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A textural view of the distinction between uniformities and quasi-uniformities

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Abstract

In this paper the authors consider the interplay between di-uniformities on a texture and a complementation on that texture. It is shown that with each di-uniformity corresponds a second di-uniformity, called its complement. Di-uniformities that coincide with their complement are then called complemented, and it is verified that the uniform ditopology of a complemented di-uniformity is a complemented ditopology. The connection with the uniform bicontinuity of the complement of a difunction is also considered.

The relation between quasi-uniformities and uniformities on a set X in the classical sense is then investigated in the setting of di-uniformities on the complemented discrete texture on X. It is shown that di-uniformities on this discrete texture correspond in a one-to-one way with quasi-uniformities on X, a quasi-uniformity being a uniformity if and only if the corresponding di-uniformity is complemented. This shows that while the difference between quasi-uniformities and uniformities in the classical description is a question of symmetry, this becomes a matter of complementation in the di-uniform case.

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1. Introduction

In [16] the authors laid the foundations of a theory of uniformities on textures, giving descriptions in terms of direlations, dicovers and dimetrics. In that paper no account was taken of any possible complementation on the texture in question, and this topic is taken up in the present paper. It is shown that with each direlational uniformity on a complemented texture corresponds a second direlational uniformity, called its complement. Direlational uniformities that coincide with their complement are then said to be complemented, and it is verified that the ditopology of a complemented direlational uniformity is a complemented ditopology. The uniform bicontinuity of the comple-

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ment of a difunction is studied, and the notion of complement characterized in terms of dicovering uniformities on a complemented texture.

Direlational uniformities have a base of symmetric direlations, and it might therefore be conjectured that they would correspond to uniformities in the classical sense. However, by restricting attention to the discrete complemented texture $(X, \mathcal{P}(X), \pi_X)$ on a set X it is shown that in fact direlational uniformities correspond in a one-to-one way with quasi-uniformities on X. Moreover, a direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$ corresponds to a uniformity if and only if it is complemented. Hence the distinction between quasi-uniformities and uniformities, which is one of symmetry in the classical representation, becomes a matter of complementation in the description using direlations. This shows the very pervasive role played by complementation, since without a complementation it would not seem possible to formulate the quasi-uniform/uniform dichotomy for di-uniformities on general textures.

The paper ends with an analysis of the link between dual covers and dicovers, which parallels that between point relations and direlations used in establishing the above mentioned relationship between quasi-uniformities and direlational uniformities on $(X, \mathcal{P}(X),\ \pi_X)$.

Constant reference will be made to [16] for definitions and results relating to di-uniformities, none of which will be repeated here. General references on ditopological texture spaces include [4–10], and a major part of the theory of direlations and difunctions, first described in the preprint [3], may be found in [8] with largely new proofs. It should be noted that if (r, R) is a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, and $A \in \mathcal{S}$, then the A-sections of r and R are denoted by $r^{\to}A$, $R^{\to}A$, respectively, in [8], in preference to the former notation $r(A)$, $R(A)$. We will use this new notation here since it is more consistent with the notation $r^{\leftarrow}B$, $R^{\leftarrow}B$ for the B-presections, $B \in \mathcal{T}$, which is maintained as a short form for $(r^{\leftarrow})^{\to}B$ and $(R^{\leftarrow})^{\to}B$, respectively. General references on quasi-uniformities and uniformities include [11,14,15,18], while terms relating to lattice theory not given here may be found in [13].

A preliminary version of part of this work is given in [17].

We end this introduction with a few words of a general nature which aim at motivating the study of ditopological texture spaces, and which explain their relation to other topics of General Topology.

Ditopological texture spaces were conceived as a point-set setting for the study of fuzzy topology, and provide a unified setting for the study of topology, bitopology and fuzzy topology. Some of the links with Hutton spaces, $\mathbb{L}$-fuzzy sets and topologies are expressed in a categorical setting in [6].

Despite the close links with fuzzy sets and topologies, the development of the theory of ditopological texture spaces has proceeded largely independently, and the work on di-uniformities [16], in particular, has shown that it has much closer ties with mainstream topological ideas than might be expected. This paper provides further confirmation of this by giving firm evidence that di-uniformities provide a more unified setting for the study of quasi-uniformities and uniformities than does the classical approach (see especially the discussion following Theorem 3.5). Here the notion of difunction plays an important role, and we see additional evidence that despite its very minimal nature, the unit texture $(\mathbb{I}, \mathcal{I}, \iota)$ under its natural di-uniformity $\mathcal{U}_{\mathbb{I}}$ plays an extremely important role in the theory.

As argued in the introduction to [16], it is precisely the existence of such minimal but powerful structures that makes them of potential importance as computational models. However, as also pointed out in [16], this power is only realized through the use of appropriate concepts, and much of the effort in the development of the theory of ditopological texture spaces has been devoted to the establishment of such important concepts as dicover, direlation and difunction.

The formal duality [16, comments following Lemma 1.1], occurring in a texture, is an important element in defining such concepts. When applied to ditopologies it often gives rise to pairs of properties, such as compact–cocompact, regular–coregular. Here there is a close parallel with bitopological concepts, and indeed the links with bitopology are naturally strong. In the presence of a complementation a property and its co-property often coincide, but in the case of a direlational uniformity it is actually the symmetry of that structure that causes the uniform ditopology to be simultaneously completely regular and completely coregular. The results in this paper therefore represent an interesting interplay between duality and complementation in the context of di-uniformities on a texture.

A form of duality also plays a role in Giovanni Sambin's basic picture for formal topology [18]. There are clear parallels here which warrant further study. Likewise, links with the theory of locales and with domain theory have yet to be worked out. Finally, complement free textural concepts can be expected to find applications in negation free logics, and indeed (ditopological) textures themselves could well prove to be useful models for certain classes of such logics.

2. Complementation in di-uniform spaces

In this section we consider di-uniformities in relation to complementation.

By a *complementation* on the texture $(S, \mathcal{S})$ we mean a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Longrightarrow \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$. Set difference $\pi(Y) = X \setminus Y$ is a complementation on $(X, \mathcal{P}(X))$. It is the existence of this complementation that is responsible for many of the special properties of this texture, particularly in relation to symmetry. We can define a complementation ι on the unit interval texture $(I, \mathcal{I})$ (see, for example, [8, Definition 1.1(5)], or [16]) by setting

$$\iota\big([0,r]\big) = [0, 1-r) \quad \text{and} \quad \iota\big([0,r)\big) = [0, 1-r] \quad \text{for all } r \in I,$$

and indeed many of the naturally occurring textures can be equipped with a complementation.

If (τ, κ) is a ditopology on the texture $(S, \mathcal{S})$ we say (τ, κ) *is complemented* if $\kappa = \sigma[\tau]$.

In order to consider the effect of a complementation on a direlational uniformity it is necessary to define the action of a complementation on a direlation. Let us recall the following definition from [3], see also [8].

Definition 2.1. Let (d, D) be a direlation between the complemented textures $(S, \mathcal{S}, \sigma)$ and $(T, \mathcal{T}, \theta)$.

(1) The *complement d' of the relation d* is the co-relation

$$d' = \bigcap\big\{\overline{Q}_{(s,t)} \mid \exists u, v \text{ with } d \not\subseteq \overline{Q}_{(u,v)},\ \sigma(Q_s) \not\subseteq Q_u \text{ and } P_v \not\subseteq \theta(P_t)\big\}.$$

(2) The *complement D' of the co-relation D* is the relation

$$D' = \bigvee\big\{\overline{P}_{(s,t)} \mid \exists u, v \text{ with } \overline{P}_{(u,v)} \not\subseteq D,\ P_u \not\subseteq \sigma(P_s) \text{ and } \theta(Q_t) \not\subseteq Q_v\big\}.$$

(3) The *complement $(d, D)'$ of the direlation (d, D)* is the direlation

$$(d, D)' = (D', d').$$

The direlation (d, D) is said to be *complemented* if $(d, D)' = (d, D)$.

We summarize below some properties of the complementation operator which will be needed later.

Proposition 2.2. (Cf. [3]) *For direlations (d, D), (e, E) between complemented textures we have*:

(1) $\sigma(d^{\to} A) = (d')^{\to}\sigma(A)$ *and* $\sigma(D^{\to} A) = (D')^{\to}\sigma(A)$ *for all* $A \in \mathcal{S}$.
(2) $((d, D)')' = (d, D)$.
(3) $((d, D)')^{\leftarrow} = ((d, D)^{\leftarrow})'$.
(4) $((d, D) \circ (d, D))' = (d, D)' \circ (d, D)'$.
(5) *Each identity direlation (i, I) is complemented.*
(6) $(d, D) \sqsubseteq (e, E) \Longrightarrow (d, D)' \sqsubseteq (e, E)'$.
(7) (d, D) *is complemented if and only if* $(d, D)' \sqsubseteq (d, D)$.
(8) $((d, D) \sqcap (e, E))' \sqsubseteq (d, D)' \sqcap (e, E)'$.

Proof. The proofs of (1)–(5) are sketched in [8], while (6) and (7) are straightforward. For (8) note that $((d, D) \sqcap (e, E))' = ((D \sqcup E)', (d \sqcap e)')$, $(d, D)' \sqcap (e, E)' = (D' \sqcap E', d' \sqcup e')$ and assume that $(D \sqcup E)' \not\subseteq D' \sqcap E'$. Since $D \sqcup E$ is a correlation, by Definition 2.1(2) there exist $s \in S$, $t \in T$ with $\overline{P}_{(s,t)} \not\subseteq D' \sqcap E'$ for which we have $u \in S$, $v \in T$ satisfying $\overline{P}_{(u,v)} \not\subseteq D \sqcup E$, $P_u \not\subseteq \sigma(P_s)$ and $\sigma(Q_t) \not\subseteq Q_v$. By the definition of $D \sqcup E$ (see [3] or [16, Definition 1.8]) there exists $v' \in T$ with $\overline{P}_{(u,v)} \not\subseteq \overline{Q}_{(u,v')}$ and $u' \in S$ with $P_{u'} \not\subseteq Q_u$ and $\overline{P}_{(u',v')} \not\subseteq D, E$. Since $P_v \not\subseteq Q_{v'}$ we have $Q_{v'} \subseteq Q_v$ and so $\theta(Q_t) \not\subseteq Q_{v'}$, i.e., $\theta(Q_{v'}) \not\subseteq Q_t$. Hence, by [8, Theorem 1.2(5)], we have $t' \in T$ satisfying $\theta(Q_{v'}) \not\subseteq Q_{t'}$, whence $\theta(Q_{t'}) \not\subseteq Q_{v'}$, and $P_{t'} \not\subseteq Q_t$. Likewise we have $s' \in S$ satisfying $P_s \not\subseteq Q_{s'}$ and $P_{u'} \not\subseteq \sigma(P_{s'})$. We now have $\overline{P}_{(s',t')} \subseteq D', E'$ by Definition 2.1(2), whence $D', E' \not\subseteq \overline{Q}_{(s',t)}$. This, together with $P_s \not\subseteq Q_{s'}$ and the fact that D', E' are relations, gives the contradiction $\overline{P}_{(s,t)} \subseteq D' \sqcap E'$ (see [3] or [16, Definition 1.8]).

This establishes that $(D \sqcup E)' \subseteq D' \sqcap E'$. The proof of $d' \sqcup e' \subseteq (d \sqcap e)'$ is dual to this, and is omitted. □

Now let $\mathcal{U}$ be a direlational uniformity on $(S, \mathcal{S}, \sigma)$ and set

$$\mathcal{U}' = \{(d, D)' \mid (d, D) \in \mathcal{U}\}.$$

We have:

Theorem 2.3. *For each direlational uniformity $\mathcal{U}$ the family $\mathcal{U}'$ is also a direlational uniformity. The uniform ditopology of $\mathcal{U}'$ is $(\sigma[\kappa_{\mathcal{U}}], \sigma[\tau_{\mathcal{U}}])$.*

Proof. We must verify conditions (1)–(5) of [16, Definition 3.1].

(1) $(d, D)' \in \mathcal{U}' \Longrightarrow (d, D) \in \mathcal{U} \Longrightarrow (i, I) \sqsubseteq (d, D) \Longrightarrow (i, I) = (i, I)' \sqsubseteq (d, D)'$, by Proposition 2.2(2), (5), and (6).

(2) $(d, D)' \in \mathcal{U}',\ (d, D)' \sqsubseteq (e, E) \Longrightarrow (d, D) \in \mathcal{U},\ (d, D) \sqsubseteq (e, E)' \Longrightarrow (e, E)' \in \mathcal{U} \Longrightarrow (e, E) \in \mathcal{U}'$, by Proposition 2.2(2) and (6).

(3) $(d, D)', (e, E)' \in \mathcal{U}' \Longrightarrow (d, D), (e, E) \in \mathcal{U} \Longrightarrow (d, D) \sqcap (e, E) \in \mathcal{U} \Longrightarrow ((d, D) \sqcap (e, E))' \in \mathcal{U}$. By Proposition 2.2(8) we now have $(d, D)' \sqcap (e, E)' \in \mathcal{U}'$.

(4) $(d, D)' \in \mathcal{U}' \Longrightarrow (d, D) \in \mathcal{U} \Longrightarrow \exists (e, E) \in \mathcal{U}$ satisfying $(e, E) \circ (e, E) \sqsubseteq (d, D)$. Now we have $(e, E)' \in \mathcal{U}'$ and by Proposition 2.2(4) and (5), $(e, E)' \circ (e, E)' = ((e, E) \circ (e, E))' \sqsubseteq (d, D)'$.

(5) $(d, D)' \in \mathcal{U}' \Longrightarrow (d, D) \in \mathcal{U} \Longrightarrow \exists (c, C) \in \mathcal{U}$ with $(c, C)^{\leftarrow} \sqsubseteq (d, D)$. Hence $(c, C)' \in \mathcal{U}'$ and $((c, C)')^{\leftarrow} = ((c, C)^{\leftarrow})' \sqsubseteq (d, D)'$ by Proposition 2.2(3) and (6).

To show $\tau_{\mathcal{U}'} = \sigma[\kappa_{\mathcal{U}}]$, take $K \in \kappa_{\mathcal{U}}$ and $s \in S$ with $\sigma(K) \nsubseteq Q_s$. By [8, Lemma 2.19(2)] we have $u \in S$ with $\sigma(K) \nsubseteq \sigma(P_u)$ and $\sigma(Q_u) \nsubseteq Q_s$. Now $P_u \nsubseteq K$, so there exists $(d, D) \in \mathcal{U}$ with $K \subseteq D^{\rightarrow} Q_u$. Since $P_s \subseteq \sigma(Q_u)$ we have $(D')^{\rightarrow} P_s \subseteq (D')^{\rightarrow} \sigma(Q_u) = \sigma(D^{\rightarrow} Q_u) \subseteq \sigma(K)$ by [8, Lemma 2.20(1)]. This verifies $\sigma(K) \in \tau_{\mathcal{U}'}$, so $\sigma[\kappa_{\mathcal{U}}] \subseteq \tau_{\mathcal{U}'}$.

Dually it may be verified that $G \in \tau_{\mathcal{U}} \Longrightarrow \sigma(G) \in \kappa_{\mathcal{U}'}$, so applying this to $\mathcal{U}'$ and noting $(\mathcal{U}')' = \mathcal{U}$ we have $\tau_{\mathcal{U}'} \subseteq \sigma[\kappa_{\mathcal{U}}]$. This shows $\tau_{\mathcal{U}'} = \sigma[\kappa_{\mathcal{U}}]$, and likewise $\kappa_{\mathcal{U}'} = \sigma[\tau_{\mathcal{U}}]$. □

This leads to the following definition.

Definition 2.4. For a given direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S}, \sigma)$ the direlational uniformity $\mathcal{U}' = \{(d, D)' \mid (d, D) \in \mathcal{U}\}$ is called the *complement* of $\mathcal{U}$. The di-uniformity $\mathcal{U}$ is said to be *complemented* if $\mathcal{U}' = \mathcal{U}$.

Proposition 2.5. *Let $\mathcal{U}$ be a direlational uniformity on $(S, \mathcal{S}, \sigma)$.*

(1) *$\mathcal{U}$ is complemented if and only if it has a base of complemented direlations.*
(2) *The uniform ditopology of a complemented direlational uniformity is a complemented ditopology.*

Proof. (1) Suppose $\mathcal{U}' = \mathcal{U}$ and take $(d, D) \in \mathcal{U}$. Then $(d, D) = (e, E)'$ for some $(e, E) \in \mathcal{U}$. Let $(c, C) = (d, D) \sqcap (e, E)$. Then $(c, C) \in \mathcal{U}$ and $(c, C) \sqsubseteq (d, D)$. Using Proposition 2.2(8), (2) we obtain $(c, C)' \sqsubseteq (c, C)$, whence (c, C) is complemented by Proposition 2.2(7). Thus $\mathcal{U}$ has a base of complemented direlations.

If, conversely, $\mathcal{U}$ has a base of complemented direlations it is trivial to verify $\mathcal{U}' = \mathcal{U}$.

(2) If $\mathcal{U}$ is complemented then $\mathcal{U}' = \mathcal{U}$ so $\tau_{\mathcal{U}} = \tau_{\mathcal{U}'} = \sigma[\kappa_{\mathcal{U}}]$ by Theorem 2.3. Hence the uniform ditopology of $\mathcal{U}$ is complemented. □

Example 2.6. Consider the usual direlational uniformity $\mathcal{U}_I$ on $(I, \mathcal{I}, \iota)$ defined in [16]. For $\varepsilon > 0$ we have $d'_\varepsilon = D_\varepsilon$. Indeed, suppose $d'_\varepsilon \nsubseteq D_\varepsilon$. Then we have $s, t \in I$ with $d'_\varepsilon \nsubseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \nsubseteq D_\varepsilon$. From $\overline{P}_{(s,t)} \nsubseteq D_\varepsilon$ we obtain $t > s - \varepsilon$, so if we let $u = 1 - s$ and $v = 1 - t$ we have $v < u + \varepsilon$ and so $d_\varepsilon \nsubseteq \overline{Q}_{(u,v)}$, while $\iota(Q_s) \nsubseteq Q_u$ and $P_v \nsubseteq \iota(P_t)$. This gives the contradiction $d'_\varepsilon \subseteq \overline{Q}_{(s,t)}$, so $d'_\varepsilon \subseteq D_\varepsilon$. The reverse inclusion is proved likewise, so $(d_\varepsilon, D_\varepsilon)' = (d_\varepsilon, D_\varepsilon)$, which proves that $\mathcal{U}_I$ has a base of complemented direlations. Hence $\mathcal{U}_{\mathbb{I}}$ is a complemented direlational uniformity by Proposition 2.5.

Proposition 2.7. *Let $\mathcal{U}$ be a direlational uniformity on $(S, \mathcal{S}, \sigma)$ with uniform ditopology $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$. Then*

$$\mathcal{U} \vee \mathcal{U}' = \{(d, D) \mid \exists (e, E) \in \mathcal{U} \text{ with } (e, E) \sqcap (e, E)' \sqsubseteq (d, D)\}$$

is a complemented direlational uniformity on $(S, \mathcal{S}, \sigma)$. Moreover, the uniform ditopology of $\mathcal{U} \vee \mathcal{U}'$ has base $\{G \cap \sigma(K) \mid G \in \tau_{\mathcal{U}},\ K \in \kappa_{\mathcal{U}}\}$ and cobase $\{K \cup \sigma(G) \mid K \in \kappa_{\mathcal{U}},\ G \in \tau_{\mathcal{U}}\}$.

Proof. The proof that $\mathcal{U} \vee \mathcal{U}'$ is a complemented direlational uniformity is a straightforward application of Proposition 2.2, and is omitted. As suggested by the notation $\mathcal{U} \vee \mathcal{U}'$ is the smallest di-uniformity containing $\mathcal{U}$ and $\mathcal{U}'$. In particular, the open sets of the uniform ditopologies of $\mathcal{U}$, $\mathcal{U}'$ are open for the uniform ditopology of $\mathcal{U} \vee \mathcal{U}'$, so $\beta = \{G \cap \sigma(K) \mid G \in \tau_{\mathcal{U}},\ K \in \kappa_{\mathcal{U}}\} \subseteq \tau_{\mathcal{U} \vee \mathcal{U}'}$. To show that β is a base, let us first note that for relations d, e in $(S, \mathcal{S})$ and $s, s' \in S$ with $P_{s'} \not\subseteq Q_s$ we have

$$d^{\to} P_s \cap e^{\to} P_s \subseteq (d \sqcap e)^{\to} P_{s'}.$$

To see this, suppose the contrary and take $t, t' \in S$ with $d^{\to} P_s \cap e^{\to} P_s \not\subseteq Q_t$, $P_t \not\subseteq Q_{t'}$ and $P_{t'} \not\subseteq (d \sqcap e)^{\to} P_{s'}$. By [8, Lemma 2.6(1)] we have $d, e \not\subseteq \overline{Q}_{(s,t)}$ and so $\overline{P}_{(s',t)} \subseteq d \sqcap e$ by [16, Definition 1.8]. Hence $d \sqcap e \not\subseteq \overline{Q}_{(s',t')}$, which gives the contradiction $(d \sqcap e)^{\to} P_{s'} \not\subseteq Q_{t'}$.

Now take $H \in \tau_{\mathcal{U} \vee \mathcal{U}'}$ and $s \in S^{\flat}$ with $H \not\subseteq Q_s$. Choose $s' \in S$ with $H \not\subseteq Q_{s'}$, $P_{s'} \not\subseteq Q_s$. By [16, Lemma 4.3(i)] we have $(d, D) \in \mathcal{U} \vee \mathcal{U}'$ with $d^{\to} P_{s'} \subseteq H$, and by definition we have $(e, E) \in \mathcal{U}$ with $(e, E) \sqcap (e, E)' \sqsubseteq (d, D)$. Applying the above inclusion to the relations e, E' now gives

$$e^{\to} P_s \cap (E')^{\to} P_s \subseteq (e \sqcap E')^{\to} P_{s'} \subseteq d^{\to} P_{s'} \subseteq H. \tag{1}$$

It is easy to verify that for $L \in \mathcal{S}$,

$$G(L) = \bigvee \{P_u \mid \exists (r, R) \in \mathcal{U} \text{ with } r^{\to} P_u \subseteq L\} \in \tau_{\mathcal{U}},$$
$$K(L) = \bigcap \{Q_u \mid \exists (r, R) \in \mathcal{U} \text{ with } L \subseteq R^{\to} Q_u\} \in \kappa_{\mathcal{U}}$$

(cf. [16, Lemma 4.6]). We may also note that $L \subseteq G(e^{\to} L) \subseteq e^{\to} L$ and $E^{\to} L \subseteq K(E^{\to} L) \subseteq L$. Hence, setting $G = G(e^{\to} P_s)$ gives $P_s \subseteq G \subseteq e^{\to} P_s$, $G \in \tau_{\mathcal{U}}$, while setting $K = K(E^{\to} \sigma(P_s))$ gives $E^{\to} \sigma(P_s) \subseteq K \subseteq \sigma(P_s)$, whence on taking the complement we obtain $P_s \subseteq \sigma(K) \subseteq \sigma(E^{\to} \sigma(P_s)) = (E')^{\to} P_s$, $K \in \kappa_{\mathcal{U}}$, by [8, Lemma 2.20(1)]. Hence $G \cap \sigma(K) \in \beta$ and $P_s \subseteq G \cap \sigma(K) \subseteq H$ by (1), which completes the proof that β is a base of $(\tau_{\mathcal{U} \vee \mathcal{U}'}, \kappa_{\mathcal{U} \vee \mathcal{U}'})$ by [9, Theorem 3.2(1)].

The proof that $\gamma = \{K \cup \sigma(G) \mid K \in \kappa_{\mathcal{U}},\ G \in \tau_{\mathcal{U}}\}$ is a cobase is dual to the above, and is omitted. $\square$

Let us next consider the uniform bicontinuity of difunctions in relation to complementation. The following lemma is fundamental.

Lemma 2.8. *Let $(f, F) : (S, \mathcal{S}, \sigma) \to (T, \mathcal{T}, \theta)$ be a difunction between complemented textures. If (r, R) is a direlation on $(T, \mathcal{T}, \theta)$ then*

$$(f, F)^{-1}(r, R) = \left(\left((f, F)'\right)^{-1}\left((r, R)'\right)\right)'.$$

Proof. To establish $((F', f')^{-1}(r'))' \subseteq (f, F)^{-1}(r)$, assume the contrary. Then we have $s_1, s_2 \in S$ with $\overline{P}_{(s_1,s_2)} \not\subseteq (f, F)^{-1}(r)$ for which there exist $u, v \in S$ satisfying

$$\overline{P}_{(u,v)} \not\subseteq (F', f')^{-1}(r'), \qquad P_u \not\subseteq \sigma(P_{s_1}) \quad \text{and} \quad \sigma(Q_{s_2}) \not\subseteq Q_v.$$

By [8, Lemma 2.19] we have $s_1' \in S$ with $P_u \not\subseteq \sigma(P_{s_1'})$ and $\sigma(Q_{s_1'}) \not\subseteq \sigma(P_{s_1})$. Since $P_{s_1} \not\subseteq Q_{s_1'}$, $\overline{P}_{(u,v)} \not\subseteq (F', f')^{-1}(r')$ now gives the existence of $t_1, t_2 \in S$ satisfying

$$\overline{P}_{(s_1',t_1)} \not\subseteq F, \qquad f \not\subseteq \overline{Q}_{(s_2,t_2)} \quad \text{and} \quad \overline{P}_{(t_1,t_2)} \not\subseteq r.$$

On the other hand, since r' is a correlation, from $\overline{P}_{(u,v)} \not\subseteq (F', f')^{-1}(r')$ we obtain $v', u' \in S$ with $P_v \not\subseteq Q_{v'}$, $P_{u'} \not\subseteq Q_u$ for which

$$\forall z_1, z_2 \in S, \quad F' \not\subseteq \overline{Q}_{(u',z_1)} \quad \text{and} \quad f' \not\subseteq \overline{Q}_{(v',z_2)} \Longrightarrow \overline{P}_{(z_1,z_2)} \not\subseteq r'. \tag{2}$$

(See the note following [16, Definition 5.1].) From $\overline{P}_{(s_1',t_1)} \not\subseteq F$ we have $t_1' \in S$ with $P_{t_1} \not\subseteq Q_{t_1'}$ and $\overline{P}_{(s_1',t_1')} \not\subseteq F$. By [8, Lemma 2.19] there exists $w_1 \in S$ with $P_{t_1} \not\subseteq \sigma(P_{w_1})$, $\sigma(Q_{w_1}) \not\subseteq Q_{t_1'}$, and then $w_1' \in S$ with $\sigma(Q_{w_1}) \not\subseteq \sigma(P_{w_1'})$, $\sigma(Q_{w_1'}) \not\subseteq Q_{t_1'}$. Clearly $\overline{P}_{(u',w_1')} \subseteq F'$ and $\overline{P}_{(u',w_1')} \not\subseteq \overline{Q}_{(u',w_1)}$ so $F' \not\subseteq \overline{Q}_{(u',w_1)}$.

In the same way, working from $f \not\subseteq \overline{Q}_{(s_2,t_2)}$, we obtain $w_2 \in S$ satisfying $\overline{P}_{(v',w_2)} \not\subseteq f'$ and $\sigma(Q_{w_2}) \not\subseteq Q_{t_2}$. Hence, setting $z_1 = w_1$, $z_2 = w_2$ in the implication (2) gives $\overline{P}_{(w_1,w_2)} \not\subseteq r'$. Now we have $w_2' \in S$ with $P_{w_2} \not\subseteq Q_{w_2'}$ for which there are $\alpha, \beta \in S$ satisfying

$$r \not\subseteq \overline{Q}_{(\alpha,\beta)}, \qquad \sigma(Q_{w_1}) \not\subseteq Q_\alpha \quad \text{and} \quad P_\beta \not\subseteq \sigma(P_{w_2'}).$$

Note that $P_{w_2'} \subseteq P_{w_2}$, whence $\sigma(P_{w_2}) \subseteq \sigma(P_{w_2'})$ and so $P_\beta \not\subseteq \sigma(P_{w_2})$.

From $\sigma(Q_{w_1}) \not\subseteq Q_\alpha$ we obtain $\sigma(Q_\alpha) \not\subseteq Q_{w_1}$ and so $P_{w_1} \subseteq \sigma(Q_\alpha)$. On the other hand, $P_{t_2} \not\subseteq \sigma(P_{w_1})$ implies $P_{w_1} \not\subseteq \sigma(P_{t_1})$, so $\sigma(Q_\alpha) \not\subseteq \sigma(P_{t_1})$ and therefore $P_{t_1} \not\subseteq Q_\alpha$. Since $r \not\subseteq \overline{Q}_{(\alpha,\beta)}$ we obtain $r \not\subseteq \overline{Q}_{(t_1,\beta)}$ by condition *R1* for r. Likewise, using $P_\beta \not\subseteq \sigma(P_{w_2})$ and $\sigma(Q_{w_2}) \not\subseteq Q_{t_2}$ we deduce $P_\beta \not\subseteq Q_{t_2}$, whence $Q_{t_2} \subseteq Q_\beta$ and we obtain $r \not\subseteq \overline{Q}_{(t_1,t_2)}$ and so the contradiction $\overline{P}_{(t_1,t_2)} \subseteq r$.

This completes the proof of $((F', f')^{-1}(r'))' \subseteq (f, F)^{-1}(r)$, and the reader may easily provide the proof of the dual inclusion $(f, F)^{-1}(R) \subseteq ((F', f')^{-1}(R'))'$. Applying this latter result to the difunction (F', f') and the correlation r' gives $(F', f')^{-1}(r') \subseteq ((f, F)^{-1}(r))'$ by [8, Proposition 2.21(1)], so taking the complement of both sides gives $(f, F)^{-1}(r) \subseteq ((F', f')^{-1}(r'))'$. Combined with the earlier inclusion this shows $(f, F)^{-1}(r) = ((F', f')^{-1}(r'))'$. The same method gives $(f, F)^{-1}(R) = ((F', f')^{-1}(R'))'$, so completing the proof. □

Now we have:

Theorem 2.9. *Let $(f, F) : (S, \mathcal{S}, \sigma) \to (T, \mathcal{T}, \theta)$ be a difunction between complemented textures, $\mathcal{U}$ a direlational uniformity on $(S, \mathcal{S})$ and $\mathcal{V}$ on $(T, \mathcal{T})$. Then (f, F) is $\mathcal{U}$–$\mathcal{V}$ uniformly bicontinuous if and only if $(f, F)'$ is $\mathcal{U}'$–$\mathcal{V}'$ uniformly bicontinuous.*

Proof. Let (f, F) be $\mathcal{U}$–$\mathcal{V}$ uniformly bicontinuous and take $(e, E) \in \mathcal{V}'$. Then $(e, E) = (d, D)'$, $(d, D) \in \mathcal{V}$, whence $(f, F)^{-1}(d, D) \in \mathcal{U}$. Hence, by Lemma 2.7, $((f, F)')^{-1}(e, E) = ((f, F)')^{-1}((d, D)') = ((f, F)^{-1}(d, D))' \in \mathcal{U}'$, showing $(f, F)'$ to be $\mathcal{U}'$–$\mathcal{V}'$ uniformly bicontinuous.

Since the operation of taking the complement is idempotent the reverse implication is now clear. □

This immediately gives:

Corollary 2.10. *With the notation as in Theorem 2.9, if $\mathcal{U}$ and $\mathcal{V}$ are complemented then (f, F) is $\mathcal{U}$–$\mathcal{V}$ uniformly bicontinuous if and only if $(f, F)'$ is $\mathcal{U}$–$\mathcal{V}$ uniformly bicontinuous.*

Theorem 2.11. *Let $(S, \mathcal{S}, \sigma)$ be a complemented texture and (τ, κ) a ditopology on $(S, \mathcal{S})$. Then there exists a complemented direlational uniformity on $(S, \mathcal{S}, \sigma)$ compatible with (τ, κ) if and only if (τ, κ) is complemented and completely biregular.*

Proof. Necessity is clear by Proposition 2.5(2) and [16, Theorem 4.14]. For sufficiency suppose that (τ, κ) is complemented and completely biregular. By [16, proof of Theorem 5.16] we know that the initial direlational uniformity $\mathcal{U}$ generated by the family D of all bicontinuous difunctions from $(S, \mathcal{S}, \sigma, \tau, \kappa)$ to $(\mathbb{I}, \mathcal{I}, \iota, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$, and the usual direlational uniformity $\mathcal{U}_{\mathbb{I}}$ on $(\mathbb{I}, \mathcal{I}, \iota)$, is compatible with (τ, κ). Let us note the following:

Lemma 2.12. *For $k = 1, 2$, let $(S_k, \mathcal{S}_k, \sigma_k, \tau_k, \kappa_k)$ be complemented ditopological texture spaces. Then the difunction $(f, F) : (S_1, \mathcal{S}_1, \sigma_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \sigma_2, \tau_2, \kappa_2)$ is continuous (cocontinuous) if and only if its complement $(f, F)' : (S_1, \mathcal{S}_1, \sigma_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \sigma_2, \tau_2, \kappa_2)$ is cocontinuous (continuous).*

Proof. Immediate from [8, Lemma 2.20(2)]. □

Returning to the proof of the theorem, for $(f,F)\in D$ and $\varepsilon>0$ we now have

$$\big((f,F)^{-1}(d_\varepsilon,D_\varepsilon)\big)'=\big((f,F)'\big)^{-1}\big((d_\varepsilon,D_\varepsilon)'\big)=\big((f,F)'\big)^{-1}(d_\varepsilon,D_\varepsilon)\in\mathcal{U}$$

by Lemmas 2.8 and 2.12. By Proposition 2.2 we see that $(f,F)^{-1}(d_\varepsilon,D_\varepsilon)\sqcap((f,F)^{-1}(d_\varepsilon,D_\varepsilon))'$ is a complemented element of $\mathcal{U}$ with

$$(f,F)^{-1}(d_\varepsilon,D_\varepsilon)\sqcap\big((f,F)^{-1}(d_\varepsilon,D_\varepsilon)\big)'\sqsubseteq(f,F)^{-1}(d_\varepsilon,D_\varepsilon),$$

whence $\mathcal{U}$ has a base of complemented direlations. Hence $\mathcal{U}$ is complemented, as required. □

It is natural to ask if the complemented di-uniformity $\mathcal{U}$ in the proof of Theorem 2.11 may also be generated by the complemented bicontinuous difunctions. The following example shows that this is not the case in general.

Example 2.13. Consider the discrete complemented texture $(X,\mathcal{P}(X),\pi_X)$. Clearly any function $\varphi:X\to\mathbb{I}$ satisfies conditions (a), (b) and (c) of [9, Lemma 3.8] so the corresponding difunction (f_φ,F_φ) is given by

$$f_\varphi=\bigvee\{\overline{P}_{(s,\varphi(s))}\mid s\in S\}\quad\text{and}\quad F_\varphi=\bigcap\{\overline{Q}_{(s,\varphi(s))}\mid s\in S^\flat\}.$$

Moreover, since $(X,\mathcal{P}(X))$ and $(\mathbb{I},\mathcal{J})$ are plain textures, all difunctions have the form (f_φ,F_φ) for some $\varphi:X\to\mathbb{I}$ by [8, Proposition 3.7]. Now,

$$\begin{aligned}F'_\varphi&=\bigvee\big\{\overline{P}_{(x,\alpha)}\mid \exists y,\beta\text{ with }\overline{P}_{(y,\beta)}\not\subseteq F_\varphi,\ P_y\not\subseteq\pi_X(P_x)\text{ and }\iota(Q_\alpha)\not\subseteq Q_\beta\big\}\\&=\bigcup\big\{\{x\}\times[0,\alpha]\mid\varphi(x)\leqslant 1-\alpha\big\}\\&=f_{1-\varphi},\end{aligned}$$

and so $(f_\varphi,F_\varphi)'=(f_{1-\varphi},F_{1-\varphi})$. The difunction (f_φ,F_φ) is complemented if and only if $f_\varphi=f_{1-\varphi}$, and so if and only if $\varphi=1-\varphi$. This shows that the only complemented difunction from $(X,\mathcal{P}(X),\pi_X)$ to $(\mathbb{I},\mathcal{J},\iota)$ is given by the constant function $\frac{1}{2}:X\to\mathbb{I}$. Hence, in this case, there are certainly insufficient complemented difunctions to support the generation of complemented di-uniformities on $(X,\mathcal{P}(X),\pi_X)$.

To conclude this section we consider complementation from the point of view of dicovering uniformities. Let $\mathcal{C}=\{(A_j,B_j)\mid j\in J\}$ be a dicover on the complemented texture $(S,\mathcal{S},\sigma)$. It is easy to see that $\mathcal{C}'=\{(\sigma(B_j),\sigma(A_j))\mid j\in J\}$ is also a dicover. The following lemma shows how this complementation on dicovers is related to the complementation of direlations defined above.

Lemma 2.14. *Let $\mathcal{C}$ be a dicover on the complemented texture $(S,\mathcal{S},\sigma)$. Then $\delta(\mathcal{C})'=\delta(\mathcal{C}')$.*

Proof. We must verify $(D(\mathcal{C})',d(\mathcal{C})')=(d(\mathcal{C}'),D(\mathcal{C}'))$. Since clearly $(\mathcal{C}')'=\mathcal{C}$, it will be sufficient to prove that $D(\mathcal{C})'=d(\mathcal{C}')$.

Suppose that $D(\mathcal{C})'\not\subseteq d(\mathcal{C}')$. From Definition 2.1(2) we have $s,t\in S$ with $\overline{P}_{(s,t)}\not\subseteq d(\mathcal{C}')$ and $u,v\in S$ satisfying $\overline{P}_{(u,v)}\not\subseteq D(\mathcal{C})$, $P_u\not\subseteq\sigma(P_s)$ and $\sigma(Q_t)\not\subseteq Q_v$. Now from the definition of $D(\mathcal{C})$ (see [16, Proposition 2.5]) we have $v'\in S$ with $\overline{P}_{(u,v)}\not\subseteq\overline{Q}_{(u,v')}$, and $j\in J$ satisfying $P_{v'}\not\subseteq B_j$ and $A_j\not\subseteq Q_u$. It is now easy to verify that $\sigma(B_j)\not\subseteq Q_t$ and $P_s\not\subseteq\sigma(A_j)$, which leads to the contradiction $\overline{P}_{(s,t)}\subseteq d(\mathcal{C}')$.

Now suppose that $d(\mathcal{C}')\not\subseteq D(\mathcal{C})'$. Then we have $s,t\in S$ with $\overline{P}_{(s,t)}\not\subseteq D(\mathcal{C})'$ and $j\in J$ satisfying $\sigma(B_j)\not\subseteq Q_t$ and $P_s\not\subseteq\sigma(A_j)$. Now $\sigma(Q_t)\not\subseteq B_j$, $A_j\not\subseteq\sigma(P_s)$ so we may take $u,v\in S$ with $A_j\not\subseteq Q_u$, $P_u\not\subseteq\sigma(P_s)$ and $\sigma(Q_t)\not\subseteq Q_v$, $P_v\not\subseteq B_j$. Finally we may take $v'\in S$ satisfying $P_v\not\subseteq Q_{v'}$ and $P_{v'}\not\subseteq B_j$. From the definition of $D(\mathcal{C})$ we obtain $D(\mathcal{C})\subseteq\overline{Q}_{(u,v')}$, whence $\overline{P}_{(u,v)}\not\subseteq D(\mathcal{C})$. However this, together with $P_v\not\subseteq B_j$ and $A_j\not\subseteq Q_u$, now gives the contradiction $\overline{P}_{(s,t)}\subseteq D(\mathcal{C})'$ by Definition 2.1(2). □

We may now describe what we should mean by the complement of a dicovering uniformity υ by passing to the direlational uniformity $\Delta(\upsilon)$, taking the complement and then applying Γ (see [16]). This leads to the following:

Proposition 2.15. *For a dicovering uniformity υ on $(S, \mathcal{S}, \sigma)$,*

$$\Gamma(\Delta(\upsilon)') = \{\mathcal{D} \in \mathcal{DC} \mid \exists \mathcal{C} \in \upsilon \text{ with } (\mathcal{C}')^{\Delta} \prec \mathcal{D}\}.$$

Proof. Clearly $\Gamma(\Delta(\upsilon)') = \{\mathcal{D} \in \mathcal{DC} \mid \exists \mathcal{C} \in \upsilon \text{ with } \gamma(\delta(\mathcal{C})') \prec \mathcal{D}\}$. However, $\gamma(\delta(\mathcal{C})') = \gamma(\delta(\mathcal{C}')) = (\mathcal{C}')^{\Delta}$ by Lemma 2.14 and [8, Theorem 2.7(2)]. □

Definition 2.16. Let υ be a dicovering uniformity on $(S, \mathcal{S}, \sigma)$. Then the *complement of* υ is the covering di-uniformity υ' with base $(\mathcal{C}')^{\Delta}$, $\mathcal{C} \in \upsilon$. The dicovering uniformity υ is said to be *complemented* if $\upsilon = \upsilon'$.

Clearly Proposition 2.15 ensures that on a complemented texture, the notions of complement and of being complemented for dicovering uniformities correspond precisely to the corresponding notions for direlational uniformities.

3. Uniformities and quasi-uniformities

In this section we investigate how the relation between quasi-uniformities and uniformities appears in the context of di-uniformities. That is, we consider di-uniformities on the discrete texture $(X, \mathcal{P}(X), \pi_X)$ and look at their relation to classical uniformities and quasi-uniformities.

We begin by specializing various concepts concerning relations, correlations and difunctions to discrete textures, and relate these to known concepts for binary point relations and point functions. Several of these results will be needed in the sequel.

Lemma 3.1. *Let $(X, \mathcal{P}(X), \pi_X)$, $(Y, \mathcal{P}(Y), \pi_Y)$, $(Z, \mathcal{P}(Z), \pi_Z)$ be discrete textures.*

(1) *Any binary point relation φ from X to Y is both a relation and a correlation from $(X, \mathcal{P}(X))$ to $(Y, \mathcal{P}(Y))$.*
(2) *If we regard $\varphi \in \mathcal{P}(X \times Y)$ as a relation or as a correlation from $(X, \mathcal{P}(X), \pi_X)$ to $(Y, \mathcal{P}(Y), \pi_Y)$ then $\varphi' = (X \times Y) \setminus \varphi$.*
(3) *If we regard $\varphi \in \mathcal{P}(X \times Y)$ as a relation or as a correlation from $(X, \mathcal{P}(X))$ to $(Y, \mathcal{P}(Y))$ then $\varphi^{\leftarrow} = (Y \times X) \setminus \varphi^{-1} = ((X \times Y) \setminus \varphi)^{-1}$. In view of* (2) *we may therefore write*

$$\varphi^{\leftarrow} = (\varphi^{-1})' = (\varphi')^{-1}.$$

(4) *The identity direlation (i_X, I_X) on $(X, \mathcal{P}(X))$ is given by*

$$i_X = \{(x, x) \mid x \in X\} = \Delta_X,$$
$$I_X = \{(x, x') \mid x, x' \in X,\ x \neq x'\} = \Delta'_X.$$

(5) *If $\varphi, \psi \in \mathcal{P}(X \times Y)$ are regarded as relations then $\varphi \sqcap \psi = \varphi \cap \psi$.*
(6) *If $\varphi, \psi \in \mathcal{P}(X \times Y)$ are regarded as correlations then $\varphi \sqcup \psi = \varphi \cup \psi$.*
(7) *If $\varphi \in \mathcal{P}(X \times Y)$ and $\psi \in \mathcal{P}(Y \times Z)$ are regarded as relations then*

$$\psi \circ \varphi = \{(x, z) \mid \exists y \in Y,\ (x, y) \in \varphi,\ (y, z) \in \psi\},$$

the usual composition of the binary point relations φ, ψ.
(8) *If $\varphi \in \mathcal{P}(X \times Y)$ and $\psi \in \mathcal{P}(Y \times Z)$ are regarded as correlations then*

$$\psi \circ \varphi = (\psi' \circ \varphi')',$$

the complement of the usual composition of the binary relations φ', ψ'.
(9) *If $\varphi \in \mathcal{P}(X \times Y)$ is regarded as a relation and $A \subseteq X$ then*

$$\varphi^{\rightarrow} A = \{y \mid \exists x \in A \text{ with } (x, y) \in \varphi\} = \varphi[A].$$

(10) *If $\varphi \in \mathcal{P}(X \times Y)$ is regarded as a correlation and $A \subseteq X$ then*

$$\varphi^{\rightarrow} A = \{y \mid (x, y) \notin \varphi \Longrightarrow x \in A\} = Y \setminus \varphi'[X \setminus A].$$

(11) *If* $\varphi \in \mathcal{P}(X \times Y)$ *is regarded as a relation and* $B \subseteq Y$ *then*

$$\varphi^{\leftarrow} B = X \setminus \varphi^{-1}[Y \setminus B].$$

(12) *If* $\varphi \in \mathcal{P}(X \times Y)$ *is regarded as a correlation and* $B \subseteq Y$ *then*

$$\varphi^{\leftarrow} B = (\varphi')^{-1}[B].$$

(13) *If* $\varphi \in \mathcal{P}(X \times Y)$ *is regarded as a relation and* $\psi \in \mathcal{P}(X \times Y)$ *is regarded as a correlation then* (φ, ψ) *is a difunction from* $(X, \mathcal{P}(X))$ *to* $(Y, \sigma(P(Y))$ *if and only if* $\varphi : X \to Y$ *is a point function and* $\psi = \varphi'$.
In particular, every difunction $(S, \mathcal{S}, \sigma) \to (T, \mathcal{T}, \theta)$ *is complemented.*

Proof. We content ourselves with establishing (1), (11) and (13), leaving the proofs of the remaining results to the interested reader. Note that for the discrete texture $(X, \mathcal{P}(X))$, $x \in X$, we have $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$, whence $P_x \nsubseteq Q_{x'} \iff x = x'$. Also, if φ is a binary point relation from X to Y then $\varphi \nsubseteq \overline{Q}_{(x,y)} \iff \overline{P}_{(x,y)} \subseteq \varphi \iff (x, y) \in \varphi$.

(1) If $\varphi \subseteq X \times Y$ then certainly $\varphi \in \mathcal{P}(X \times Y) = \mathcal{P}(X) \otimes \mathcal{P}(Y)$. Suppose $\varphi \nsubseteq \overline{Q}_{(x,y)}$. In $(X, \mathcal{P}(X))$, $P_x \nsubseteq Q_{x'} \implies x = x' \implies \varphi \nsubseteq \overline{Q}_{(x',y)}$, which verifies *R1*, while *R2* follows from $P_x \nsubseteq Q_x$ and $\varphi \nsubseteq \overline{Q}_{(x,y)}$. Hence φ is a relation from $(X, \mathcal{P}(X))$ to $(Y, \mathcal{P}(Y))$. Likewise, it is a correlation.

(11) If φ is regarded as a relation, [16, Lemma 1.4(1)] gives,

$$\begin{aligned} r^{\leftarrow} B &= \bigvee \bigl\{P_x \mid \varphi \nsubseteq \overline{Q}_{(x,y)} \implies P_y \subseteq B\bigr\} \\ &= \bigl\{x \mid (x, y) \in \varphi \implies y \in B\bigr\} \\ &= X \setminus \varphi^{-1}[Y \setminus B]. \end{aligned}$$

(13) Let φ, ψ be binary point relations from X to Y, and regard (φ, ψ) as a direlation from $(X, \mathcal{P}(X))$ to $(Y, \mathcal{P}(Y))$.

($\Rightarrow$) Suppose that (φ, ψ) is a difunction, and take $x \in X$. Since $P_x \nsubseteq Q_x$, by *DF1* there exists $y \in Y$ satisfying $\varphi \nsubseteq \overline{Q}_{(x,y)}$ and $\overline{P}_{(x,y)} \nsubseteq \psi$. In particular, $(x, y) \in \varphi$. On the other hand, if we also have $(x, y') \in \varphi$ for some $y' \in Y$ then $\varphi \nsubseteq \overline{Q}_{(x,y')}$ and this, together with $\overline{P}_{(x,y)} \nsubseteq \psi$ gives $P_y \nsubseteq Q_{y'}$ by *DF2*, and so $y' = y$. This shows that φ is a point function, that is a function in the ordinary sense. The above argument also shows that for the unique y for which $(x, y) \in \varphi$ we have $(x, y) \notin \psi$, that is $(x, y) \in \varphi \implies (x, y) \notin \psi$.

On the other hand, suppose that for some $x \in X$, $y \in Y$ we have $(x, y) \notin \varphi$ and $(x, y) \notin \psi$. Then, as above, we have $w \in Y$ satisfying $(x, w) \in \varphi$, and now *DF2* gives $w = y$, which is a contradiction. Hence $(x, y) \notin \psi \implies (x, y) \in \varphi$, and we have verified that $\psi = X \times Y \setminus \varphi = \varphi'$ by (1).

($\Leftarrow$) Left to the interested reader.

Since every difunction $(S, \mathcal{S}, \sigma) \to (T, \mathcal{T}, \theta)$ has the form (φ, φ'), and $(\varphi, \varphi')' = ((\varphi)')', \varphi') = (\varphi, \varphi')$ we deduce that every such difunction is complemented. □

Definition 3.2. Let $d \subseteq X \times X$ be a point relation. We define

$$u(d) = (d, d^{\leftarrow})$$

and regard this as a direlation on $(X, \mathcal{P}(X))$.

According to Lemma 3.1(1) we may indeed regard d as a relation on $(X, \mathcal{P}(X))$, whence $d^{\leftarrow}$ is a correlation and the definition is consistent.

Theorem 3.3. *Let* $\mathcal{Q}$ *be a diagonal quasi-uniformity on* X. *Then the family*

$$u(\mathcal{Q}) = \bigl\{(e, E) \mid \exists d \in \mathcal{Q} \text{ and } u(d) \sqsubseteq (e, E)\bigr\}$$

is a direlational uniformity on the discrete texture $(X, \mathcal{P}(X))$.

Proof. We must show that conditions 1–5 of [16, Definition 3.1] hold for $u(\mathcal{Q})$.

(1) Take $(e, E) \in u(\mathcal{Q})$. Then we have $d \in \mathcal{Q}$ with $u(d) \sqsubseteq (e, E)$. Since $\mathcal{Q}$ is a quasi-uniformity $\Delta_X \subseteq d$, and $\Delta_X = i_X$ by Lemma 3.1(4), so $i_X \subseteq d$. Hence $I_X = i_X^{\leftarrow} \supseteq d^{\leftarrow}$, so $(i_X, I_X) \sqsubseteq (d, d^{\leftarrow}) = u(d) \sqsubseteq (e, E)$, that is (e, E) is reflexive.

(2) Clear from the definition.

(3) Take $(e, E), (f, F) \in u(\mathcal{Q})$ and $d, h \in \mathcal{Q}$ with $u(d) = (d, d^{\leftarrow}) \sqsubseteq (e, E)$ and $u(h) = (h, h^{\leftarrow}) \sqsubseteq (f, F)$. Now $d \cap h \in \mathcal{Q}$ since $\mathcal{Q}$ is a quasi-uniformity, $d \cap h = d \sqcap h$ by Lemma 3.1(5) and $(d \sqcap h)^{\leftarrow} = d^{\leftarrow} \sqcup h^{\leftarrow}$ by [16, Proposition 1.9(4)] so

$$u(d \cap h) = (d \sqcap h, d^{\leftarrow} \sqcup h^{\leftarrow}) = (d, d^{\leftarrow}) \sqcap (h, h^{\leftarrow}) = u(d) \sqcap u(h) \sqsubseteq (e, E) \sqcap (f, F).$$

Hence $(e, E) \sqcap (f, F) \in u(\mathcal{Q})$.

(4) Take $(e, E) \in u(\mathcal{Q})$ and $d \in \mathcal{Q}$ with $u(d) \sqsubseteq (e, E)$. Since $\mathcal{Q}$ is a quasi-uniformity there exists $h \in \mathcal{Q}$ with $h \circ h \subseteq d$. Here $\circ$ denotes the composition of point relations, which coincides with the composition of relations in $(X, \mathcal{P}(X))$ by Lemma 3.1(7). Hence $u(h) \circ u(h) = (h, h^{\leftarrow}) \circ (h, h^{\leftarrow}) = (h \circ h, h^{\leftarrow} \circ h^{\leftarrow}) = (h \circ h, (h \circ h)^{\leftarrow}) \sqsubseteq (d, d^{\leftarrow}) = u(d) \sqsubseteq (e, E)$ by [8, Lemma 2.17(2) and Lemma 2.4(3)]. Finally, $u(h) \in u(\mathcal{Q})$.

(5) Clear since for $d \in \mathcal{Q}$ we have $u(d)^{\leftarrow} = (d, d^{\leftarrow})^{\leftarrow} = ((d^{\leftarrow})^{\leftarrow}, d^{\leftarrow}) = (d, d^{\leftarrow}) = u(d)$, whence $u(\mathcal{Q})$ has a base of symmetric direlations. □

Since u is clearly a bijection between the binary point relations on X and the symmetric direlations on $(X, \mathcal{P}(X))$, it is clear by Theorem 3.3 that it also sets up a bijection between the diagonal quasi-uniformities on X and the direlational uniformities on $(X, \mathcal{P}(X))$.

Let $\mathcal{Q}$ be a quasi-uniformity on X and recall [11,15] that $\mathcal{Q}^{-1} = \{d^{-1} \mid d \in \mathcal{Q}\}$ is also a quasi-uniformity on X, called the *conjugate* of $\mathcal{Q}$.

Proposition 3.4. *Let $\mathcal{Q}$ be a quasi-uniformity on X and $\mathcal{Q}^{-1}$ its conjugate. Then the direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$ corresponding to $\mathcal{Q}^{-1}$ is the complement of the direlational uniformity corresponding to $\mathcal{Q}$. That is,*

$$u(\mathcal{Q}^{-1}) = u(\mathcal{Q})'.$$

Proof. For any $d \subseteq X \times X$ let us show that

$$(d, d^{\leftarrow})' = (d^{-1}, (d^{-1})^{\leftarrow}).$$

Clearly $(d, d^{\leftarrow})' = ((d^{\leftarrow})', d')$ by the definition of complement. If we take $\varphi = d$ in Lemma 3.1(3) we obtain $d^{\leftarrow} = (d^{-1})'$, whence $(d^{\leftarrow})' = ((d^{-1})')' = d^{-1}$ by [8, Proposition 2.21(1)]. Also, taking $\varphi = d^{-1}$ in Lemma 3.1(3) gives $(d^{-1})^{\leftarrow} = ((d^{-1})^{-1})' = d'$ and so $((d^{\leftarrow})', d') = ((d^{-1}, (d^{-1})^{\leftarrow})$ as stated. Since $\{u(d) \mid d \in \mathcal{Q}\}$ is a base for $u(\mathcal{Q})$, $\{u(d)' \mid d \in \mathcal{Q}\}$ is a base for $u(\mathcal{Q})'$. On the other hand $\{u(d^{-1}) \mid d \in \mathcal{Q}\}$ is a base for $u(\mathcal{Q}^{-1})$ since $\{d^{-1} \mid d \in \mathcal{Q}\}$ is a base for $\mathcal{Q}^{-1}$, so the equality $u(d)' = u(d^{-1})$ proved above shows that $u(\mathcal{Q})' = u(\mathcal{Q}^{-1})$. □

Theorem 3.5. *Let $\mathcal{Q}$ be a quasi-uniformity on X. Then $\mathcal{Q}$ is a uniformity if and only if the corresponding di-uniformity $u(\mathcal{Q})$ on $(X, \mathcal{P}(X), \pi_X)$ is complemented.*

Proof. If $\mathcal{Q}$ is a uniformity then $\mathcal{Q} = \mathcal{Q}^{-1}$ and so $u(\mathcal{Q})' = u(\mathcal{Q}^{-1}) = u(\mathcal{Q})$ by Proposition 3.4, that is $u(\mathcal{Q})$ is complemented.

Conversely suppose that $u(\mathcal{Q})$ is complemented. Then by Proposition 3.4, $u(\mathcal{Q}^{-1}) = u(\mathcal{Q})' = u(\mathcal{Q})$. Hence if $d \in \mathcal{Q}$ we have $u(d) \in u(\mathcal{Q}) = u(\mathcal{Q}^{-1})$ and so there exists $e \in \mathcal{Q}$ satisfying $u(e^{-1}) \sqsubseteq u(d)$. In particular $e^{-1} \subseteq d$, which establishes that $\mathcal{Q}$ is a uniformity. □

Theorem 3.3 shows that direlational uniformities are the textural analogue of diagonal quasi-uniformities, while Theorem 3.5 says that complemented direlational uniformities are the textural analogue of diagonal uniformities. Hence the distinction between quasi-uniformities and uniformities, which is a matter of symmetry in the classical representation, becomes a question of complementation in the textural case, even for discrete textures.

This difference between the two representations has important consequences. Let us recall that a quasi-uniformity $\mathcal{Q}$ on X gives rise to a bitopological space $(X, \mathcal{T}_{\mathcal{Q}}, \mathcal{T}_{\mathcal{Q}^{-1}})$, where $\mathcal{T}_{\mathcal{Q}}$ is the topology generated by $\mathcal{Q}$ and $\mathcal{T}_{\mathcal{Q}^{-1}}$ that generated by $\mathcal{Q}^{-1}$. It follows easily that $(\mathcal{T}_{\mathcal{Q}}, \mathcal{T}^c_{\mathcal{Q}^{-1}})$, $\mathcal{T}^c_{\mathcal{Q}^{-1}} = \pi_X[\mathcal{T}_{\mathcal{Q}^{-1}}]$, is the uniform ditopology of $u(\mathcal{Q})$, so a bitopological space (X, u, v) is pairwise completely regular if and only if the ditopological texture space $(X, \mathcal{P}(X), u, v^c)$ is completely biregular. We know from [16, proof of Theorem 5.16] that a compatible di-uniformity on $(X, \mathcal{P}(X), \pi_X, u, v^c)$ may be obtained as an initial di-uniformity generated by bicontinuous difunctions mapping to the complemented di-uniform space $(\mathbb{I}, \mathcal{I}, \iota, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ and the natural di-uniformity $\mathcal{U}_{\mathbb{I}}$ on $(\mathbb{I}, \mathcal{I}, \iota)$. Moreover, by Theorem 2.11, if (u, v^c) is complemented, that is $u = v$, we obtain a compatible complemented di-uniformity. This means that both a quasi-uniformity compatible with a pairwise completely regular bitopology (u, v), and a uniformity compatible with a completely regular topology u, may be obtained from the same space $(\mathbb{I}, \mathcal{I}, \iota, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}}, \mathcal{U}_{\mathbb{I}})$. This is because of the presence of sufficient non-complemented difunctions from $(X, \mathcal{P}(X), \pi_X)$ to $(\mathbb{I}, \mathcal{I}, \iota)$ (see Example 2.13). On the other hand, by Lemma 3.1(13), all difunctions between discrete textures are complemented, which explains why different structures are needed to generate the above quasi-uniformities and uniformities in the classical theory. This illustrates quite clearly that the textural representation can provide a more uniform treatment of quasi-uniformities and uniformities.

We conclude this paper by considering this relationship between the textural and classical cases in terms of covers. It is well known that uniformities on a set X may be described in terms of covers of X [14,19], but ordinary covers cannot be used to describe quasi-uniformities since they generate symmetric relations or entourages. Gantner and Steinlage [12] presented a description of quasi-uniformities in terms of pairs of covers with a common index set, and Aydın [1] independently developed a theory of quasi-uniformities using the essentially equivalent concept of dual cover. Dual covers were also studied extensively in a bitopological setting by the second author [2]. Since dual covers are not so well known we outline the development of the theory of dual-covering quasi-uniformities and show formally its equivalence with that of diagonal quasi-uniformities.

Let X be a set. We recall [1,2] that a family $U = \{(A_j, B_j) \mid j \in J\}$ of subsets of X is called a *dual cover of* X if $\bigcup\{A_j \cap B_j \mid j \in J\} = X$. We may usually assume without loss of generality that $A_j \cap B_j \neq \emptyset$ for all $j \in J$.

If U, V are dual covers of X we say U *refines* V, and write $U \prec V$ if whenever AUB there exists CVD satisfying $A \subseteq C$ and $B \subseteq D$. For $Y \subseteq X$ there are two different stars of Y with respect to U, namely

$$\mathrm{St}(U, Y) = \bigcup\{A \mid \exists B \text{ with } AUB \text{ and } Y \cap B \neq \emptyset\},$$
$$\mathrm{St}(Y, U) = \bigcup\{B \mid \exists A \text{ with } AUB \text{ and } A \cap Y \neq \emptyset\}.$$

In case Y is a single point set $\{x\}$ we write $\mathrm{St}(U, x)$, $\mathrm{St}(x, U)$ in place of $\mathrm{St}(U, \{x\})$, $\mathrm{St}(\{x\}, U)$ respectively. Now U is called a *delta refinement* (*star refinement*) of V, and we write $U \prec_{(\Delta)} V$ ($U \prec_{(\star)} V$) if $U^{\Delta} = \{(\mathrm{St}(U, x), \mathrm{St}(x, U)) \mid x \in X\} \prec V$ ($U^* = \{(\mathrm{St}(U, A), \mathrm{St}(B, U)) \mid AUB\} \prec V$). We recall that $U \prec_{(\Delta)} V \prec_{(\Delta)} W \Longrightarrow U \prec_{(\star)} W$ [1,2].

We also note that if U, V are dual covers of X then $U \wedge V = \{(A \cap C, B \cap D) \mid AUB, CVD\}$ is also a dual cover of X—the greatest lower bound of U and V in the family of dual covers of X ordered by refinement.

Given a binary point relation φ on X we may associate with φ the family

$$\gamma^*(\varphi) = \{(\varphi[x], \varphi^{-1}[x]) \mid x \in X\},$$

where, as usual, $\varphi[x] = \{y \in X \mid (x, y) \in \varphi\}$ and $\varphi^{-1}[x] = \{y \in X \mid (y, x) \in \varphi\}$. It is clear that if $\Delta_X \subseteq \varphi$, i.e. φ is reflexive in the classical sense, then $\gamma^*(\varphi)$ is a dual cover of X. If we call a dual cover U of X *uniform* for the diagonal quasi-uniformity $\mathcal{Q}$ if $\gamma^*(d) \prec U$ for some $d \in \mathcal{Q}$, and denote the family of uniform dual covers of $\mathcal{Q}$ by $\Gamma^*(\mathcal{Q})$, then:

Proposition 3.6. *For a given diagonal quasi-uniformity $\mathcal{Q}$ the family $\Gamma^*(\mathcal{Q})$ of uniform dual covers has the properties,*

(1) *$\Gamma^*(\mathcal{Q})$ has a base of dual covers U satisfying $AUB \Longrightarrow A \cap B \neq \emptyset$.*
(2) *$U \in \Gamma^*(\mathcal{Q})$, $U \prec V \Longrightarrow V \in \Gamma^*(\mathcal{Q})$.*
(3) *U, $V \in \Gamma^*(\mathcal{Q}) \Longrightarrow U \wedge V \in \Gamma^*(\mathcal{Q})$.*
(4) *$U \in \Gamma^*(\mathcal{Q}) \Longrightarrow \exists V \in \Gamma(\mathcal{Q})$ with $V \prec_{(\star)} U$.*

Proof. (1) Clear since the dual covers $\gamma^*(\varphi)$, $\varphi \in \mathcal{Q}$ have this property because $x \in \varphi[x] \cap \varphi^{-1}[x] \neq \emptyset$.

(2) Immediate from the definition of $\Gamma^*(\mathcal{Q})$.

(3) For U, $V \in \Gamma^*(\mathcal{Q})$ we have $d, e \in \mathcal{Q}$ with $\gamma^*(d) \prec U$ and $\gamma^*(e) \prec V$. But now $d \cap e \in \mathcal{Q}$ and $\gamma^*(d \cap e) \prec \gamma^*(d) \wedge \gamma^*(e)$ since for $x \in X$, $(d \cap e)[x] = d[x] \cap e[x]$, $(d \cap e)^{-1}[x] = d^{-1}[x] \cap e^{-1}[x]$ and $d[x] \cap e[x](\gamma^*(d) \wedge \gamma^*(e))d^{-1}[x] \cap e^{-1}[x]$. Hence $\gamma^*(d \cap e) \prec U \wedge V$, that is $U \wedge V \in \Gamma^*(\mathcal{Q})$.

(4) Take $U \in \Gamma^*(\mathcal{Q})$ and $d \in \mathcal{Q}$ with $\gamma^*(d) \prec U$. Since $\mathcal{Q}$ is a quasi-uniformity there exists $e \in \mathcal{Q}$ satisfying $e \circ e \subseteq d$. Now, for $x \in X$,

$$\mathrm{St}\big(\gamma^*(e), x\big) = \bigcup\big\{e[u] \mid x \in e^{-1}[u]\big\} = d[x],$$

and likewise $\mathrm{St}(x, \gamma^*(e)) \subseteq d^{-1}[x]$. Thus $\gamma^*(e) \prec_{(\Delta)} \gamma^*(d)$, and if we take $f \in \mathcal{Q}$ with $f \circ f \subseteq e$ then also $\gamma^*(f) \prec_{(\Delta)} \gamma^*(e)$, whence $\gamma^*(f) \prec_{(\star)} \gamma^*(d)$. Setting $V = \gamma^*(f)$ now gives $V \in \Gamma^*(\mathcal{Q})$ and $V \prec_{(\star)} U$, as required. □

These conditions on $\Gamma^*(\mathcal{Q})$ are precisely what are needed for $\Gamma^*(\mathcal{Q})$ to be a *dual-covering quasi-uniformity* in the sense of [1,2][2] (cf. [12]). Hence:

Corollary 3.7. *For every diagonal quasi-uniformity $\mathcal{Q}$ on X, $\Gamma^*(\mathcal{Q})$ is a dual-covering quasi-uniformity on X.*

Working in the opposite direction, for a dual cover U on X define

$$\delta^*(U) = \bigcup\{B \times A \mid AUB\}.$$

Since U is a dual cover, given $x \in X$ there exists AUB with $x \in A \cap B$ and so $(x, x) \in \delta^*(U)$. Thus $\Delta_X \subseteq \delta^*(U)$, i.e. $\delta^*(U)$ is a reflexive binary point relation on X in the classical sense. If $\mathfrak{U}$ is a dual-covering quasi-uniformity on X we may therefore set

$$\Delta^*(\mathfrak{U}) = \big\{d \mid \exists U \in \mathfrak{U} \text{ with } \delta^*(U) \subseteq d \subseteq X \times X\big\}.$$

Proposition 3.8. *For every dual-covering quasi-uniformity $\mathfrak{U}$ on X, $\Delta^*(\mathfrak{U})$ is a diagonal quasi-uniformity on X.*

Proof. (1) $d \in \Delta^*(\mathfrak{U})$, $d \subseteq e \subseteq X \times X \Longrightarrow e \in \Delta^*(\mathfrak{U})$ is immediate from the definition.

(2) Given $d \in \Delta^*(\mathfrak{U})$ we have $U \in \mathfrak{U}$ with $\delta^*(U) \subseteq d$. Since $\Delta_X \subseteq \delta^*(U)$, as noted above, we have $\Delta_X \subseteq d$.

(3) Take $d, e \in \Delta^*(\mathfrak{U})$ and $U, V \in \mathfrak{U}$ with $\delta^*(U) \subseteq d$ and $\delta^*(V) \subseteq e$. Then it is easy to verify that $\delta^*(U \wedge V) \subseteq \delta^*(U) \cap \delta^*(V) \subseteq d \cap e$, whence $d \cap e \in \Delta^*(\mathfrak{U})$ since $U \wedge V \in \mathfrak{U}$.

(4) Take $d \in \Delta^*(\mathfrak{U})$, $U \in \mathfrak{U}$ with $\delta^*(U) \subseteq d$ and $V \in \mathfrak{U}$ satisfying $V \prec_{(\star)} U$. The reader may easily verify that $\delta^*(V) \circ \delta^*(V) \subseteq \delta^*(U)$, whence if we take $e = \delta^*(V)$ we have $e \in \Delta^*(\mathfrak{U})$ and $e \circ e \subseteq d$. □

Theorem 3.9. *The mappings Γ^* and Δ^* are dual to one another. That is, for each diagonal quasi-uniformity $\mathcal{Q}$ and dual-covering quasi-uniformity $\mathfrak{U}$ we have*

(1) $\mathcal{Q} = \Delta^*(\Gamma^*(\mathcal{Q}))$, *and*
(2) $\mathfrak{U} = \Gamma^*(\Delta^*(\mathfrak{U}))$.

Proof. We establish (1), leaving the proof of (2) to the interested reader.

Take $d \in \mathcal{Q}$ and $e \in \mathcal{Q}$ satisfying $e \circ e \subseteq d$. Clearly

$$\delta^*(\gamma^*(e)) = \bigcup\big\{e^{-1}[x] \times e[x] \mid x \in X\big\} \subseteq d,$$

while $\delta^*(\gamma^*(e)) \in \Delta^*(\Gamma^*(\mathcal{Q}))$, and so $d \in \Delta^*(\Gamma^*(\mathcal{Q}))$.

Conversely take $d \in \Delta^*(\Gamma^*(\mathcal{Q}))$. Now we have $U \in \Gamma^*(\mathcal{Q})$ with $\delta^*(U) \subseteq d$, and then $e \in \mathcal{Q}$ satisfying $\delta^*(e) \prec U$. For $(x, y) \in e$ take AUB satisfying $e[x] \subseteq A$ and $e^{-1}[x] \subseteq B$. Then, since e is reflexive, $(x, y) \in e^{-1}[x] \times e[x] \subseteq B \times A \subseteq \delta^*(U) \subseteq d$, which gives $e \subseteq d$ and hence $d \in \mathcal{Q}$. □

Clearly Γ^*, Δ^* are the counterparts for diagonal and dual-covering quasi-uniformities of the mappings Γ, Δ, respectively, defined in [16].

Dual covers formed the inspiration for the notion of dicover, and we now make the connection between dual covers and dicovers explicit in the case of discrete textures.

[2] Condition (1) does not appear in [1,2] explicitly because there is an overall assumption that every pair in a dual cover has non-empty intersection. Subsequently it was found convenient to permit dual covers not satisfying this condition, as is done here.

Proposition 3.10. *Let $U = \{(A_j, B_j) \mid j \in J\}$ be a dual cover on X. Then*

$$u^\star(U) = \{(A_j, X \backslash B_j) \mid j \in J\}$$

is a dicover on $(X, \mathcal{P}(X))$. Moreover, if U satisfies $A_j \cap B_j \neq \emptyset\ \forall j \in J$ then $u^*(U)$ is anchored.

Proof. To show $u^*(U)$ is a dicover it will suffice to show that $\mathcal{P} = \{(P_x, Q_x) \mid x \in X\} \prec u^*(U)$. Take $x \in X$. Since U is a dual cover there exists $j \in J$ satisfying $x \in A_j \cap B_j$. Hence $P_x = \{x\} \subseteq A_j$ and $X \setminus B_j \subseteq X \setminus \{x\} = Q_x$, as required.

Secondly, suppose that U satisfies $A_j \cap B_j \neq \emptyset$ for all $j \in J$. We have already verified $\mathcal{P} \prec u^*(U)$, as required by [16, Definition 2.1(1)]. To verify [16, Definition 2.1(2)], associate with $j \in J$ an element $x = x(j) \in X$ satisfying $x \in A_j \cap B_j$. Now for (a) take $A_j \nsubseteq Q_u$. If we choose $A' = A_j$, $B' = X \setminus B_j$ then $A' u^*(U) B'$, $A' \nsubseteq Q_u$ and $P_s \nsubseteq B'$. Condition (b) may be verified in the same way, so $u^*(U)$ is an anchored dicover, as stated. □

For any dual covering quasi-uniformity $\mathfrak{U}$ on X let us set

$$u^*(\mathfrak{U}) = \{\mathcal{C} \mid \mathcal{C} \text{ is a dicover of } (X, \mathcal{P}(X)) \text{ and } \exists U \in \mathfrak{U} \text{ with } u^*(U) \prec \mathcal{C}\}.$$

Theorem 3.11. *For any diagonal quasi-uniformity $\mathcal{Q}$ on X, $u^*(\Gamma^*(\mathcal{Q})) = \Gamma(u(\mathcal{Q}))$.*

Proof. First take $\mathcal{C} \in u^*(\Gamma^*(\mathcal{Q}))$, so we have $U \in \Gamma^*(\mathcal{Q})$ with $u^*(U) \prec \mathcal{C}$ and then $d \in \mathcal{Q}$ satisfying $\gamma^*(d) \prec U$. Now $u(d) = (d, d^{\leftarrow})$, where d is regarded as a relation on $(X, \mathcal{P}(X))$, and so

$$\gamma(u(d)) = \{(d^{\rightarrow} P_x, (d^{\leftarrow})^{\rightarrow} Q_x) \mid x \in X\}.$$

By Lemma 3.1(9) we have $d^{\rightarrow} P_x = d^{\rightarrow}\{x\} = d[\{x\}] = d[x]$ and by [8] and Lemma 3.1(11), $(d^{\leftarrow})^{\rightarrow} Q_x = d^{\leftarrow}(X \setminus \{x\}) = X \setminus d^{-1}[\{x\}] = X \setminus d^{-1}[x]$. Hence

$$\gamma(u(d)) = \{(d[x], X \setminus d^{-1}[x]) \mid x \in X\} = u^*(\gamma^*(d)) \prec \mathcal{C},$$

which shows that $u^*(\Gamma^*(\mathcal{Q})) \subseteq \Gamma(u(\mathcal{Q}))$.

The proof of the opposite inclusion is left to the reader. □

Corollary 3.12.

(a) *For every dual covering quasi-uniformity $\mathfrak{U}$, $u^*(\mathfrak{U})$ is a dicovering uniformity on $(X, \mathcal{P}(X))$.*
(b) *The mapping u^* is a bijection between the dual covering quasi-uniformities on X and the dicovering uniformities on $(X, \mathcal{P}(X))$.*

Proof. If we set $\mathcal{Q} = \Delta^*(\mathfrak{U})$ in the above identity and use Theorem 3.9(2) we obtain $u^*(\mathfrak{U}) = \Gamma(u(\Delta^*(\mathfrak{U})))$. Since the right-hand side is known to be a dicovering uniformity, so is $u^*(\mathfrak{U})$. Also, Γ, u and Δ^* are bijective, so u^* is bijective too. □

Finally, we characterize those quasi-uniformities which are uniformities in terms of dual covering quasi-uniformities and dicovering uniformities.

Theorem 3.13. *Let $\mathcal{Q}$ be a diagonal quasi-uniformity on X. Then the following are equivalent*:

(1) *$\mathcal{Q}$ is a uniformity.*
(2) *$\mathfrak{U} = \Gamma^*(\mathcal{Q})$ satisfies $U \in \mathfrak{U} \Longrightarrow \exists V \in \mathfrak{U}$ with $V^{-1} = \{(B, A) \mid A V B\} \prec U$.*
(3) *$u^*(\mathfrak{U})$ is a complemented dicovering uniformity on $(X, \mathcal{P}(X), \pi_X)$.*

Proof. (1) ⇒ (2). Take $U \in \mathfrak{U} = \Gamma^*(\mathcal{Q})$, $d \in \mathcal{Q}$ with $\gamma^*(d) \prec U$ and $e \in \mathcal{Q}$ with $e^{-1} \subseteq d$. Let $V = \gamma^*(e)$. Then $V \in \mathfrak{U}$ and $V^{-1} = \{(e^{-1}[x], e[x]) \mid x \in X\} \prec \{(d[x], d^{-1}[x]) \mid x \in X\} = \gamma^*(d) \prec U$.

(2) ⇒ (3). According to Definition 2.16 we must show that the family $(\mathcal{C}')^{\Delta}$, $\mathcal{C} \in u^*(\mathfrak{U})$, is a base for $u^*(\mathfrak{U})$. For $\mathcal{C} \in u^*(\mathfrak{U})$ we have $U \in \mathfrak{U}$ with $u^*(U) \prec \mathcal{C}$. By (2) there exists $V \in \mathfrak{U}$ with $V^{-1} \prec U$, and by Proposition 3.6 we may assume that $AVB \Longrightarrow A \cap B \neq \emptyset$. Finally, there exists $W \in \mathfrak{U}$ with $W \prec_{(\star)} V$. We verify that

$$u^*(V) \prec \left(u^*(U)'\right)^{\Delta} \prec (\mathcal{C}')^{\Delta} \quad \text{and} \quad \left(u^*(W)'\right)^{\Delta} \prec u^*(U) \prec \mathcal{C},$$

which gives the required result. First let us note that $u^*(U) = \{(A, X \setminus B) \mid AUB\}$, whence $u^*(U)' = \{(B, X \setminus A) \mid AUB\}$ and so for any $x \in X$,

$$\mathrm{St}\left(u^*(U)', P_x\right) = \bigcup \{B \mid P_x \not\subseteq X \setminus A,\ AUB\} = \mathrm{St}(x, U),$$

and likewise $\mathrm{CSt}(u^*(U)', Q_x) = X \setminus \mathrm{St}(U, x)$. Here we recall that CSt denotes the co-star with respect to a dicover, which is defined dually to the star St [16].

Now take AVB and $x \in A \cap B$. Then $B \subseteq C$, $A \subseteq D$ for some CUD and so $A \subseteq D \subseteq \mathrm{St}(x, U) = \mathrm{St}(u^*(U)', P_x)$, $\mathrm{CSt}(u^*(U), Q_x) = X \setminus \mathrm{St}(U, x) \subseteq X \setminus C \subseteq X \setminus B$ which gives $u^*(V) \prec (u^*(U)')^{\Delta} \prec (\mathcal{C}')^{\Delta}$. The remaining refinements may be shown in the same way.

(3) ⇒ (1). Clear from Proposition 2.8 and Theorem 3.5. □

References

[1] Y. Aydın, Quasi-uniform spaces and dual covers, PhD thesis, Hacettepe University, 1976 (in Turkish).
[2] L.M. Brown, Dual covering theory, confluence structures and the lattice of bicontinuous functions, PhD thesis, Glasgow, 1980.
[3] L.M. Brown, Relations and functions on textures, Preprint.
[4] L.M. Brown, M. Diker, Ditopological texture spaces and intuitionistic sets, Fuzzy Sets and Systems 98 (1998) 217–224.
[5] L.M. Brown, M. Diker, Paracompactness and full normality in ditopological texture spaces, J. Math. Anal. Appl. 227 (1998) 144–165.
[6] L.M. Brown, R. Ertürk, Fuzzy sets as texture spaces, I. Representation theorems, Fuzzy Sets and Systems 110 (2) (2000) 227–236.
[7] L.M. Brown, R. Ertürk, Fuzzy sets as texture spaces, II. Subtextures and quotient textures, Fuzzy Sets and Systems 110 (2) (2000) 237–245.
[8] L.M. Brown, R. Ertürk, Ş. Dost, Ditopological texture spaces and fuzzy topology, I. Basic concepts, Fuzzy Sets and Systems 147 (2) (2004) 171–199.
[9] L.M. Brown, R. Ertürk, Ş. Dost, Ditopological texture spaces and fuzzy topology, II. Topological considerations, Fuzzy Sets and Systems 147 (2) (2004) 201–231.
[10] M. Diker, Connectedness in ditopological texture spaces, Fuzzy Sets and Systems 108 (1999) 223–230.
[11] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, Marcel Dekker, New York, 1982.
[12] T.E. Gantner, R.C. Steinlage, Characterization of quasi-uniformities, J. London Math. Soc. (Ser. 5) 11 (1972) 48–52.
[13] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer, Berlin, 1980.
[14] J.R. Isbell, Uniform Spaces, Mathematical Surveys, vol. 12, American Mathematical Society, Providence, RI, 1964.
[15] M.G. Murdeshwar, S.A. Naimpally, Quasi-Uniform Topological Spaces, Noordhoff, Groningen, 1966.
[16] S. Özçağ, L.M. Brown, Di-uniform texture spaces, Appl. Gen. Topology 4 (1) (2003) 157–192.
[17] S. Özçağ, Uniform texture spaces, PhD thesis, Hacettepe University, 2004 (in Turkish).
[18] G. Sambin, Some points in formal topology, in: Proc. Dagstuhl Seminar Topology in Computer Science, June 2000, Theoretical Computer Science 305 (1–3) (2003) 347–408.
[19] J.W. Tukey, Convergence and Uniformity in Topology, Ann. of Math. Stud., vol. 2, Princeton University Press, Princeton, NJ, 1940.