

## The productively Lindelöf property in the remainders of topological spaces

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**Abstract:** A topological space  $X$  is called *productively Lindelöf* if  $X \times Y$  is Lindelöf for every Lindelöf space  $Y$ . We study with remainders and investigate topological spaces with productively Lindelöf remainders.

**Key words:** Remainder, compactification, productively Lindelöf, of countable type, topological group, Čech complete, Ohio complete, continuum hypothesis

### 1. Introduction

All spaces are assumed to be Tychonoff. All undefined notions can be found in [20, 26]. A space  $X$  is of *countable type* if every compact subspace  $P$  of  $X$  is contained in a compact subspace  $F \subset X$  that has a countable base of open neighborhoods in  $X$ . All metrizable spaces, all Čech-complete spaces, and, more generally, all  $p$ -spaces are contained in the class of spaces of countable type [3].

A *topological group*  $G$  is a group with a topology such that the multiplication mapping of  $G \times G$  into  $G$  is continuous and the inverse mapping of  $G$  onto itself associating  $x^{-1}$  with arbitrary  $x \in G$  is continuous. For more details, see [6].

Recall that a topological space is *productively Lindelöf* if its product with every Lindelöf space is Lindelöf. Since the Cartesian product of a compact space and a Lindelöf space is Lindelöf, any  $\sigma$ -compact space is productively Lindelöf (see, e.g., [20]).

In 1971 Michael proved that:

**Theorem 1.1** ([29]) *CH implies every productively Lindelöf metrizable space is  $\sigma$ -compact.*

More recently Alas et al. proved the following result.

**Theorem 1.2** ([1]) *CH implies productively Lindelöf regular  $p$ -spaces are  $\sigma$ -compact.*

Even though the class of projectively Lindelöf spaces has been extensively studied, it is still not well understood. For more details see, e.g., [2, 14, 36].

For a space  $X$  and its compactification  $bX$ , the complement  $bX \setminus X$  is called a *remainder* of  $X$ .  $X$  is called *nowhere locally compact* if no point of  $X$  has a compact neighborhood. Notice that if  $X$  is nowhere locally compact, then any remainder of  $X$  is also dense in any compactification  $bX$  of  $X$ . The theory on

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remainders of compactifications has a long history and its root goes back to Čech [16]. A famous classical result in this theory is the following due to Henriksen and Isbell [21]: a space  $X$  is of countable type if and only if the remainder in any (or some) compactification of  $X$  is Lindelöf. Later, Arhangel'skii conducted a systematic study of the theory and he has made many significant contributions to this topic (see, e.g., [7–13]).

Let  $\mathbb{Q}$  denote the space of rationals. It is well known that there exists a compactification  $b\mathbb{Q}$  in which the remainder of  $\mathbb{Q}$  is the space  $\mathbb{P}$  of irrationals. Clearly,  $\mathbb{Q}$  and  $\mathbb{P}$  are  $p$ -space, since they are metrizable. Michael proved the following well-known result.

**Theorem 1.3** ([29]) *CH implies there is a Lindelöf space  $X$  such that  $X \times \mathbb{P}$  is not Lindelöf.*

It follows that, under CH, the remainder of  $\mathbb{Q}$  is not productively Lindelöf, since  $\mathbb{P}$  is not  $\sigma$ -compact [23, 32]. However, without additional assumptions it is still an open problem whether  $\mathbb{P}$  is productively Lindelöf or not. A Lindelöf space  $Y$  is called a *Michael space* if  $\mathbb{P} \times Y$  is not Lindelöf. Indeed, one of the classical problem of Michael is: does there exist a Michael space? It is known that under some assumptions such as  $\mathfrak{b} = \aleph_1$  or  $\mathfrak{d} = \text{cov}(\mathcal{M})$  there is a Michael space (see, e.g., [27, 30]). Thus, this example shows that remainders of a productively Lindelöf space need not be productively Lindelöf.

In this sense, it is natural to ask:

**Question 1.4** *When does a productively Lindelöf space have a productively Lindelöf remainder?*

Despite much effort, the theory of productively Lindelöf spaces is still not clear. Therefore, Question 1.4 may be rewritten as follows:

**Question 1.5** *How can we characterize topological spaces with a productively Lindelöf remainder?*

It is known that any compactification  $bX$  of a space  $X$  is the image of the Stone–Čech compactification  $\beta X$  under a (unique) continuous mapping  $f$  that keeps  $X$  pointwise fixed; furthermore,  $f(\beta X \setminus X) = bX \setminus X$  [21, Lemma 1.1]. Note that  $f$  and its restriction to  $\beta X \setminus X$  are perfect. On the other hand, the class of productively Lindelöf spaces is preserved under perfect maps [35].

Therefore we have the following:

**Lemma 1.6** *If the Stone–Čech remainder  $\beta X \setminus X$  of  $X$  is productively Lindelöf, then every remainder of  $X$  is productively Lindelöf.*

In this paper we are mainly interested in the remainders of topological spaces with productively Lindelöf property. In Section 2, we present some examples around the productively Lindelöfness. In Section 3, we focus on the characterizing of remainders that have productively Lindelöf property or not. Finally, in Section 4, we discuss the remainders of the space of all continuous real-valued functions on a space  $X$  with the topology of pointwise convergence.

## 2. Examples

In this section we analyze some topological spaces with productively Lindelöf remainder.

**Example 2.1** *The space  $\mathbb{S}$  of the set of all real numbers with the topology generated by the base consisting of all intervals  $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ , where  $a, b \in \mathbb{R}$  and  $a < b$ , is called the Sorgenfrey line. It*

is known that  $\mathbb{S}$  is Lindelöf, and  $\mathbb{S} \times \mathbb{S}$  is not Lindelöf (see [20]). Then  $\mathbb{S}$  is not productively Lindelöf. Let  $Z = X \cup Y$  be the double arrows space (also called, two arrows space), where  $X = \{(x, 0) : 0 < x \leq 1\}$  and  $Y = \{(x, 1) : 0 \leq x < 1\}$ . The subspace  $X$  is the arrow space that is homeomorphic to  $\mathbb{S}$  [20, Exercise 3.10.C].  $Z$  is a Hausdorff compactification of  $\mathbb{S}$ , and its remainder  $Y$  is still a copy of  $\mathbb{S}$  (see [8]). Thus,  $\mathbb{S}$  has a nonproductively Lindelöf remainder. This implies that the Stone–Čech remainder  $\beta\mathbb{S} \setminus \mathbb{S}$  cannot be productively Lindelöf by Lemma 1.6. Let us note that  $\mathbb{S}$  is not a  $p$ -space (see [37]). Assuming CH,  $\mathbb{S}$  has no metrizable productively Lindelöf remainder. However, it is not clear whether there is a nonmetrizable productively Lindelöf remainder of  $\mathbb{S}$ .

**Example 2.2** *There is a productively Lindelöf space such that the remainder in any (or some) compactification of it is productively Lindelöf.*

**Proof** Let  $\beta\mathbb{N}$  be the Stone–Čech compactification of the discrete space  $\mathbb{N}$ .  $\beta\mathbb{N} \setminus \mathbb{N}$  is compact since it is a closed subspace of  $\beta\mathbb{N}$  (see, e.g., [39, pp. 74]). Then  $\beta\mathbb{N} \setminus \mathbb{N}$  is productively Lindelöf. By following Lemma 1.6, every remainder of  $\mathbb{N}$  is productively Lindelöf.  $\square$

**Example 2.3** *There is a nonproductively Lindelöf space  $X$  such that every remainder of  $X$  is productively Lindelöf.*

**Proof** Let  $X$  be the space of the ordinal numbers that are less than the first uncountable ordinal  $\omega_1$  in the order topology. Since  $X$  is not Lindelöf (see, e.g., [41]) it cannot be productively Lindelöf. Now consider the one-point compactification of  $X$  denoted by  $bX = X \cup \{\omega_1\}$ . Clearly,  $bX \setminus X$  is productively Lindelöf. Indeed,  $\beta X$  coincides with  $bX$  (see, e.g., [5]), and then every remainder of  $X$  is productively Lindelöf.  $\square$

We should also note that we do not have a Lindelöf but not productively Lindelöf space such that every remainder of it is productively Lindelöf. In fact, Todorćević [38] constructs a stationary Aronszajn line that is Lindelöf and not productively Lindelöf (see [19]). Moreover, in [14] Barr et al. give an example of a space that is Lindelöf and not productively Lindelöf under CH. However, we do not know whether Stone–Čech remainders of these spaces are productively Lindelöf.

**Example 2.4** (CH) *There is a nonproductively Lindelöf Čech-complete space with a productively Lindelöf remainder.*

**Proof** By following an example in [11], let  $X$  be the product space  $G \times B$ , where  $B$  is a compact topological group such that  $w(B) > 2^c$  and  $G$  is the countable power of the usual space  $\mathbb{R}$ .  $X$  is Čech-complete (see, e.g., [20]) and then any remainder of  $X$  is  $\sigma$ -compact. Therefore, any remainder of  $X$  is productively Lindelöf.

Note that  $X$  is productively Lindelöf if and only if  $G$  is productively Lindelöf.

*Claim:*  $G$  is not productively Lindelöf. If  $G$  were productively Lindelöf, then  $G$  would be  $\sigma$ -compact by Theorem 1.2, but  $G$  is not  $\sigma$ -compact. To see this suppose that  $G$  is  $\sigma$ -compact. Note that  $G$  is a Polish space; then by the Baire category theorem (see, e.g., [25], p. 41) it is a Baire space, i.e.  $G$  cannot be represented as a union of countable family of nowhere dense subspaces. Let  $G = \bigcup_n F_n$ , where each  $F_n$  is compact. Then there is a natural number  $n_0$  such that  $F_{n_0}$  has a nonempty interior. Let  $g_0$  be a point in  $F_{n_0}$ . Let  $g$  be any point in  $G$ . Observe that  $gg_0^{-1}F_{n_0}$  is a compact neighborhood of  $g$ . Thus,  $G$  is locally compact. However  $G$  cannot be locally compact, since it is a nowhere locally compact topological group.  $\square$

**Example 2.5** *There is a countable subgroup  $G$  such that no remainder of  $G$  is productively Lindelöf.*

**Proof** Consider the Cantor cube  $D^{\omega_1} = \{0, 1\}^{\omega_1}$  with its standart Boolean group structure. Since it is separable, there is a countable dense subspace of  $D^{\omega_1}$ , called  $G$ . Clearly,  $G$  is productively Lindelöf. Note that  $G$  cannot be of countable type [9, p. 169]. Therefore, any remainder of  $G$  is not Lindelöf [21], and then none of the remainder of  $G$  is productively Lindelöf.  $\square$

### 3. Some results on spaces with productively Lindelöf remainders

Now we discuss certain restrictions on remainders of topological spaces that guarantee that these remainders are productively Lindelöf or not. We have the following partial results in this direction.

**Lemma 3.1** *Every Čech-complete space has a productively Lindelöf remainder.*

Recall from [11] a space  $X$  is *locally Čech-complete* if for each  $x \in X$  there exists an open neighborhood  $V$  of  $x$  such that the closure of  $V$  in  $X$  is Čech-complete.

**Corollary 3.2** *Every remainder of a locally Čech-complete topological group is productively Lindelöf.*

**Proof** Since every locally Čech-complete topological group is Čech-complete [28, Lemma 4.1], the proof is immediate by Lemma 3.1.  $\square$

A *Lindelöf  $p$ -space* is a preimage of a separable metrizable space under a perfect mapping [3].

**Theorem 3.3** (CH) *A non-Čech-complete Lindelöf  $p$ -space has no productively Lindelöf remainder.*

**Proof** Let  $X$  be a non-Čech-complete Lindelöf  $p$ -space. Suppose there is a productively Lindelöf remainder  $Y = bX \setminus X$  of  $X$  in some compactification  $bX$ . Then  $Y$  is a Lindelöf  $p$ -space [7, Theorem 2.1]. Following Theorem 1.2,  $Y$  is  $\sigma$ -compact, but this implies  $X$  is Čech-complete.  $\square$

In what follows, we show that the assumption “Lindelöf  $p$ -space” in Theorem 3.3 cannot be dropped.

**Example 3.4** *There is a non-Čech-complete space with a productively Lindelöf remainder (here we do not need CH).*

**Proof** Let  $T(\omega_1 + 1)$  be the space of all ordinal numbers not exceeding the first uncountable ordinal  $\omega_1$  and  $Z$  be the subspace of  $T(\omega_1 + 1)$  consisting of all nonisolated points of  $T(\omega_1 + 1)$ . Now define  $Y_0$  as the subspace of  $Z$  consisting of all isolated points of  $Z$ . Set  $Y = Y_0 \cup \{\omega_1\}$  and  $X = T(\omega_1 + 1) \setminus Y$ . Note that  $T(\omega_1 + 1)$  is compact [5] and since  $X$  is dense in  $T(\omega_1 + 1)$ ,  $T(\omega_1 + 1)$  is a compactification  $bX$  of  $X$ , and  $Y$  is a remainder of  $X$  in  $bX$ . Observe that  $Y_0$  is uncountable and all points of  $Y_0$  are isolated in  $Y$ . Moreover, any open neighborhood of  $\omega_1$  contains all but countably many points of  $Y$  due to the topology on  $T(\omega_1 + 1)$ .

Following Example 2.2 in [11]  $X$  is not Čech-complete and  $Y$  is not a Lindelöf  $\Sigma$ -space. The class of Lindelöf  $\Sigma$ -space is introduced by Nagami [31]. Recall that a space is a *Lindelöf  $\Sigma$ -space* if it is the image of a Lindelöf  $p$ -space under a continuous mapping. Since all Lindelöf  $p$ -spaces are Lindelöf  $\Sigma$ -spaces [10],  $Y$  cannot be a Lindelöf  $p$ -space.

Now we show that  $Y$  is productively Lindelöf. Let  $K$  be any Lindelöf space. Take any open cover  $\mathcal{U}$  of  $Y \times K$ . Without loss of generality, we may assume  $\mathcal{U} = \{V_Y \times O_K : V_Y \text{ is open in } Y, O_K \text{ is open in } K\}$ . Since  $\{\omega_1\} \times K$  is Lindelöf (see, e.g., [20]), there is a countable subfamily  $\mathcal{W}$  of  $\mathcal{U}$  such that  $\bigcup \mathcal{W}$  covers the subspace  $\{\omega_1\} \times K$  of  $Y \times K$ .

Set  $\mathcal{W} = \{V_1 \times O_1, V_2 \times O_2, \dots\}$ , where  $V_i = \{\omega_1\} \cup (Y_0 \setminus M_i)$  and each  $M_i \subseteq Y_0$  is countable for  $i = 1, 2, \dots$ . Note that  $(Y \times K) \setminus (\bigcup \mathcal{W}) = \bigcup_{i \in \mathbb{N}} (M_i \times K)$ . Clearly, each  $M_i \times K$  is Lindelöf, and so  $\bigcup_{i \in \mathbb{N}} (M_i \times K)$  is Lindelöf. Then there is a countable subfamily  $\mathcal{W}'$  of  $\mathcal{U}$  such that  $\bigcup \mathcal{W}'$  covers  $\bigcup_{i \in \mathbb{N}} (M_i \times K)$ . Therefore we have a countable subfamily  $\mathcal{W}' \cup \mathcal{W}$  of  $\mathcal{U}$  that contains  $Y \times K$ .  $\square$

Ohio completeness was introduced by Arhangel'skii [7], who has shown that it is a useful tool in the study of remainders of compactifications. Recall that a space  $X$  is *Ohio complete* if in every compactification  $bX$  of  $X$  there exists a  $G_\delta$ -subset  $Z$  such that  $X \subset Z$  and every  $y \in Z \setminus X$  is separated from  $X$  by a  $G_\delta$ -subset of  $Z$ . All Čech-complete spaces, all Lindelöf spaces, and all p-spaces are examples of Ohio-complete spaces.

It is obvious that every Čech-complete space has a productively Lindelöf remainder. If we extend it to the class of Ohio-complete spaces, the following question naturally arises: does every Ohio-complete space have a productively Lindelöf remainder? The answer is “no”.

As we discussed in Section 2, the Sorgenfrey line  $\mathbb{S}$  is Lindelöf and so it is Ohio complete. However, there is a remainder of  $\mathbb{S}$  that is homeomorphic to  $\mathbb{S}$ , and  $\mathbb{S}$  is not productively Lindelöf.

The next statement shows how Ohio complete spaces are related to productively Lindelöf remainders.

**Theorem 3.5** *Let  $G$  be a not of countable type topological group. If some remainder (or any remainder) is Ohio complete, then  $G$  is productively Lindelöf and  $G$  has no productively Lindelöf remainder.*

**Proof** Let  $G$  be a not of countable type topological group. Take any remainder  $bG \setminus G$  that is Ohio complete in some compactification  $bG$  of  $G$ . Since every topological group is paratopological group, by following Corollary 4.2 in [12],  $G$  is  $\sigma$ -compact. Hence,  $G$  is productively Lindelöf. Clearly, none of the remainder of  $G$  is Lindelöf by the classical result of Henriksen and Isbell. Thus, there is no productively Lindelöf remainder of  $G$ .  $\square$

Since there is a generalization of the topological group that is called a *rectifiable space* (see section 2 in [12] for more information) by following Corollary 3.7 in [12] one can generalize Theorem 3.5 to a larger class of rectifiable spaces.

The following example shows that the assumption  $bG \setminus G$  is Ohio complete in Theorem 3.5 cannot be dropped.

**Example 3.6** *There is a not of countable type topological group  $G$  such that neither  $G$  nor  $bG \setminus G$  is productively Lindelöf.*

**Proof** Let  $G$  be the product space  $\mathbb{R}^c$ . It is well known that it is a nonnormal, nowhere locally compact topological group (see [7]). Take any remainder  $bG \setminus G$  of  $G$ . Note that  $G$  is not of countable type, since it is not a paracompact p-space [7, Theorem 4.1 and Example 4.2]. By Theorem 4.3 in [7]  $bG \setminus G$  cannot be Ohio complete, since  $G$  is neither a  $\sigma$ -compact nor a paracompact p-space. On the other hand,  $G$  cannot be productively Lindelöf since it is not normal. Furthermore, it is clear that  $bG \setminus G$  is not productively Lindelöf.  $\square$

Recall from [34] that a topological space  $X$  is *Hurewicz* if given any sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of open covers of  $X$  one may pick finite set  $\mathcal{V}_n \subset \mathcal{U}_n$  in such a way that  $\{\bigcup \mathcal{V}_n : n \in \mathbb{N}\}$  is a  $\gamma$ -cover of  $X$ . An infinite open cover  $\mathcal{U}$  is a  $\gamma$ -cover if for each  $x \in X$  the set  $\{U \in \mathcal{U} : x \notin U\}$  is finite. It is known that

$$\sigma\text{-compact} \Rightarrow \text{Hurewicz} \Rightarrow \text{Lindelöf}.$$

We have the following result by Theorem 1.3 in [15] and Lemma 1.6.

**Corollary 3.7** *Let  $G$  be a topological group. If  $\beta G \setminus G$  is Hurewicz, then every remainder of  $G$  is productively Lindelöf.*

We denote by  $(MA+\neg CH)$  that we assume Martin's axiom and the negation of the continuum hypothesis (see [24]).

Recall that a topological group  $G$  is *precompact* if for every neighborhood  $U$  of the identity element  $e \in G$  there is a finite subset  $F$  of  $G$  such that  $FU = G$ . It is known that a topological group  $G$  is precompact if and only if it is a dense subgroup of a compact group  $\overline{G}$  [33, 40]. Arhangel'skii and van Mill give a dichotomy for precompact topological groups in [13]. Then we have:

**Corollary 3.8**  $(MA+\neg CH)$  *Let  $G$  be a nonlocally compact precompact topological group with  $\omega_1 \leq w(G) < \mathfrak{c}$ . If  $G$  is not a Lindeöf  $p$ -space, then no remainder of  $G$  is productively Lindelöf.*

**Proof** Since  $G$  is a nonlocally compact topological group,  $G$  is nowhere locally compact, and then any remainder  $Y = bG \setminus G$  is also dense in  $bG$  (see, e.g., [8]). Therefore,  $G$  is a remainder of  $Y$ . By using Theorem 2.1 in [7]  $Y$  is not a Lindeöf  $p$ -space. Following Theorem 3.1 in [13] there is no productively Lindelöf remainder of  $G$ .  $\square$

#### 4. Remainders of $C_p(X)$

We denote by  $C_p(X, \mathbb{R})$  the space of all continuous real-valued functions on  $X$  with the topology of pointwise convergence, i.e. the topology of  $C_p(X, \mathbb{R})$  is inherited from the Tychonoff product  $\mathbb{R}^X$ . We write  $C_p(X)$  instead of  $C_p(X, \mathbb{R})$ , as usual.

It is well known that  $C_p(X)$  is metrizable if and only if  $X$  is countable [4]. Then we have:

**Corollary 4.1** *If  $X$  is a countable discrete space, then any remainder of  $C_p(X)$  is productively Lindelöf.*

**Proof** Let  $X$  be a countable discrete space. Since  $C_p(X)$  is Čech-complete [37, p. 31], any remainder of  $C_p(X)$  is productively Lindelöf.  $\square$

**Corollary 4.2** *Any remainder of  $C_p(\omega)$  is productively Lindelöf.*

Note that if  $C_p(X)$  is normal, then it is countably paracompact [37, p. 33]. Therefore we have the following result.

**Corollary 4.3** *If  $C_p(X)$  is not countably paracompact, then neither  $C_p(X)$  nor any remainder of  $C_p(X)$  is productively Lindelöf.*

**Proof** Since  $C_p(X)$  is not countably paracompact, it is not normal, and it cannot be productively Lindelöf (see also, Theorem 4 in [18]). Moreover,  $X$  is uncountable, since  $C_p(X)$  is not metrizable. Then following Corollary 4.10 in [13] every remainder of  $C_p(X)$  is nonproductively Lindelöf.  $\square$

Recall from [22] an infinite family  $\mathcal{A} \subset \mathcal{P}(\omega)$  is an *almost disjoint (AD)* if the intersection of any two distinct elements of  $\mathcal{A}$  is finite. It is *maximal almost disjoint (MAD)* if it is not properly included in any larger almost disjoint family.

A topological space is a *Mrówka space* (or  *$\psi(\mathcal{A})$  space*) if it is of the form  $\omega \cup \mathcal{A}$ , where  $\mathcal{A}$  is an almost disjoint family, and its topology is generated by the following base: every point  $n$  in  $\omega$  is isolated, and basic neighborhoods of  $A \in \mathcal{A}$  are of the form  $\{A\} \cup (A \setminus F)$ , where  $F$  is a finite subset of  $\omega$  (see [22]).

**Corollary 4.4** *If  $\mathcal{A}$  is a MAD family on  $\omega$ , then neither  $C_p(\psi(\mathcal{A}))$  nor any remainder of  $C_p(\psi(\mathcal{A}))$  is productively Lindelöf.*

**Proof** It is well known that every MAD family is uncountable (see, e.g., [22], Proposition 1). By Corollary 4.10 in [13] and Proposition 1 in [17], the proof is immediate.  $\square$

Let us note that if one is restricted to the subspace of two-valued continuous functions denoted by  $C_p(\psi(\mathcal{A}), \{0, 1\})$ , then  $C_p(\psi(\mathcal{A}), \{0, 1\})$  may be Lindelöf under some set-theoretic assumptions; see [17] for more details. Thus, we cannot guarantee that the Lindelöf property of  $C_p(\psi(\mathcal{A}), \{0, 1\})$  always fails for a MAD family  $\mathcal{A}$ . It follows that Corollary 4.4 could be not true for  $C_p(\psi(\mathcal{A}), \{0, 1\})$ .

We also have the following result.

**Corollary 4.5** *Assume  $\mathfrak{b} > \omega_1$ . If  $\mathcal{A}$  is an almost disjoint family on  $\omega$  of size  $\omega_1$ , then neither  $C_p(\psi(\mathcal{A}))$  nor any remainder of  $C_p(\psi(\mathcal{A}))$  is productively Lindelöf.*

**Proof** By Corollary 4.10 in [13] and Proposition 6 in [17],  $C_p(\psi(\mathcal{A}))$  and its remainder cannot be productively Lindelöf.  $\square$

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