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *relation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies
  R1 $r \not\subseteq \overline{Q}_{(u,v)}, P_{u'} \not\subseteq Q_u \implies r \not\subseteq \overline{Q}_{(u',v)}$.
  R2 $r \not\subseteq \overline{Q}_{(u,v)} \implies \exists u' \in U$ such that $P_u \not\subseteq Q_{u'}$ and $r \not\subseteq \overline{Q}_{(u',v)}$.
(ii) $R \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *corelation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies
  CR1 $\overline{P}_{(u,v)} \not\subseteq R, P_u \not\subseteq Q_{u'} \implies \overline{P}_{(u',v)} \not\subseteq R$.
  CR2 $\overline{P}_{(u,v)} \not\subseteq R \implies \exists u' \in U$ such that $P_{u'} \not\subseteq Q_u$ and $\overline{P}_{(u',v)} \not\subseteq R$.

A pair (r, R), where r is a relation and R a corelation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ is called a *direlation from* $(U, \mathcal{U})$ *to* $(V, \mathcal{V})$.

If textures are discrete, then there is a close relation between direlations and ordinary relations. Indeed, if r is an ordinary relation from U to V, then the pair $(r, (U \times V) \setminus r)$ may be regarded as a complemented direlation between discrete textures $(U, \mathcal{P}(U))$ and $(V, \mathcal{P}(V))$. Conversely, if (r, R) is a complemented direlation from $(U, \mathcal{P}(U))$ to $(V, \mathcal{P}(V))$, then r and R are already ordinary relations where $R = (U \times V) \setminus r$. Hence, direlations between textures may be considered as natural generalizations of ordinary relations between sets. On the other hand, direlations are abstract approximation operators in rough set theory and this is the essential connection between rough sets and textures. Further, note that if (r, R) is a direlation from $(U, \mathcal{P}(U))$ to $(V, \mathcal{P}(V))$, then r and R are point relations from U to V, that is, $r, R \subseteq U \times V$ since $\mathcal{P}(U) \otimes \mathcal{P}(V) = \mathcal{P}(U \times V)$.

The identity direlation (i, I) on $(U, \mathcal{U})$ is defined by

$$i = \bigvee\{\overline{P}_{(u,u)} \mid u \in U\} \text{ and } I = \bigcap\{\overline{Q}_{(u,u)} \mid u \in U^\flat\}$$

where $U^\flat = \{u \mid U \not\subseteq Q_u\}$. Recall that if (r, R) is a direlation on $(U, \mathcal{U})$, then r is *reflexive* if $i \subseteq r$ and R is *reflexive* if $R \subseteq I$. Then we say that (r, R) is *reflexive* if r and R are reflexive.

Now let (r, R) be a direlation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ where $(U, \mathcal{U})$ and $(V, \mathcal{V})$ are any two texture spaces. *The inverses* of r and R are defined by

$$r^\leftarrow = \bigcap\{\overline{Q}_{(v,u)} \mid r \not\subseteq \overline{Q}_{(u,v)}\} \text{ and } R^\leftarrow = \bigvee\{\overline{P}_{(v,u)} \mid \overline{P}_{(u,v)} \not\subseteq R\},$$

respectively. One can prove that $r^\leftarrow$ is a corelation and $R^\leftarrow$ is a relation. Then the direlation $(r, R)^\leftarrow = (R^\leftarrow, r^\leftarrow)$ from $(V, \mathcal{V})$ to $(U, \mathcal{U})$ is called the *inverse* of the direlation (r, R) and (r, R) is called *symmetric* if $r = R^\leftarrow$ and $R = r^\leftarrow$.

The A-sections and the B-presections with respect to relation and corelation are given as

$$r^\rightarrow A = \bigcap\{Q_v \mid \forall u, r \not\subseteq \overline{Q}_{(u,v)} \Rightarrow A \subseteq Q_u\}$$
$$R^\rightarrow A = \bigvee\{P_v \mid \forall u, \overline{P}_{(u,v)} \not\subseteq R \Rightarrow P_u \subseteq A\}$$
$$r^\leftarrow B = \bigvee\{P_u \mid \forall v, r \not\subseteq \overline{Q}_{(u,v)} \Rightarrow P_v \subseteq B\}, \text{ and}$$
$$R^\leftarrow B = \bigcap\{Q_u \mid \forall v, \overline{P}_{(u,v)} \not\subseteq R \Rightarrow B \subseteq Q_v\}$$

for all $A \in \mathcal{U}$ and $B \in \mathcal{V}$, respectively.

Now let $(U, \mathcal{U})$, $(V, \mathcal{V})$, $(W, \mathcal{W})$ be texture spaces. For any relation p from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and for any relation q from $(V, \mathcal{V})$ to $(W, \mathcal{W})$ their *composition* $q \circ p$ from $(U, \mathcal{U})$ to $(W, \mathcal{W})$ is defined by

$$q \circ p = \bigvee\{\overline{P}_{(u,w)} \mid \exists\, v \in V \text{ with } p \not\subseteq \overline{Q}_{(u,v)} \text{ and } q \not\subseteq \overline{Q}_{(v,w)}\}$$

and any corelation P from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and for any corelation Q from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ their *composition* $Q \circ P$ from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ defined by

$$Q \circ P = \bigcap\{\overline{Q}_{(u,w)} \mid \exists\, v \in V \text{ with } \overline{P}_{(u,v)} \not\subseteq P \text{ and } \overline{P}_{(v,w)} \not\subseteq Q\}.$$

Finally, the *composition* of the direlations (p, P), (q, Q) is the direlation

$$(q, Q) \circ (p, P) = (q \circ p, Q \circ P).$$

Further, r is *transitive* if $r \circ r \subseteq r$ and R is *transitive* if $R \subseteq R \circ R$. Then (r, R) is called *transitive* if r and R are transitive. Finally, if (r, R) is reflexive, symmetric and transitive, then it is called *an equivalence direlation*.

Now let c_U and c_V be the complementations on $(U, \mathcal{U})$ and $(V, \mathcal{V})$, respectively. The complement r' of the relation r is the corelation

$$r' = \bigcap\{\overline{Q}_{(u,v)} \mid \exists w, z \text{ with } r \not\subseteq \overline{Q}_{(w,z)}, c_U(Q_u) \not\subseteq Q_w \text{ and } P_z \not\subseteq c_V(P_v)\}.$$

The complement R' of the corelation R is the relation

$$R' = \bigvee\{\overline{P}_{(u,v)} \mid \exists w, z \text{ with } \overline{P}_{(w,z)} \not\subseteq R, P_w \not\subseteq c_U(P_u) \text{ and } c_V(Q_v) \not\subseteq Q_z\}.$$

The complement $(r, R)'$ of the direlation (r, R) is the direlation $(r, R)' = (R', r')$. A direlation (r, R) is called *complemented* if $r = R'$ and $R = r'$. It is easy to see that if (r, R) is a complemented direlation from $(U, \mathcal{P}(U))$ to $(V, \mathcal{P}(V))$, then we have

$$r' = (U \times V) \setminus r \text{ and } R' = (U \times V) \setminus R.$$

Therefore, if (r, R) is a complemented direlation, then we obtain $r = (U \times V) \setminus R$.

5. Category of rough set approximation operators

For the basic motivation of rough sets in terms of equivalence relations, we refer to [28]. Here, we consider rough set models on two universes [10,33]. Let U and V be any two sets and r be any relation from U to V. Recall that *a generalized rough set based on* r is given by a pair $(\underline{apr}_r(A), \overline{apr}_r(A))$ where the approximation operators $\underline{apr}_r, \overline{apr}_r : \mathcal{P}(V) \to \mathcal{P}(U)$ are defined by

$$\forall A \subseteq V,\ \underline{apr}_r(A) = \{x \in U \mid \forall y \in V,\ (x, y) \in r \implies y \in A\} \text{ and}$$
$$\overline{apr}_r(A) = \{x \in U \mid \exists y \in V,\ (x, y) \in r \text{ and } y \in A\},$$

respectively.

The following result will be useful in the sequel:

Theorem 5.1. *If r is a relation from U to V, then for any subset $A \subseteq V$,*

$$\underline{apr}_r(A) = U \setminus r^{-1}(V \setminus A) \text{ and } \overline{apr}_r A = r^{-1}(A)$$

where r^{-1} is the inverse relation of r.

Proof. Suppose that $\underline{apr}_r A \not\subseteq U \setminus r^{-1}(V \setminus A)$. Let us choose a point $u \in U$ where $u \in \underline{apr}_r A$ and $u \notin U \setminus r^{-1}(V \setminus A)$. Then $u \in r^{-1}(V \setminus A)$ and so we have $v \in V \setminus A$ such that $(u, v) \in r$. But $v \notin A$ is a contradiction since $u \in \underline{apr}_r A$. Now let $U \setminus r^{-1}(V \setminus A) \not\subseteq \underline{apr}_r A$ and take a point $u \in U$ where $u \in U \setminus r^{-1}(V \setminus A)$ and $u \notin \underline{apr}_r A$. Then $u \notin r^{-1}(V \setminus A)$ and for some $v \in V$ we have $(u, v) \in r$ and $v \notin A$. However, $v \in V \setminus A$ and this contradicts $u \notin r^{-1}(V \setminus A)$. The proof of the second equality follows from the definition of the inverse image of a relation. □

The lower and upper approximation operators satisfy the following properties [33] which can be easily proved in view of Theorem 5.1.

(L1) $\underline{apr}_r(A) = U \setminus (\overline{apr}_r(V \setminus A))$,
(L2) $\underline{apr}_r(A \cap B) = \underline{apr}_r(A) \cap \underline{apr}_r(B)$,
(L3) $A \subseteq B \implies \underline{apr}_r(A) \subseteq \underline{apr}_r(B)$,
(U1) $\overline{apr}_r(A) = U \setminus (\underline{apr}_r(V \setminus A))$,
(U2) $\overline{apr}_r(A \cup B) = \overline{apr}_r(A) \cup \overline{apr}_r(B)$,
(U3) $A \subseteq B \implies \overline{apr}_r(A) \subseteq \overline{apr}_r(B)$.

The system $(\mathcal{P}(U), \mathcal{P}(V), \cap, \cup, \setminus, \underline{apr}_r, \overline{apr}_r)$ defines a rough set model on two universes.
Now we may give the following lemma:

Proposition 5.2. Let U, V, W and Z be sets, and let $r \subseteq U \times V$, $q \subseteq V \times W$ and $p \subseteq W \times Z$. Then we have the following statements:

(i) For any subset $C \subseteq W$, $\underline{apr}_{q\circ r}(C) = \underline{apr}_r(\underline{apr}_q(C))$ and $\overline{apr}_{q\circ r}(C) = \overline{apr}_r(\overline{apr}_q(C))$.
(ii) Let $\Delta_U = \{(u, u) \mid u \in U\} \subseteq U \times U$. Then for any subset $A \subseteq U$,

$$\overline{apr}_{r\circ\Delta_U}(A) = \overline{apr}_r(A) \text{ and } \forall B \subseteq V,\ \underline{apr}_{\Delta_V\circ r}(B) = \underline{apr}_r(B).$$

(iii) For any subset $D \subseteq Z$, $\underline{apr}_{p\circ(q\circ r)}(D) = \underline{apr}_{(p\circ q)\circ r}(D)$ and $\overline{apr}_{p\circ(q\circ r)}(D) = \overline{apr}_{(p\circ q)\circ r}(D)$.

Proof. We give the proof using Theorem 5.1.

$$\begin{aligned}
\text{(i)}\quad \underline{apr}_{q\circ r}(C) &= U \setminus ((q \circ r)^{-1}(W \setminus C)) = U \setminus (r^{-1}(q^{-1}(W \setminus C)) \\
&= U \setminus (\overline{apr}_r(\overline{apr}_q(W \setminus C))) = \underline{apr}_r(V \setminus (\overline{apr}_q(W \setminus C))) \\
&= \underline{apr}_r(\underline{apr}_q(C)), \text{ and}
\end{aligned}$$

$$\overline{apr}_{q\circ r}(C) = (q \circ r)^{-1}(C) = r^{-1}(q^{-1}(C)) = r^{-1}(\overline{apr}_q(C)) = \overline{apr}_r(\overline{apr}_q(C)).$$

(ii) It is immediate since $r \circ \Delta_U = \Delta_V \circ r = r$.
(iii)

$$\underline{apr}_{p\circ(q\circ r)}(D) = \underline{apr}_{q\circ r}(\underline{apr}_p(D)) = \underline{apr}_r(\underline{apr}_q(\underline{apr}_p(D))) = \underline{apr}_r(\underline{apr}_{p\circ q}(D)) = \underline{apr}_{(p\circ q)\circ r}(D)$$

and

$$\overline{apr}_{p\circ(q\circ r)}(D) = \overline{apr}_{q\circ r}(\overline{apr}_p(D)) = \overline{apr}_r(\overline{apr}_q(\overline{apr}_p(D))) = \overline{apr}_r(\overline{apr}_{p\circ q}(D)) = \overline{apr}_{(p\circ q)\circ r}(D). \quad \square$$

Note that Proposition 5.2(ii) is also true for $\Delta_U \circ r$ and $r \circ \Delta_V$.

Corollary 5.3. (i) *The composition of the pair of rough set approximation operators defined by*

$$(\underline{apr}_q, \overline{apr}_q) \circ (\underline{apr}_r, \overline{apr}_r) = (\underline{apr}_{r\circ q}, \overline{apr}_{r\circ q})$$

is associative.

(ii) $(\underline{apr}_r, \overline{apr}_r) \circ (\underline{apr}_{\Delta_U}, \overline{apr}_{\Delta_U}) = (\underline{apr}_{\Delta_V}, \overline{apr}_{\Delta_V}) \circ (\underline{apr}_r, \overline{apr}_r) = (\underline{apr}_r, \overline{apr}_r)$.

Proof. (i) By Proposition 5.2(ii), we have

$$(\underline{apr}_p, \overline{apr}_p) \circ ((\underline{apr}_q, \overline{apr}_q) \circ (\underline{apr}_r, \overline{apr}_r)) = (\underline{apr}_p, \overline{apr}_p) \circ (\underline{apr}_{r\circ q}, \overline{apr}_{r\circ q})$$
$$= (\underline{apr}_{(r\circ q)\circ p}, \overline{apr}_{(r\circ q)\circ p}) = (\underline{apr}_{r\circ(q\circ r)}, \overline{apr}_{r\circ(q\circ p)}) = (\underline{apr}_{q\circ p}, \overline{apr}_{q\circ p}) \circ (\underline{apr}_r, \overline{apr}_r)$$
$$= ((\underline{apr}_p, \overline{apr}_p) \circ (\underline{apr}_q, \overline{apr}_q)) \circ (\underline{apr}_r, \overline{apr}_r).$$

(ii) It is immediate by Proposition 5.2(iii). $\square$

Corollary 5.4. *Power sets and the pairs of rough set approximation operators form a category which is denoted by* **R-APR***.*

Theorem 5.5. *The functor* $\mathfrak{T}$*:* **REL**$\to$ **R-APR** *defined by*

$$\mathfrak{T}(U) = \mathcal{P}(U) \quad and \quad \mathfrak{T}(r) = (\underline{apr}_r, \overline{apr}_r)$$

for all sets U, V *and* $r \subseteq U \times V$ *is contravariant, and an isomorphism.*

Proof. For any object U, the pair $\mathrm{id}_U = (\underline{apr}_{\Delta_U}, \overline{apr}_{\Delta_U})$ is an identity morphism in the category of **R-APR** and $\mathfrak{T}(\Delta_U) = (\underline{apr}_{\Delta_U}, \overline{apr}_{\Delta_U})$. Further,

$$\mathfrak{T}(q \circ r) = (\underline{apr}_{q\circ r}, \overline{apr}_{q\circ r)}) = (\underline{apr}_r, \overline{apr}_r) \circ (\underline{apr}_q, \overline{apr}_q) = \mathfrak{T}(r) \circ \mathfrak{T}(q)$$

and so indeed $\mathfrak{T}$ is a contravariant functor. Let U and V be any two sets, and r, q be direlations from U to V where $r \neq q$. Suppose that $(u, v) \in r$ and $(u, v) \notin q$ for some $(u, v) \in U \times V$. Then we have $u \in r^{-1}(\{v\}) = \overline{apr}_r(\{v\})$ and $u \notin q^{-1}(\{v\}) = \overline{apr}_q(\{v\})$ and this gives $(\underline{apr}_r, \overline{apr}_r) \neq (\underline{apr}_q, \overline{apr}_q)$. Conversely, if $(\underline{apr}_r, \overline{apr}_r) \neq (\underline{apr}_q, \overline{apr}_q)$, then we have $\underline{apr}_r(B) \neq \underline{apr}_q(B)$ or $\overline{apr}_r(B) \neq \overline{apr}_q(B)$ for some $B \subseteq V$. With no loss of generality, if $\overline{apr}_r(B) \neq \overline{apr}_q(B)$, then $r^{-1}(B) \neq q^{-1}(B)$ and so clearly, $r \neq q$. Therefore, the functor $\mathfrak{T}$ is bijective on hom-sets. Clearly, it is also bijective on objects. $\square$

6. Category of textures and direlations

By Proposition 2.14 in [8], direlations are closed under compositions and the composition is associative. By Theorem 2.17(1) in [8], for any texture $(U, \mathcal{U})$, we have the identity direlation (i_U, I_U) on $(U, \mathcal{U})$ and if (r, R) is a direlation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$, then

$$(i_V, I_V) \circ (r, R) = (r, R) \quad \text{and} \quad (r, R) \circ (i_U, I_U) = (r, R).$$

Now we may claim:

Theorem 6.1. *Texture spaces and direlations form a category which is denoted by* **drTex***.*

Let $(U, \mathcal{U}, c_U)$ and $(V, \mathcal{V}, c_V)$ be complemented textures, and (r, R) a complemented direlation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$. If (q, Q) is a complemented direlation from $(V, \mathcal{V}, c_V)$ to $(Z, \mathcal{Z}, c_Z)$, then by Proposition 2.21(3) in [8], we have

$$(q \circ r)' = q' \circ r' = Q \circ R \text{ and } (Q \circ R)' = Q' \circ R' = q \circ r.$$

Hence,

$$((q, Q) \circ (r, R))' = (q \circ r, Q \circ R)' = ((Q \circ R)', (q \circ r)') = (q \circ r, Q \circ R) = (q, Q) \circ (r, R),$$

that is the composition of (r, R) and (q, Q) is also complemented. Since the identity direlation (i_U, I_U) is also complemented, we have the following result:

Theorem 6.2. *Complemented texture spaces and complemented direlations form a category which is denoted by* **cdrTex***.*

Now let r be a relation from U to V. Then the pair $(r, \ (U \times V) \setminus r)$ can be regarded as a complemented direlation from $(U, \mathcal{P}(U))$ to $(V, \mathcal{P}(V))$ where $R = U \times V \setminus r$ (for detail see Proposition 3.1(11) and (12) in [26]). Conversely, if (r, R) is a complemented direlation from $(U, \mathcal{P}(U))$ to $(V, \mathcal{P}(V))$, then

$$r, R \subseteq \mathcal{P}(U) \otimes \mathcal{P}(V) = \mathcal{P}(U \times V),$$

that is, r and R are ordinary relations from U to V where $R = (U \times V) \setminus r$. For discrete textures $(U, \mathcal{P}(U))$ and $(V, \mathcal{P}(V))$, we have the following facts:

(1) $\overline{Q}_{(u,v)} = ((U \setminus \{u\}) \times V) \cup (U \times (V \setminus \{v\}))$.
(2) $\overline{P}_{(u,v)} = P_u \times P_v = \{u\} \times \{v\} = \{(u, v)\}$.
(3) $r \not\subseteq \overline{Q}_{(u,v)} \iff (u, v) \in r$.
(4) $\overline{P}_{(u,v)} \not\subseteq R \iff (u, v) \notin R$.

By definition of A-presections, we may easily see that

$$(r^{\leftarrow}A, \ R^{\leftarrow}A) = (\underline{apr}_r A, \ \overline{apr}_r A)$$

for every set $A \in \mathcal{P}(V)$. To see the equality, it is enough to observe that

$$\begin{aligned} r^{\leftarrow}A &= \bigvee\{P_u \mid \forall v, r \not\subseteq \overline{Q}_{(u,v)} \Longrightarrow P_v \subseteq A\} \\ &= \bigcup\{\{u\} \mid \forall v, (u, v) \in r \Longrightarrow v \in A\} \\ &= \{u \mid \forall v, (u, v) \in r \Longrightarrow v \in A\} = \underline{apr}_r A \end{aligned}$$

and

$$\begin{aligned} R^{\leftarrow}A &= \bigcap\{Q_u \mid \forall v, \overline{P}_{(u,v)} \not\subseteq R \Longrightarrow A \subseteq Q_v\} \\ &= \bigcap\{U \setminus \{u\} \mid \forall v, (u, v) \in r \Longrightarrow A \subseteq U \setminus \{v\}\} \\ &= U \setminus \left(\bigcup\{\{u\} \mid \forall v, (u, v) \in r \Longrightarrow v \notin A\}\right) \\ &= U \setminus \{u \mid \forall v, (u, v) \in r \Longrightarrow v \notin A\} \\ &= \{u \mid \exists v, (u, v) \in r \text{ and } v \in A\} = \overline{apr}_r A \end{aligned}$$

for all $A \in \mathcal{P}(V)$. Therefore, presections are very natural generalizations of approximation operators of rough sets. Further, we have

$$\begin{aligned} r^{\leftarrow} &= \bigcap\{\overline{Q}_{(v,u)} \mid r \not\subseteq \overline{Q}_{(u,v)}\} \\ &= \bigcap\{((V \setminus \{v\}) \times U) \cup (V \times (U \setminus \{u\})) \mid (u, v) \in r\} \\ &= (V \times U) \setminus \left(\bigcup\{(\{v\} \times U) \cap (V \times \{u\}) \mid (u, v) \in r\}\right) \\ &= (V \times U) \setminus \left(\bigcup\{\{(v, u)\} \mid (u, v) \in r\}\right) \\ &= (V \times U) \setminus \{(v, u) \mid (u, v) \in r\} = (V \times U) \setminus r^{-1}. \end{aligned}$$

By a similar argument, we find

$$R^{\leftarrow} = \bigvee\{\overline{P}_{(v,u)} \mid \overline{P}_{(u,v)} \not\subseteq R\} = r^{-1}.$$

Now we can prove the following.

Theorem 6.3.

(i) *The functor* $\mathfrak{L}$: **R-APR**$\to$ **cdrTex** *defined by*

$$\mathfrak{L}(\mathcal{P}(U)) = (U, \mathcal{P}(U)), \ \ \mathfrak{L}(\underline{apr}_r, \ \overline{apr}_r) = (R^{\leftarrow}, r^{\leftarrow})$$

for every morphism $(\underline{apr}_r, \overline{apr}_r) : \mathcal{P}(U) \to \mathcal{P}(V)$ *in* **R-APR** *where*

$$R^{\leftarrow} = r^{-1} \text{ and } r^{\leftarrow} = (U \times V) \setminus r^{-1}$$

is a full embedding.

(ii) *The functor* $\mathfrak{N}$: **REL**$\to$ **cdrTex** *defined by*

$$\mathfrak{N}(U) = (U, \mathcal{P}(U)), \qquad \mathfrak{N}(r) = (r, R)$$

for every morphism $r : U \to V$ *in* **REL** *where* $R = (U \times V) \setminus r$ *is a full embedding.*

Proof. First let us show that $\mathfrak{L}$ is a functor leaving the proof of $\mathfrak{N}$. By Corollary 5.3(ii), the pair $(\underline{apr}_{\Delta_U}, \overline{apr}_{\Delta_U})$ is the identity morphism in **R-APR** for an object $\mathcal{P}(U)$. Then

$$\mathfrak{L}(\underline{apr}_{\Delta_U}, \overline{apr}_{\Delta_U}) = (\Delta_U^{-1}, (U \times U) \setminus \Delta_U^{-1}) = (\Delta_U, (U \times U) \setminus \Delta_U)$$

is the identity direlation on the texture $(U, \mathcal{P}(U))$. Now let $(\underline{apr}_r, \overline{apr}_r) : \mathcal{P}(W) \to \mathcal{P}(V)$ and $(\underline{apr}_q, \overline{apr}_q) : \mathcal{P}(V) \to \mathcal{P}(U)$ be morphisms in **R-APR** where $q : U \to V$ and $r : V \to W$ are relations. Then by Proposition 3.1(7) and (8) in [26], we have

$$\begin{aligned}\mathfrak{L}((\underline{apr}_q, \overline{apr}_q) \circ (\underline{apr}_r, \overline{apr}_r)) &= \mathfrak{L}(\underline{apr}_{r\circ q}, \overline{apr}_{r\circ q}) \\ &= ((r \circ q)^{-1}, (W \times U) \setminus (r \circ q)^{-1}) \\ &= (q^{-1} \circ r^{-1}, ((V \times U) \setminus q^{-1}) \circ ((W \times V) \setminus r^{-1})) \\ &= (q^{-1}, (V \times U) \setminus q^{-1}) \circ (r^{-1}, (W \times V) \setminus r^{-1}) \\ &= \mathfrak{L}(\underline{apr}_q, \overline{apr}_q) \circ \mathfrak{L}(\underline{apr}_r, \overline{apr}_r).\end{aligned}$$

The functors $\mathfrak{L}$ and $\mathfrak{N}$ are injective on objects and hom-sets. Further, if $(R^{\leftarrow}, r^{\leftarrow})$ is a complemented direlation from $(U, \mathcal{P}(U))$ to $(V, \mathcal{P}(V))$, then r is a relation from V to U. Hence, $(\underline{apr}_r, \overline{apr}_r)$ is a pair of approximation operators from $\mathcal{P}(U)$ to $\mathcal{P}(V)$. Therefore, $\mathfrak{L}$ is full. Likewise, $\mathfrak{N}$ is also full. □

7. Textural isomorphisms

Definition 7.1 (*[6]*). Let $(U, \mathcal{U})$ and $(V, \mathcal{V})$ be texture spaces. A function $\psi : U \to V$ is *a textural isomorphism* if

(i) ψ is bijective,
(ii) $\forall A \in \mathcal{U}$, $\psi(A) \in \mathcal{V}$, and
(iii) the mapping $\psi : \mathcal{U} \to \mathcal{V}, A \mapsto \psi(A)$ is bijective.

We say $(U, \mathcal{U})$ and $(V, \mathcal{V})$ are *isomorphic* if there exists an isomorphism between them. We denote this by $(U, \mathcal{U}) \cong (V, \mathcal{V})$. If c_U and c_V are complementations on $(U, \mathcal{U})$ and $(V, \mathcal{V})$, respectively, and ψ satisfies the additional property

$$\forall A, \psi(c_U(A)) = c_V(\psi(A)),$$

then ψ is called *a complemented textural isomorphism*. When such an isomorphism exists we write $(U, \mathcal{U}, c_U) \cong (V, \mathcal{V}, c_V)$.

Proposition 7.2 (*[6]*). Let ψ be a textural isomorphism from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and $\{A_j \mid j \in J\} \subseteq \mathcal{U}$. Then

(i) $\psi(\bigvee_{j\in J} A_j) = \bigvee_{j\in J} \psi(A_j)$.
(ii) $\psi(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \psi(A_j)$.

Proposition 7.3. (i) Let $(U, \mathcal{U})$, $(V, \mathcal{V})$ and $(W, \mathcal{W})$ be texture spaces. Then

$$((U \times V) \times W, (\mathcal{U} \otimes \mathcal{V}) \otimes \mathcal{W}) \cong (U \times (V \times W), (\mathcal{U} \otimes (\mathcal{V} \otimes \mathcal{W})).$$

(ii) Take the texture $(E, \mathcal{E})$ where $E = \{e\}$ and $\mathcal{E} = \{\{e\}, \emptyset\}$. Then for any texture $(U, \mathcal{U})$ we have

$$(U, \mathcal{U}) \cong (E \times U, \mathcal{E} \otimes \mathcal{U}) \text{ and } (U, \mathcal{U}) \cong (U \times E, \mathcal{U} \otimes \mathcal{E}).$$

(iii) $(U \times V, \mathcal{U} \otimes \mathcal{V}) \cong (V \times U, \mathcal{V} \otimes \mathcal{U})$.

Proof. (i) For the sake of shortness, we denote the product textures

$$((U \times V) \times W, (\mathcal{U} \otimes \mathcal{V}) \otimes \mathcal{W}) \text{ and } (U \times (V \times W), \mathcal{U} \otimes (\mathcal{V} \otimes \mathcal{W}))$$

by $(S, \mathcal{S})$ and $(T, \mathcal{T})$, respectively. Then the function $\psi : S \to T$ defined by

$$\forall((u, v), w) \in S, \psi(((u, v), w)) = (u, (v, w)) \in T$$

is one-to-one and onto. Further, the mapping

$$\psi : \mathcal{S} \to \mathcal{T}, A \mapsto \psi(A), A \in \mathcal{S}$$

is also one-to-one and onto. Hence, ψ is a textural isomorphism.
(ii) It is enough to consider the mappings defined by

$$\varphi : U \to E \times U, \varphi(u) = (e, u) \text{ and } \varphi' : U \to U \times E, \varphi'(u) = (u, e)$$

for all $u \in U$, respectively.

(iii) The mapping $\gamma : U \times V \to V \times U$ defined by $\gamma(u, v) = (v, u)$ for all $(u, v) \in U \times V$ is a textural isomorphism. □

Proposition 7.4. If $(U, \mathcal{U})$, $(V, \mathcal{V})$ and $(W, \mathcal{W})$ are complemented, then the textural mappings ψ, φ and γ in the proof of Proposition 7.3 are complemented.

Proof. Let us show that the mapping ψ is complemented leaving the mappings φ, γ to the interested reader. Let $c_{(U\times V)\times W} = c_S$ and $c_{U\times(V\times W)} = c_T$ be the complementations on S and T, respectively. By Proposition 7.2, for the proof it is enough to consider a set $(E \times G) \times H$ where $E \in \mathcal{U}$, $G \in \mathcal{V}$ and $H \in \mathcal{W}$. Then

$$\begin{aligned} c_T\psi((E\times G)\times H) &= c_T(E\times(G\times H)) \\ &= c_T(E\times(V\times W)\cap(U\times(G\times H)) \\ &= (c_U(E)\times(V\times W))\cup(U\times(c_{V\times W}(G\times H)) \\ &= (c_U(E)\times(V\times W))\cup(U\times(c_{V\times W}((G\times W)\cap(V\times H)) \\ &= (c_U(E)\times(V\times W))\cup(U\times(c_V(G)\times W))\cup(U\times(V\times c_W(H))) \\ &= \psi((c_U(E)\times V)\times W)\cup((U\times c_V(G))\times W)\cup((U\times V)\times c_W(H)) \\ &= \psi((c_{U\times V}(E\times V)\times W)\cup(c_{U\times V}(U\times G)\times W)\cup((U\times V)\times c_W(H) \\ &= \psi(c_S((E\times V)\times W)\cup c_S((U\times G)\times W)\cup c_S((U\times V)\times H))) \\ &= \psi(c_S((E\times V)\times W))\cap((U\times G)\times W)\cap((U\times V)\times H))) \\ &= \psi(c_S((E\times G)\times H)). \quad \square \end{aligned}$$

Proposition 7.5. (i) Let ψ be a textural isomorphism from $(U, \mathcal{U})$ to $(V, \mathcal{V})$. Then the direlation (r_ψ, R_ψ) from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ defined by

$$r_\psi = \bigvee\{\overline{P}_{(u,v)} \mid P_{\psi(u)} \not\subseteq Q_v\} \text{ and } R_\psi = \bigcap\{\overline{Q}_{(u,v)} \mid P_v \not\subseteq Q_{\psi(u)}\}$$

is an isomorphism in **drTex**.

(ii) If $(U, \mathcal{U})$ and $(V, \mathcal{V})$ are complemented textures and ψ is a complemented textural isomorphism, then (r_ψ, R_ψ) is also complemented.

(iii) Let φ be a textural morphism from $(V, \mathcal{V})$ to a texture $(W, \mathcal{W})$. Then $\varphi\circ\psi$ is also a textural isomorphism from $(U, \mathcal{U})$ to $(W, \mathcal{W})$. If φ and ψ are complemented isomorphisms, then $\varphi\circ\psi$ is also complemented. Further, we have

$$r_{\varphi\circ\psi} = r_\varphi\circ r_\psi \text{ and } R_{\varphi\circ\psi} = R_\varphi\circ R_\psi.$$

Proof. (i) It can be easily checked that (r_ψ, R_ψ) is a direlation, that is, it satisfies conditions R1 and R2. To show that the direlation (r_ψ, R_ψ) is an isomorphism in **drTex**, it is enough to prove the equalities

$$(r_\psi, R_\psi)\circ(r_\psi, R_\psi)^{\leftarrow} = (i_V, I_V) \text{ and } (r_\psi, R_\psi)^{\leftarrow}\circ(r_\psi, R_\psi) = (i_U, I_U),$$

respectively. For the first equality let us suppose that $r_\psi\circ R_\psi^{\leftarrow} \not\subseteq i_V$. Then we may choose $v, v' \in V$ such that

$$r_\psi\circ R_\psi^{\leftarrow} \not\subseteq \overline{Q}_{(v,v')} \text{ and } \overline{P}_{(v,v')} \not\subseteq i_V.$$

Hence, for some $u \in U$, there exist $v_1, v_2 \in V$ such that

$$R_\psi^{\leftarrow} \not\subseteq \overline{Q}_{(v_1,u)} \text{ and } r_\psi \not\subseteq \overline{Q}_{(u,v_2)}$$

and $\overline{P}_{(v_1,v_2)} \not\subseteq \overline{Q}_{(v,v')}$. Note that $v_1 = v$, $P_{v_2} \not\subseteq Q_{v'}$ and $P_{v'} \not\subseteq Q_v$. Further, by Proposition 2.4(1) in [8] we may obtain that $\overline{P}_{(u,v_1)} \not\subseteq R_\psi$, that is, $\overline{P}_{(u,v)} \not\subseteq R_\psi$. Then for some $w \in U$ and $z \in V$, $\overline{P}_{(u,v)} \not\subseteq \overline{Q}_{(w,z)}$ and $P_z \not\subseteq Q_{\psi(w)}$. Clearly, we have $u = w$ and $P_v \not\subseteq Q_z$ and so we obtain $P_v \not\subseteq Q_{\psi(u)}$. On the other hand, since $r_\psi \not\subseteq \overline{Q}_{(u,v_2)}$, for some $w_1 \in U$ and $z_1 \in V$ we have

$$\overline{P}_{(w_1,z_1)} \not\subseteq \overline{Q}_{(u,v_2)} \text{ and } P_{\psi(w_1)} \not\subseteq Q_{z_1}.$$

It is easy to see that $w_1 = u$ and $P_{z_1} \not\subseteq Q_{v_2}$. Hence, we have $P_{\psi(u)} \not\subseteq Q_{v_2}$. Since $P_{v_2} \not\subseteq Q_{v'}$, $P_{\psi(u)} \not\subseteq Q_{v'}$ and so $P_{v'} \not\subseteq Q_v$ implies that $P_{\psi(u)} \not\subseteq Q_v$. But this is a contradiction. The reverse inclusion $i_V \subseteq r_\psi\circ R_\psi^{\leftarrow}$ and the second equality can be proved in a similar way.

(ii) Let c_U and c_V be complementations on the textures $(U, \mathcal{U})$ and $(V, \mathcal{V})$. If ψ is a textural isomorphism, then by Proposition 3.15 in [8] for all $u \in U$ we have $\psi(P_u) = P_{\psi(u)}$ and $\psi(Q_u) = Q_{\psi(u)}$. Suppose that for some $w \in U$ and $z \in V$, we have $r_\psi \not\subseteq \overline{Q}_{(w,z)}$ such that $c_U(Q_u) \not\subseteq Q_w$ and $P_z \not\subseteq c_V(P_v)$. The function ψ preserves the inclusion and so since $P_w \subseteq c_U(Q_u)$, $\psi(P_w) \subseteq \psi(c_U(Q_u))$ and hence, we find $P_{\psi(w)} \subseteq c_V\psi(Q_u) = c_V(Q_{\psi(u)})$. If we apply the complementation to the both sides of the inclusion, we obtain $Q_{\psi(u)} \subseteq c_V(P_{\psi(w)})$. Further, since $r_\psi \not\subseteq \overline{Q}_{(w,z)}$, $P_{\psi(w)} \not\subseteq Q_z$. Then by the inclusion $c_V(P_v) \subseteq Q_z$, we conclude that $P_{\psi(w)} \not\subseteq c_V(P_v)$. Therefore, we have $P_v \not\subseteq c_V P_{\psi(w)}$, that is, $P_v \not\subseteq Q_{\psi(u)}$. Finally, by definition of the corelation R_ψ, we have $R_\psi \subseteq \overline{Q}_{(u,v)}$ and so we obtain $R_\psi \subseteq r'_\psi$. Similarly, one can show that $r'_\psi \subseteq R_\psi$.

(iii) It is easy to see that $\varphi\circ\psi$ is a textural isomorphism. Let us show that $r_{\varphi\circ\psi} = r_\varphi\circ r_\psi$. The second equality is similar. Suppose that $r_\varphi\circ r_\psi \not\subseteq r_{\varphi\circ\psi}$. Let us choose $u \in U$ and $w \in W$ such that

$$r_\varphi\circ r_\psi \not\subseteq \overline{Q}_{(u,w)} \text{ and } \overline{P}_{(u,w)} \not\subseteq r_{\varphi\circ\psi}.$$

Then for some $w' \in W$, we have

$$r_\psi \not\subseteq \overline{Q}_{(u,w)} \text{ and } r_\varphi \not\subseteq \overline{Q}_{(u,w')}$$

where $v \in V$. By (i), $r_\psi \not\subseteq \overline{Q}_{(u,w)}$ and $r_\varphi \not\subseteq \overline{Q}_{(u,w')}$ implies that $P_{\psi(u)} \not\subseteq Q_v$ and $P_{\varphi(v)} \not\subseteq Q_{w'}$, respectively. Further, since $\overline{P}_{(u,w)} \not\subseteq r_{\varphi\circ\psi}$ and $P_{w'} \not\subseteq Q_w$, $P_{(\varphi\circ\psi)(u)} \subseteq Q_{w'}$. By the proof of Proposition 3.15 in [8], textural isomorphisms preserve the p-sets and q-sets and so we have

$$\varphi(P_{\psi(u)}) = P_{\varphi(\psi(u))} \not\subseteq \varphi(Q_v) = Q_{\varphi(v)}.$$

On the other hand, $P_{\varphi(v)} \not\subseteq Q_{w'}$ gives that $P_{(\varphi\circ\psi)(u)} \not\subseteq Q_{w'}$ which is a contradiction. The reverse inclusion is similar. □

8. Product of direlations

Let (r, R) be a direlation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and (q, Q) be a direlation from $(W, \mathcal{W})$ to $(Z, \mathcal{Z})$. Then the product of (r, R) and (q, Q) is defined by

$$(r \times q,\ R \times Q) : (U \times W,\ \mathcal{U} \otimes \mathcal{W}) \to (V \times Z,\ \mathcal{V} \otimes \mathcal{Z})$$

where

$$r \times q = \bigvee\{\overline{P}_{((u,w),(v,z))} \mid r \not\subseteq \overline{Q}_{(u,v)} \text{ and } q \not\subseteq \overline{Q}_{(w,z)}\}, \text{ and}$$
$$R \times Q = \bigcap\{\overline{Q}_{((u,w),(v,z))} \mid \overline{P}_{(u,v)} \not\subseteq R \text{ and } \overline{Q}_{(w,z)} \not\subseteq Q\} \text{ [5].}$$

Proposition 8.1. (i) If the above textures are complemented, then
$(r \times q)' = r' \times q',\ (R \times Q)' = R' \times Q'$.
(ii) $(r \times q)^{\leftarrow} = r^{\leftarrow} \times q^{\leftarrow},\ (R \times Q)^{\leftarrow} = R^{\leftarrow} \times Q^{\leftarrow}$.
(iii) $(r \times p) \circ (q \times k) = (r \circ q) \times (p \circ k)$,
$(R \times P) \circ (Q \times K) = (R \circ Q) \times (P \circ K)$.

Proof. (i) Assume that $r' \times q' \not\subseteq (r \times q)'$. Let us choose $(u, w) \in U \times W$ and $(v, z) \in V \times Z$ such that

$$(r \times q)' \not\subseteq \overline{Q}_{((u,w),(v,z))} \text{ and } \overline{P}_{((u,w),(v,z))} \not\subseteq r' \times q'.$$

From the first statement, for all $(u', w'), (v', z')$, we have

$$\sigma(Q_{(u,v)}) \subseteq Q_{(u',w')} \text{ and } P_{(v',z')} \subseteq \eta(P_{(v,z)}) \implies r \times q \subseteq \overline{Q}_{((u',w'),(v',z'))} \qquad (*)$$

where $c_U \times c_W = \sigma$ and $c_V \times c_Z = \eta$. From the latter, it is easy to show that

$$\overline{P}_{(u,v)} \not\subseteq r' \text{ and } \overline{P}_{(w,z)} \not\subseteq q'.$$

Then by definition of complementation, for some $v_1 \in V$ we have

$$P_v \not\subseteq Q_{v_1},\ r \not\subseteq \overline{Q}_{(u_2,v_2)},\ P_v \not\subseteq c_V(P_{v_2}) \text{ and } c_U(Q_u) \not\subseteq Q_{u_2}$$

where $u_2 \in U$ and $v_2 \in V$. Similarly, for some $z_1 \in Z$, we have

$$P_z \not\subseteq Q_{z_1},\ q \not\subseteq \overline{Q}_{(w_2,z_2)},\ P_z \not\subseteq c_Z(P_{z_2}) \text{ and } c_W(Q_w) \not\subseteq Q_{w_2}$$

where $w_2 \in W$ and $z_2 \in Z$. Now let us choose $(u_2', v_2'), (w_2', z_2')$ such that

$$r \not\subseteq \overline{Q}_{(u_2',v_2')},\ \overline{P}_{(u_2',v_2')} \not\subseteq \overline{Q}_{(u_2,v_2)} \text{ and } q \not\subseteq \overline{Q}_{(w_2',z_2')},\ \overline{P}_{(w_2',z_2')} \not\subseteq \overline{Q}_{(w_2,z_2)}.$$

Then we have

$$r \not\subseteq \overline{Q}_{(u_2,v_2')},\ P_{v_2'} \not\subseteq Q_{v_2} \text{ and } q \not\subseteq \overline{Q}_{(w_2,z_2')},\ P_{z_2'} \not\subseteq Q_{z_2}.$$

Therefore, $\overline{P}_{((u_2,w_2),(v_2',z_2'))} \subseteq r \times q$. On the other hand, if $c_U(Q_u) \not\subseteq Q_{u_2}$ and $c_W(Q_w) \not\subseteq Q_{w_2}$, then $(c_U \times c_W)(Q_{(u_2,w_2)}) \not\subseteq Q_{(u,w)}$ and similarly, if $P_v \not\subseteq c_V(P_{v_2})$ and $P_z \not\subseteq c_Z(P_{z_2})$, then $P_{(v,z)} \not\subseteq (c_V \times c_Z)(P_{(v_2,z_2)})$. Hence, by $(*)$ we have $r \times q \subseteq \overline{Q}_{((u_2,w_2),(v_2,z_2))}$. But $P_{v_2'} \not\subseteq Q_{v_2}$ and $P_{z_2'} \not\subseteq Q_{z_2}$ give a contradiction. The reverse inclusion is similar.
(ii) Let $(r \times q)^{\leftarrow} \not\subseteq r^{\leftarrow} \times q^{\leftarrow}$. Let us choose $(u, w) \in U \times W$ and $(v, z) \in V \times Z$ such that

$$(r \times q)^{\leftarrow} \not\subseteq \overline{Q}_{((v,z),(u,w))} \text{ and } \overline{P}_{((v,z),(u,w))} \not\subseteq r^{\leftarrow} \times q^{\leftarrow}.$$

From the first statement, we have $\overline{P}_{((u,w),(v,z))} \not\subseteq r \times q$ and by definition of product of direlations we have $r \subseteq \overline{Q}_{(u,v)}$ or $q \subseteq \overline{Q}_{(w,z)}$. Further, if we consider the latter statement, then for some $u_1 \in U$ and $w_1 \in W$ we have

$$\overline{P}_{(v,u_1)} \not\subseteq r^{\leftarrow} \text{ and } \overline{P}_{(z,w_1)} \not\subseteq q^{\leftarrow}$$

where $P_{(u,w)} \not\subseteq Q_{(u_1,w_1)}$. Hence, $r \not\subseteq \overline{Q}_{(u_1,v)}$ and $q \not\subseteq \overline{Q}_{(w_1,z)}$. However, since $P_u \times P_w \not\subseteq (Q_{u_1} \times W) \cup (U \times Q_{w_1})$, $P_u \not\subseteq Q_{u_1}$ and $P_w \not\subseteq Q_{w_1}$. As a result, by condition R1, we obtain $r \not\subseteq \overline{Q}_{(u,v)}$ and $q \not\subseteq \overline{Q}_{(w,z)}$ which is a contradiction. The reverse inclusion and the proof of second equality is similar.

(iii) For the first equality, let us choose $(u, z) \in U \times Z$ and $(w, n) \in W \times N$ such that

$$(r \times p) \circ (q \times k) \not\subseteq \overline{Q}_{((u,z),(w,n))} \text{ and } \overline{P}_{((u,z),(w,n))} \not\subseteq (r \circ q) \times (p \circ k).$$

From the first statement for some (v_1, m_1), there exist $(u_1, z_1) \in U \times Z$ and $(w_1, n_1) \in W \times N$ such that

$$q \times k \not\subseteq \overline{Q}_{((u_1,z_1),(v_1,m_1))} \text{ and } r \times p \not\subseteq \overline{Q}_{((v_1,m_1),(w_1,n))}$$

with $\overline{P}_{((u_1,z_1),w_1,n_1)} \not\subseteq \overline{Q}_{((u,z),(w,n))}$. Further, it is clear that $u_1 = u$, $z_1 = z$, $P_{w_1} \not\subseteq Q_w$ and $P_{n_1} \not\subseteq Q_n$. Therefore, we obtain that

$$q \times k \not\subseteq \overline{Q}_{((u,z),(v_1,m_1))} \text{ and } r \times p \not\subseteq \overline{Q}_{((v_1,m_1),(w,n))}.$$

Since $q \times k \not\subseteq \overline{Q}_{((u,z),(v_1,m_1))}$, for some $(u_2, z_2) \in U \times Z$ and $(v_2, m_2) \in V \times M$ we have

$$\overline{P}_{((u_2,z_2),(v_2,m_2))} \not\subseteq \overline{Q}_{((u,z),(v_1,m_1))},\ q \not\subseteq \overline{Q}_{(u_2,v_2)} \text{ and } k \not\subseteq \overline{Q}_{(z_2,m_2)}.$$

This gives that $u = u_2$, $z = z_2$, $P_{v_2} \not\subseteq Q_{v_1}$, and $P_{m_2} \not\subseteq Q_{m_1}$. Now we have

$$q \not\subseteq \overline{Q}_{(u,v_1)} \text{ and } k \not\subseteq \overline{Q}_{(z,m_1)}. \tag{1}$$

On the other hand, since $r \times p \not\subseteq \overline{Q}_{((v_1,m_1),(w,n))}$, for some $(v_3, m_3) \in V \times M$ and $(w_2, n_2) \in W \times N$, we have $r \not\subseteq \overline{Q}_{(v_3,w_2)}$ and $p \not\subseteq \overline{Q}_{(m_3,n_2)}$. Since $v_1 = v_3$, $m_1 = m_3$, $P_{w_2} \not\subseteq Q_{w_1}$ and $P_{n_2} \not\subseteq Q_{n_1}$, where $\overline{P}_{((v_3,m_3),(w_2,n_2))} \not\subseteq \overline{Q}_{((v_1,m_1),(w_1,n_1))}$, we obtain

$$r \not\subseteq \overline{Q}_{(v_1,w_1)} \text{ and } p \not\subseteq \overline{Q}_{(m_1,n_1)}. \tag{2}$$

Hence, by (1) and (2), we conclude that $\overline{P}_{(u,w_1)} \subseteq r \circ q$ and $\overline{P}_{(z,n_1)} \subseteq p \circ k$. By the assumption $r \circ q \subseteq \overline{Q}_{(u,w)}$ or $k \not\subseteq \overline{Q}_{(z,m_2)}$. Then $\overline{P}_{(u,w_1)} \subseteq \overline{Q}_{(u,w)}$ or $\overline{P}_{(z,n_1)} \subseteq \overline{Q}_{(z,n_2)}$. However,

$$P_{w_1} \subseteq Q_w \text{ or } P_{n_1} \subseteq Q_{n_2}$$

is a contradiction. The reverse inclusion and the proof of the second equality is similar. □

Corollary 8.2. *If (r, R) and (q, Q) are complemented direlations, then*

$$(r \times q, R \times Q)$$

is also a complemented direlation.

Proof. By Proposition 8.1(i), it is immediate. □

Corollary 8.3. *The mapping $\otimes$:* **cdrTex** $\times$ **cdrTex** $\longrightarrow$ **cdrTex** *defined by*

$$\otimes((U, \mathcal{U}), (V, \mathcal{V})) = (U \times V,\ \mathcal{U} \otimes \mathcal{V}) \text{ and } \otimes((r, R), (q, Q)) = (r \times q, R \times Q),$$

is a functor.

Proof. Let $(r, R) : (V, \mathcal{V}) \to (W, \mathcal{W})$, $(q, Q) : (U, \mathcal{U}) \to (V, \mathcal{V})$, $(p, P) : (M, \mathcal{M}) \to (N, \mathcal{N})$ and $(k, K) : (Z, \mathcal{Z}) \to (M, \mathcal{M})$ be direlations in **cdrTex**. By Proposition 8.1(iii), we have

$$\begin{aligned}
\otimes(((r, R), (q, Q)) \circ ((p, P), (k, K))) &= \otimes((r, R) \circ (p, P), (q, Q) \circ (k, K)) \\
&= \otimes((r \circ p, R \circ P), (q \circ k, Q \circ K)) \\
&= ((r \circ p) \times (q \circ k), (R \circ P) \times (Q \circ K)) \\
&= ((r \times q) \circ (p \times k), (R \times Q) \circ (P \times K)) \\
&= (r \times q, R \times Q) \circ (p \times k, P \times K) \\
&= \otimes((r, R), (q, Q)) \circ \otimes((p, P), (k, K)).
\end{aligned}$$

Further, if (i_U, I_U) and (i_V, I_V) are the identity direlations of some objects $(U, \mathcal{U})$ and $(V, \mathcal{V})$, respectively in **cdrTex**, then the identity of object $((U, \mathcal{U}), (V, \mathcal{V}))$ in **cdrTex**×**cdrTex** is $((i_U, I_U), (i_V, I_V))$. Now let us show that $(i_U \times i_V, I_U \times I_V) = (i_{U\times V}, I_{U\times V})$ where $(i_{U\times V}, I_{U\times V})$ is the identity direlation of $U \times V$. Let $i_U \times i_V \not\subseteq \overline{Q}_{((u,v),(u',v'))}$. By definition of product, we have

$$\overline{P}_{((u_1,v_1),(u_2,v_2))} \not\subseteq \overline{Q}_{((u,v),(u',v'))},\ i_U \not\subseteq \overline{Q}_{(u_1,u_2)} \text{ and } i_V \not\subseteq \overline{Q}_{(v_1,v_2)}$$

for $(u_1, v_1), (u_2, v_2) \in U \times V$. This follows that $\overline{P}_{((u,v),(u,v))} \not\subseteq \overline{Q}_{((u,v),(u_2,v_2))}$, that is, $i_{U\times V} \not\subseteq \overline{Q}_{((u,v),(u_2,v_2))}$. This implies that $i_U \times i_V \subseteq i_{U\times V}$. For the reverse inclusion, let $i_{U\times V} \not\subseteq \overline{Q}_{((u,v),(u',v'))}$. Then for some $(u_1, v_1) \in U \times V$, we have $P_{u_1} \not\subseteq Q_{u'}$ and $P_{v_1} \not\subseteq Q_{v'}$. Now let us choose $u_2 \in U$ and $v_2 \in V$ such that

$$P_{u_1} \not\subseteq Q_{u_2},\ P_{u_2} \not\subseteq Q_{u'},\ P_{v_1} \not\subseteq Q_{v_2} \text{ and } P_{v_2} \not\subseteq Q_{v'}.$$

Then clearly,

$$\overline{P}_{((u_1,v_1),(u_2,v_2))} \not\subseteq \overline{Q}_{((u_1,v_1),(u',v'))},\ P_{u_1} \not\subseteq Q_{u_2} \text{ and } P_{v_1} \not\subseteq Q_{v_2}.$$

This means that $i_U \times i_V \not\subseteq \overline{Q}_{((u,v),(u',v'))}$ and so we obtain $i_{U\times V} \subseteq i_{U\times V}$. The second equality $I_{U\times V} = I_{U\times V}$ can be proved by a similar way. Hence, we have

$$\otimes((i_U, I_U), (i_V, I_V)) = (i_U \times i_V, I_U \times I_V) = (i_{U\times V}, I_{U\times V}).$$

Since $(i_{U\times V}, I_{U\times V})$ is the identity of the object $(U \times V, \mathcal{U} \otimes \mathcal{V})$ in **cdrTex**, the proof is complete. □

Proposition 8.4. Let

$$(p, P) : (U, \mathcal{U}) \to (U', \mathcal{U}'),\ (q, Q) : (V, \mathcal{V}) \to (V', \mathcal{V}') \text{ and } (r, R) : (W, \mathcal{W}) \to (W', \mathcal{W}')$$

be direlations where $(U, \mathcal{U}), (V, \mathcal{V}), (W, \mathcal{W}), (U', \mathcal{U}'), (V', \mathcal{V}')$ and $(W', \mathcal{W}')$ are arbitrary texture spaces. Then for all $u \in U, v \in V, w \in W, u' \in U', v' \in V', w' \in W'$, we have the following conditions:

(i) $(p \times q) \times r \not\subseteq \overline{Q}_{(((u,v),\ w),\ ((u',v'),\ w'))} \iff p \times (q \times r) \not\subseteq \overline{Q}_{((u,\ (v,w)),\ (u',\ (v',w')))}$.

(ii) $(P \times Q) \times R \not\subseteq \overline{Q}_{(((u,v),w),((u',v'),w'))} \iff P \times (Q \times R) \not\subseteq \overline{Q}_{((u,(v,w)),(u',(v',w')))}$.

(iii) $p \times q \not\subseteq \overline{Q}_{((u,v),\ (u',v'))} \implies q \times p \not\subseteq \overline{Q}_{((v,u),(v',u'))}$.

(iv) $P \times Q \not\subseteq \overline{Q}_{((u,v),\ (u',v'))} \implies Q \times P \not\subseteq \overline{Q}_{((v,u),(v',u'))}$.

Proof. (i) Let $(p \times q) \times r \not\subseteq \overline{Q}_{(((u,v),\ w),\ ((u',v'),\ w'))}$. Then there exist $(u_1, v_1, w_1) \in U \times V \times W$ and $(u_1', v_1', w_1') \in U' \times V' \times W'$ such that

$$\overline{P}_{((((u_1,v_1),\ w_1),(((u_1',v_1'),\ w_1')) } \not\subseteq \overline{Q}_{(((u,v),\ w),\ ((u',v'),\ w'))}$$

and

$$p \times q \not\subseteq \overline{Q}_{((u_1,v_1),\ (u_1',v_1'))} \text{ and } r \not\subseteq \overline{Q}_{(w_1,w_1')}.$$

Then we have $u_1 = u, v_1 = v, w_1 = w$ and $P_{u_1'} \not\subseteq Q_{u'}, P_{v_1'} \not\subseteq Q_{v'}, P_{w_1'} \not\subseteq Q_{w'}$ and so by definition p-sets and q-sets in textural product we obtain

$$\overline{P}_{((u_1,\ (v_1,w_1)),\ (u_1',\ (v_1',w_1')))} \not\subseteq \overline{Q}_{((u,\ (v,w)),\ (u',\ (v',w')))}.$$

Then we must show that

$$q \times r \not\subseteq \overline{Q}_{((v_1,w_1),\ (v_1',w_1'))} \text{ and } p \not\subseteq \overline{Q}_{(u_1,u_1')}. \tag{$*$}$$

By definition of $p \times q$, there exist $(u_2, v_2) \in U \times V$ and $(u_2', v_2') \in U' \times V'$ such that

$$\overline{P}_{((u_2,v_2)),(u_2',v_2'))} \not\subseteq \overline{Q}_{((u_1,v_1),(u_1',v_1'))},\ p \not\subseteq \overline{Q}_{(u_2,u_2')} \text{ and } q \not\subseteq \overline{Q}_{(v_2,v_2')}.$$

However, since $u_2 = u_1$, $p \not\subseteq \overline{Q}_{(u_1,u_2')}$. Further, since $P_{u_2'} \not\subseteq Q_{u_1'}$, by definition of p-sets and q-sets, we obtain $p \not\subseteq \overline{Q}_{(u_1,u_1')}$. Now we show that $q \times r \not\subseteq \overline{Q}_{((v_1,w_1),\ (v_1',w_1'))}$. First, let us choose $w_1^*, w_2^* \in W$ such that $r \not\subseteq Q_{(w_1^*,w_2^*)}$ and $\overline{P}_{(w_1^*,w_2^*)} \not\subseteq \overline{Q}_{(w_1,w_1')}$. Then $r \not\subseteq Q_{(w_1,w_2^*)}$ and $P_{w_2^*} \not\subseteq Q_{w_1'}$. By definition of product of relations, we have write $\overline{P}_{((v_1,w_1),\ (v_2',w_2^*))} \subseteq q \times r$. On the other hand, $P_{w_2'} \not\subseteq Q_{v_1'}$ and $P_{w_2^*} \not\subseteq Q_{w_1'}$ implies that

$$\overline{P}_{((v_1,w_1),\ (v_2',w_2^*))} \not\subseteq \overline{Q}_{((v_1,w_1),\ (v_1',w_1'))}.$$

As a result, we obtain $q \times r \not\subseteq \overline{Q}_{((v_1,w_1),\ (v_1',w_1'))}$. From ($*$), we have

$$\overline{P}_{((u_1,\ (v_1,w_1)),(u_1',\ (v_1',w_1')))} \subseteq p \times (q \times r).$$

We conclude that $p \times (q \times r) \not\subseteq \overline{Q}_{(u,\ (v,w)),\ (u',\ (v',w'))}$. The second part of the equivalence is similar.
(ii) Similar to (i).
(iii) Let $p \times q \not\subseteq \overline{Q}_{((u,v),\ (u',v'))}$. By definition of product, there exists $(u_1', v_1') \in U' \times V'$ such that

$$P_{u_1'} \not\subseteq Q_{u'},\ P_{v_1'} \not\subseteq Q_{v'},\ p \not\subseteq \overline{Q}_{(u,u_1')} \text{ and } q \not\subseteq \overline{Q}_{(v,v_1')}.$$

Then we have $\overline{P}_{((v,u),(v_1',u_1'))} \subseteq q \times p$. However, since $\overline{P}_{(v_1',u_1')} \not\subseteq \overline{Q}_{(v',u)}, \overline{P}_{((v,u),\ (v_1',u_1'))} \not\subseteq \overline{Q}_{((v,u),\ (v',u'))}$ and this gives that

$$q \times p \not\subseteq \overline{Q}_{((v,u),\ (v',u'))}.$$

(iv) Similar to (iii). □

9. Dagger symmetric monoidal categories

Dagger symmetric monoidal categories are used in abstract quantum mechanics [1,30]. The primary examples are the categories **REL** of relations and sets, and **FdHilb** of finite dimensional Hilbert spaces and linear mappings. Since **REL** and **R-APR** are isomorphic categories, **R-APR** is also a dagger symmetric monoidal category. In this section, we show that the categories **drTex** and **cdrTex** are also dagger symmetric monoidal categories.

Definition 9.1. (i) *A dagger category* [9,22] is a category **C** together with an involutive, identity-on-objects, contravariant functor $\dagger : \mathbf{C} \to \mathbf{C}$. In other words, every morphism $f : A \to B$ in **C** corresponds to a morphism $f^\dagger : B \to A$ such that for all $f : A \to B$ and $g : B \to C$ the following conditions hold:

$$\mathrm{id}_A^\dagger = \mathrm{id}_A : A \to A,\ \ (g \circ f)^\dagger = f^\dagger \circ g^\dagger : C \to A,\ \text{ and } f^{\dagger\dagger} = f : A \to B.$$

(ii) *A symmetric monoidal category* [25] is a category **C** together with a bifunctor $\otimes$, a distinguished object I, and natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$
$$\lambda_A : A \to I \otimes A,\ \rho_A : A \to A \otimes I \text{ and} \sigma_{A,B} : A \otimes B \to B \otimes A$$

subject to Mac Lane's standard coherence conditions.

(iii) A dagger symmetric monoidal category [30] is a symmetric monoidal category **C** with a dagger structure preserving the symmetric monoidal structure:

For all $f : A \to B$ and $g : C \to D$, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger : B \otimes D \to A \otimes C$,
$\alpha^\dagger_{A,B,C} = \alpha^{-1}_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $\lambda^\dagger = \lambda^{-1} : I \otimes A \to A$, and
$\sigma^\dagger_{A,B} = \sigma^{-1}_{A,B} : B \otimes A \to A \otimes B$.

Theorem 9.2. *The categories* **drTex** *and* **cdrTex** *are dagger categories.*

Proof. First let us determine the dagger structure on **drTex**. By Proposition 2.17 in [8], note that for any texture $(U, \mathcal{U})$,

$$(i_U, I_U)^{\leftarrow} = (i_U, I_U) \text{ and } ((q, Q) \circ (r, R))^{\leftarrow} = (r, R)^{\leftarrow} \circ (q, Q)^{\leftarrow}$$

where (r, R) is a direlation from $(U, \mathcal{U})$ to $V, \mathcal{V})$ and (q, Q) is a direlation from $(V, \mathcal{V})$ to $(Z, \mathcal{Z})$. Therefore, $\dagger$: **drTex** $\to$ **drTex** is a functor defined by

$$\dagger(U, \mathcal{U}) = (U, \mathcal{U}) \text{ and } \dagger(r, R) = (r, R)^{\leftarrow}$$

for all $(U, \mathcal{U}) \in$ob(**drTex**) and $(r, R) \in$hom(**drTex**). Further, we have $((r, R)^{\leftarrow})^{\leftarrow} = (r, R)$.
Therefore, **drTex** is a dagger category. On the other hand, if (r, R) is complemented, then $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ is also complemented. Indeed, by Proposition 2.21 in [8],

$$(R^{\leftarrow})' = (R')^{\leftarrow} = r^{\leftarrow} \text{ and } (r^{\leftarrow})' = (r')^{\leftarrow} = R^{\leftarrow}.$$

As a result, the category **cdrTex** is also a dagger category. □

Corollary 9.3. *The diagram*

$$\begin{array}{ccc} \mathbf{REL} & \xrightarrow{\ \mathfrak{T}\ } & \mathbf{R-APR} \\ \downarrow{\scriptstyle \mathfrak{L}} & & \downarrow{\scriptstyle \mathfrak{N}} \\ \mathbf{cdrTex} & \xrightarrow{\ \dagger\ } & \mathbf{cdrTex} \end{array}$$

commutes.

Proof. Let $r : U \to V$ be a morphism in **REL**. If we take $R = (U \times V) \setminus r$, then

$$(\dagger \circ \mathfrak{L})(r) = \dagger(\mathfrak{L}(r)) = \dagger(r, R) = (r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$$
$$= \mathfrak{N}(\underline{apr}_r, \overline{apr}_r) = \mathfrak{N}(\mathfrak{T}(r)) = (\mathfrak{N} \circ \mathfrak{T})(r). \quad \square$$

Corollary 9.4. (i) *For the functors*

$$\mathfrak{F}, \mathfrak{B} : \mathbf{cdrTex} \times \mathbf{cdrTex} \times \mathbf{cdrTex} \to \mathbf{cdrTex}$$

defined by

$$\mathfrak{F}((U, \mathcal{U}), (V, \mathcal{V}), (W, \mathcal{W})) = ((U \times V) \times W,\ (\mathcal{U} \otimes \mathcal{V}) \otimes \mathcal{W}),$$

$$\mathfrak{F}((p, P), (q, Q), (r, R)) = ((p \times q) \times r,\ (P \times Q) \times R)$$

and

$$\mathfrak{B}((U,\mathcal{U}),(V,\mathcal{V}),(W,\mathcal{W}))=(U\times(V\times W),\ \mathcal{U}\otimes(\mathcal{V}\otimes\mathcal{W})),$$

$$\mathfrak{B}((p,P),(q,Q),(r,R))=(p\times(q\times r),\ P\times(Q\times R)),$$

respectively, there exists a natural transformation $\alpha:\mathfrak{F}\to\mathfrak{B}$ *with the component*

$$\alpha_{(\mathcal{U},\mathcal{V},\mathcal{W})}:((U\times V)\times W,\ (\mathcal{U}\otimes\mathcal{V})\otimes\mathcal{W})\cong(U\times(V\times W),\ \mathcal{U}\otimes(\mathcal{V}\otimes\mathcal{W}))$$

which is a natural isomorphism.
(ii) *Take the functors* $\mathfrak{R},\mathfrak{D}$: **cdrTex** $\to$ **cdrTex** *defined by*

$$\mathfrak{R}((U,\mathcal{U}))=(U\times E,\ \mathcal{U}\otimes\mathcal{E})\qquad\mathfrak{D}((U,\mathcal{U}))=(E\times U,\ \mathcal{E}\otimes\mathcal{U})$$

$\mathfrak{R}((r,R))=(r\times i_E,\ R\times I_E)$*, and* $\mathfrak{D}((r,R))=(i_E\times r,\ I_E\times R)$ *where* $(E,\mathcal{E})$ *is the texture given in Proposition 7.3(ii).*
Then there exist the natural transformations $\lambda:\mathfrak{R}\to\mathfrak{I}_{\mathbf{cdrTex}}$ *and* $\rho:\mathfrak{D}\to\mathfrak{I}_{\mathbf{cdrTex}}$ *such that for all* $(U,\mathcal{U})$*, the components*

$$\lambda_{(U,\mathcal{U})}:(U,\mathcal{U})\cong(E\times U,\ \mathcal{E}\otimes\mathcal{U})\text{ and }\rho_{(U,\mathcal{U})}:(U,\mathcal{U})\cong(U\times E,\ \mathcal{U}\otimes\mathcal{E}).$$

are natural isomorphisms where $\mathfrak{I}_{\mathbf{cdrTex}}$: **cdrTex** $\to$ **cdrTex** *is the unit functor.*
(iii) *Consider the functors* $\mathfrak{S},\mathfrak{U}$: **cdrTex** $\times$ **cdrTex** $\to$ **cdrTex** *defined by*

$$\mathfrak{S}((U,\mathcal{U}),(V,\mathcal{V}))=(U\times V,\ \mathcal{U}\otimes\mathcal{V})\qquad\mathfrak{U}((U,\mathcal{U}),(V,\mathcal{V}))=(V\times U,\ \mathcal{V}\otimes\mathcal{U})$$
$$\mathfrak{S}((r,R),(q,Q))=(r\times q,\ R\times Q),\quad\text{and}\quad\mathfrak{U}((r,R),(q,Q))=(q\times r,\ Q\times R).$$

Then there exists a natural transformation $\sigma:\mathfrak{S}\to\mathfrak{U}$ *such that for all* $(U,\mathcal{U})$*, the component*

$$\sigma_{(\mathcal{U},\mathcal{V})}:(U\times V,\ \mathcal{U}\otimes\mathcal{V})\cong(V\times U,\mathcal{V}\otimes\mathcal{U})$$

is a natural isomorphism.

Proof. (i) Using a similar argument as in the proof of Corollary 8.3, it is easy to show that the mappings $\mathfrak{F}$ and $\mathfrak{B}$ are indeed functors. Now let $((U,\mathcal{U}),(V,\mathcal{V}),(W,\mathcal{W}))$ be an object in **cdrTex** $\times$ **cdrTex** $\times$ **cdrTex** and consider the complemented textural isomorphism $\psi:(U\times V)\times W\to U\times(V\times W)$ defined by $\psi((u,v),w))=(u,(v,w))$ for all $((u,v),w)\in(U\times V)\times W$. By Proposition 7.5(i), the corresponding isomorphism (r_ψ,R_ψ) in **cdrTex** can be given by the equalities

$$r_\psi=\bigvee\{\overline{P}_{(((u,v),w),\ (a,(b,c)))}\mid P_{(u,(v,w))}\not\subseteq Q_{(a,(b,c))}\},$$
$$R_\psi=\bigcap\{\overline{Q}_{(((u,v),w),(a,(b,c)))}\mid P_{(a,(b,c))}\not\subseteq Q_{(u,(v,w))}\}.$$

We prove that (r_ψ,R_ψ) is the desired natural isomorphism $\alpha_{(\mathcal{U},\mathcal{V},\mathcal{W})}$ in **cdrTex**. Now for all objects $((U',\mathcal{U}'),(V',\mathcal{V}'),(W',\mathcal{W}'))$ in **cdrTex** $\times$ **cdrTex** $\times$ **cdrTex** and for all morphisms

$$((p,P),(q,Q),(r,R)):((U,\mathcal{U}),(V,\mathcal{V}),(W,\mathcal{W}))\to((U',\mathcal{U}'),(V',\mathcal{V}'),(W',\mathcal{W}')),$$

we show that the diagram

$$\begin{array}{ccc}
(\mathcal{U}\otimes\mathcal{V})\otimes\mathcal{W} & \xrightarrow{((p\times q)\times r,\ (P\times Q)\times R)} & (\mathcal{U}'\otimes\mathcal{V}')\otimes\mathcal{W}' \\
\big\downarrow{\scriptstyle (r_\psi,R_\psi)} & & \big\downarrow{\scriptstyle (r_\varphi,R_\varphi)} \\
\mathcal{U}\otimes(\mathcal{V}\otimes\mathcal{W}) & \xrightarrow{(p\times(q\times r),\ P\times(Q\times R))} & \mathcal{U}'\otimes(\mathcal{V}'\otimes\mathcal{W}')
\end{array}$$

is commutative, where (r_φ,R_φ) is a direlation corresponding to the mapping

$$\varphi:(U'\times V')\times W'\to U'\times(V'\times W')$$

defined by $\varphi((u',v'),w')=(u',(v',w'))$ for all $((u',v'),w')\in(U'\times V')\times W'$. In other words, we check the equalities

$$(p\times(q\times r))\circ r_\psi=r_\varphi\circ((p\times q)\times r)\text{ and }(P\times(Q\times R))\circ R_\psi=R_\varphi\circ((P\times Q)\times R).$$

For the first equality, let us suppose that $(p\times(q\times r))\circ r_\psi\not\subseteq r_\varphi\circ((p\times q)\times r)$ and let us choose $((a,b),c)\in(U\times V)\times W$ and $(a',(b',c'))\in U'\times(V'\times W')$ such that

$$(p\times(q\times r))\circ r_\psi\not\subseteq\overline{Q}_{(((a,b),c),(a',(b',c')))}\ \text{ and }\ \overline{P}_{(((a,b),c),(a',(b',c')))}\not\subseteq r_\varphi\circ((p\times q)\times r).\qquad(*)$$

Then for some $((a_1,b_1),c_1)\in(U\times V)\times W$, $(a_1',(b_1',\ c_1'))\in U'\times(V'\times W')$ and $(u^*,(v^*,w^*))\in U\times(V\times W)$, we have

$$\overline{P}_{(((a_1,b_1),c_1),(a_1',(b_1',c_1')))}\not\subseteq\overline{Q}_{(((a,b),c),(a',(b',c')))}$$

where

$$r_\psi \not\subseteq \overline{Q}_{(((a_1,b_1),c_1),(u^*,(v^*,w^*)))} \text{ and } p \times (q \times r) \not\subseteq \overline{Q}_{((u^*,(v^*,\ w^*)),\ (a'_1,(b'_1,c'_1)))}.$$

Therefore, by Proposition 8.4(i), we conclude

$$(p \times q) \times r \not\subseteq \overline{Q}_{(((u^*,v^*),w^*),\ ((a'_1,b'_1),c'_1))}.$$

Further, $r_\psi \not\subseteq \overline{Q}_{(((a,b),c),(u^*,(v^*,w^*)))}$ and $P_{(a'_1,(b'_1,\ c'_1))} \not\subseteq Q_{(a',(b',c'))}$. On the other hand, by the second part of $(*)$, for all $((u',v'),w') \in (U' \times V') \times W'$ we have

$$(p \times q) \times r \not\subseteq Q_{((a,b),c)),((u',(v',w'))} \implies r_\varphi \subseteq \overline{Q}_{(((u',v'),w'),(a',(b',c'))}. \qquad (**)$$

Now let us choose $((a'_3,b'_3),c'_3) \in (U' \times V') \times W'$ such that

$$(p \times q) \times r \not\subseteq \overline{Q}_{((u^*,v^*),w^*),\ ((a'_3,b'_3),c'_3)} \text{ and } P_{((a'_3,b'_3),c'_3)} \not\subseteq Q_{((a'_1,b'_1),c'_1)}.$$

It is easy to see that since $r_\psi \not\subseteq \overline{Q}_{(((a,b),c),((u^*,(v^*,w^*))}$, $P_{((a,b),c)} \not\subseteq Q_{(u^*,(v^*,w^*))}$. Hence, by R1 we find

$$(p \times q) \times r \not\subseteq \overline{Q}_{(((a,b),c),\ ((a'_3,b'_3),c'_3))}.$$

Thus by $(**)$, we obtain $r_\varphi \subseteq \overline{Q}_{(((a'_3,b'_3),c'_3),(a',(b',c'))}$. Further, since $P_{((a'_3,b'_3),c'_3))} \not\subseteq Q_{((a'_1,b'_1),c'_1))}$, $P_{((a'_3,b'_3),c'_3),((a'_1,b'_1),c'_1))} \subseteq r_\varphi$. But this is a contradiction since $P_{(a'_1,(b'_1,\ c'_1))} \not\subseteq Q_{(a',(b',c'))}$. Therefore,

$$(p \times (q \times r)) \circ r_\psi \subseteq r_\varphi \circ ((p \times q) \times r).$$

The reverse inclusion can be proved using a similar argument.
(ii) Let us prove the existence of the natural isomorphism $\lambda : \mathfrak{R} \to \mathfrak{I}_{\mathbf{cdrTex}}$. Consider the textural isomorphism $\psi : (U, \mathcal{U}) \to (E \times u, \mathcal{E} \otimes \mathcal{U})$ defined by

$$\forall u \in U,\ \psi(u) = (e, u).$$

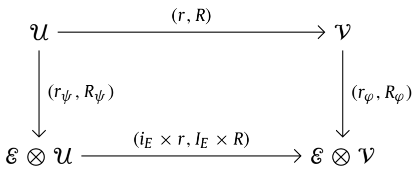
It is easy to see that by Proposition 7.5(i), the corresponding isomorphism (r_ψ, R_ψ) can be given by the equalities

$$r_\psi = \bigvee\{\overline{P}_{(u,(e,u'))} \mid P_u \not\subseteq Q_{u'}\} \text{ and } R_\psi = \bigcap\{\overline{Q}_{(u,(e,u'))} \mid P_{u'} \not\subseteq Q_u\}.$$

Now let (r, R) be a morphism from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ in **cdrTex**. By the definition of product of direlations, we have

$$i_E \times r = \bigvee\{\overline{P}_{((e,u),(e,v))} \mid r \not\subseteq \overline{Q}_{(u,v)}\} \text{ and } I_E \times R = \bigcap\{\overline{Q}_{((e,u),(e,v))} \mid P_{(u,v)} \not\subseteq R\}.$$

Take the isomorphism (r_φ, R_φ) in **cdrTex** corresponding to the textural isomorphism $\varphi : (V, \mathcal{V}) \to (V \times E, \mathcal{V} \otimes \mathcal{E})$. Now we show that the diagram



is commutative, that is, the equalities

$$(i_E \times r) \circ r_\psi = r_\varphi \circ r \text{ and } (I_E \times R) \circ R_\psi = R_\varphi \circ R$$

hold. Let $(i_E \times r) \circ r_\psi \not\subseteq r_\varphi \circ r$. Let us choose $u_1 \in U$, $v_1 \in V$ such that

$$(i_E \times r) \circ r_\psi \not\subseteq \overline{Q}_{(u_1,(e,v_1))} \text{ and } \overline{P}_{(u_1,(e,v_1))} \not\subseteq r_\varphi \circ r.$$

From the first statement, for some $u \in U$ and $v \in V$, we have $\overline{P}_{(u,(e,v))} \not\subseteq \overline{Q}_{(u_1,(e,v_1))}$ such that $r_\psi \not\subseteq \overline{Q}_{(u,(e,u'))}$ and $r \times i_E \not\subseteq \overline{Q}_{((e,u'),(e,v))}$ where $u' \in U$. On the other hand, since $u = u_1$ and $P_v \not\subseteq Q_{v_1}$, $r_\psi \not\subseteq \overline{Q}_{(u_1,(e,u'))}$ and $r \times i_E \not\subseteq \overline{Q}_{((e,u'),(e,v_1))}$. Hence, $r_\psi \not\subseteq \overline{Q}_{(u_1,(e,u'))}$ implies that $\overline{P}_{(u_3,(e,u_4))} \not\subseteq \overline{Q}_{(u_1,(e,u'))}$ and $P_{\psi(u_3)} \not\subseteq Q_{(e,u_4)}$ for some $u_3, u_4 \in U$. Therefore, it is easy to see that $P_{u_1} \not\subseteq Q_{u'}$. Further, $i_E \times r \not\subseteq \overline{Q}_{((e,u'),(e,v_1))}$ gives that $\overline{P}_{((e,u_5),(e,v_2))} \not\subseteq \overline{Q}_{((e,u'),(e,v_1))}$ and $r \not\subseteq \overline{Q}_{(u_5,v_2)}$ for some $u_5 \in U$ and $v_2 \in V$. Then we have $r \not\subseteq \overline{Q}_{(u',v_1)}$. As a result, $P_{u_1} \not\subseteq Q_{u'}$ and condition R1 implies that $r \not\subseteq \overline{Q}_{(u_1,v_1)}$. Further, since $\overline{P}_{(u_1,(e,v_1))} \not\subseteq r_\varphi \circ r$, we have that

$$\forall v^* \in V, r \not\subseteq \overline{Q}_{(u_1,v^*)} \implies r_\varphi \subseteq \overline{Q}_{(v^*,(e,v_1))}. \qquad (1)$$

Now let us choose $u^* \in U$ and $v^* \in V$ such that

$$r \not\subseteq \overline{Q}_{(u^*,v^*)} \text{ and } \overline{P}_{(u^*,v^*)} \not\subseteq \overline{Q}_{(u_1,v_1)}.$$

Then we have $u_1 = u^*$ and $P_{v^*} \not\subseteq Q_{v_1}$. Hence, $r \not\subseteq \overline{Q}_{(u_1,v^*)}$ and so by (1) we find $r_\varphi \subseteq \overline{Q}_{(v^*,(e,v_1))}$. Let us choose $a \in V$ such that $P_{v^*} \not\subseteq Q_a$ and $P_a \not\subseteq Q_{v_1}$. Clearly, $P_{\varphi(v^*)} \not\subseteq Q_{(e,a)}$. Therefore, we obtain $\overline{P}_{(v^*,(e,a))} \subseteq r_\varphi$, that is, $\overline{P}_{(v^*,(e,a))} \subseteq \overline{Q}_{(v^*,(e,v_1))}$. By the inclusion

$$\{v^*\} \times P_{(e,a)} \subseteq (V \setminus \{v^*\} \times (\{e\} \times V) \cup (V \times Q_{(e,v_1)})$$

we conclude that $\{v^*\} \times P_{(e,a)} \subseteq V \times Q_{(e,v_1)}$. Hence,

$$\{e\} \times P_a \subseteq (\{e\} \times Q_{v_1}) \cup (Q_e \times V) = (\{e\} \times Q_{v_1}) \cup (\emptyset \times V) = \{e\} \times Q_{v_1}$$

so that one obtains the contradiction $P_a \subseteq Q_{v_1}$. The reverse inclusion and the second equality can be proved using a similar argument. The proof of the existence of the natural transformation $\rho : \mathfrak{D} \to \mathfrak{I}_{\mathbf{cdrTex}}$ is similar.
(iii) Let $((U, \mathcal{U}), (V, \mathcal{V}))$ be an object in **cdrTex**$\times$ **cdrTex**. The mapping $\psi : U \times V \to V \times U$ defined by $\psi(u, v) = (v, u)$ for all $(u, v) \in U \times V$ is a complemented textural isomorphism. Using Proposition 7.5(i), let us consider the corresponding isomorphism (r_ψ, R_ψ) from $(U \times V, \mathcal{U} \otimes \mathcal{V})$ to $(V \times U, \mathcal{V} \otimes \mathcal{U})$ where

$$r_\psi = \bigvee\{\overline{P}_{((u,v),(v_1,u_1))} \mid P_{(v,u)} \not\subseteq Q_{(v_1,u_1)}\} \text{ and } R_\psi = \bigcap\{\overline{Q}_{((u,v),v_1,u_1))} \mid P_{(v_1,u_1)} \not\subseteq Q_{(v,u)}\}.$$

We show that the diagram

$$\begin{array}{ccc}
\mathcal{U} \otimes \mathcal{V} & \xrightarrow{(p \times q,\, P \times Q)} & \mathcal{U}' \otimes \mathcal{V}' \\
\Big\downarrow{\scriptstyle (r_\psi, R_\psi)} & & \Big\downarrow{\scriptstyle (r_\varphi, R_\varphi)} \\
\mathcal{V} \otimes \mathcal{U} & \xrightarrow{(q \times p,\, Q \times P)} & \mathcal{V}' \otimes \mathcal{U}'
\end{array}$$

is commutative, that is, the equalities

$$(q \times p) \circ r_\psi = r_\phi \circ (p \times q) \text{ and } (Q \times P) \circ r_\psi = r_\varphi \circ (P \times Q)$$

hold for all morphisms $((p, P), (q, Q)) : ((U, \mathcal{U}), (V, \mathcal{V})) \to ((U', \mathcal{U}'), (V', \mathcal{V}'))$ where (r_φ, R_φ) is defined as (r_ψ, r_ψ). Suppose that $(q \times p) \circ r_\psi \not\subseteq r_\varphi \circ (p \times q)$. Then we may choose $(u, v) \in U \times V$ and $(v', u') \in V' \times U'$ such that

$$(q \times p) \circ r_\psi \not\subseteq \overline{Q}_{((u,v),(v',u'))} \text{ and } \overline{P}_{((u,v),(v',u'))} \not\subseteq r_\varphi \circ (p \times q).$$

From the first statement, for some $(v_1, u_1) \in V \times U$ we may find $(v_1', u_1') \in V' \times U'$ such that

$$r_\psi \not\subseteq \overline{Q}_{((u,v),(v_1,u_1))} \text{ and } q \times p \not\subseteq \overline{Q}_{((v_1,u_1),(v_1',u_1'))}$$

where $P_{(v_1',u_1')} \not\subseteq Q_{(v',u')}$. On the other hand, since $r_\psi \not\subseteq \overline{Q}_{((u,v),(v_1,u_1))}$, we find $P_{(u,v)} \not\subseteq Q_{(u_1,v_1)}$ or equivalently, $P_{(v,u)} \not\subseteq Q_{(v_1,u_1)}$. Hence, if we consider condition R1, then we obtain $q \times p \not\subseteq \overline{Q}_{((v,u),(v_1',u_1'))}$. Therefore, by Proposition 8.4(iii), we conclude that $p \times q \not\subseteq \overline{Q}_{((u,v),(u_1',v_1'))}$. Now let us choose $u_2' \in U'$ and $v_2' \in V'$ such that

$$p \times q \not\subseteq \overline{Q}_{((u,v),(u_2',v_2'))} \text{ and } P_{(u_2',v_2')} \not\subseteq Q_{(u_1',v_1')}.$$

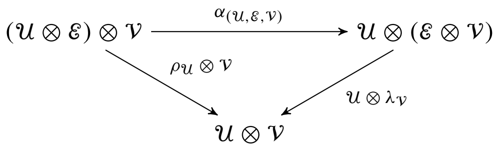
Since $\overline{P}_{((u,v),(v',u'))} \not\subseteq r_\varphi \circ (p \times q)$, we have $r_\varphi \subseteq \overline{Q}_{((u_2',v_2'),(v',u'))}$. Further, $P_{(u_2',v_2')} \not\subseteq Q_{(u_1',v_1')}$ implies that $P_{(v_2',u_2')} \not\subseteq Q_{(v_1',u_1')}$. Then we see that $\overline{P}_{((u_2',v_2'),\,(v_1',u_1'))} \subseteq r_\varphi$. But $P_{(v_1',u_1')} \not\subseteq Q_{(v',u')}$ gives a contradiction. We have showed that $(q \times p) \circ r_\psi \subseteq r_\varphi \circ (p \times q)$. The reverse inclusion and the second equality can be proved in a similar way. □

Proposition 9.5. Mac Lane's associativity and unit coherence conditions hold [25]:

(i) The following pentagonal diagram commutes:

$$\begin{array}{ccccc}
((\mathcal{U} \otimes \mathcal{V}) \otimes \mathcal{W}) \otimes \mathcal{Z} & \xrightarrow{\alpha_{(\mathcal{U},\mathcal{V},\mathcal{W})\otimes\mathcal{Z}}} & (\mathcal{U} \otimes (\mathcal{V} \otimes \mathcal{W})) \otimes \mathcal{Z} & \xrightarrow{\alpha_{(\mathcal{U},\mathcal{V}\otimes\mathcal{W},\mathcal{Z})}} & \mathcal{U} \otimes ((\mathcal{V} \otimes \mathcal{W}) \otimes \mathcal{Z}) \\
\Big\downarrow{\scriptstyle \alpha_{(\mathcal{U}\otimes\mathcal{V},\mathcal{W},\mathcal{Z})}} & & & & \Big\downarrow{\scriptstyle \mathcal{U} \otimes \alpha_{(\mathcal{V},\mathcal{W},\mathcal{Z})}} \\
(\mathcal{U} \otimes \mathcal{V}) \otimes (\mathcal{W} \otimes \mathcal{Z}) & & \xrightarrow{\alpha_{(\mathcal{U},\mathcal{V},\mathcal{W}\otimes\mathcal{Z}))}} & & \mathcal{U} \otimes (\mathcal{V} \otimes (\mathcal{W} \otimes \mathcal{Z}))
\end{array}$$

(ii) The following diagram is commutative.



Proof. (i) Consider the complemented textural isomorphisms defined by

$$\psi : ((U \times V) \times W) \times Z \to (U \times (V \times W)) \times Z,\ \psi((((u, v), w), z)) = ((u, (v, w)), z),$$
$$\varphi : (U \times (V \times W)) \times Z \to U \times ((V \times W) \times Z,)\ \varphi((u, (v, w)), z)) = (u, ((v, w), z)),$$
$$\gamma : (U \times ((V \times W) \times Z) \to U \times (V \times (W \times Z)),\ \gamma((u, ((v, w)), z))) = (u, (v, (w, z)))$$
$$\psi' : ((U \times V) \times W) \times Z \to (U \times V) \times (W \times Z),\ \psi'(((u, v), w), z)) = ((u, v),\ (w, z)),$$
$$\varphi' : (U \times V) \times (W \times Z) \to U \times (V \times (W \times Z)),\ \varphi'(((u, v),\ (w, z))) = (u, (v, (w, z))$$

for all $u \in U$, $v \in V$, $w \in W$, $z \in Z$, respectively. By Proposition 7.5(iii),

$$\gamma \circ (\varphi \circ \psi) \text{ and } \gamma' \circ \psi'$$

are also complemented textural isomorphisms. Again by Proposition 7.5(iii), for the corresponding isomorphisms in **cdrTex**, we have

$$r_{\gamma\circ(\varphi\circ\psi)} = r_\gamma \circ (r_\varphi \circ r_\psi) \ \text{ and } \ R_{\gamma\circ(\varphi\circ\psi)} = R_\gamma \circ (R_\varphi \circ R_\psi)$$

and

$$r_{\varphi'\circ\psi'} = r_{\varphi'} \circ r_{\psi'} \ \text{ and } \ R_{\varphi'\circ\psi'} = R_{\varphi'} \circ R_{\psi'}.$$

It is easy to see that we have

$$\begin{aligned} r_{\gamma\circ(\varphi\circ\psi)} &= \bigvee\{\overline{P}_{(((u,v),w),z),\ (u_1,(v_1,(w_1,z_1)))} \mid P_{((u,(v,(w,z))} \not\subseteq Q_{(u_1,(v_1,(w_1,z_1))}\} \\ &= \bigvee\{\overline{P}_{(((u,v),w),z),\ (u_1,(v_1,(w_1,z_1)))} \mid P_{\varphi'((u,v),(w,z))} \not\subseteq Q_{(u_1,(v_1,(w_1,z_1))}\} \\ &= \bigvee\{\overline{P}_{(((u,v),w),z),\ (u_1,(v_1,(w_1,z_1)))} \mid P_{\varphi'(\psi'((u,(v,(w,z))))} \not\subseteq Q_{(u_1,(v_1,(w_1,z_1))}\} \\ &= \bigvee\{\overline{P}_{(((u,v),w),z),\ (u_1,(v_1,(w_1,z_1)))} \mid P_{(\varphi\circ\psi')((u,(v,(w,z))))} \not\subseteq Q_{(u_1,(v_1,(w_1,z_1))}\} \\ &= r_{\varphi'\circ\psi'} \end{aligned}$$

and similarly

$$\begin{aligned} R_{\gamma\circ(\varphi\circ\psi)} &= \bigcap\{\overline{Q}_{(((u,v),w),z),(u_1,\ (v_1,(w_1,z_1)))} \mid P_{((u_1,(v_1,(w_1,z_1))} \not\subseteq Q_{(u,(v,(w,z))}\} \\ &= \bigcap\{\overline{Q}_{(((u,v),w),z),\ (u_1,(v_1,(w_1,z_1)))} \mid P_{(u_1,(v_1,(w_1,z_1))} \not\subseteq Q_{\varphi'((u,v),(w,z))}\} \\ &= \bigcap\{\overline{Q}_{(((u,v),w),z),\ (u_1,(v_1,(w_1,z_1)))} \mid P_{(u_1,(v_1,(w_1,z_1))} \not\subseteq Q_{\varphi'(\psi'((u,(v,(w,z))))}\} \\ &= \bigcap\{\overline{Q}_{(((u,v),w),z),\ (u_1,(v_1,(w_1,z_1)))} \mid P_{(u_1,(v_1,(w_1,z_1))} \not\subseteq Q_{(\varphi\circ\psi')((u,(v,(w,z))))}\} \\ &= R_{\varphi'\circ\psi'}. \end{aligned}$$

As a result, the isomorphisms

$$(r_\psi, R_\psi), (r_\varphi, R_\varphi), (r_\gamma, R_\gamma), (r_{\psi'}, R_{\psi'}) \ \text{ and } (r_{\varphi'} R_{\varphi'})$$

in **cdrTex** are the desired morphisms

$$\alpha_{(\mathcal{U},\mathcal{V},\mathcal{W})\otimes\mathcal{Z}},\ \alpha_{(\mathcal{U}\otimes\mathcal{V},\mathcal{W},\mathcal{Z})},\ \alpha_{(\mathcal{U},\mathcal{V}\otimes\mathcal{W},\mathcal{Z})},\ \mathcal{U} \otimes \alpha_{(\mathcal{V},\mathcal{W},\mathcal{Z})} \text{ and } \mathcal{U} \otimes \alpha_{(\mathcal{V},\mathcal{W},\mathcal{Z})}$$

satisfying the pentagonal diagram, respectively.

(ii) Let us consider the complemented textural isomorphisms defined by

$$\psi : (U \times E) \times V \to U \times (E \times V),\ \psi((u, e), v) = (u, (e, v)),$$
$$\varphi : U \times (E \times V) \to U \times V,\ \varphi((u, (e, v)) = (u, v),$$
$$\gamma : (U \times E) \times V \to U \times V,\ \varphi((u, e), v)) = (u, v).$$

Then $\varphi \circ \gamma$ is also a complemented textural isomorphism and we clearly have $\gamma = \varphi \circ \psi$. Further, by Proposition 7.5(iii), we obtain that

$$\begin{aligned} r_\gamma &= \bigvee \{\overline{P}_{(((u,e),v),(u_1,v_1))} \mid P_{(u,v)} \not\subseteq Q_{(u_1,v_1)}\} \\ &= \bigvee \{\overline{P}_{(((u,e),v),(u_1,v_1))} \mid P_{\varphi((u,(e,v))} \not\subseteq Q_{(u_1,v_1)}\} \\ &= \bigvee \{\overline{P}_{(((u,e),v),(u_1,v_1))} \mid P_{\varphi(\psi((u,e),v))} \not\subseteq Q_{(u_1,v_1)}\} \\ &= \bigvee \{\overline{P}_{(((u,e),v),(u_1,v_1))} \mid P_{(\varphi\circ\psi)(((u,e),v)))} \not\subseteq Q_{(u_1,v_1)}\} \\ &= r_{\varphi\circ\psi} \\ &= r_\varphi \circ r_\psi \end{aligned}$$

and

$$\begin{aligned} R_\gamma &= \bigcap \{\overline{Q}_{(((u,e),v),(u_1,v_1))} \mid P_{(u_1,v_1)} \not\subseteq Q_{(u,v)}\} \\ &= \bigcap \{\overline{Q}_{(((u,e),v),(u_1,v_1))} \mid P_{(u_1,v_1)} \not\subseteq Q_{\varphi((u,(e,v))}\} \\ &= \bigcap \{\overline{Q}_{(((u,e),v),(u_1,v_1))} \mid P_{(u_1,v_1)} \not\subseteq Q_{\varphi(\psi((u,e),v))}\} \\ &= \bigcap \{\overline{Q}_{(((u,e),v),(u_1,v_1))} \mid P_{(u_1,v_1)} \not\subseteq Q_{(\varphi\circ\psi)(((u,e),v))}\} \\ &= R_{\varphi\circ\psi} \\ &= R_\varphi \circ R_\psi . \end{aligned}$$

Then the corresponding isomorphisms (r_ψ, R_ψ), (r_φ, R_φ) and (r_γ, R_γ) in **cdrTex** are the desired morphisms

$$\alpha_{(\mathcal{U},\mathcal{E},\mathcal{V})},\ \rho_{\mathcal{U}} \otimes \mathcal{V},\ \mathcal{U} \otimes \lambda_{\mathcal{V}}$$

satisfying the diagram, respectively. □

Corollary 9.6. *The categories* **drTex** *and* **cdrTex** *are dagger symmetric monoidal categories.*

Proof. It is immediate from Proposition 8.1(ii) and 9.5, and Corollary 9.4. □

10. Conclusions

In this paper, we have considered a rough set model on two universes. We have determined the position of the theory of rough sets with respect to category **REL** of sets and relations. We have shown that the categories **REL** and **R-APR** are isomorphic. In view of this argument, we have obtained that **R-APR** and **REL** are a full subcategories of the category **cdrTex** of complemented textures and complemented direlations. Further, we have shown that **cdrTex** and **R-APR** are new examples of dagger symmetric monoidal categories.

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