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Asymptotic calculation of inviscidly absolutely unstable modes of the compressible boundary layer on a rotating disk

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Abstract

In this work a long-wavelength asymptotic approach is used to analyze the region of absolute instability in the compressible rotating disk boundary layer flow. Theoretically determined values of branch points for the occurrence of absolute instability in the compressible flow are shown to match onto the ones which are obtained via a numerical solution of the linear inviscid compressible Rayleigh equations.

Keywords: Long-wavelength analysis; Compressible rotating disk flow; Absolute instability

1. Introduction

The flow over a rotating disk has attracted a great deal of attention over the last decade as regards the determination of the character of instabilities in terms of their absolute as well as convective nature. This aspect is of crucial significance since it is expected to shed light on the similar instabilities existing over real aircraft wings.

The convective nature of the instability of the rotating disk boundary layer flow has been investigated by a broad investigator group; see for instance the experimental studies of [1] and [2], and the numerical and theoretical works of [1,3] and [4], amongst many others. Since the pioneering work of [5] with regard to the existence of absolute instability in fluid dynamic problems, see also [6,7] and [8], attention has been focused on research into the absolute type of instability. Absolute instability occurs simply when the small amplitude disturbances introduced into a system start growing exponentially in time at every fixed position in space, while in convective instability growing wave packets desert the place of excitation. Plane Poiseuille flow is an example of convective instability and Taylor–Couette flow is an example of absolute instability.

The current study is devoted to the absolute rather than the convective instability of the rotating disk flow. This instability was first explored by [9–11] and [12]; see also the recent study of [13]. Making use of the Briggs–Bers criterion and assuming that the flow is parallel, the latter authors were able to show that the flow becomes both inviscidly and viscously absolutely unstable. The main conclusion from this research is that the absolute instability

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mechanism found in this flow causes the disturbances to grow exponentially at a fixed radius, leading to an unbounded linear response that would promote non-linearity followed by transition.

In the work of [14] the absolutely unstable inviscid regime was identified with the use of analytical tools, which adequately matched with the numerical calculations of [10] and [12] when the compressibility was ignored. Therefore, our main interest in the present research is to extend the findings of [14] to cases in which the compressibility is important. For this reason the analytical branch points generated using the long-wavelength analysis of [15] are compared with the ones obtained from the Rayleigh solver in the case of wall insulation as well as heat transfer. The criteria used for absolute or convective instability and the terminology used throughout closely follow the derivation presented in [16]. The branch points where the group velocity tends to zero are searched for in the relevant eigenvalue planes. The numerical calculations are done using a spectral collocation technique as well as a fourth-order Runge–Kutta integrator. The stability characteristics of the Von Kármán velocity profile are then examined and the absolute instability range associated with this profile is determined.

We adopt the following strategy in this work. First, governing equations for the fluid motion are given in Section 2 followed by the asymptotic expansion of flow quantities and their analysis in the regions identified. Second, analytical and numerical results are compared in Section 3. Conclusions are finally drawn in Section 4.

2. Governing equations and the dispersion relation

Our concern here is with the motion of a three-dimensional inviscid compressible boundary layer flow adjacent to a disk rotating about its axis of rotation z with a constant angular velocity ω_a. The flow has kinematic viscosity ν_∞, and the cylindrical polar coordinates (r, θ, z) are made dimensionless with respect to a reference length scale l, followed by non-dimensionalization of other flow variables using $l\omega_a$. The characterizing parameter, the Reynolds number of the flow, which is based on the local angular velocity, is defined as $R_e = \omega_a l^2/\nu_\infty$, and it is assumed to be large for the following analysis. The inviscid Rayleigh equations governing the evolution of long-wavelength perturbations are then given by

$$\begin{aligned}
&\mathrm{i}[\alpha r F + \beta G - \Omega]\rho + \mathrm{i}\rho_B\left[\alpha u + \frac{\beta}{r}v\right] + (\rho_B w)' = \mathrm{O}\left(\frac{1}{Re}\right),\\
&\rho_B\left[\mathrm{i}\,[\alpha r F + \beta G - \Omega]\,u + rF'w\right] + \mathrm{i}\alpha p = \mathrm{O}\left(\frac{1}{Re}\right),\\
&\rho_B\left[\mathrm{i}\,[\alpha r F + \beta G - \Omega]\,v + rG'w\right] + \frac{\mathrm{i}\beta}{r}p = \mathrm{O}\left(\frac{1}{Re}\right),\\
&\rho_B\,[\mathrm{i}\,[\alpha r F + \beta G - \Omega]\,w] + p' = \mathrm{O}\left(\frac{1}{Re}\right),\\
&\rho_B\,[\mathrm{i}\,[\alpha r F + \beta G - \Omega]\,p] = \Gamma P_B\left[\mathrm{i}\,[\alpha r F + \beta G - \Omega]\,\rho + \rho_B' w\right] + \mathrm{O}\left(\frac{1}{Re}\right),\\
&\Gamma M_\infty^2 p = T_B\rho + \rho_B T.
\end{aligned} \tag{1}$$

Here a prime denotes differentiation with respect to a scaled parameter $Y = Re^{\frac{1}{2}}z$ that defines the boundary layer coordinate. The perturbation quantities ρ, u, v, w, p and T are respectively the density, instantaneous three-velocity components, pressure and temperature. Moreover, the parameters Γ and M_∞ are respectively the ratio of specific heats and the free stream Mach number. Finally, the mean flow (the terms in Eq. (1) given by the suffix B) is disturbed by perturbations proportional to $\mathrm{e}^{\mathrm{i}(\alpha r + \beta\theta - \Omega t)}$, where α and β are disturbance wavenumbers and Ω the disturbance frequency.

The generalized Von Kármán solution for the compressible three-dimensional mean velocity U_B, pressure P_B and density ρ_B are given in the form

$$U_B = (rF, rG, Re^{-\frac{1}{2}}H), \qquad P_B = \frac{1}{\Gamma M_\infty^2}, \qquad \rho_B = \frac{1}{T_B},$$

where the functions F, G and H satisfy the following ordinary differential equations and boundary conditions:

$$
\begin{aligned}
&F^2-(G+1)^2+F'H-F''=0, \qquad 2F(G+1)+G'H-G''=0,\\
&2F+H'=0, \qquad F(0)=G(0)=H(0)=0, \qquad F(\infty)=0,\ G(\infty)=-1.
\end{aligned} \tag{2}
$$

The basic temperature field can be written in terms of a viscous dissipation term f and a heat conducting term q; see [17] and [18]. Assuming that an ideal fluid enables us to express f and q in terms of the velocity components, see for instance [19], the basic temperature field can be represented by

$$
T_B(Y)=1-\frac{(\Gamma-1)}{2}M^2 f(Y)+(T_w-1)q(Y), \tag{3}
$$

where $M=rM_\infty$ is the local Mach number and T_w is the wall temperature value. The viscous dissipation term f is found to be $f=F^2+G^2-1$ for an insulated surface ($q\equiv 0$), and $f=F^2+G^2+G$ for surface heat conduction.

The asymptotic analysis depends crucially on the properties of the basic flow near the wall where generally a critical layer is situated. Therefore, we can write the basic velocity profiles near the surface of the rigid disk employing the Taylor expansion

$$
(F,G,\rho_B,T_B)=(0,0,R_0,S_0)+(\lambda_1,\mu_1,R_1,S_1)Y+(\lambda_2,\mu_2,R_2,S_2)Y^2+\cdots, \tag{4}
$$

where the coefficients λ_1, μ_1, etc., depend upon r and the Mach number. For insulated wall conditions, we simply have $R_1=S_1=0$, and if there is heat transfer through the surface, then $S_1\neq 0$. For further details of the basic flow see [20].

Via a technique of dominant balance between the unsteady terms and viscous terms (of order of magnitude R_e^{-1} on the right hand side of Eq. (1)), it can be concluded that a viscous layer of thickness of $O(R_e^{-1/2})$ exists, which we denote by h in what follows. Following next the works of [15,21], the wavenumbers and frequency can be sought for in terms of this small parameter h as follows:

$$
(\alpha,\beta,\Omega)=(\alpha_0,\beta_0,0)h+(\alpha_1,\beta_1,\Omega_0)h^2+(\alpha_2,\beta_2,\Omega_1)h^3+\cdots. \tag{5}
$$

As implemented in [14], the long-wavelength analysis is based on the method of matched asymptotic expansions for obtaining solutions to Eq. (1) as $R_e\to\infty$. Thus, in the main part of the boundary layer in which the flow is viscous, flow variables are expanded in accordance with the boundary layer scale Y as

$$
(u,v,w,p,\rho,T)=(\tilde{u}_0,\tilde{v}_0,0,0,\tilde{\rho}_0,\tilde{T}_0)+h(\tilde{u}_1,\tilde{v}_1,\tilde{w}_0,\tilde{p}_0,\tilde{\rho}_1,\tilde{T}_1)+\cdots. \tag{6}
$$

Substituting these expansions into the linearized governing Eq. (1), we obtain the leading-order solutions depending upon a displacement function $\tilde{A}$,

$$
\tilde{p}_0=\tilde{P}_0(r,\theta),\ \tilde{u}_0=r\tilde{A}F',\ \tilde{v}_0=r\tilde{A}G',\ \tilde{w}_0=-\mathrm{i}\tilde{A}\bar{U}_B(Y),\ \tilde{\rho}_0=\tilde{A}\rho_B',\ \tilde{T}_0=-\frac{T_B}{\rho_B}\tilde{A}\rho_B'. \tag{7}
$$

To avoid the singularity occurring at the next-order solutions in the viscous layer, a wall layer is required whose thickness is $O(h)$, yielding a new coordinate $\bar{Y}$ such that $Y=h\bar{Y}$. Thus, in this zone the appropriate expansions are

$$
(u,v,w,p,\rho,T)=(\bar{u}_0,\bar{v}_0,0,0,\bar{\rho}_0,\bar{T}_0)+h(\bar{u}_1,\bar{v}_1,0,\bar{p}_0,\bar{\rho}_1,\bar{T}_1)+\cdots. \tag{8}
$$

Upon substitution of (8) into the governing Eq. (1) the leading-order solutions p_0 and w_0 are obtained as follows:

$$
\bar{p}_0=\bar{P}_0(r,\theta),\ \bar{w}_0=-\frac{\mathrm{i}\gamma^2}{R_0\Lambda_1}\bar{p}_0-\mathrm{i}\tilde{A}(\Lambda_1\bar{Y}-\Omega_0), \tag{9}
$$

where $\gamma^2=\alpha_0^2+\frac{\beta_0^2}{r^2}$ and $\Lambda_1=\alpha_0 r\lambda_1+\beta_0\mu_1$. The boundary layer flow must also match with the potential flow outside the boundary layer region, introducing a new coordinate $Y=\frac{\hat{Y}}{h}$, and thus the flow quantities here expand as

$$
(u,v,w,p,\rho,T)=h(\hat{u}_0,\hat{v}_0,\hat{w}_0,\hat{p}_0,\hat{\rho}_0,\hat{T}_0)+\cdots, \tag{10}
$$

Again substituting these into the linearized governing Eq. (1) gives solutions, some of which are

$$\hat{p}_1 = \left[\hat{P}_1(r,\theta) - \frac{1}{\nu}\left(\frac{\gamma^2-\nu^2}{\bar{U}_\infty}\Omega_0\right)\hat{Y}\,\hat{p}_0(r,\theta)\right]\mathrm{e}^{-\nu\hat{Y}},$$
$$\hat{w}_1 = -\mathrm{i}\left[\left[\frac{\Omega_0\nu}{\bar{U}_\infty^2} + \left(\frac{\gamma^2-\nu^2}{\bar{U}_\infty^2}\Omega_0\right)\left(\frac{1}{\nu}-\hat{Y}\right)\right]\hat{P}_0(r,\theta) + \frac{\nu}{\bar{U}_\infty}\hat{P}_1(r,\theta)\right]\mathrm{e}^{-\nu\hat{Y}}, \tag{11}$$

in which $U_\infty = \alpha_0 r F_\infty + \beta_0 G_\infty$ and $\nu^2 = \gamma^2 - U_\infty^2 M_\infty^2$ (a subscript ∞ indicates a free-stream value). For further details on the solutions one can refer to [14].

Having found the solutions in each of the asymptotic regions, a analysis of matching between the pressure p and the normal velocity w is pursued, resulting in

$$\Omega_0 = \frac{\gamma^2 U_\infty^2}{R_0 \Lambda_1 \nu}, \qquad \Omega_1 = (\mathrm{i}c - k + \Phi)\Omega_0^2, \tag{12}$$

where $\Phi = \frac{\Lambda_1}{\Omega_0^2}[\frac{\Omega_0 R_0}{\mu(S_0)}]^{1/2}(-\mathrm{i})^{1/2}$ comes in due to viscous correction through the Stokes layer, Chapman's viscosity law is adopted with $\mu(S_0) = S_0$, and also k and c are functions of (α_0, Ω_0) found to be

$$k = \frac{1}{\bar{U}_\infty}\left[1+\frac{\gamma^2}{\nu^2}\right] + \frac{1}{\Lambda_1}\left[\frac{2\Lambda_2}{\Lambda_1} + \frac{R_1}{R_0}\right][\ln|\Omega_0|+1]$$
$$+ R_0\Lambda_1\left[I_2^\infty - I_2^0 - \frac{\nu^2}{\gamma^2}\frac{I_1}{\bar{U}_\infty^4} - \frac{1}{\bar{U}_\infty^2} + \frac{\Lambda_2-\Lambda_1}{\Lambda_1^3}\right], \tag{13}$$
$$c = \frac{\pi}{\Lambda_1}\left[\frac{2\Lambda_2}{\Lambda_1} + \frac{R_1}{R_0}\right]\mathrm{sign}(\Lambda_1).$$

It should be remarked here that the term $\mathrm{sign}(\Lambda_1)$ arises from the jump condition in the linear critical layer theory. Moreover, the parameters appearing in (13) involve integrals and are respectively given by $I_1 = \int_0^\infty(\rho_B\bar{U}_B^2 - \rho_\infty\bar{U}_\infty^2)\,\mathrm{d}Y$,

$$I_2^0 = \int_1^0\left[\frac{1}{\rho_B\bar{U}_B^2} - \frac{1}{R_0\Lambda_1^2Y^2} + \frac{1}{R_0\Lambda_1^2Y}\left[\frac{2\Lambda_2}{\Lambda_1} + \frac{R_1}{R_0}\right]\right]\mathrm{d}Y \text{ and } I_2^\infty = \int_1^\infty\left(\frac{1}{\rho_B\bar{U}_B^2} - \frac{1}{\rho_\infty\bar{U}_\infty^2}\right)\mathrm{d}Y.$$

Finally, from the condition that the group velocity, which can be expanded from (5) as $\frac{\partial\Omega}{\partial\alpha} = \frac{\partial\Omega_0}{\partial\alpha} + \frac{\partial\Omega_1}{\partial\alpha} + \mathrm{O}(h^3)$, vanishes whenever absolute instability is present, a dispersion relation for the occurrence of branch points is obtained, written implicitly in the form

$$F(\alpha, \beta, M, T_w) = 0. \tag{14}$$

3. Results and discussion

Although some interesting cases associated with the transonic ($M \to 1$) and also with the hypersonic flow regimes ($M \gg 1$) can be further investigated analytically through the relations (12)–(14), since the corresponding expressions are rather lengthy, we prefer instead a straightforward numerical treatment to locate the branch points satisfying the Briggs–Bers pinching criterion. By setting $M = 0$, $R_0 = 1$ and $R_1 = 0$ in (14), the results of [14] for the incompressible flow case are easily recovered, which will also be shown graphically later.

Eq. (14) was next treated numerically with a Newton iteration technique. The effects of compression arising by means of wall insulation and heat transfer on the long-wavelength perturbations which give rise to the absolute instability are shown in Figs. 1 and 2. In these figures solid lines correspond to the numerical inviscid flow calculations (which compare excellently with the numerical calculations of [19]) and dashed lines denote the asymptotic results. The wave angle ϵ is defined by $\epsilon = \tan^{-1}(\frac{\beta}{\mathrm{Real}(\alpha)})$. It can be immediately seen that the long-wavelength limit of the eigenvalues leading to absolute instability is captured to a considerable extent. In addition to this, in the case of the incompressible limit perfect agreement is observed with the figures displayed in [14]. Fig. 1 emphasizes that the

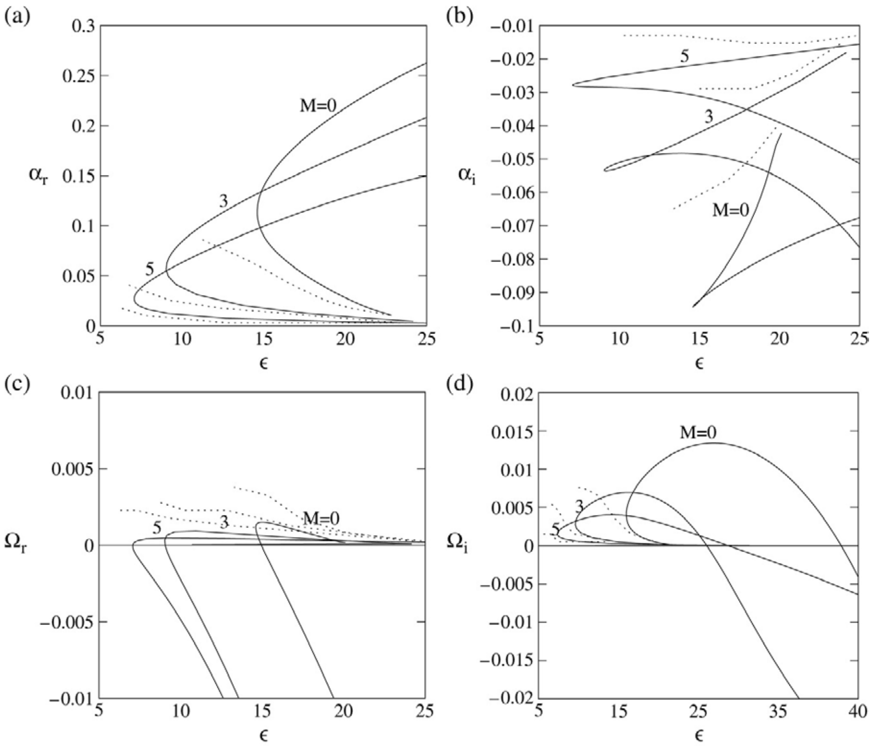

Fig. 1. Branch points are shown in the (a) α_r, (b) α_i, (c) Ω_r, (d) Ω_i versus the wave angle ϵ planes, all for the insulated wall case and drawn for $M = 0$, 3 and 5. Solid lines correspond to numerical data, broken lines to asymptotic values.

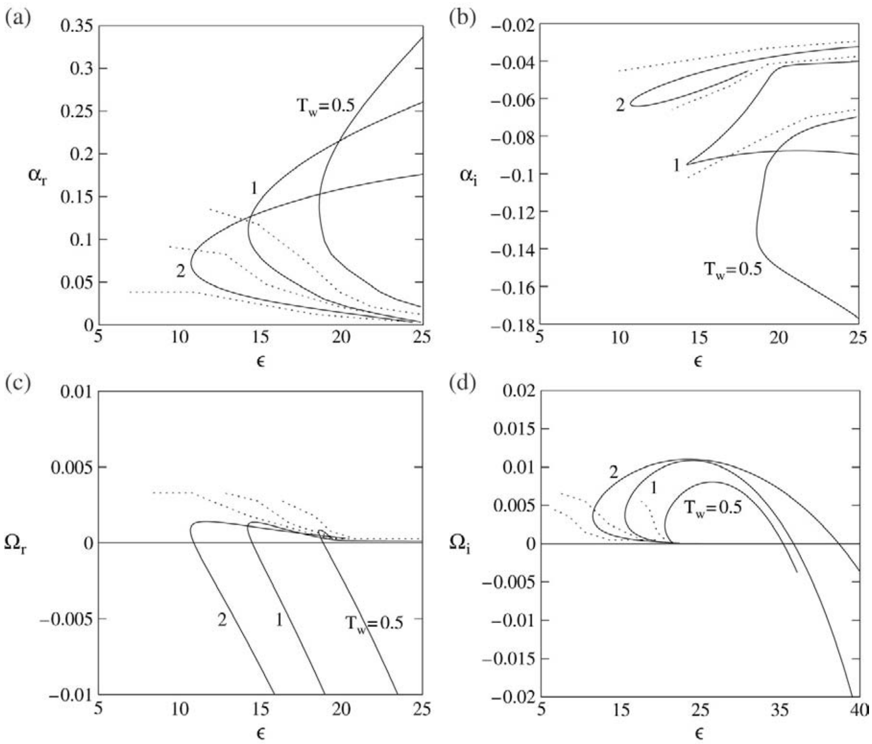

Fig. 2. Branch points are shown in the (a) α_r, (b) α_i, (c) Ω_r, (d) Ω_i versus the wave angle ϵ planes, when the heat transfer is taken into account at a fixed Mach number of $M = 1$. Solid and broken lines are as in Fig. 1.

compressibility within the wall insulation has a suppressive effect on the amplitude of the absolute growth, though the range of the instability is seen to enlarge. A similar effect can also be seen in Fig. 2. Moreover, it is seen that the wall cooling has a stabilizing effect, unlike the wake flows in the vicinity of trailing edges, see for instance [22], and also the mixing layer flows, see for instance [23]; however, wall heating enhances the instability.

4. Conclusions

In this work we have analyzed the analytical dispersion relation which was previously obtained in [14] in the long-wavelength limit. Using the singularities in the dispersion relationship, the modes that cause local absolute instability have been located and good agreement with the ones obtained from the numerical calculations of the inviscid Rayleigh equations has been shown to exist. The current work carries importance owing to the fact that the presence of a local absolute instability may contaminate the entire mean flow field, leading to a nonlinearity and a possible transition to turbulence. The results found in this work clearly indicate that the rotating disk boundary layer flow is subject to a local absolute instability in some regions of the eigenvalues for insulated wall cases and, in particular, for the wall heating.

Even though the present research emphasizes the local analysis with the effects of non-parallelism disregarded, the results obtained here may be used in further study of a global analysis requiring the solution of full linearized stability equations.

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