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Analytic approximate solutions of parameterized unperturbed and singularly perturbed boundary value problems

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ABSTRACT

A novel approach is presented in this paper for approximate solution of parameterized unperturbed and singularly perturbed two-point boundary value problems. The problem is first separated into a simultaneous system regarding the unknown function and the parameter, and then a methodology based on the powerful homotopy analysis technique is proposed for the approximate analytic series solutions, whose convergence is guaranteed by optimally chosen convergence control parameters via square residual error. A convergence theorem is also provided. Several nonlinear problems are treated to validate the applicability, efficiency and accuracy of the method. Vicinity of the boundary layer is shown to be adequately treated and satisfactorily resolved by the method. Advantages of the method over the recently proposed conventional finite-difference or Runga–Kutta methods are also discussed.

1. Introduction

The prime motivation of the present article is prompted by the recent publications [1–3], where the papers consider the subsequent singular perturbation boundary value problem depending on a parameter λ:

$$\epsilon u'(t) + g(t, u, \lambda) = 0, \quad t \in (0, 1], \tag{1.1}$$

$$u(0) = A, \quad u(1) = B, \tag{1.2}$$

where a prime denotes derivative with respect to t, ϵ is a small parameter, A and B are some prescribed constants. Under the assumptions on u, g and λ as stated in [1], problems (1.1) and (1.2) have a unique solution pair $\{u(t), \lambda\}$. For $\epsilon = 1$ the problem considered is known as unperturbed, whereas for $\epsilon \ll 1$ it is so-called singularly perturbed. We should remark that although the two end points enter into the problem in exactly the same manner so that the boundary layers are possible both near $t = 0$ and near $t = 1$, we consider the present problem only with a boundary layer of width $O(\epsilon)$ near the point $t = 0$, see [4]. It should also be reminded that the parameter λ has no connection with the eigenvalue of the nonlinear differential equation under consideration, since there are two unknowns in (1.1) that can be determined exactly by the conditions given in (1.2).

Parameterized problems of above-mentioned kind have been dealt with for many years. A variety of examples exist within boundary layer flows in fluid mechanics, for a discussion of existence and uniqueness results and also for applications of the parameterized equations one can refer to the frequently cited bibliography [5–7], and the references therein.

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The boundary layer behavior of the solution inherent in the singularly perturbed problems always made difficult the numerical analysis of singular perturbation cases. Such problems undergo rapid changes within thin layers near the boundary or inside the problem domain, see for instance [8]. It is well known that standard numerical methods for solving such problems are unstable and thus fail to give accurate results when the perturbation parameter is reasonably small enough. Therefore, it is important to develop suitable numerical methods and taking this need into account [1] gave a uniform finite difference method on a Shishkin mesh for problems (1.1) and (1.2). They proved that the scheme is almost first-order convergent in the discrete maximum norm. In [3] a hybrid difference scheme on generalized Shishkin meshed was later proposed. The scheme uses a midpoint difference method whenever the local mesh size allows to do so this without losing stability, but employs an upwind difference method away from the boundary layer. [3] showed that the scheme is second-order convergent in the discrete maximum norm, independent of singular perturbation parameter. The approach [2] used was based on the boundary layer correction technique. By constructing a modified problem with a boundary layer correction, the original problem was converted into two non-singularly perturbed problems which were then solved by using classical numerical methods, such as Runge–Kutta methods.

Another classical approach for approximate solutions of singular boundary layer problems is to pursue perturbation methods. However, solutions obtained within perturbation techniques may not be uniform, restricting the applicability of results [9,10]. To overcome the limitations of the perturbative techniques Liao [11] proposed a new analytic method for highly nonlinear problems, namely homotopy analysis method. Unlike the perturbative and non-perturbative methods, this technique allows more than a uniformly valid analytic solution of nonlinear equations with no possible small parameters. In this method, according to the homotopy technique, a homotopy with an embedding parameter is constructed, and the embedding parameter is considered as a small parameter. Thus the original nonlinear problem is converted into an infinite number of linear problems without using the perturbation techniques. Further advantages of this new technique have been severally outlined in the literature as also briefly addressed in Section 2. After the introduction of the method, several problems of science and engineering were revisited. For example, Liao successfully applied the method for the analytical solution of Falkner–Skan equation [12]. Foundations and fundamentals of the method have been recently summarized in [13]. The homotopy analysis method keeps evolving steadily and the recent research clearly shows that it may replace the place of traditional perturbation or numerical methods in a near future. The exact analytic solutions of magnetohydrodynamic swirling boundary layer flow over a porous rotating disk was recently presented in [14], which reveals the success of homotopy analysis method even to highly nonlinear systems.

As opposed to the conventional finite-difference or Runge–Kutta methods laid out in Refs. [1–3], we propose here a scheme for the parameterized two-point boundary value problems. This technique is designed both for the regular case, that is when the boundary layers are absent, as well as for the singularly perturbed case. The methodology that we develop relies essentially upon the recently fashionable and powerful homotopy analysis method. Within this aim, suitable auxiliary linear operators, optimal convergence control parameters (via square residual error concept) and initial guesses are suggested that generate explicit analytic form of the solution of nonlinear unperturbed/perturbed two-point boundary value problems. The proposed homotopy analysis technique provides uniformly valid approximate series solutions which is further shown to be particularly advantageous in the case of singular perturbation.

The following strategy is adopted in the rest of the paper. In Section 2 the methodology of homotopy analysis approach is presented. Convergence of the method is given in Section 3 Application of the method to unperturbed and singularly perturbed nonlinear problems is implemented in Section 4, in which analytic expressions are derived and compared with the numerical ones. Finally conclusions follow in Section 5.

2. The method

Liao [11] proposed a new kind of analytic technique for the nonlinear problems, namely the homotopy analysis method. This method is based on the homotopy and has several advantages. To underline, firstly its validity does not depend upon whether or not nonlinear equations under consideration contain small or large parameters, hence it can solve more of strongly nonlinear equations than the perturbation techniques. Secondly, it provides us with a great freedom to select proper auxiliary linear operators and initial guesses so that uniformly valid approximations can be obtained. Thirdly, it gives a family of approximations which are convergent in a larger region. Liao successfully applied the homotopy analysis method to solve some nonlinear problems in mechanics. For example, Liao [15] gave a purely analytic solution of 2D Blasius's viscous flow over a semi-infinite flat plate, which is uniformly valid in the whole physical region. Further examples are provided within Ref. [16]. A recent interesting application was given in [17].

Prior to an outline of the homotopy analysis method and keeping also in mind the singular perturbation theory, for our analysis we initially modify the parameterized differential equation system (1.1) and (1.2) into the form

$$\epsilon u'(t) + g(t, u, \lambda) = 0, \quad u(0) = A, \tag{2.3}$$

$$\epsilon u'(1) + g(1, B, \lambda) = 0. \tag{2.4}$$

In the case of a singularly perturbed problem, a boundary layer is presumed to take place near the point $t = 0$. Such a split of the original system enables us to construct a homotopy

$$(1-p)L(u-u_0)+ph_uN=0, \quad u(0)=A, \tag{2.5}$$
$$(1-p)F+ph_\lambda f=0. \tag{2.6}$$

In the homotopy system (2.5) and (2.6), $p \in [0,1]$ is an embedding parameter (h_u, h_λ) are the parameters to adjust the convergence of the homotopy series to be defined later. Moreover, L is an auxiliary linear differential operator whose proper shape depends on the particular example considered, F is defined by

$$g(1,B,\lambda)-g(1,B,\lambda_0),$$

$(u_0(t), \lambda_0)$ are initial guesses for the solutions, and furthermore N and f are, respectively, given by

$$N=\epsilon u'(t)+g(t,u,\lambda), \quad f=\epsilon u'(1)+g(1,B,\lambda).$$

It is obvious from Eqs. (2.5) and (2.6) that for $p = 0$ we have the initial approximations $u_0(t) = u(t,0)$ and $\lambda_0 = \lambda(0)$ to the solution, and when $p = 1$ we have the exact solution pair $u(t) = u(t,1)$ and $\lambda = \lambda(1)$ to Eqs. (2.3) and (2.4). It can be deduced that the deformation process of p from zero to unity is just that of from $(u(t,0), \lambda(0))$ to $(u(t,1), \lambda(1))$. The zeroth-order deformations to homotopies (2.5) and (2.6) are thus basically the linear differential equation with the boundary conditions in (2.5) satisfied exactly and $F = 0$. Next, the kth-order deformation equations follow as:

$$L(u_k-\kappa_k u_{k-1})=-h_uN_k, \quad u_k(0)=0 \tag{2.7}$$

and

$$F_k-\kappa_kF_{k-1}=-h_\lambda f_k, \tag{2.8}$$

where $\kappa_k = 0$ for $k \leqslant 1$ and $\kappa_k = 1$ otherwise. In addition to this, N_k and f_k are obtained as a result of differentiating Eqs. (2.5) and (2.6) with respect to p (note that $u(t,p)$ and $\lambda(p)$ in these equations), imposing the resulting equations at $p = 0$ and hence are defined by

$$N_k=\epsilon u_k'+\frac{1}{k!}\frac{\partial^k g}{\partial p^k}\bigg|_{p=0}, \quad f_k=\frac{1}{k!}\partial^k f\partial p^k\bigg|_{p=0}.$$

Further taking into account Taylor series expansion of the solutions $u(t,p)$ and $\lambda(p)$ at $p = 0$ and later imposition of the expansion at $p = 1$ we obtain respectively

$$u(t)=u_0(t)+\sum_{k=1}^{\infty}u_k(t), \tag{2.9}$$

and

$$\lambda=\lambda_0+\sum_{k=1}^{\infty}\lambda_k, \tag{2.10}$$

where u_k and λ_k are also defined by

$$u_k=\frac{1}{k!}\frac{\partial^k u}{\partial p^k}\bigg|_{p=0}, \quad \lambda_k=\frac{1}{k!}\frac{\partial^k \lambda}{\partial p^k}\bigg|_{p=0}.$$

The Mth-order finite truncation of the series (2.9) and (2.10) yields the approximations to the solutions

$$u(t)=\sum_{k=0}^{M}u_k(t), \tag{2.11}$$
$$\lambda=\sum_{k=0}^{M}\lambda_k, \tag{2.12}$$

which represent the approximate solution to the desired degree of accuracy.

We should also emphasize that the region of validity of the parameters h_u and h_v for the convergence of the corresponding homotopy series might be identified by drawing constant-h curves for particular values of the solution, such as $u'(0)$, $u''(0)$, *etc.* [16,13]. In addition to this, a better and optimal value of the convergence control parameters can be determined at the Mth-order of approximations by the square residual errors [18]

$$Res(h_u,h_\lambda)=\int_0^1\{\epsilon u'(t)+g(t,u,\lambda)\}^2dt, \tag{2.13}$$

owing to (1.1). Actually, $Res(h_u,h_\lambda)$ is a polynomial in terms of h_u and h_λ and one can easily minimize (2.13) by the requirement

$$\frac{\partial Res}{\partial h_u} = \frac{\partial Res}{\partial h_\lambda} = 0,$$

obviously by further checking out the second derivative test for two variable functions. It is strongly suggested in [18] on the classical equation of Blasius flow, the use of the minimum of square residual error to find out the optimal value of the two convergence-control parameters. However, in the case when the exact square residual error defined by (2.13) needs too much CPU time to calculate, even if the order of approximation is not very high, and thus is useless in practice, then to avoid the time-consuming computation we employ the case $h_u = h_\lambda$ in (2.13) reducing the unknown optimal parameters by one, which is committed in the subsequent analysis.

3. Convergence of the method

A proof for the analytical convergence of the homotopy analysis method given in Section 2 is outlined here. To do this, let us state the following theorem which gives sufficient conditions for the convergence/divergence of the homotopy series.

Theorem. *Suppose that $A \subset R$ be a Banach space donated with the L^2 norm, over which the sequence $u_k(t)$ of the homotopy series $u(t,p) = \sum_{k=0}^{\infty} u_k(t)p^k$ is defined for a prescribed value of h, where $u_k(t)$ represent both $u_k(t)$ of (2.9) and λ_k as given in (2.10). Assume also that the initial approximation $u_0(t)$ remains inside the ball of the solution $u(t)$. Taking $r \in R$ be a constant, the following statements hold true:*

(i) *If $\|v_{k+1}(t)\| \leqslant r\|v_k(t)\|$ for all k, given some $0 < r < 1$, then the series solution converges absolutely at $p = 1$ over the domain of definition of t,*
(ii) *If $\|v_{k+1}(t)\| \geqslant r\|v_k(t)\|$ for all k, given some $r > 1$, then the series solution diverges at $p = 1$ over the domain of definition of t.*

Proof. (i) If $S_n(t)$ denote the sequence of partial sum of the homotopy series, we need to show that $S_n(t)$ is a Cauchy sequence in A. For this purpose, consider,

$$\|S_{n+1}(t) - S_n(t)\| = \|u_{n+1}(t)\| \leqslant r\|u_n(t)\| \leqslant r^2\|u_{n-1}(t)\| \leqslant \cdots \leqslant r^{n+1}\|u_0(t)\|. \tag{3.14}$$

It should be remarked that owing to (3.14), all the approximations produced by the homotopy method will lie within the ball of $u(t)$. For every m, $n \in N$, $n \geqslant m$, making use of (3.14) and the triangle inequality successively, we have,

$$\|S_n(t) - S_m(t)\| = \left\|(S_n(t) - S_{n-1}(t)) + (S_{n-1}(t) - S_{n-2}(t)) + \cdots + (S_{m+1}(t) - S_m(t))\right\| \leqslant \frac{1 - r^{n-m}}{1 - r} r^{m+1}\left\|u_0(t)\right\|. \tag{3.15}$$

Since $0 < r < 1$, we get from (3.15)

$$\lim_{n,m\to\infty} \|S_n(t) - S_m(t)\| = 0. \tag{3.16}$$

Therefore, $S_n(t)$ is a Cauchy sequence in the Banach space A, and this implies that the series solution is convergent. This completes the proof (i). □

The proof of (ii) follows from the fact that under the hypothesis supplied in (ii), there exist a number l, $l > r > 1$, so that the interval of convergence of the power series is $|p| < 1/l < 1$, which obviously excludes the case of $p = 1$. □

Remark. Since the finite number of terms does not affect the convergence, Theorem is equally valid if the inequalities stated in (i–ii) are true for sufficiently large k's. Hence, the ratio $\beta = \frac{\|v_{k+1}(t)\|}{\|v_k(t)\|}$ needs to be pursued in the convergence analysis.

4. Results and discussion

In this section we apply the above outlined homotopy algorithm to some selected unperturbed ($\epsilon = 1$) and singularly perturbed two-point boundary value problems. Approximate analytical solutions obtained from the method have been compared with those obtained from the numerical computations using conventional finite-difference schemes in the literature. The accumulated error between the approximate and exact numerical solutions (calculated using the MATHEMATICA software) can eventually be estimated via

$$er_1 = \int_0^1 |u(t) - u_e(t)|\,dt, \tag{4.17}$$

$$er_2 = |\lambda - \lambda_e| \tag{4.18}$$

in which a subscript is to denote the numerically calculated value.

First example that is considered here is the subsequent parameterized singularly perturbed nonlinear system

$$\epsilon u' + 2u + u^2 + \lambda^2 - 2\lambda = 0, \quad u(0) = 1, \quad u(1) = 0. \tag{4.19}$$

For clarity, we propose the following homotopy parameters for evaluation of the solutions $u(t,p)$ and $\lambda(p)$, that are to be substituted into Eqs. (2.7) and (2.8)

$$L = \epsilon\frac{d}{dt} + 2, \quad u_0 = e^{-2x/\epsilon}, \quad \lambda_0 = 0, \quad f(1,0,\lambda) = \epsilon u'(1) + \lambda^2 - 2\lambda,$$
$$N_k = \epsilon u'_{k-1} + 2u_{k-1} - 2\lambda_{k-1} + \sum_{j=0}^{k-1}(u_j u_{k-1-j} + \lambda_j\lambda_{k-1-j}).$$

Together with these, taken into account the unperturbed case initially, i.e. $\epsilon = 1$, optimal values of the convergence control parameters at the $M = 10$th-order of approximation are found to be respectively $h_u = -0.833$ and $h_\lambda = -0.985$. However, as aforementioned, since it receives much computational cost to evaluate these, we prefer initially $h_u = h_\lambda = h$ in (2.13) and hence the optimal value of the convergence control parameter h evaluated through (2.13) appears to be $h = -0.886$ at the $M = 20$th-order of approximation, yielding a residual error 1.677×10^{-10}. Using this, the homotopy solutions in (2.11) and (2.12) of order $M = 0$, 1, 3 and 20 are compared with the numerical solution in Fig. 1. It can be seen that as the order of homotopy series solution in equations (2.11) and (2.12) increases, a fast convergence takes place to bring our approximate solution in excellent agreement with the numerical solution, so, the approximation to the present order can be used to represent the exact solution. Table 1 tabulates the accumulated errors (4.18) and (4.17) occurred at different orders. It is quite remarkable that errors between our homotopy solution and numerically calculated one decay quite fast as the order of iteration in the homotopy series (2.11) and (2.12) increases. This Table and Fig. 2(a) and (b) clearly imply the fact of convergence of the homotopy series solution to the true solution of system (4.19). We used L^1 norm and absolute value respectively while evaluating β's for u and λ. Classical finite difference solution of Eq. (4.19) were also performed for a further comparison. Using 1000 number of uniformly spaced grid points, we found that the maximum error is 5.77601×10^{-6} which is fairly above the maximum error illustrated in Table 1.

If we consider now the singularly perturbed case of (4.19) with $\epsilon = 10^{-5}$, we obtain the following expression for the second-order ($M = 2$) homotopy solution

$$u(t) = \frac{e^{-\alpha(1+3t)}\left(289e^{\alpha} + 1156e^{2\alpha t} - 1156e^{3\alpha t} + 204e^{\alpha(1+t)} + 1107e^{\alpha(1+2t)}\right)}{1600}, \tag{4.20}$$

where $\alpha = 200{,}000$. As also demonstrated in Fig. 3, Eq. (4.20) represents the solution to (4.19) with a boundary layer of thickness $O(\epsilon)$. Even this fascinating approximate analytic solution can be used in place of the exact singularly perturbed solution since maximum of the errors comes out to be only 2.45162×10^{-8}. On the other hand, to obtain this accuracy numerically with a finite-difference scheme introduced in [1], over 2500 points were needed.

Second example that we consider is the subsequent singularly perturbed parameterized system given in [3]

$$\epsilon u' + 2u - e^{-u} + xe^{\lambda} + x^2 = 0, \quad u(0) = 1, \quad u(1) = 0. \tag{4.21}$$

For this set of parameterized singularly perturbed equations, involving a much stronger nonlinearity due to the exponentials than (4.19), the corresponding auxiliary homotopy parameters are as listed

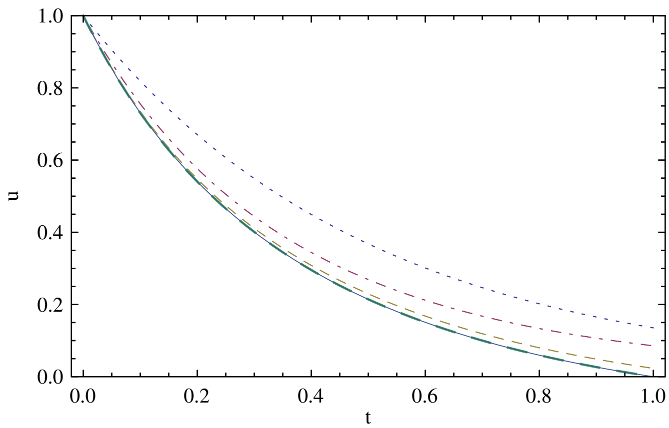

Fig. 1. Solution of parameterized Eq. (4.19) with $\epsilon = 1$: straight curve from the numerical solution and homotopy solutions are thick-dashed curve from the 20th-order, dashed curve from the third-order, dash-dotted curve from the first-order and dotted curve is the initial approximation.

Table 1
Illustrating the accumulated errors computed at the orders written for the problem (4.19). The exact value of λ is -0.11402189.

	$M = 1$	$M = 5$	$M = 10$	$M = 20$	$M = 30$
er_1	5.019×10^{-2}	3.209×10^{-3}	3.418×10^{-4}	6.115×10^{-6}	1.201×10^{-8}
er_2	5.885×10^{-3}	3.271×10^{-3}	3.110×10^{-4}	5.638×10^{-6}	1.041×10^{-8}

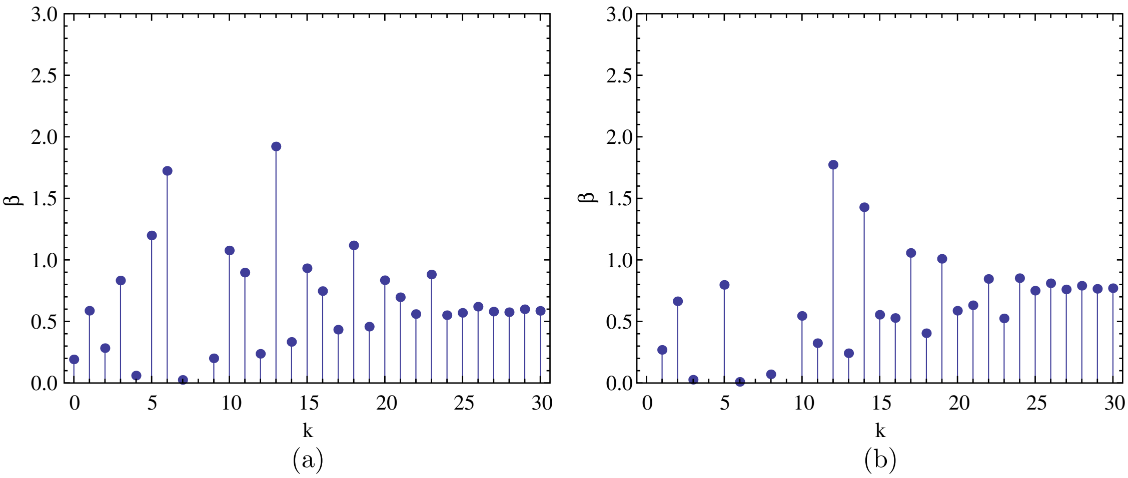

Fig. 2. A list plot of the ratio β from the theorem to reveal the convergence of the homotopy solutions for (4.19): (a) u and (b) λ.

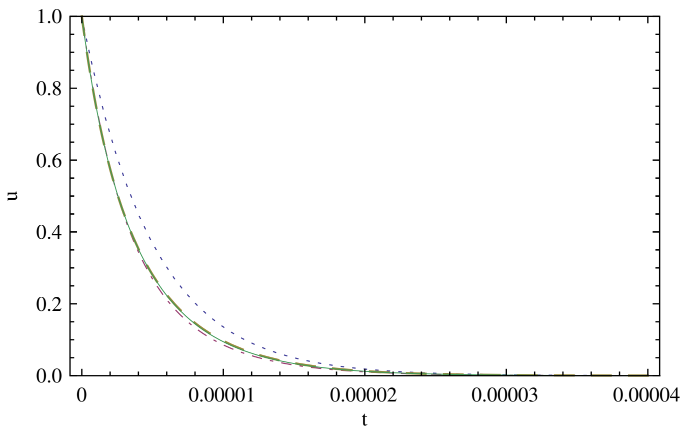

Fig. 3. Solution of parameterized Eq. (4.19) with $\epsilon = 10^{-5}$: straight curve from the numerical solution and homotopy solutions are thick-dashed curve from the second-order, dash-dotted curve from the first-order and dotted curve is the initial approximation.

$$L = \epsilon\frac{d}{dt} + 2, \quad u_0 = e^{-2x/\epsilon}, \quad \lambda_0 = 18, \quad f(1,0,\lambda) = \epsilon u'(1) + e^{\lambda},$$
$$N_1 = \epsilon u_0' + 2u_0 - 1 + xe^{\lambda_0} + x^2,$$
$$N_k = \epsilon u_{k-1}' + 2u_{k-1} - Du_{k-1} + xDl_{k-1}, \quad (k \neq 1),$$

where to be able to carry out integrations in Eq. (2.7) rapidly and without any difficulty, the exponential functions appearing in (4.21) were evaluated from the recursion relation

$$Du_0 = 1, \quad Du_k = -\sum_{m=0}^{k-1}\left(1 - \frac{m}{k}\right)u_{k-m-1}Du_m,$$

and similar for Dl_k.

Homotopy solution in this case is performed with an optimal convergence control parameter $h = -0.555$ obtained at the order of approximation $M = 6$, yielding a residual error 5.209×10^{-6}. To bring out the singularly perturbed nature of the problem we take a sufficiently small $\epsilon = 10^{-8}$ as in the paper [3]. Fig. 4(a) and (b) show a comparison between the sixth-order homotopy series solution and the numerically computed one. Fig. 4(b) is for displaying how well the homotopy methodology adopted here resolves very thin boundary layer region near $t = 0$ for the considered singular problem. Table 2 presents a list of errors calculated at different orders of approximation, which clearly illustrate the convergence of homotopy series results (2.11) and (2.12) to the exact one. Employing a second-order hybrid finite-difference scheme, [3] gets a minimum error of only 6.8202×10^{-7} with 1024 Bakhvalov–Shishkin grid points for the problem (4.21). This almost coincides with our 20th-order homotopy series solution. To achieve smaller and smaller errors, it is not known how the mesh size will be reduced in the scheme of [3], but as tabulated in Table 2, it is adequate to increase the order of approximation in our method.

The final example that we consider here is

$$\begin{aligned} &\epsilon u' + u - e^{-u} + (x+\lambda)e^{-1/\epsilon} + e^{(xe^{-1/\epsilon} - e^{-x/\epsilon})} + e^{\lambda} + \lambda - 1 = 0, \\ &u(0) = 1, \quad u(1) = 0, \end{aligned} \tag{4.22}$$

which is first posed in [2]. For this parameterized singularly perturbed two-point boundary value problem, the corresponding auxiliary homotopy parameters are as follows:

$$\begin{aligned} &L = \epsilon\frac{d}{dt} + 1, \quad u_0 = e^{-x/\epsilon}, \quad \lambda_0 = 0, \quad f(1,0,\lambda) = \epsilon u'(1) + (1+\lambda)e^{-1/\epsilon} + e^{\lambda} + \lambda - 1, \\ &N_1 = \epsilon u_0' + u_0 - 1 + (x+\lambda_0)e^{-1/\epsilon} + e^{xe^{-1/\epsilon}} + e^{\lambda_0} + \lambda_0 - 1, \\ &N_k = \epsilon u_{k-1}' + u_{k-1} - Du_{k-1} + (1+e^{-1/\epsilon})\lambda_{k-1} + (-1)^{k-1} e^{(xe^{-1/\epsilon} - (k-1)x/\epsilon)} + Dl_{k-1}. \end{aligned}$$

Homotopy solution in this case is performed with the convergence control parameters $h = -0.4$ with the singular perturbation $\epsilon = 10^{-2}$ as in [2]. For this particular example even the initial approximation is well capable of resolving all the solution field with a maximum error of 1.56496×10^{-9}, which indicates a fairly sharp convergence. Smaller values of ϵ yield much smaller errors. Since no numerical values were presented in [2] in which a boundary layer correction combined with a Runge–Kutta integrator was used, no a direct comparison can be made.

It can be readily deduced from Figs. 1–4 and Tables 1 and 2 that the homotopy analysis solutions for the parameterized singularly perturbed two-point boundary value problems considered are uniformly valid approximate solutions and hence they reliably represent the exact solutions. It is furthermore worthwhile to state that although specific values of ϵ are chosen as examples for the homotopy analysis solutions obtained here, the technique introduced can be applied to a vast variety of parameter ϵ without repeatedly running the algorithm for different ϵ.

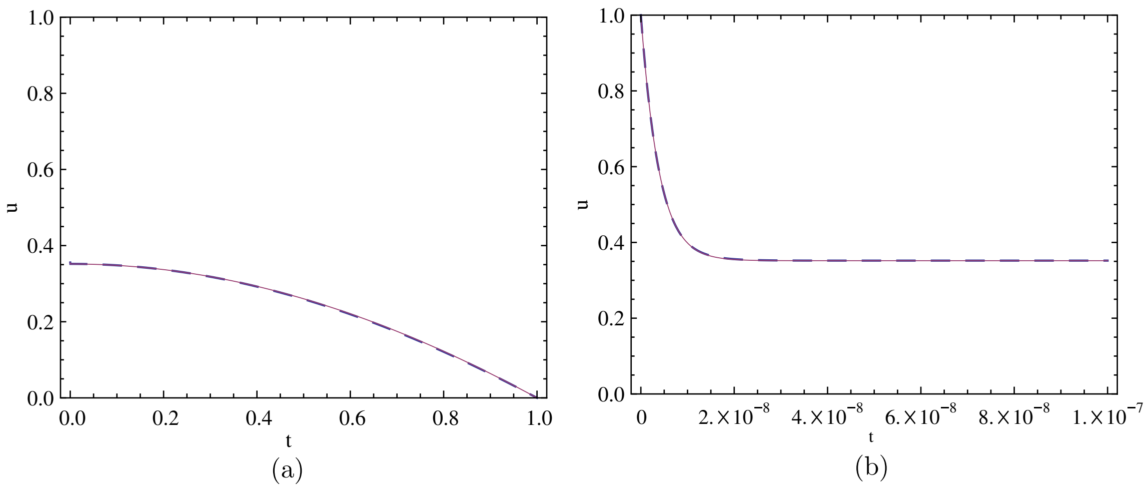

Fig. 4. Solution of parameterized Eq. (4.21) with $\epsilon = 10^{-8}$: straight curve from the numerical solution and thick-dashed curve from the sixth-order homotopy solution. (a) The full solution, (b) to demonstrate an enlarged view of the boundary layer region.

Table 2
Illustrating the accumulated errors computed at the orders written for the problem (4.21). The exact value of λ is 18.8248.

	$M = 1$	$M = 6$	$M = 10$	$M = 20$	$M = 30$
er_1	4.701×10^{-2}	3.571×10^{-4}	4.202×10^{-6}	6.444×10^{-8}	1.089×10^{-10}
er_2	2.698×10^{-1}	3.597×10^{-2}	2.306×10^{-3}	1.009×10^{-6}	1.707×10^{-9}

5. Concluding remarks

In this paper the nonlinear parameterized unperturbed and singularly perturbed two-point boundary layer problems have been considered by means of the homotopy analysis technique. First, the equation system has been modified and then, a methodology based on the homotopy has been developed with a proper proposal of auxiliary parameters involved.

The success of the method has later been tested by applying it to several unperturbed and singularly perturbed cases taken from the literature. The convergence of the corresponding homotopy series has been ensured by using optimal convergence control parameters obtained from the square residual error, supported by a mathematical proof of convergence. Even sufficiently low-order uniformly valid approximate analytic homotopy solutions whose forms have been explicitly written here reveal excellent agreement with the numerical solutions. Surprisingly, it has been found that as singularity gets stronger, the methodology proposed results in much more accurate solutions by satisfactorily resolving the boundary layers. The presented approach has clearly shown its advantage over the recently introduced conventional numerical methods for the singularly perturbed parameterized boundary value problems.

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