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An extension of the incomplete beta function for negative integers ☆

Emin Özçağ *, İnci Ege, Haşmet Gürçay

Department of Mathematics, University of Hacettepe, 06532-Beytepe, Ankara, Turkey

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Abstract

The incomplete beta function $B_x(a,b)$ is defined for $a,b>0$ and $0<x<1$. Its definition can be extended, by regularization, to negative non-integer values of a and b. In this paper we define the incomplete beta function $B_x(a,b)$ for negative integer values of a and b. Further we prove that the function $\frac{\partial^{m+n}}{\partial a^m \partial b^n} B_x(a,b)$ exists for $m,n=0,1,2,\ldots$ and all a and b.

Keywords: Beta function; Incomplete beta function; Gamma function; Incomplete gamma function; Distribution; Neutrix; Neutrix limit

1. Introduction

The incomplete beta function $B_x(a,b)$ is defined by

$$B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}\,dt, \quad a,b>0,\ 0<x<1, \tag{1}$$

see [14]. While best known for its applications in Statistics, it is also widely used in many other fields such as actuarial science, economics, finance, survival analysis, life testing and telecommunications.

Temme [15] obtained three asymptotic representations of the incomplete beta function with $a+b\to\infty$ valid in some cases. An asymptotic expansion of $B_x(a,b)$ for large a, small b and $x>0.5$ was considered by Doman [2]. Lopez and Sesma used the Laplace transform representation of $B_x(a,b)$ to obtain a very simple asymptotic expansion of it in ascending powers of $1/a$, see [10].

In this paper we aim to extend the definition of $B_x(a,b)$ to negative values of a and b, especially, to negative integer values of a and b.

The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part.

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* Corresponding author.
E-mail address: ozcag1@hacettepe.edu.tr (E. Özçağ).

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Using the concepts of neutrix and neutrix limit due to van der Corput [1], Fisher gave general principles for the discarding of unwanted infinite quantities from asymptotic expansions and this has been exploited in the context of distributions, see [3,4,9]. Recently, Y. Jack Ng and H. van Dam applied the neutrix calculus, in conjunction with the Hadamard integral, developed by van der Corput, to quantum field theories, in particular, to obtain finite results for the coefficients in the perturbation series. They also applied neutrix calculus to quantum field theory, obtaining finite renormalization in the loop calculations, see [11,12].

The following definitions were given by van der Corput [1].

Definition 1. A neutrix N is defined as a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some ν in N, $\nu(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.

Definition 2. Let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in N$, then c is called the neutrix limit of f as ξ tends to b and we write $N\text{-}\lim_{\xi \to b} f(\xi) = c$.

Note that if $f(\xi)$ tends to c in the normal sense as ξ tends to zero, it converges to c in the neutrix sense.

Example 1. The incomplete gamma function $\gamma(\alpha, x)$ is defined for $\alpha > 0$ and $x \geqslant 0$ by

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1} e^{-u}\, du$$

and more generally if $-m < \alpha < -m + 1$ and $x > 0$,

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1} \left[e^{-u} - \sum_{i=0}^{m-1} \frac{(-u)^i}{i!} \right] du + \sum_{i=0}^{m-1} \frac{(-1)^i x^{\alpha+i}}{(\alpha+i)i!}.$$

It has been shown in [6] that the incomplete gamma function $\gamma(\alpha, x)$ is defined by the neutrix limit

$$\gamma(\alpha, x) = \underset{\epsilon \to 0}{N\text{-}\lim} \int_\epsilon^x u^{\alpha-1} e^{-u}\, du$$

for $\alpha \neq 0, -1, -2, \ldots,$ $x > 0$, and the function $\gamma(-m, x)$ is also defined by the neutrix limit

$$\begin{aligned} \gamma(-m, x) &= \underset{\epsilon \to 0}{N\text{-}\lim} \int_\epsilon^x u^{-m-1} e^{-u}\, du \\ &= \int_1^x u^{-m-1} e^{-u}\, du + \int_0^1 u^{-m-1} \left[e^{-u} - \sum_{i=0}^{m} \frac{(-u)^i}{i!} \right] du - \sum_{i=0}^{m-1} \frac{(-1)^i}{i!(m-i)} \end{aligned}$$

for $m = 1, 2, \ldots,$ see [7], where N is the neutrix having domain $N' = \{\epsilon\colon 0 < \epsilon < \infty\}$ and range N'' the real numbers, with the negligible functions being finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1} \epsilon, \quad \ln^r \epsilon \quad (\lambda < 0,\ r = 1, 2, \ldots)$$

and all functions $o(\epsilon)$ which converge to zero in the normal sense as ϵ tends to zero, see [3,4].

It was proved in [13] that the rth derivative of the incomplete gamma function is

$$\gamma^{(r)}(\alpha, x) = \underset{\epsilon \to 0}{N\text{-}\lim} \int_\epsilon^x u^{\alpha-1} \ln^r u e^{-u}\, du$$

for all α and $r = 0, 1, 2, \ldots.$

Example 2. The beta function $B(a,b)$ is defined by

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\,dt$$

for $a,b>0$. By regularization

$$\begin{aligned}B(a,b)={}&\int_0^{1/2} t^{a-1}\left[(1-t)^{b-1}-\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)}t^i\right]dt+\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{2^{a+i}i!\Gamma(b-i)(a+i)}\\&+\int_{1/2}^{1}(1-t)^{b-1}\left[t^{a-1}-\sum_{i=0}^{s-1}\frac{(-1)^i\Gamma(a)}{i!\Gamma(a-i)}(1-t)^i\right]dt+\sum_{i=0}^{s-1}\frac{(-1)^i\Gamma(a)}{2^{b+i}i!\Gamma(a-i)(b+i)}\end{aligned}\tag{2}$$

for $a>-r$, $b>-s$, and $a\neq 0,-1,\dots,-r+1$, $b\neq 0,-1,\dots,-s+1$, see [8], where Γ denotes gamma function.

Now if we let N be a neutrix having domain the open interval $\{\epsilon\colon\ 0<\epsilon<\frac{1}{2}\}$ with the same negligible functions as in Example 1, then

$$B(a,b)=N\text{-}\lim_{\epsilon\to 0}\int_{\epsilon}^{1-\epsilon} t^{a-1}(1-t)^{b-1}\,dt\tag{3}$$

and in general

$$\frac{\partial^{m+n}}{\partial a^m\partial b^n}B(a,b)=N\text{-}\lim_{\epsilon\to 0}\int_{\epsilon}^{1-\epsilon} t^{a-1}\ln^m t(1-t)^{b-1}\ln^n(1-t)\,dt\tag{4}$$

for $a,b\neq 0,-1,-2,\dots$ and $m,n=0,1,2,\dots,$ see [5].

2. Incomplete beta function $B_x(a,b)$

Equation (2) suggests that the integral (1) can be regularized for negative non-integer values of a and b so that we can express the incomplete beta function $B_x(a,b)$ $(0<x<1)$ in the form

$$\begin{aligned}B_x(a,b)={}&\int_0^{x/2} t^{a-1}\left[(1-t)^{b-1}-\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)}t^i\right]dt\\&+\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)(a+i)}(x/2)^{a+i}+\int_{x/2}^{x} t^{a-1}(1-t)^{b-1}\,dt\end{aligned}\tag{5}$$

for all b and $a>-r$, $a\neq 0,-1,-2,\dots,-r+1$.

Now consider

$$\begin{aligned}B_x(a,b)={}&\int_{\epsilon}^{x/2} t^{a-1}\left[(1-t)^{b-1}-\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)}t^i\right]dt+\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)(a+i)}(x/2)^{a+i}\\&-\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)(a+i)}\epsilon^{a+i}+\int_{x/2}^{x} t^{a-1}(1-t)^{b-1}\,dt\end{aligned}\tag{6}$$

for all b and $a>-r$, $a\neq 0,-1,-2,\dots,-r+1$.

Letting N be the neutrix having domain the open interval $(0, x/2)$ with the same negligible functions as in Example 1, we see that

$$B_x(a,b) = \underset{\epsilon\to 0}{N\text{-lim}} \int_\epsilon^x t^{a-1}(1-t)^{b-1}\,dt \tag{7}$$

for all values of a and b, $a \neq 0, -1, -2, \ldots$.

If we differentiate Eq. (5) partially with respect to a, we see that the right-hand side contains terms of the form $\epsilon^a \ln\epsilon$ and it follows that

$$\frac{\partial}{\partial a} B_x(a,b) = \underset{\epsilon\to 0}{N\text{-lim}} \int_\epsilon^x t^{a-1}\ln t(1-t)^{b-1}\,dt \tag{8}$$

for all values of a and b, $a \neq 0, -1, -2, \ldots$.

Similarly

$$\frac{\partial}{\partial b} B_x(a,b) = \underset{\epsilon\to 0}{N\text{-lim}} \int_\epsilon^x t^{a-1}(1-t)^{b-1}\ln(1-t)\,dt$$

and in general

$$\frac{\partial^{m+n}}{\partial a^m \partial b^n} B_x(a,b) = \underset{\epsilon\to 0}{N\text{-lim}} \int_\epsilon^x t^{a-1}\ln^m t(1-t)^{b-1}\ln^n(1-t)\,dt \tag{9}$$

for $m, n = 0, 1, 2, \ldots$ and all values of a and b, $a \neq 0, -1, -2, \ldots$.

We will first of all prove the existence of $B_x(0,0)$. We have

$$\begin{aligned}\int_\epsilon^x t^{-1}(1-t)^{-1}\,dt &= \int_\epsilon^x \left[t^{-1} + (1-t)^{-1}\right]dt = \ln x - \ln(1-x) - \ln\epsilon + \ln(1-\epsilon)\\ &= \ln\frac{x}{1-x} - \ln\epsilon + o(\epsilon)\end{aligned}$$

and it follows that

$$B_x(0,0) = \underset{\epsilon\to 0}{N\text{-lim}} \int_\epsilon^x t^{-1}(1-t)^{-1}\,dt = \ln\frac{x}{1-x}.$$

Next

$$\int_\epsilon^x t^{-1}(1-t)^{b-1}\,dt = \int_\epsilon^x (1-t)^{b-1}\,d\ln t = (1-x)^{b-1}\ln x - (1-\epsilon)^{b-1}\ln\epsilon + (b-1)\int_\epsilon^x \ln t(1-t)^{b-2}\,dt.$$

Thus

$$B_x(0,b) = \underset{\epsilon\to 0}{N\text{-lim}} \int_\epsilon^x t^{-1}(1-t)^{b-1}\,dt = (1-x)^{b-1}\ln x + (b-1)\frac{\partial}{\partial a}B_x(1,b-1)$$

on using Eq. (8). This proves the existence of $B_x(0,b)$ for $b \neq 0, -1, -2, \ldots$.

In particular,

$$\int_{\epsilon}^{x} t^{-1}(1-t)^{n-1}\,dt = \sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i \int_{\epsilon}^{x} t^{i-1}\,dt = \ln x - \ln\epsilon + \sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^i \frac{x^i}{i} + o(\epsilon)$$

and

$$B_x(0,n) = \underset{\epsilon\to 0}{N\text{-lim}} \int_{\epsilon}^{x} t^{-1}(1-t)^{n-1}\,dt = \ln x - \sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^i \frac{x^i}{i} \tag{10}$$

for $n = 1, 2, \ldots$.

Similarly

$$\begin{aligned} B_x(n,0) &= \int_{0}^{x} t^{n-1}(1-t)^{-1}\,dt = \int_{0}^{x}\left[(1-t)^{-1} - \sum_{i=0}^{n-2} t^i\right]dt \\ &= -\ln(1-x) - \sum_{i=1}^{n-1}\frac{x^i}{i}. \end{aligned} \tag{11}$$

Further we have

$$\begin{aligned} \int_{\epsilon}^{x} t^{-n-1}(1-t)^{-1}\,dt &= \int_{\epsilon}^{x}\left[(1-t)^{-1} + \sum_{i=1}^{n+1} t^{-i}\right]dt = \left[-\ln(1-t) + \ln t - \sum_{i=1}^{n}\frac{t^{-i}}{i}\right]_{\epsilon}^{x} \\ &= \ln x - \ln(1-x) - \sum_{i=1}^{n}\frac{x^{-i}}{i} + \ln(1-\epsilon) - \ln\epsilon + \sum_{i=1}^{n}\frac{\epsilon^{-i}}{i} \end{aligned}$$

and it follows that

$$B_x(-n,0) = \underset{\epsilon\to 0}{N\text{-lim}} \int_{\epsilon}^{x} t^{-n-1}(1-t)^{-1}\,dt = \ln\frac{x}{1-x} - \sum_{i=1}^{n}\frac{x^{-i}}{i} \tag{12}$$

for $n = 1, 2, \ldots$.

Next

$$\begin{aligned} \int_{\epsilon}^{x} t^{-1}(1-t)^{-n-1}\,dt &= \int_{1-x}^{1-\epsilon} t^{-n-1}(1-t)^{-1}\,dt = \int_{1-x}^{1-\epsilon}\left[(1-t)^{-1} + \sum_{i=1}^{n+1} t^{-i}\right]dt \\ &= -\ln\epsilon + \ln x + \ln(1-\epsilon) - \ln(1-x) - \sum_{i=1}^{n}\frac{1}{i}\left[\frac{1}{(1-\epsilon)^i} - \frac{1}{(1-x)^i}\right]. \end{aligned}$$

Thus

$$\begin{aligned} B_x(0,-n) &= \underset{\epsilon\to 0}{N\text{-lim}} \int_{\epsilon}^{x} t^{-1}(1-t)^{-n-1}\,dt \\ &= \ln\frac{x}{1-x} + \sum_{i=1}^{n}\frac{(1-x)^{-i}}{i} - \sum_{i=1}^{n}\frac{[(i-1)!]^2}{i}, \end{aligned} \tag{13}$$

where

$$\underset{\epsilon\to 0}{N\text{-lim}} \frac{1}{(1-\epsilon)^k} = \underset{\epsilon\to 0}{N\text{-lim}} \frac{\partial^k}{\partial\epsilon^k}\sum_{i=0}^{\infty}\epsilon^i = \left[(k-1)!\right]^2.$$

We will now assume the existence of $B_x(-n+1,b)$ for some positive integer n. This is certainly true when $n=1$. Then

$$\begin{aligned}\int_\epsilon^x t^{-n-1}(1-t)^{b-1}\,dt &= -\frac{1}{n}\int_\epsilon^x (1-t)^{b-1}\,dt^{-n}\\ &= -\frac{1}{n}\big[t^{-n}(1-t)^{b-1}\big]_\epsilon^x - \frac{b-1}{n}\int_\epsilon^x t^{-n}(1-t)^{b-2}\,dt\\ &= -\frac{1}{n}x^{-n}(1-x)^{b-1} + \frac{1}{n}\epsilon^{-n}(1-\epsilon)^{b-1} - \frac{b-1}{n}\int_\epsilon^x t^{-n}(1-t)^{b-2}\,dt.\end{aligned}$$

We have

$$\epsilon^{-n}(1-\epsilon)^{b-1} = \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)}\epsilon^{i-n} = \sum_{i=0}^{n}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)}\epsilon^{i-n} + o(\epsilon).$$

It follows from our assumption that

$$\operatorname*{N\text{-}lim}_{\epsilon\to 0}\int_\epsilon^x t^{-n-1}(1-t)^{b-1}\,dt = -\frac{1}{n}x^{-n}(1-x)^{b-1} + \frac{(-1)^n\Gamma(b)}{nn!\Gamma(b-n)} - \frac{b-1}{n}B_x(-n+1,b-1).$$

The existence of $B_x(-n,b)$ follows by induction for all $b \neq 0, \pm 1, \pm 2, \dots$ and $n = 1, 2, \dots$.

Next we have

$$\int_\epsilon^x t^{-n-1}\,dt = -\frac{1}{n}\big(x^{-n} - \epsilon^{-n}\big)$$

and it follows that $B_x(-n,1)$ exists and

$$B_x(-n,1) = -\frac{1}{n}x^{-n} \tag{14}$$

for $n = 1, 2, \dots$.

Now assume the existence of $B_x(-n+1,m)$ for some positive integer n and $m = 1, 2, \dots$. Equation (10) shows that this is true when $n=1$. Then

$$\begin{aligned}\int_\epsilon^x t^{-n-1}(1-t)^{m-1}\,dt &= -\frac{1}{n}\int_\epsilon^x (1-t)^{m-1}\,dt^{-n}\\ &= -\frac{1}{n}\big[t^{-n}(1-t)^{m-1}\big]_\epsilon^x - \frac{m-1}{n}\int_\epsilon^x t^{-n}(1-t)^{m-2}\,dt\\ &= \frac{1}{n}\sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^i\big[\epsilon^{i-n} - x^{i-n}\big] - \frac{m-1}{n}\int_\epsilon^x t^{-n}(1-t)^{m-2}\,dt \end{aligned}\tag{15}$$

and it follows from our assumption that

$$\operatorname*{N\text{-}lim}_{\epsilon\to 0}\int_\epsilon^x t^{-n-1}(1-t)^{m-1}\,dt = -\frac{1}{n}x^{-n}(1-x)^{m-1} - \frac{m-1}{n}B_x(-n+1,m-1)$$

for $m = 1, 2, \dots, n$. The existence of $B_x(-n,m)$ follows by induction.

Using Eqs. (14), it now follows easily by induction that

$$B_x(-n,m)=\sum_{i=1}^{m}\frac{(-1)^i(m-1)!(n-i)!}{n!(m-i)!}x^{-n+i-1}(1-x)^{m-i} \tag{16}$$

for $m=1,2,\dots,n$.

When $m>n$, it follows from Eqs. (15) and (16) and our assumption that

$$\operatorname*{N\text{-}lim}_{\epsilon\to 0}\int_\epsilon^x t^{-n-1}(1-t)^{m-1}\,dt=-\frac{1}{n}x^{-n}(1-x)^{m-1}+\frac{(-1)^n(m-1)!}{nn!(m-n-1)!}-\frac{m-1}{n}B_x(-n+1,m-1).$$

The existence of $B_x(-n,m)$ follows by induction and it follows from Eq. (16) that

$$B_x(-n,m)=\sum_{i=1}^{m}\frac{(-1)^i(m-1)!(n-i)!}{n!(m-i)!}x^{-n+i-1}(1-x)^{m-i}+\frac{(-1)^n(m-1)!}{n!(m-n)!}\big[\phi(n)-\phi(m-n-1)\big] \tag{17}$$

for $m=n+1,n+2,\dots$ and $n=1,2,\dots,$ where $\phi(n)=\sum_{i=1}^{n}i^{-1}$.

Finally we assume the existence of $B_x(-n+1,-m)$ for some positive integer n and $m=1,2,\dots$. Equation (13) shows that this is certainly true when $n=1$. Then

$$\begin{aligned}\int_\epsilon^x t^{-n-1}(1-t)^{-m-1}\,dt&=-\frac{1}{n}\int_\epsilon^x(1-t)^{-m-1}\,dt^{-n}\\&=-\frac{1}{n}x^{-n}(1-x)^{-m-1}+\frac{1}{n}\epsilon^{-n}(1-\epsilon)^{-m-1}+\frac{m+1}{n}\int_\epsilon^x t^{-n}(1-t)^{-m-2}\,dt\\&=-\frac{1}{n}x^{-n}(1-x)^{-m-1}+\frac{1}{n}\sum_{i=0}^{n}\frac{(m+i)!}{i!m!}\epsilon^{i-n}+\frac{m+1}{n}\int_\epsilon^x t^{-n}(1-t)^{-m-2}\,dt+o(\epsilon).\end{aligned}$$

It follows from our assumption that

$$\operatorname*{N\text{-}lim}_{\epsilon\to 0}\int_\epsilon^x t^{-n-1}(1-t)^{-m-1}\,dt=\frac{(m+n)!}{nn!m!}-\frac{1}{n}x^{-n}(1-x)^{-m-1}+\frac{m+1}{n}B_x(-n+1,-m-1) \tag{18}$$

for $m,n=1,2,\dots$. The existence of $B_x(-n,-m)$ follows by induction.

It now follows easily by induction and on using Eqs. (13) and (18) that

$$\begin{aligned}B_x(-n,-m)=&-\sum_{i=1}^{n}\frac{(m+i-1)!(n-i)!}{n!m!}x^{-n+i-1}(1-x)^{-m-i}\\&+\frac{(m+n)!}{n!m!}\left[\phi(n)+\ln\frac{x}{1-x}+\sum_{i=1}^{m+n}\frac{(1-x)^{-i}}{i}-\sum_{i=1}^{m+n}\frac{[(i-1)!]^2}{i}\right]\end{aligned}$$

for $m,n=1,2,\dots$.

Fisher and Kuribayashi give the definition of $B_x(a,b)$ as $x\to 1$ for all values of a and b in [5]. For instance they define

$$\begin{aligned}&B(0,n)=B(n,0)=-\phi(n-1),\\&B(0,-n)=B(-n,0)=-\phi(n)\end{aligned}$$

for $n=1,2,\dots,$ and

$$B(-n,m)=\frac{(-1)^m(m-1)!(n-m)!}{n!}$$

for $m = 1, 2, \dots, n$ and $n = 1, 2, \dots$ and

$$B(-n, m) = \frac{(-1)^n (m-1)!}{n!(m-n)!}\big[\phi(n) - \phi(m-n-1)\big]$$

for $m = n+1, n+2, \dots$.

Equation (9) suggests the following definition.

Definition. The function $\frac{\partial^{m+n}}{\partial a^m \partial b^n} B_x(a, b)$ is defined by

$$\frac{\partial^{m+n}}{\partial a^m \partial b^n} B_x(a, b) = N\text{-}\lim_{\epsilon \to 0} \int_\epsilon^x t^{a-1} \ln^m t (1-t)^{b-1} \ln^n(1-t)\, dt \tag{19}$$

for $m, n = 0, 1, 2, \dots$ and all a and b.

It is not immediately obvious that the neutrix limit in Eq. (19) exists and it has been so far proved that this neutrix limit exists only for the case $m = n = 0$. In the following, we prove that this neutrix limit exists for $m = n = 0, 1, 2, \dots$ and all a, b so that $\frac{\partial^{m+n}}{\partial a^m \partial b^n} B_x(a, b)$ is well defined. We first of all need the following.

Lemma. *The neutrix limit as ϵ tends to zero of the function*

$$\int_\epsilon^{x/2} t^a \ln^m t \ln^n(1-t)\, dt$$

exists for $m, n = 0, 1, 2, \dots$ and all a.

Proof. Suppose first of all that $m = n = 0$. Then

$$\int_\epsilon^{x/2} t^a\, dt = \begin{cases} \frac{(x/2)^{a+1} - \epsilon^{a+1}}{a+1}, & a \neq -1, \\ \ln(x/2) - \ln\epsilon, & a = -1 \end{cases}$$

and so $N\text{-}\lim_{\epsilon \to 0} \int_\epsilon^{x/2} t^a\, dt$ exists for all a.

Now suppose that $n = 0$ and that $N\text{-}\lim_{\epsilon \to 0} \int_\epsilon^{x/2} t^a \ln^m t\, dt$ exists for some non-negative integer m and all a. Then

$$\int_\epsilon^{x/2} t^a \ln^{m+1} t\, dt = \begin{cases} \frac{(x/2)^{a+1} \ln^{m+1}(x/2) - \epsilon^{a+1} \ln^{m+1}\epsilon}{a+1} - \frac{m+1}{a+1} \int_\epsilon^{x/2} t^a \ln^m t\, dt, & a \neq -1, \\ \frac{\ln^{m+2}(x/2) - \ln^{m+2}\epsilon}{m+2}, & a = -1 \end{cases}$$

and it follows by induction that $N\text{-}\lim_{\epsilon \to 0} \int_\epsilon^{x/2} t^a \ln^m t\, dt$ exists for $m = 0, 1, 2, \dots$ and all a.

Finally we note that we can write

$$\ln^n(1-t) = \sum_{i=1}^\infty \alpha_{in} t^i$$

for $n = 1, 2, \dots$, the expansion being valid for $|t| < 1$. Choosing a positive integer k such that $a + k > -1$, we have

$$\int_\epsilon^{x/2} t^a \ln^m t \ln^n(1-t)\, dt = \sum_{i=1}^{k-1} \alpha_{in} \int_\epsilon^{x/2} t^{a+i} \ln^m t\, dt + \sum_{i=k}^\infty \alpha_{in} \int_\epsilon^{x/2} t^{a+i} \ln^m t\, dt.$$

It follows from what we have just proved that

$$N\text{-}\lim_{\epsilon \to 0} \sum_{i=1}^{k-1} \alpha_{in} \int_\epsilon^{x/2} t^{a+i} \ln^m t\, dt$$

exists and further

$$N\text{-}\lim_{\epsilon\to 0}\sum_{i=k}^{\infty}\alpha_{in}\int_{\epsilon}^{x/2} t^{a+i}\ln^m t\,dt=\lim_{\epsilon\to 0}\sum_{i=k}^{\infty}\alpha_{in}\int_{\epsilon}^{x/2} t^{a+i}\ln^m t\,dt=\sum_{i=k}^{\infty}\alpha_{in}\int_{0}^{x/2} t^{a+i}\ln^m t\,dt,$$

proving that $\int_\epsilon^{x/2} t^a \ln^m t \ln^n(1-t)\,dt$ exists for $m, n = 0, 1, 2, \ldots$ and all a. □

Theorem. *The function* $\frac{\partial^{m+n}}{\partial a^m \partial b^n} B_x(a, b)$ *exists for* $m, n = 0, 1, 2, \ldots$ *and all* a, b.

Proof. Choose positive integer r such that $a > -r$. Then we can write

$$\int_\epsilon^x t^{a-1}\ln^m t(1-t)^{b-1}\ln^n(1-t)\,dt$$
$$=\int_\epsilon^{x/2} t^{a-1}\ln^m t\ln^n(1-t)\left[(1-t)^{b-1}-\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)}t^i\right]dt$$
$$+\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)}\int_\epsilon^{x/2} t^{a-1}\ln^m t\ln^n(1-t)\,dt+\int_{x/2}^{x} t^{a-1}\ln^m t(1-t)^{b-1}\ln^n(1-t)\,dt.$$

We have the first integral on the right-hand side being convergent. Further, from the lemma we see that the neutrix limit of the function

$$\sum_{i=0}^{r-1}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)}\int_\epsilon^{x/2} t^{a-1}\ln^m t\ln^n(1-t)\,dt$$

exists, implying that

$$N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^x t^{a-1}\ln^m t(1-t)^{b-1}\ln^n(1-t)\,dt$$

exists. This proves the existence of the function $\frac{\partial^{m+n}}{\partial a^m \partial b^n} B_x(a, b)$ which exists for $m, n = 0, 1, 2, \ldots$ and all a, b. □

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