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WEAK $(C_{11}{}^{+})$ MODULES WITH ACC OR DCC ON ESSENTIAL SUBMODULES

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Abstract. In this note we study modules with $(WC_{11}{}^{+})$ property. We prove that if M satisfies $(WC_{11}{}^{+})$ and $M/(SocM)$ has finite uniform dimension then $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 with finite uniform dimension. In particular, if M satisfies $(WC_{11}{}^{+})$ and ascending chain (respectively, descending chain) condition on essential submodules then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2.

1. Introduction

Assume that all rings are associative and have identity elements and all modules are unital right modules. Let R be any ring. A right R-module M is called *CS-module* if every submodule is essential in a direct summand. The module M has *finite uniform* (*Goldie*) *dimension* if M does not contain an infinite direct sum of non-zero submodules. It is well known that a module M has finite uniform dimension if and only if there exist a positive integer n and uniform submodules $U_i (1 \leq i \leq n)$ of M such that $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ is an essential submodule of M and in this case n is an invariant of the module called the *uniform dimension* of M (see, for example, [1, p. 294 ex. 2]).

Armendariz [2, Proposition 1.1] proved that the module M satisfies DCC (descending chain condition) on essential submodules if and only if $M/(SocM)$ is an Artinian module. On the other hand, Goodearl [5, Proposition 3.6] proved that the module M satisfies ACC (ascending chain condition) on essential submodules if and only if $M/(SocM)$ is a Noetherian module.

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It is proved in [10, Theorem 2.1] that the following statements are equivalent for a module M: (i) M/N has finite uniform dimension for every essential submodule N of M; (ii) every homomorphic image of $M/(SocM)$ has finite uniform dimension.

Camillo and Yousif [3, Corollary 3] proved that if M is a CS-module and $M/(SocM)$ has finite uniform dimension then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 of M and submodule M_2 with finite uniform dimension, and in this case M is a direct sum of uniform modules. They deduced in [3, Proposition 5] that if M is a CS-module then M has ACC (respectively, DCC) on essential submodules if and only if $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2 of M.

A module M is called a *weak CS-module* if, for each semisimple submodule S of M, there exists a direct summand K of M such that S is essential in K. Clearly CS-modules are weak CS-modules. Smith [9, Corollary 2.7, Theorem 2.8] showed that the result of [3] mentioned above can be extended to weak CS-modules. A module M is called (C_{11})-*module* if, every submodule of M has a complement which is a direct summand of M. Smith and Tercan [12, Theorem 5.2, Corollary 5.3] extended the result of [3] to modules with $({C_{11}}^{+})$.

Following [4], a module is called a *weak* (C_{11})-*module* if each of its semisimple submodules has a complement which is a direct summand and denoted (WC_{11}). Note that the following implications hold for a module M:

$$\begin{array}{ccc} CS & \Longrightarrow & \text{Weak}CS \\ \Downarrow & & \Downarrow \\ (C_{11}) & \Longrightarrow & (WC_{11}). \end{array}$$

No other implications can be added to this table in general. In particular, [11, Example 10] show that (WC_{11}) does not imply (C_{11}). Also Zhou [14, Counter example 3] makes it clear that there exists a module with (C_{11}) which is not weak CS.

The purpose of this note is to try to extend the result of [12, Theorem 5.2, Corollary 5.3] to modules with $({WC_{11}}^{+})$.

For any unexplained terminology please, see [1], [8].

Weak $({C_{11}}^{+})$- modules

Definition 1. Let (P) be some module property of modules. Then we shall say that a module M *satisfies* (P^{+}) if every direct summand of M satisfies (P).

For example, if a module M has injective socle then M satisfies $({WC_{11}}^{+})$. In particular, if R is a (commutative) Dedekind domain, then any finitely generated R-module is a $({WC_{11}}^{+})$- module. Moreover, we have the following.

Corollary 2. *Let R be a Dedekind domain and M an R-module with finite* *uniform dimension. Then M is a $(WC_{11}{}^{+})$-module.*

Proof. Let $M = M_1 \oplus M_2$ be the direct sum of submodules M_1 and M_2. Let us show that M_1 is (WC_{11})-module. First assume M_1 is torsion-free then $SocM_1 = 0$ and, in this case M_1 is (WC_{11}). Next we assume that M_1 is not torsion-free. By [6, Theorem 9], it follows that $M_1 = N_1 \oplus N_2 \oplus N_3$ for some finitely generated module N_1, injective module N_2 and torsion free module N_3. By [4, Theorem 2.10], M_1 is (WC_{11}). Hence M is $(WC_{11}{}^{+})$.

Lemma 3. *Let $M = U \oplus V$ where U and V are uniform modules. Then M has $(WC_{11}{}^{+})$.*

Proof. Let $0 \neq K$ be a direct summand of M. If $K = M$ then K has (WC_{11}). If $K \neq M$ then K is uniform hence K has (WC_{11}). Thus M has $(WC_{11}{}^{+})$ as required.

Corollary 4. *Let $M = U \oplus V$ where U and V are uniform modules. Then M has $(C_{11}{}^{+})$.*

Recall that every direct summand of a non-zero $(C_{11}{}^{+})$-module with finite uniform dimension is a (finite) direct sum of uniform modules (see [12, Proposition 4.4]). However this is not true for $(WC_{11}{}^{+})$-modules, in general.

Example 5. Let R be a principal ideal domain. If R is not a complete discrete valuation ring then there exists an indecomposable torsion-free R-module M of rank 2 by [7, Theorem 19]. For M, $SocM = 0$ so that M satisfies $(WC_{11}{}^{+})$ and M_R has finite uniform dimension, namely 2. But M is not a direct sum of uniform modules.

Before proving a theorem we should note the following example.

Example 6. Let K be a field and V an infinite dimensional vector space over K. Let

$$R = \left\{ \begin{bmatrix} k & v \\ 0 & k \end{bmatrix} \quad : \quad k \in K, \quad v \in V \right\}.$$

Then R is a commutative indecomposable ring with respect to the usual matrix operations. Moreover R_R is not a $(WC_{11}{}^{+})$-module and contains an semisimple submodule I is such that R/I has finite uniform dimension but I is not finitely generated.

Proof. It is straightforward to check that R_R is not a $(WC_{11}{}^{+})$-module. Let

$$I = SocR = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}.$$

Define $\varphi : R \to K$ by

$$\varphi\left(\begin{bmatrix} k & v \\ 0 & k \end{bmatrix}\right) = k \quad (k \in K, \quad v \in V).$$

Then φ is an epimorphism with kernel I. Thus R/I has uniform dimension 1. Since V is infinite dimensional then I is not finitely generated.

Theorem 7. *Let R be any ring and let M be a finitely generated $({WC_{11}}^+)$ R-module. Let N be a semisimple submodule of M such that M/N has finite uniform dimension. Then N is finitely generated.*

Proof. Let $n < \infty$ be the uniform dimension of M/N. Suppose that N is not finitely generated. Then there exist non-finitely generated submodules N_1 and N_2 such that $N = N_1 \oplus N_2$. By hypothesis, there exist submodules M_1, M' of M such that $M = M_1 \oplus M'$, $N_1 \cap M' = 0$ and $N_1 \oplus M'$ is essential in M. Now let $\pi_1 : M \to M_1$ be the canonical projection. Since $N_1 \cap M' = 0$ then $\pi_1(N_1) \oplus M' = N_1 \oplus M'$. Thus $\pi_1(N_1)$ is essential in M_1. Also $SocM = Soc(N_1 \oplus M') = N_1 \oplus SocM'$ by [1, Propositions 9.7 and 9.19]. Hence $N = N_1 \oplus (N \cap SocM')$. Now $N_2 \cong N \cap SocM'$ so that $N \cap SocM'$ is not finitely generated. Repeating this argument there exist $\pi_i(N_i) \leq M_i \leq M$ $(2 \leq i \leq n+1)$ such that for each $2 \leq i \leq n+1$, N_i is not finitely generated, $M = M_1 \oplus M_2 \oplus \cdots \oplus M_{n+1}$. Let $L = \pi_1(N_1) \oplus \cdots \oplus \pi_{n+1}(N_{n+1})$. Then

$$M/L \cong \Big(M_1/\pi_1(N_1)\Big) \oplus \Big(M_2/\pi_2(N_2)\Big) \oplus \cdots \oplus \Big(M_{n+1}/\pi_{n+1}(N_{n+1})\Big).$$

Since M/L has finite uniform dimension then there exists $1 \leq i \leq n+1$ such that $M_i = \pi_i(N_i)$. But M_i is finitely generated and hence so is $\pi_i(N_i)$, a contradiction. Thus N is finitely generated.

Corollary 8. *Let R be any ring and M be finitely generated $({C_{11}}^+)$ R-module. If $M/(SocM)$ has finite uniform dimension then $SocM$ is finitely generated.*

It is clear that if M is a semisimple right R-module then $({C_{11}}^+)$ and $({WC_{11}}^+)$ properties are the same. We may conjecture whether a (WC_{11})-module with essential socle is a (C_{11})-module? However we shall provide a negative answer. The following example is taken from [13, Example 3.5].

Example 9. Let p be a prime integer and

$$R = \begin{bmatrix} \mathbb{Z}/\mathbb{Z}p^2 & \mathbb{Z}/\mathbb{Z}p^2 \\ 0 & \mathbb{Z}/\mathbb{Z}p^2 \end{bmatrix}.$$

Then the right R-module R has essential socle and satisfies (WC_{11}) but does not satisfy (C_{11}).

Proof. It is clear that $\mathbb{Z}/\mathbb{Z}p^2$ has a unique composition series :

$$0 = \mathbb{Z}p^2/\mathbb{Z}p^2 \leq \mathbb{Z}p/\mathbb{Z}p^2 \leq \mathbb{Z}/\mathbb{Z}p^2.$$

Since $\mathbb{Z}/\mathbb{Z}p^2$ is faithful $\mathbb{Z}/\mathbb{Z}p^2$-module then

$$SocR = \begin{bmatrix} 0 & \mathbb{Z}p/\mathbb{Z}p^2 \\ 0 & \mathbb{Z}p/\mathbb{Z}p^2 \end{bmatrix},$$

which is essential in the right R-module R (see [13, Lemma 2.1]). It is easy to check that R satisfies (WC_{11}). Now, let

$$A = \begin{bmatrix} p + \mathbb{Z}p^2 & 0 \\ 0 & 0 \end{bmatrix} R + \begin{bmatrix} 0 & 1 + \mathbb{Z}p^2 \\ 0 & p + \mathbb{Z}p^2 \end{bmatrix}$$
$$R = \left\{ \begin{bmatrix} pa + \mathbb{Z}p^2 & pb + c + \mathbb{Z}p^2 \\ 0 & pc + \mathbb{Z}p^2 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

Clearly A is a right ideal of R. Note that A is a complement in R (see [13, Example 3.5]) and there is no non-zero direct summand of R which has zero intersection with A. If R were a right (C_{11})-module then A would be essential in R, a contradiction. It follows that R does not satisfy (C_{11}).

Now we return to general modules over arbitrary rings. We prove first.

Lemma 10. *Let M be a module such that M satisfies $(WC_{11}{}^{+})$ and $M/(SocM)$ has finite uniform dimension. Suppose that $SocM$ is contained in a finitely generated submodule of M. Then M has finite uniform dimension.*

Proof. Suppose M does not have finite uniform dimension. Then $SocM$ is not finitely generated. There exist submodules S_1, S_2 of $SocM$ such that S_i is not finitely generated for $i = 1, 2$, and $SocM = S_1 \oplus S_2$. By hypothesis, there exist submodules K, K' of M such that $M = K \oplus K'$, $S_1 \cap K = 0$ and $S_1 \oplus K$ is essential in M. Note that, by [1, Proposition 9.7 and 9.19],

$$S_1 \oplus S_2 = SocM = Soc(S_1 \oplus K) = S_1 \oplus (SocK).$$

Thus $SocK \cong S_2$ and hence $SocK$ is not finitely generated. Also,

$$SocK \oplus SocK' = SocM = S_1 \oplus (SocK)$$

so that $SocK' \cong S_1$, and hence $SocK'$ is not finitely generated. By hypothesis, there exists a finitely generated submodule N of M such that $SocM \leq N$. Suppose that $K = SocK$. Then $SocK$ is a direct summand of M and hence also a direct summand of N. It follows that $SocK$ is finitely generated which is a contradiction. Thus $K \neq SocK$. Similarly, $K' \neq SocK'$. Now, by [1, Proposition 9.19],

$$M/SocM \cong \Big[K/(SocK)\Big] \oplus \Big[K'/(SocK')\Big].$$

It follows that the modules $K/(SocK)$ and $K'/(SocK')$ each have smaller uniform dimension than $M/(SocM)$. By induction on the uniform dimension of $M/(SocM)$, we conclude that K and K' both have finite uniform dimension, and hence so does $M = K \oplus K'$, a contradiction. Thus M has finite uniform dimension.

Next we prove a theorem which was pointed out in the introduction.

Theorem 11. *Let M be a module such that M satisfies $({WC_{11}}^{+})$ and $M/(SocM)$ has finite uniform dimension. Then M contains a semisimple submodule M_1 and a submodule M_2 with finite uniform dimension such that $M = M_1 \oplus M_2$.*

Proof. If $M = SocM$ then there is nothing to prove. Suppose that $M \neq SocM$. Let $m \in M$, $m \notin SocM$. By hypothesis, there exist submodules K, K' of M such that $M = K \oplus K'$, $Soc(mR) \cap K = 0$ and $Soc(mR) \oplus K$ is essential in M. Let $\pi : M \to K'$ be the canonical projection. Then

$$Soc(mR) \oplus K = \pi(Soc(mR) \oplus K).$$

It follows that $\pi(Soc(mR))$ is an essential submodule of K' and hence

$$SocK' = \pi(Soc(mR)) \leq \pi(mR) = \pi(m)R$$

and so $SocK' \leq \pi(m)R$ (see [1, Proposition 9.7]). By Lemma 10, K' has finite uniform dimension. Note that $\pi(m) \in K'$ and $\pi(m) \notin SocK'$. Thus $K' \neq SocK'$. Now

$$M/SocM \cong \Big[K/(SocK)\Big] \oplus \Big[K'/(SocK')\Big]$$

implies that the module $K/(SocK)$ has smaller uniform dimension than $M/(SocM)$. By induction on the uniform dimension of $M/SocM$, there exist submodules K_1, K_2 of K such that $K = K_1 \oplus K_2$, K_1 is semisimple and K_2 has finite uniform dimension. Then M is the direct sum of the semisimple submodule K_1 and the submodule $K_2 \oplus K'$, which has finite uniform dimension.

Corollary 12. *Let M be a module which satisfies $({WC_{11}}^{+})$ and ACC (respectively, DCC) on essential submodules. Then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2.*

Proof. We prove the result in the ACC case, the DCC case is similar. Suppose M satisfies ACC on essential submodules. By [5, Proposition 3.6], $M/(SocM)$ is Noetherian. Hence by Theorem 11, $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and submodule M_2 with finite uniform dimension. Now $SocM = M_1 \oplus (SocM_2)$ by [1, Proposition 9.19] and hence $M/(SocM) \cong M_2/(SocM_2)$. Thus $M_2/(SocM_2)$ is Noetherian. But $SocM_2$ is Noetherian, because M_2 has finite uniform dimension. Thus M_2 is Noetherian.

Corollary 13. *Let M be a module which satisfies $(C_{11}{}^{+})$ and ACC (respectively, DCC) on essential submodules. Then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2.*

Proof. It is trivial by Corollary 12.

Remark. Note that M is a direct sum of uniform modules [12, Theorem 5.2]. However M need not to be a direct sum of uniform modules in Theorem 11 (see, Example 5).

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