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A class of uniquely (strongly) clean rings

Orhan GÜRGÜN[1,*], Ayşe Çiğdem ÖZCAN[2]
[1]Department of Mathematics, Ankara University, Ankara, Turkey
[2]Department of Mathematics, Hacettepe University, Beytepe Ankara, Turkey

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Abstract: In this paper we call a ring R δ_r*-clean* if every element is the sum of an idempotent and an element in $\delta(R_R)$ where $\delta(R_R)$ is the intersection of all essential maximal right ideals of R. If this representation is unique (and the elements commute) for every element we call the ring *uniquely (strongly)* δ_r*-clean*. Various basic characterizations and properties of these rings are proved, and many extensions are investigated and many examples are given. In particular, we see that the class of δ_r-clean rings lies between the class of uniquely clean rings and the class of exchange rings, and the class of uniquely strongly δ_r-clean rings is a subclass of the class of uniquely strongly clean rings. We prove that R is δ_r-clean if and only if $R/\delta_r(R_R)$ is Boolean and $R/Soc(R_R)$ is clean where $Soc(R_R)$ is the right socle of R.

Key words: Clean ring, strongly clean ring, uniquely clean ring, strongly J-clean ring

1. Introduction

Clean rings have been studied by many ring and module theorists since 1977, and it is still a very popular subject. They were defined by Nicholson as a subclass of exchange rings. An associative ring with unity is called *clean* if every element is the sum of an idempotent and a unit [14]. If this representation is unique for every element, Nicholson and Zhou [17] call the ring *uniquely clean*. They proved that a ring R is uniquely clean if and only if for all $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in J(R)$ where $J(R)$ is the Jacobson radical of R (we call the ring with this property *uniquely* J*-clean*). Chen et al. [7] call a ring *uniquely strongly clean* if every element can be written uniquely as the sum of an idempotent and a unit that commute. They proved that R is uniquely strongly clean if and only if for every $a \in R$, there exists a unique idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$ (we call the ring with this property *uniquely strongly* J*-clean*). Recently, Chen [6] defined strongly J-clean rings. A ring R is called *strongly* J*-clean* if for all $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ea = ae$ [6]. Note that strongly J-clean rings are strongly clean but the converse need not be true [6, Proposition 2.1 and Example 2.2].

These results motivate us to define the class of uniquely $\delta(R_R)$-clean and uniquely strongly $\delta(R_R)$-clean rings where $\delta(R_R)$ is the ideal defined by Zhou [21]. These classes of rings give some new classes of uniquely clean and uniquely strongly clean rings and also give some ideas on the cleanness of $R/Soc(R_R)$ where $Soc(R_R)$ is the right socle of R. Firstly basic properties of $\delta(R_R)$-clean rings are given in Section 2. Interestingly we see that the class of $\delta(R_R)$-clean rings lies between the class of uniquely clean rings and exchange rings. We also

*Correspondence: orhangurgun@gmail.com
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prove that if R is $\delta(R_R)$-clean, then $R/Soc(R_R)$ is clean and partially unit regular, i.e. every regular element is unit regular. In Section 3, uniquely $\delta(R_R)$-clean rings are studied. We see that any uniquely $\delta(R_R)$-clean ring is uniquely clean. Contrary to the result in [17] saying that R is uniquely clean if and only if $R[[x]]$ is uniquely clean, just the necessity is true for uniquely $\delta(R_R)$-clean rings. Section 4 is devoted to uniquely strongly $\delta(R_R)$-clean rings (USDC for short). Any uniquely $\delta(R_R)$-clean ring is USDC, and any USDC ring is uniquely strongly clean. We prove that if R is a commutative ring, then R is USDC if and only if the ring of 2×2 upper triangular matrices, $T_2(R)$, is USDC. In the last section $\delta(R_R)$-cleanness of the formal triangular matrix ring is investigated.

Recall some definitions. Following [21], a submodule N of a module M is called δ*–small* in M (denoted by $N \ll_\delta M$) if $N+K \neq M$ for any submodule K of M with M/K singular. Denote $\delta(M)$ to be the sum of all δ–small submodules of M (see [21, Lemma 1.5]). We use δ_r (or $\delta_r(R)$) for $\delta(R_R)$ for a ring R. Clearly $J(R) \subseteq \delta_r(R) \ll_\delta R_R$. If S is simple and M is essential, then $S\cap M$ must equal S (as it cannot be zero). Since every simple right ideal is contained in every essential right ideal, then $S_r := Soc(R_R) \subseteq \delta_r(R)$ (see also [21, Lemma 1.9]). By view of [21, Corollary 1.7], $J(R/S_r) = \delta_r/S_r$; in particular, R is semisimple if and only if $\delta(R_R) = R$.

A ring R is an *exchange ring* if, for every $a \in R$, there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$ (see [14]). For example, (von Neumann) regular rings and clean rings are exchange. If I is a left ideal of a ring R, *idempotents lift modulo* I if, given $a \in R$ with $a^2 - a \in I$, there exists $e^2 = e \in R$ such that $a - e \in I$ [14]. Note that R is an exchange ring if and only if idempotents lift modulo every left ideal of R [14, Corollary 1.3]. A ring R is called δ*-semiregular* if R/δ_r is a regular ring and idempotents lift modulo δ_r [21, Theorem 3.5]. A ring R is called *abelian* if every idempotent of R is central.

Throughout this article, all rings are associative with unity and all modules are unitary. We denote $S_r = Soc(R_R)$ and $Z_r = Z(R_R)$ for the right socle and the right singular ideal of a ring R. We write J (or $J(R)$) for the Jacobson radical of R. $U(R)$ is the set of all units in R. The ring of integers modulo n is denoted by $\mathbb{Z}_n$, and we write $M_n(R)$ (resp. $T_n(R)$) for the rings of all (resp., all upper triangular) $n\times n$ matrices over the ring R.

2. δ_r-clean rings

Chen [6] calls a ring R *strongly* J*-clean* if for every element $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in J$ and $ea = ae$. Call a ring R J*-clean* if for any element $a \in R$, there exists an idempotent $e \in R$ such that $a - e \in J$.

Any J-clean ring is clean. Let $a \in R$ and $a = e + w$ where $e^2 = e \in R$, $w \in J$. Then $a = (1-e) + (2e-1+w)$. Since $(2e-1)^2 = 1$ we see that $a-(1-e) \in U(R)$ (see [6, Proposition 2.1]). It is easy to give an example of a ring that is clean but not J-clean (e.g., $\mathbb{Z}_3$). Now we introduce the notion of δ_r-clean rings.

Definition 2.1 A ring R is called δ_r*-clean* if for every element $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in \delta_r$.

The class of δ_r-clean rings contains Boolean rings, semisimple rings, and J-clean rings. Clearly, R is δ_r-clean if and only if R/δ_r is Boolean and idempotents lift modulo δ_r. Note that there exists a ring R with R/δ_r is Boolean but such that idempotents do not lift modulo δ_r. There is a ring R with $R/J(R)$ Boolean but such that idempotents do not lift modulo $J(R)$ (see [13, Example 15]). In this ring, idempotents do not lift modulo δ_r, for, if they did, then R would be δ_r-clean and therefore exchange, by Theorem 2.2 below. Then idempotents would lift modulo $J(R)$, a contradiction.

On the other hand, if R is δ_r-clean, then R/J need not be a Boolean ring. For example, $\mathbb{Z}_3$ is semisimple but not Boolean.

Theorem 2.2 *If R is a δ_r-clean ring, then*

1) R/S_r is a semiregular ring, i.e. R is δ_r-semiregular;

2) R is an exchange ring;

3) R/S_r is a clean ring;

4) $Z_r \subseteq J$.

Proof 1) Since R/δ_r is a Boolean ring and idempotents lift modulo δ_r, R is δ-semiregular. By [19, Theorem 1.4], R is δ_r-semiregular if and only if R/S_r is semiregular.
2) If R/S_r is semiregular, then R is exchange by [19, Corollary 1.5].
3) If R is δ_r-clean, then R/S_r is $J(R/S_r)$-clean since $J(R/S_r) = \delta_r/S_r$. Any J-clean ring is clean. We thus conclude that R/S_r is a clean ring.
4) Since R is δ_r-semiregular, $Z_r \subseteq \delta_r$ by [16, Theorem 1.2]. Then Z_r is δ-small in R. This gives that Z_r is small in R. Hence, $Z_r \subseteq J$. □

Example 2.3 *If R is a semisimple ring that is not a Boolean ring (e.g., $\mathbb{Z}_3$), then R is δ_r-clean but not J-clean since $J = 0$ and $\delta_r = R$.*

Example 2.4 *There exist clean rings that are not δ_r-clean.*

Proof 1) Let V_D be a nonzero vector space over a division ring D and let $R = \mathrm{End}_D(V)$. Then R is regular (see [1, Exercise 15.13]) and clean [15, Lemma 1] (see also [3, Lemma 3.1]) and $S_r = S_l = \{f \in R \mid \mathrm{rank} f < \infty\}$ (see [1, Exercise 18.4]). Since $J(R/S_r) = \delta_r/S_r$ and R is regular, we have that $\delta_r = S_r$.

Now assume that V_D is a countably infinite dimensional vector space and let $\{v_1, v_2, \ldots\}$ be a basis of V. Define the shift operator f on V by $f(v_n) = v_{n+1}$ for $n = 1, 2, 3, \ldots$. Then $f^2 - f \notin S_r$. This shows that $R/S_r = R/\delta_r$ is not Boolean. Hence, R is not δ_r-clean.

2) Let p be a prime integer and consider the local ring $\mathbb{Z}_{(p)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (m, n) = 1, \ p \nmid n\}$. Since $\mathbb{Z}_{(p)}$ is not semisimple, $J = \delta_r = p\mathbb{Z}_{(p)}$. Then $\mathbb{Z}_{(p)}$ is clean but not δ_r-clean, because $\mathbb{Z}_{(p)}/\delta_r$ is not Boolean. □

Note that any clean ring is exchange [14, Proposition 1.8]. Bergman's example is an example of an exchange ring that is not clean. We prove below that this ring is not δ_r-clean, and so we pose the following question.

Question: Is any δ_r-clean ring clean?

Example 2.5 (Bergman) Let F be a field with $\mathrm{char}(F) \neq 2$, and $A = F[[x]]$. Let Q be the field of fractions of A. Define

$$R = \{r \in \mathrm{End}_F(A) \mid \exists q \in Q \text{ and } \exists n > 0 \text{ with } r(a) = qa \text{ for all } a \in x^n A\}.$$

Then R is a regular (so exchange) ring [10], but not clean [4]. There is also an epimorphism $\theta : R \to Q$ given by $r \mapsto q$, where r agrees with q on $x^n A$ for some $n > 0$ with $\mathrm{Ker}\theta = S_r = \delta_r$ (see [12, Example 1]). Now assume that R is δ_r-clean. Then, for any $r \in R$, there exists an idempotent $e \in R$ such that $r - e \in \delta_r$. This gives that $\theta(r - e) = \theta(r) - \theta(e) = 0$ and $\theta(r) = \theta(e)$ is an idempotent in Q. Since Q is a field, $\theta(r) = 0$ or 1, which contradicts the fact that θ is an epimorphism. Therefore, R is not δ_r-clean.

Thus we conclude that

$$\{ \text{ Boolean } \} \underset{\neq}{\subsetneq} \{ \ J\text{-clean}\} \underset{\neq}{\subsetneq} \{ \ \delta_r\text{-clean } \} \underset{\neq}{\subsetneq} \{ \text{ exchange } \}.$$

Now we give a few conditions for a δ_r-clean ring to be clean or J-clean. First note that Baccella [2] proved the important fact that idempotents lift modulo S_r for any ring R.

Proposition 2.6 *Any δ_r-clean ring R is J-clean if*

1) R/J is Boolean, or 2) $S_r \subseteq J$.

Proof 1) Assume that R is δ_r-clean and R/J is Boolean. Let $a \in R$. Then $a^2 - a \in J$. By Theorem 2.2, idempotents lift modulo J. Hence, there exists an idempotent $e \in R$ such that $a - e \in J$.
2) Assume that R is δ_r-clean. If $S_r \subseteq J$, then $J/S_r = J(R/S_r) = \delta_r/S_r$, and we have that $J = \delta_r$. Hence, R is J-clean. □

Proposition 2.7 *If R is δ_r-clean and R/J is abelian, then R is clean.*

Proof Assume that R is δ_r-clean. According to Theorem 2.2, R is exchange and so R/J is exchange and idempotents lift modulo J by [14, Corollary 1.3]. Thus, R/J is abelian exchange and it is clean by [14, Proposition 1.8]. By [9, Proposition 6], R is clean. □

Recall that a ring R is called *right quasi-duo* if every maximal right ideal is a 2-sided ideal. If R is an exchange ring, then R/J is right quasi-duo iff R/J is reduced iff R/J is abelian [20, Proposition 4.1]. Hence, the following corollary is immediate.

Corollary 2.8 *If R is δ_r-clean and right (or left) quasi-duo, then R is clean.*

Proposition 2.9 *Let R be a ring with only trivial idempotents (e.g., a local ring). Then R is δ_r-clean if and only if R is either a division ring or $R/J(R) \cong \mathbb{Z}_2$.*

Proof Assume that R is δ_r-clean. Then R is exchange by Theorem 2.2. Since R is exchange and has only trivial idempotents, R is local. Then either $J(R) = 0$ or $J(R) = \delta_r$. If $J(R) = 0$, then R is a division ring. If $J(R) = \delta_r$, then R is J-clean and so R is strongly J-clean by hypothesis. Hence, $R/J(R) \cong \mathbb{Z}_2$ by [6, Lemma 4.2]. Conversely, if R is a division ring, then R is semisimple and so R is δ_r-clean. If $R/J(R) \cong \mathbb{Z}_2$, then R is J-clean by [17, Theorem 15] and so R is δ_r-clean. □

A characterization of δ_r-clean rings can be given as follows.

Theorem 2.10 *Let R be a ring. The following statements are equivalent.*

1) R is δ_r-clean.

2) R/S_r is J-clean.

3) R/δ_r is Boolean and R/S_r is clean.

Proof Since $J(R/S_r) = \delta_r/S_r$, (1) $\Leftrightarrow$ (2). By Theorem 2.2, (1) $\Rightarrow$ (3).
(3) $\Rightarrow$ (1) Let $a \in R$. Then $a^2 - a \in \delta_r$. Since $\overline{R} = R/S_r$ is clean, idempotents of $\overline{R}/J(\overline{R})$ lift to idempotents of $\overline{R}$. By [19, Lemma 1.3], idempotents of R/δ_r lift to idempotents of R. Hence, there exists $e^2 = e \in R$ such that $a - e \in \delta_r$. Thus, R is δ_r-clean. □

Bergman's example (see Example 2.5) also shows that if R/S_r is a clean ring, then R need not be clean [12, Example 1].

Recall that a ring R is said to have *stable range 1*, written $sr(R) = 1$, if given $a, b \in R$ for which $aR + bR = R$, there exists a $y \in R$ such that $a + by \in U(R)$. It is obvious that $\mathrm{sr}(R) = 1$ if and only if $\mathrm{sr}(R/J) = 1$.

Lemma 2.11 *Let R be a ring. Then $sr(R/\delta_r) = 1$ if and only if $sr(R/S_r) = 1$.*

Proof It can be easily seen by the fact that $J(R/S_r) = \delta_r/S_r$. □

Recall that an element a of a ring R is called *regular* (resp., *unit regular*) if there exists $u \in R$ (resp., $u \in U(R)$) such that $a = aua$. A ring R is called *partially unit regular* if every regular element of R is unit regular. These rings are also called IC-ring in [11].

Theorem 2.12 *If R is a δ_r-clean ring, then R/S_r is partially unit regular.*

Proof Since R/δ_r is a Boolean ring, $\mathrm{sr}(R/\delta_r) = 1$. By Theorem 2.2, R is an exchange ring. Hence, by Lemma 2.11 and [5, Theorem 3], R/S_r is partially unit regular. □

The following example shows that if R is δ_r-clean, then R/S_r need not be a regular ring in general.

Example 2.13 *Let $R = \mathbb{Z}_8$. Then $Soc(R) = 4R$ and $J = 2R$. It is clear that R is J-clean, but since $J \not\subseteq Soc(R)$, $R/Soc(R)$ is not regular.*

3. Uniquely δ_r-clean rings

Definition 3.1 A ring R is called *uniquely δ_r-clean* if for every element $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in \delta_r$.

Let I be an ideal of R. Then *idempotents lift uniquely modulo I* if whenever $a^2 - a \in I$, there exists a unique idempotent $e \in R$ such that $e - a \in I$ [17]. This condition implies that if $e - f \in I$, $e^2 = e$, $f^2 = f$, then $e = f$; in particular, 0 is the only idempotent in I.

Clearly, R is uniquely δ_r-clean if and only if R/δ_r is Boolean and idempotents lift uniquely modulo δ_r.

Theorem 3.2 *If R is uniquely δ_r-clean, then the following hold.*

1) $\delta_r = J$.

2) R is uniquely clean.

Proof 1) Since idempotents lift uniquely modulo δ_r, by the remark above, the only idempotent in δ_r is 0. Now let $a \in \delta_r$. Then there exists a semisimple right ideal Y of R such that $R = (1-a)R \oplus Y$ by [21, Theorem 1.6]. Since $Y \subseteq S_r \subseteq \delta_r$, we have that $Y = 0$. Hence $1 - a$ is right invertible in R, and so $a \in J$.
2) It is clear by (1) and [17, Theorem 20]. □

Note that any uniquely clean ring is abelian by [17, Lemma 4].

Examples 3.3 1) *No semisimple ring is uniquely δ_r-clean, for, if R is a semisimple ring, then $\delta_r = R$ and for any $a \in R$, $a - 0 \in R$ and $a - 1 \in R$.*

2) *If $R \not\cong \mathbb{Z}_2$, then $R/J \cong \mathbb{Z}_2$ if and only if R is local uniquely δ_r-clean, for, if $R/J \cong \mathbb{Z}_2$, then $J = \delta_r$ and R is uniquely clean by [17, Theorem 15] and so R is uniquely δ_r-clean. The converse is also true by Proposition 2.9.*

Therefore, for example, the rings $R = \{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2\}$, $R = \{\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \mid x \in \mathbb{Z}_4, y \in \mathbb{Z}_4 \oplus \mathbb{Z}_4\}$, *or $R = \mathbb{Z}_{2^n}$ where $1 \neq n \in \mathbb{N}$ are uniquely δ_r-clean.*

Uniquely clean rings need not be uniquely δ_r-clean.

Example 3.4 1) *$\mathbb{Z}_2$ is uniquely clean but not uniquely δ_r-clean.*

2) *Let $R = \prod_{i=1}^{\infty} R_i$ where $R_i \cong \mathbb{Z}_2$ for all $i = 1, 2, \ldots$. Then R is a Boolean ring with $S_r = \oplus_{i=1}^{\infty} R_i$. Since R/S_r is Boolean, $J(R/S_r) = 0$ and so $S_r = \delta_r$. Clearly R is uniquely J-clean, that is, uniquely clean but not uniquely δ_r-clean.*

It is easy to see that every uniquely clean ring is δ_r-clean by the fact that R is uniquely clean if and only if R is uniquely J-clean [17, Theorem 20]. But if R is a semisimple ring that is not Boolean, then R is δ_r-clean but not uniquely clean (see Example 2.3).

Thus, we conclude that

$$\{\text{ uniquely } \delta_r\text{-clean}\} \underset{\neq}{\subsetneq} \{\text{ uniquely clean }\} \underset{\neq}{\subsetneq} \{\ \delta_r\text{-clean }\} \underset{\neq}{\subsetneq} \{\text{ exchange }\}.$$

If $S_r \subseteq J$ for a ring R, then $J/S_r = J(R/S_r) = \delta_r/S_r$ and so $J = \delta_r$. Hence, Proposition 3.5 below is obvious by Proposition 2.6.

Proposition 3.5 *If R is a uniquely clean ring with $S_r \subseteq J$, then R is uniquely δ_r-clean.*

By [17, Theorem 20] we know that R is uniquely clean if and only if R/J is Boolean, R is abelian, and idempotents lift modulo J. However, this result cannot be restated for δ_r in general. The following theorem and examples prove our claim.

Theorem 3.6 *Let R be a ring and consider the following conditions.*

1) R is uniquely δ_r-clean.

2) R/δ_r is Boolean, R is abelian, and idempotents lift modulo δ_r.

3) R/δ_r is Boolean, R/S_r is abelian, and idempotents lift modulo δ_r.

4) R/S_r is uniquely clean.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4)$.

Proof $(1) \Rightarrow (2)$ Since R is uniquely clean, it is abelian by [17, Lemma 4].
$(2) \Rightarrow (3)$ Since idempotents always lift modulo S_r, it is clear.
$(3) \Leftrightarrow (4)$ It is by [17, Theorem 20]. Note that idempotents lift modulo $J(R/S_r)$ if and only if idempotents lift modulo δ_r [19, Lemma 1.3]. □

In Theorem 3.6, (2) $\not\Rightarrow$ (1) in general.

Example 3.7 *We consider again the ring $R = \prod_{i=1}^{\infty} R_i$ where $R_i \cong \mathbb{Z}_2$, $i = 1, 2, \ldots$ (see Example 3.4). Since R is uniquely clean, R is abelian and δ_r-clean. But R is not uniquely δ_r-clean.*

In Theorem 3.6, (4) $\not\Rightarrow$ (2) in general.

Example 3.8 *Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then $S_r = \delta_r = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ and $R/S_r \cong \mathbb{Z}_2$ is Boolean. Obviously R is δ_r-clean but not abelian.*

Theorem 3.9 *If R is uniquely δ_r-clean and $e^2 = e \in R$, then eRe is uniquely δ_r-clean.*

Proof Since R is abelian, $\delta_r(eRe) = e\delta_r e$ by [18, Theorem 3.11]. By Theorem 3.2, $\delta_r = J$, so we have that $J(eRe) = eJe = \delta_r(eRe)$. If R is uniquely δ_r-clean, then R is uniquely clean by Theorem 3.2. By [17, Corollary 6], eRe is uniquely clean. By [17, Theorem 20], eRe is uniquely δ_r-clean. □

Although every factor ring of a uniquely clean ring is uniquely clean [17, Theorem 22], the same property does not hold for uniquely δ_r-clean.

Remark 3.10 1) If R is a uniquely δ_r-clean ring, then factor rings of R need not be uniquely δ_r-clean in general. For example, if $R \ncong \mathbb{Z}_2$ and $R/J \cong \mathbb{Z}_2$, then R is uniquely δ_r-clean by Example 3.3, but R/J is not uniquely δ_r-clean.

(2) Since matrix ring $M_n(R)$ and upper triangular matrix ring $T_n(R)$ are not abelian for $n \geq 2$, they are not uniquely δ_r-clean by Theorem 3.2.

Let R be a ring and V an (R,R)-bimodule that is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$, and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then the *ideal-extension* (also called the Dorroh extension) $I(R;V)$ of R by V is defined to be the additive abelian group $I(R;V) = R \oplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$.

Uniquely clean ideal-extensions are considered in [17, Proposition 7]. Now we deal with uniquely δ_r-clean ideal-extensions.

Proposition 3.11 *An ideal-extension $S = I(R;V)$ is uniquely δ_r-clean if the following conditions are satisfied:*

1) R is uniquely δ_r-clean;

2) if $e^2 = e \in R$ then $ev = ve$ for all $v \in V$;

3) if $v \in V$ then $v + w + vw = 0$ for some $w \in V$.

Proof Assume that (1), (2), and (3) are satisfied. Since R is uniquely δ_r-clean, R is uniquely clean by Theorem 3.2 and so S is uniquely clean by [17, Proposition 7]. Then S is δ_r-clean. Note by the proof of [17, Proposition 7] that any idempotent in S is of the form $(e,0)$ where $e^2 = e \in R$. Now suppose that $(e,0)+(u,v) = (e_1,0)+(u_1,v_1)$ in S where $(e,0)$ and $(e_1,0)$ are idempotents and $(u,v),(u_1,v_1) \in \delta_r(S)$. Then $e+u = e_1+u_1$ in R where e and e_1 are idempotents in R and $u,u_1 \in \delta_r(R)$ by the following result, and so $(e,0) = (e_1,0)$ by (1).

Claim. If $(u,v) \in \delta_r(S)$ then $u \in \delta_r(R)$.

Proof. Let $(u,v) \in \delta_r(S)$. Then $(u,0) \in \delta_r(S)$ because $(0,V) \subseteq J(S) \subseteq \delta_r(S)$ by (3). Let L be a right ideal of R such that $uR + L = R$. It is enough to show that L is a direct summand of R by [21, Theorem 1.6]. Since $(u,0)S + (L \oplus V) = S$ and $(u,0) \in \delta_r(S)$, we have that $L \oplus V$ is a direct summand of S and so is generated by an idempotent $(e,0) \in S$ where $e^2 = e \in R$. Then we see that $L = eR$, and hence L is a direct summand of R, as desired. □

Example 3.12 *Let R be a uniquely δ_r-clean ring and let $S = \{[a_{ij}] \in T_n(R) \mid a_{11} = \ldots = a_{nn}\}$. Then S is uniquely δ_r-clean and is noncommutative if $n \geq 3$.*

Proof If $V = \{[a_{ij}] \in T_n(R) \mid a_{11} = \ldots = a_{nn} = 0\}$, then $S \cong I(R;V)$. The conditions in Proposition 3.11 hold as in [17, Example 8]. □

If R is a ring and $\sigma : R \to R$ is a ring endomorphism, let $R[[x,\sigma]]$ denote the ring of skew formal power series over R, that is, all formal power series in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R. Since $R[[x,\sigma]] \cong I(R; < x >)$ where $< x >$ is the ideal generated by x, the proof of [17, Example 9] and Proposition 3.11 give the next results.

Corollary 3.13 *Let R be a ring and $\sigma : R \to R$ a ring endomorphism and $e = \sigma(e)$ for all $e^2 = e \in R$. If R is uniquely δ_r-clean, then $R[[x,\sigma]]$ is uniquely δ_r-clean*

Corollary 3.14 *If R is a uniquely δ_r-clean ring, then $R[[x]]$ is uniquely δ_r-clean.*

Corollary 3.14 can be proven by using Proposition 3.15 below, for, if R is uniquely δ_r-clean, then $R[[x]]$ is a uniquely clean ring by Theorem 3.2 and [17, Corollary 10]. By Proposition 3.15, $J(R[[x]]) = J(R)+ < x > \subseteq \delta_r(R[[x]]) \subseteq \delta_r(R)+ < x >$. Then since $J(R) = \delta_r(R)$ by Theorem 3.2(1), $J(R[[x]]) = \delta_r(R[[x]])$. Hence, $R[[x]]$ is a uniquely δ_r-clean ring.

Proposition 3.15 *Let R be a ring. Then $\delta_r(R[[x]]) \subseteq \delta_r(R)+ < x >$.*

Proof Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots \in \delta_r(R[[x]])$. Since $< x > \subseteq J(R[[x]])$, $a_0 \in \delta_r(R[[x]])$. Let L be a right ideal of R such that $a_0R + L = R$. It is enough to show that L is a direct summand of R by [21, Theorem 1.6]. Since $a_0R[[x]] + L[[x]] = R[[x]]$ and $a_0 \in \delta_r(R[[x]])$, we have that $L[[x]]$ is a direct summand of $R[[x]]$ and so is generated by an idempotent $e(x) = e_0 + e_1x + e_2x^2 + \ldots \in R[[x]]$. Then e_0 is an idempotent in R and it can be seen that $L = e_0R$. Thus, $a_0 \in \delta_r(R)$, as desired. □

Note that $J(\mathbb{Z}_2[[x]]) = \delta_r(\mathbb{Z}_2[[x]]) \subsetneq \delta_r(\mathbb{Z}_2)+ < x >= \mathbb{Z}_2[[x]]$.

Corollary 3.16 *If $R[[x]]$ is δ_r-clean, then R is δ_r-clean.*

Proof Let $a \in R$. Then there exist $e(x)^2 = e(x) \in R[[x]]$ and $w(x) \in \delta_r(R[[x]])$ such that $a = e(x) + w(x)$ and so $w(0) \in \delta_r(R)$ by Proposition 3.15. Thus, $a = e(0) + w(0)$ where $e(0)^2 = e(0) \in R$, as asserted. □

If $R[[x]]$ is uniquely δ_r-clean, then R need not be uniquely δ_r-clean. For example, $\mathbb{Z}_2$ is not uniquely δ_r-clean but since $\mathbb{Z}_2[[x]]/J(\mathbb{Z}_2[[x]]) \cong \mathbb{Z}_2$, $\mathbb{Z}_2[[x]]$ is uniquely δ_r-clean by Example 3.3(2).

4. Uniquely strongly δ_r-clean rings

Uniquely strongly clean rings were studied in [7]. A ring R is called *uniquely strongly clean* if for every element $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in U(R)$ and $ea = ae$. In Theorem 17 of [7] it is proven that a uniquely strongly clean ring is exactly the same as a uniquely strongly J-clean, i.e. for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in J$ and $ea = ae$.

Definition 4.1 A ring R is called *uniquely strongly δ_r-clean* if for every element $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in \delta_r$ and $ea = ae$.

Proposition 4.2 *A ring R is uniquely δ_r-clean if and only if R is an abelian USDC ring.*

Proof Since uniquely δ_r-clean rings are abelian by Theorem 3.6, the proof is obvious. □

Proposition 4.3 *Let R be a USDC ring. Then the following hold:*

1) If $e^2 = e \in \delta_r$ then $e = 0$.

2) R/J is Boolean.

3) $\delta_r = J$.

4) R is uniquely strongly clean.

Proof 1) Let $e^2 = e \in \delta_r$. Then $e + 0 = 0 + e$ and $0.e = e.0$ yield $e = 0$.
2) R is exchange by Theorem 2.2. If we show that every nonzero idempotent of R is not the sum of 2 units, then by [13, Theorem 13], R/J will be Boolean. Let e be a nonzero idempotent in R. Write $e = u + v$, where $u, v \in \mathrm{U}(R)$. Since R is USDC, R/δ_r is Boolean and so $2 \in \delta_r$. Therefore, u and v are congruent to 1, modulo δ_r, which means that their sum is in δ_r. This contradicts with (1).
3) Let $a \in \delta_r$. Since R/J is Boolean, $a^2 - a \in J$. By Theorem 2.2, R is exchange and so idempotents lift modulo J. Thus, there exist $e^2 = e \in R$ such that $a - e \in J$. Since $J \subseteq \delta_r$, $e = 0$ by (1). Hence, $a \in J$, as asserted.
4) It is clear by (3) and [7, Theorem 17]. □

However, a uniquely strongly clean ring need not be USDC. The ring $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ is uniquely strongly clean by [7, Theorem 10] but not USDC by Example 3.8.

Thus, we conclude that

$$\{ \text{ uniquely } \delta_r\text{-clean } \} \subsetneqq \{ \text{ USDC } \} \subsetneqq \{ \text{ uniquely strongly clean } \} \subsetneqq \{ \delta_r\text{-clean } \}.$$

The first and the last containments above are proper because, for example, the ring $\mathbb{Z}_p$ where $2 \neq p$ is a prime is δ_r-clean but not uniquely strongly clean because $J(\mathbb{Z}_p) = 0$ and $\mathbb{Z}_p$ is not Boolean. If R is a commutative uniquely δ_r-clean ring, then $T_n(R)$ is USDC by Theorem 4.5 for any $n \in \mathbb{N}$, but $T_n(R)$ is never uniquely δ_r-clean by Remark 3.10(2).

Any factor ring of any USDC ring need not be USDC. For example, since $\mathbb{Z}_4$ is uniquely δ_r-clean by Example 3.3, it is USDC by Proposition 4.2. However, $\mathbb{Z}_4/J(\mathbb{Z}_4) \cong \mathbb{Z}_2$ is not USDC by Proposition 4.2 and Example 3.3.

Proposition 4.4 *Let e be an idempotent of a ring R such that $eR = eRe$ (i.e. right semicentral) or $ReR = R$ (i.e. full idempotent). If R is USDC, then eRe is USDC.*

Proof Assume that R is USDC. For any idempotent e of R, eRe is uniquely strongly clean by Proposition 4.3(4) and [7, Example 5]. Since uniquely strongly clean rings are uniquely strongly J-clean, for any $a \in eRe$, there exists an idempotent $f \in eRe$ and $v \in \delta_r(eRe)$ such that $a = f + v$ and $fv = vf$. It remains to show the uniqueness. Let $a = f + v = g + w$ where f and g are idempotents in eRe and $v, w \in \delta_r(eRe)$ such that $fv = vf$ and $gw = wg$. If e is an idempotent as in the hypothesis, then $\delta_r(eRe) \subseteq e\delta_r e \subseteq \delta_r(R)$ by [18, Theorems 3.9 and 3.11]. Hence, by assumption, $f = g$. □

Since $M_n(R)$ is never uniquely strongly clean by [7, Lemma 6], $M_n(R)$ is never USDC.

Theorem 4.5 *Let R be a commutative ring. Then the following are equivalent.*

(1) R is USDC.

(2) R is uniquely δ_r-clean.

(3) $T_n(R)$ is USDC for all $n \geqslant 1$.

(4) $T_2(R)$ is USDC.

Proof (1) ⇔ (2) This follows by Proposition 4.2.

(3) ⇒ (4) It is clear.

(4) ⇒ (1) Suppose that $T_2(R)$ is USDC and let $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in T_2(R)$. Since e is right semicentral and $eT_2(R)e \cong R$, R is USDC by Proposition 4.4.

(1) ⇒ (3) If R is USDC, then $T_n(R)$ is uniquely strongly clean by Proposition 4.3(4) and [7, Theorem 10]. According to Proposition 4.3(3) and Lemma 5.1, $\delta_r(T_n(R)) = J(T_n(R))$ and so $T_n(R)$ is USDC by [7, Theorem 17]. Therefore, the proof is completed. □

5. On the formal triangular matrix rings

Let S and T be any ring, M an (S,T)-bimodule, and R the formal triangular matrix ring $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. It is well known that $J(R) = \begin{bmatrix} J(S) & M \\ 0 & J(T) \end{bmatrix}$ (e.g., [8, Corollary 2.2]), but for $\delta_r(R)$ the similar property does not hold in general. For example, if $S = M = T = F$ is a field, then $\delta_r(R) = Soc_r(R) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ since $R/Soc_r(R)$ has zero Jacobson radical, but $\begin{bmatrix} \delta_r(S) & M \\ 0 & \delta_r(T) \end{bmatrix} = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix} = R$. Now we prove the following.

Lemma 5.1 *Let* $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ *where* S, T *are any ring and* M *is an* (S,T)*-bimodule. Then* $\delta_r(R) \subseteq \begin{bmatrix} \delta_r(S) & M \\ 0 & \delta_r(T) \end{bmatrix}$.

Proof Let $r = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix} \in \delta_r(R)$ where $s \in S$, $t \in T$ and $m \in M$. We claim that $s \in \delta_r(S)$. Let I be a right ideal of S such that $sS + I = S$. It is enough to show that I is a direct summand of S by [21, Theorem 1.6]. Since $rR + \begin{bmatrix} I & M \\ 0 & T \end{bmatrix} = R$ and $r \in \delta_r(R)$, we have that $\begin{bmatrix} I & M \\ 0 & T \end{bmatrix}$ is a direct summand of R and so is generated by an idempotent $e \in R$. Let $e = \begin{bmatrix} g & n \\ 0 & f \end{bmatrix}$ where $g \in S$, $f \in T$ and $n \in M$. Then g is an idempotent in S and we see that $I = gS$, and hence I is a direct summand of S, as desired. By a similar argument we see that $t \in \delta_r(T)$. Hence, the proof is completed. □

According to [8, Proposition 6.3], $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ is clean if and only if S and T are clean. This result also holds for J-clean ring.

Proposition 5.2 *Let* $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. *Then* R *is* J*-clean if and only if* S *and* T *are* J*-clean.*

Proof Since S and T are factor rings of R, the necessity is obvious. Now assume that S and T are J-clean. Let $r = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix} \in R$ where $s \in S$, $t \in T$ and $m \in M$. Then $s = e + w$ where $e^2 = e \in S$ and $w \in J(S)$, and $t = f + v$ where $f^2 = f \in T$ and $v \in J(T)$. This gives that $\begin{bmatrix} s & m \\ 0 & t \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} + \begin{bmatrix} w & m \\ 0 & v \end{bmatrix}$ where $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ is an idempotent in R and $\begin{bmatrix} w & m \\ 0 & v \end{bmatrix} \in J(R)$. Hence, R is J-clean. □

If S and T are local rings with nonzero maximal left ideal, then $J(S) = \delta_r(S)$ and $J(T) = \delta_r(T)$. By Lemma 5.1, one can thus deduce that $J(R) = \delta_r(R)$. Hence, the following corollary is immediate from Proposition 5.2.

Corollary 5.3 *Let* $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ *where* S *and* T *are local rings with nonzero maximal left ideals. Then* R *is* δ_r*-clean if and only if* S *and* T *are* δ_r*-clean.*

If $R = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$, then $\mathbb{Z}_3$ is a δ_r-clean ring, but R is not δ_r-clean since no quotient of it is Boolean.

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