PERGAMON

Applied Mathematics Letters 12 (1999) 65–69

Applied Mathematics Letters

Blow-Up of Solutions to Quasilinear Parabolic Equations

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(Received February 1998; accepted March 1998)

Communicated by C. Bardos

Abstract—Sufficient conditions for global nonexistence (blow up) of solutions of the initial-boundary value problem for a class of second-order quasilinear parabolic equations:

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(d + |\nabla u|^{p-2} \right), \frac{\partial u}{\partial x_i} \right) + g(u, \nabla u) = f(u)$$

are established.

Keywords—Quasilinear parabolic equations, Blow-up.

In this note, we consider the following problem:

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(d + |\nabla u|^{p-2} \right), \frac{\partial u}{\partial x_i} \right) + g(u, \nabla u) = f(u), \tag{1}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{3}$$

where Ω is a domain in R^n, $p \geq 2$, g, and f are continuous functions which satisfy the following conditions:

$$(f(u), u) \geq 2(\alpha + 1)G(u), \qquad \forall\, u \in R^1, \quad \alpha > 0, \tag{4}$$

where

$$G(u) = \int_0^u f(s)\, ds, \tag{5}$$

$$|g(u,v)| \leq c_1 \left(|u| + |v| \right), \qquad c_1 > 0, \quad \forall\, u \in R^1, \quad \forall\, v \in R^n. \tag{6}$$

Our main goal is to find sufficient conditions for global nonexistence of solutions of problems (1)–(3).

 Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
PII: S0893-9659(98)00174-8

Global nonexistence theorem for problems (1)–(3) when $g \equiv 0$ and f satisfies condition (4) is proven in [1] (see also [2]). Sufficient conditions of global nonexistence of solutions of (1)–(3), when $p = 2$ with f satisfying (4) and g satisfying (6) are obtained in [3].

So our result is a generalization of corresponding result of [1,3] for second-order quasilinear parabolic equations.

THEOREM. *Suppose that all conditions formulated above are satisfied for the functions f, G, and g. Let u be the solution of problems (1)–(3). Assume that following conditions are valid:*

$$\alpha_1 = \sqrt{1+\beta} - 1, \quad \beta \in (0, \alpha), \quad \alpha > \frac{p-2}{2}, \qquad \gamma_1 = 0, \quad \gamma_2 = -2c_2,$$

where

$$c_2 = c_1 \left(1 + \frac{c_1}{2\epsilon_1}\right), \qquad \epsilon_1 > 0, \quad \alpha d > \epsilon_1,$$

and

$$j(0) \equiv -\frac{d}{2} \|\nabla u_0\|^2 - \frac{1}{p} \int_\Omega \|\nabla u_0\|^p \, dx - \frac{\lambda}{2} \|u_0\|^2 + G(u_0) > 0,$$
$$\alpha_1^2(\alpha+1)j(0) + \gamma_2(1+\alpha_1)^2 \|u_0\|^2 > 0,$$

where $\lambda \geq (1/2\alpha)(c_1^2/\epsilon_0 d)(1+d_1)$ and $d_1 = \lambda_1^{-1}$, where λ_1

$$-\Delta\varphi = \lambda_1\varphi, \qquad \varphi|_{\partial\Omega} = 0.$$

Then $\|u(.,t)\|_{L_2(\Omega)} \to \infty$ as $t \to t_1$, where

$$t_1 \leq \frac{1}{2c_2} \ln \frac{\alpha_1^2(\alpha+1)j(0)}{\gamma_2(1+\alpha_1)^2 \|u_0\|^2 + \alpha_1^2(\alpha+1)j(0)}.$$

PROOF. In order to prove this theorem, we will use the following Lemma 1.1 [3].

LEMMA 1.1. *Suppose that a positive, twice differentiable function $\Psi(t)$ satisfies on $t \geq 0$ the inequality*

$$\Psi''\Psi - (1+\alpha_1)(\Psi')^2 \geq -2M_1\Psi\Psi' - M_2\Psi^2,$$

where $\alpha_1 > 0$, $M_1, M_2 \geq 0$, if $\Psi(0) > 0$, $\Psi'(0) > -\gamma_2\alpha_1^{-1}\Psi(0)$, and $M_1 + M_2 > 0$, then $\Psi(t)$ tends to infinity $t \longrightarrow t_1 \leq t_2$.

$$t_2 = \frac{1}{2\sqrt{M_1^2 + \alpha_1 M_2}} \ln \frac{\gamma_1\Psi(0) + \alpha_1\Psi'(0)}{\gamma_2\Psi(0) + \alpha_1\Psi'(0)},$$

where $\gamma_{1,2} = -M_1 \mp \sqrt{M_1^2 + \alpha_1 M_2}$. If $\Psi(0) > 0$, $\Psi'(0) > 0$, and $M_1 = M_2 = 0$, then $\Psi(t) \to \infty$ as $t \longrightarrow t_1 \leq t_2 = \Psi(0)/\alpha_1\Psi'(0)$.

Let us set the function $v(x,t) = e^{-\lambda t}u(x,t)$ where u is the solution of problems (1)–(3). We get the following equation for v:

$$\lambda e^{\lambda t} v + e^{\lambda t} v_t - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(d e^{\lambda t} \frac{\partial v}{\partial x_i}\right)$$
$$- \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(e^{(p-2)\lambda t} |\nabla v|^{p-2} e^{\lambda t} \frac{\partial v}{\partial x_i}\right) + g\left(e^{\lambda t} v, e^{\lambda t}\nabla v\right) = f\left(e^{\lambda t} v\right).$$

We can rewrite problems (1)–(3) as follows:

$$v_t + \lambda v - d\Delta v - e^{(p-2)\lambda t} \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) + \widetilde{g}(v, \nabla v) = \widetilde{F}(t, v), \tag{7}$$

$v(x,0) = v_0(x),\ x \in \Omega,\ v(x,t) = 0,\ x \in \partial\Omega,\ t > 0$, where $\widetilde{F}(t,v) = e^{-\lambda t} f(e^{\lambda t}v)$, $\widetilde{g}(v,\nabla v) = e^{-\lambda t} g(e^{\lambda t}v, e^{\lambda t}\nabla v)$.

With the help of conditions (4),(6), we can easily obtain the following conditions for the functions $\widetilde{F}(t,v)$, $\widetilde{g}(v,\nabla v)$, and $\widetilde{G}(t,v) = e^{-2\lambda t} G(e^{\lambda t}v)$:

$$\begin{aligned}\left(\widetilde{F}(t,v),v\right) - 2(\alpha+1)\widetilde{G}(t,v) &= \left(e^{-2\lambda t} f\left(e^{\lambda t}v\right), e^{\lambda t}v\right) - 2(\alpha+1)e^{-2\lambda t} G\left(e^{\lambda t}v\right)\\ &= e^{-2\lambda t}\left[(f(u),u) - 2(\alpha+1)G(u)\right] \geq 0, \qquad (8)\\ &\text{so we have } \left(\widetilde{F}(t,v),v\right) - 2(\alpha+1)\widetilde{G}(t,v) \geq 0,\end{aligned}$$

$$\begin{aligned}\widetilde{g}(v,\nabla v) &= e^{-\lambda t} g\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right) \leq e^{-\lambda t} c_1 \left(\left|e^{\lambda t}v\right| + \left|e^{\lambda t}\nabla v\right|\right)\\ &= c_1\left(|v| + |\nabla v|\right), \quad \text{so we have } \widetilde{g}(v,\nabla v) \leq c_1\left(|v|+|\nabla v|\right), \qquad (9)\end{aligned}$$

$$\begin{aligned}\frac{d}{d\tau}\widetilde{G}\left(t,v(\tau)\right) &= \frac{\partial}{\partial\tau}\left(e^{-2\lambda t} G(e^{\lambda t}v)\right)\\ &= e^{-2\lambda t}\frac{d}{d\tau} G\left(e^{\lambda t}v(\tau)\right) = e^{-2\lambda t}\left(f\left(e^{\lambda t}v(\tau)\right), e^{\lambda t}v_\tau\right) \qquad (10)\\ &= \left(e^{-\lambda t} f\left(e^{\lambda t}v(\tau)\right), v_\tau\right) = \left(\widetilde{F}(t,v), v_\tau\right),\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\widetilde{G}(t,v(t)) &= -2\lambda e^{-2\lambda t} G\left(e^{\lambda t}v\right) + e^{-2\lambda t}\frac{d}{dt}\widetilde{G}\left(e^{\lambda t}v(t)\right)\\ &= -2\lambda e^{-2\lambda t} G\left(e^{\lambda t}v\right) + e^{-2\lambda t} f\left(e^{\lambda t}v(t)\right)\left(\lambda e^{\lambda t}v + e^{\lambda t}v_t\right) \qquad (11)\\ &= \widetilde{G}_t\left(t,v(\tau)\right) + \left(\widetilde{F}(t,v), v_t\right),\end{aligned}$$

$$\begin{aligned}\widetilde{G}_t(t,v) &= -2\lambda e^{-2\lambda t} G\left(e^{\lambda t}v\right) + e^{-2\lambda t} G_t\left(e^{\lambda t}v\right)\\ &= -2\lambda e^{-2\lambda t} G\left(e^{\lambda t}v\right) + e^{-2\lambda t}\left(f\left(e^{\lambda t}v\right), \lambda e^{\lambda t}v\right) \qquad (12)\\ &= -2\lambda\widetilde{G}(t,v) + \lambda\left(\widetilde{F}(t,v),v\right) \geq -2\lambda\widetilde{G}(t,v) + 2\lambda(\alpha+1)\widetilde{G}(t,v),\\ &\text{so we have } \widetilde{G}_t(t,v) \geq 2\lambda\alpha\widetilde{G}(t,v).\end{aligned}$$

Let us take the scaler product of (7) with v_t in $L_2(\Omega)$

$$\begin{aligned}&(v_t,v_t) + \lambda(v,v_t) - d(\Delta v, v_t)\\ &\quad - e^{(p-2)\lambda t}\left(\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right), v_t\right) + \left(\widetilde{g}(v,\nabla v), v_t\right) = \left(\widetilde{F}(t,v), v_t\right).\end{aligned} \qquad (13)$$

Using the relation

$$\int_\Omega \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) v_t\,dx = -\frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla v|^p\,dx \quad \text{and} \quad (\Delta v, v_t) = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2\,dx,$$

and (11) we get from (13) the relation

$$\|v_t\|^2 + \frac{\lambda}{2}\frac{d}{dt}\|v\|^2 + \frac{d}{2}\frac{d}{dt}\|\nabla v\|^2 + \frac{1}{p}e^{(p-2)\lambda t}\frac{d}{dt}\int_\Omega |\nabla v|^p\,dx + \left(\widetilde{g}(v,\nabla v), v_t\right) = \frac{d}{dt}\widetilde{G}(t,v) - \widetilde{G}_t(t,v).$$

By using condition (9), the Friedrichs inequality $\|v\|^2 \leq d_1\|\nabla v\|^2$ and the inequality $a.b \leq \epsilon_0 a^2 + (1/4\epsilon_0)b^2$, we can get from the previous equality

$$\begin{aligned}&\frac{d}{2}\frac{d}{dt}\|\nabla v\|^2 + \frac{1}{p}e^{(p-2)\lambda t}\frac{d}{dt}\int_\Omega |\nabla v|^p\,dx + \frac{\lambda}{2}\frac{d}{dt}\|v\|^2 - \frac{d}{dt}\widetilde{G}(t,v)\\ &\qquad \leq -\|v_t\|^2 + c_1\int_\Omega |v|\,|v_t|\,dx + c_1\int_\Omega |\nabla v|\,|v_t|\,dx - \widetilde{G}_t(t,v),\end{aligned}$$

$$\frac{d}{2}\frac{d}{dt}\|\nabla v\|^2+\frac{1}{p}e^{(p-2)\lambda t}\frac{d}{dt}\int_\Omega|\nabla v|^p\,dx+\frac{\lambda}{2}\frac{d}{dt}\|v\|^2-\frac{d}{dt}\widetilde{G}(t,v)$$
$$\leq-\|v_t\|^2+\frac{c_1^2}{4\epsilon_0}\|v\|^2+\epsilon_0\|v_t\|^2+\frac{c_1^2}{4\epsilon_0}\|\nabla v\|^2+\epsilon_0\|v_t\|^2-\widetilde{G}_t(t,v),$$

so we have

$$\frac{d}{dt}\left[\frac{d}{2}\|\nabla v\|^2+\frac{1}{p}e^{(p-2)\lambda t}\int_\Omega|\nabla v|^p\,dx+\frac{\lambda}{2}\|v\|^2-\widetilde{G}(t,v)\right]$$
$$\leq(2\epsilon_0-1)\|v_t\|^2+\frac{c_1^2}{4\epsilon_0}(1+d_1)\|\nabla v\|^2+\frac{p-2}{p}\lambda e^{(p-2)\lambda t}\int_\Omega|\nabla v|^p\,dx-\widetilde{G}_t(t,v).$$

Using (12) in the last inequality we obtain

$$\frac{d}{dt}(j(t))\geq(1-2\epsilon_0)\|v_t\|^2-\frac{c_1^2}{4\epsilon_0}(1+d_1)\|\nabla v\|^2--\frac{p-2}{p}\lambda e^{(p-2)\lambda t}\int_\Omega|\nabla v|^p\,dx+2\lambda\alpha\widetilde{G}(t,v),$$

where

$$j(t)=-\frac{d}{2}\|\nabla v\|^2-\frac{1}{p}e^{(p-2)\lambda t}\int_\Omega|\nabla v|^p\,dx-\frac{\lambda}{2}\|v\|^2+\widetilde{G}(t,v).$$

We can rewrite the last inequality as follows:

$$\frac{d}{dt}(j(t))\geq(1-2\epsilon_0)\|v_t\|^2+\frac{2c_1^2}{4\epsilon_0 d}(1+d_1)\left(-\frac{d}{2}\|\nabla v\|^2\right)$$
$$+(p-2)\lambda\left(-\frac{1}{p}e^{(p-2)\lambda t}\int_\Omega|\nabla v|^p\,dx\right)+2\lambda\alpha\widetilde{G}(t,v). \tag{14}$$

Then

$$\frac{d}{dt}(j(t))\geq(1-2\epsilon_0)\|v_t\|^2+2\lambda\alpha j(t) \tag{15}$$

is obtained.

We get following equation by taking the scalar product of (7) with v:

$$(v_t,v)+\lambda(v,v)-(d\Delta v,v)-e^{(p-2)\lambda t}\left(\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right),v\right)+(\widetilde{g}(v,\nabla v),v)=\left(\widetilde{F}(t,v),v\right). \tag{16}$$

By using the equality

$$\int_\Omega\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)v\,dx=-\int_\Omega|\nabla v|^p\,dx,$$

conditions (8),(9) and the inequality $a.b\leq\epsilon_1/2\|a\|^2+1/2\epsilon_1\|b\|^2$ in (16), we can get the following inequality:

$$\frac{1}{2}\frac{d}{dt}\|v\|^2\geq-d\|\nabla v\|^2-\lambda\|v\|^2-e^{(p-2)\lambda t}\int_\Omega|\nabla v|^p\,dx$$
$$-c_1\|v\|^2-\epsilon_1\|\nabla v\|^2-\frac{c_1^2}{2\epsilon_1}\|v\|^2+2(\alpha+1)\widetilde{G}(t,v).$$

The last inequality can be written as follows:

$$\frac{1}{2}\frac{d}{dt}\|v\|^2\geq2(\alpha+1)\left[-\frac{d}{2}\|\nabla v\|^2-\frac{e^{(p-2)\lambda t}}{p}\int_\Omega|\nabla v|^p\,dx\right.$$
$$\left.-\frac{\lambda}{2}\|v\|^2+\widetilde{G}(t,v)\right]-c_1\left(1+\frac{c_1}{2\epsilon_1}\right)\|v\|^2.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2\geq2(\alpha+1)j(t)-c_2\|v\|^2, \tag{17}$$

where $c_2 = c_1(1 + (c_1/2\epsilon_1))$. From (15) we obtain

$$\frac{d}{dt}\left(e^{-2\lambda\alpha t} \jmath(t)\right) \geq (1 - 2\epsilon_0)\, e^{-2\lambda\alpha t} \|v_t\|^2 .$$

Integrating the previous inequality over $(0, t)$, we get

$$\jmath(t) \geq (1 - 2\epsilon_0) \int_0^t \|v_s\|^2 \, ds + \jmath(0), \tag{18}$$

$$\jmath(t) \geq \frac{1+\beta}{1+\alpha} \int_0^t \|v_s\|^2 \, ds + \jmath(0), \tag{19}$$

where $2\epsilon_0 = (\alpha - \beta)/(1 + \alpha)$.

By using the inequality (19) in the inequality (17), the following result is obtained:

$$\frac{d}{dt} \|v\|^2 \geq 4(1+\beta) \int_0^t \|v_s\|^2 \, ds + 4(\alpha+1)\jmath(0) - 2c_2 \|v\|^2 . \tag{20}$$

Let us set $\Psi(t) = \int_0^t \|v\|^2 \, ds + c_3$. We will show that the function $\Psi(t)$ satisfies the conditions of Lemma 1.1 for suitable choice of positive constant c_3. We introduce the notations $A_1 = \int_0^t \|v\|^2 \, ds$ and $A_2 = \int_0^t \|v_s\|^2 \, ds$, so that

$$\Psi(t) = A_1 + c_3. \tag{21}$$

By using the Cauchy-Schwartz inequality, we can get

$$\Psi'(t) \leq 2\sqrt{A_1 A_2} + \|v_0\|^2 . \tag{22}$$

With help of (20)–(22) we can estimate the term $\Psi''(t)\Psi(t) - (1 + \alpha_1)(\Psi'(t))^2$ as follows:

$$\begin{aligned}
\Psi''(t)\Psi(t) - (1+\alpha_1)\left(\Psi'(t)\right)^2 &\geq (4(1+\beta)A_2 + 4(\alpha+1)\jmath(0) \\
&\quad - 2c_2\Psi'(t))(A_1 + c_3) - (1+\alpha_1)\left[4A_1A_2 + \|v_0\|^4 + 4\epsilon_2 A_1 A_2 + \frac{1}{\epsilon_2}\|v_0\|^4\right] \\
&\geq 4(1+\beta)A_2A_1 + 4(1+\beta)A_2c_3 + 4(\alpha+1)\jmath(0)A_1 + 4(\alpha+1)\jmath(0)c_3 \\
&\quad - 2c_2\Psi'(t)\Psi(t) - 4(1+\alpha_1)(1+\epsilon_2)A_1A_2 - (1+\alpha_1)\left(1 + \frac{1}{\epsilon_2}\right)\|v_0\|^4 \\
&\geq 4(\alpha+1)\jmath(0)c_3 - 4(1+\alpha_1)\left(1 + \frac{1}{\epsilon_2}\right)\|v_0\|^4 - 2c_2\Psi'(t)\Psi(t).
\end{aligned}$$

We get the result

$$\Psi''(t)\Psi(t) - (1+\alpha_1)\left(\Psi'(t)\right)^2 \geq -2c_2\Psi'(t)\Psi(t),$$

where

$$c_3 = \frac{(1+\alpha_1)^2 \|v_0\|^4}{\alpha_1(\alpha+1)\jmath(0)}, \quad \alpha_1 = \epsilon_2, \quad \text{and} \quad (1+\alpha_1)^2 = (1+\beta).$$

Since u_0 satisfies the condition

$$1 \geq \frac{-\gamma_2(1+\alpha_1)^2 \|u_0\|^2}{\alpha_1^2(\alpha+1)\jmath(0)},$$

we have $\Psi'(0) > -(\gamma_2/\alpha_1)\Psi(0)$. Thus, according to the Lemma 1.1, $\Psi(t)$ tends to infinity for $t \longrightarrow t_1 - 0$,

$$t_1 = \frac{1}{2c_2} \ln \frac{\alpha_1^2(\alpha+1)\jmath(0)}{\gamma_2(1+\alpha_1)^2 \|v_0\|^2 + \alpha_1^2(\alpha+1)\jmath(0)}.$$

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