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# The generalized reciprocal super Catalan matrix

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**Abstract:** The reciprocal super Catalan matrix studied by Prodinger is further generalized, introducing two additional parameters. Explicit formulae are derived for the $LU$-decomposition and their inverses, as well as the Cholesky decomposition. The approach is to use $q$-analysis and to leave the justification of the necessary identities to the $q$-version of Zeilberger's celebrated algorithm.

**Key words:** Determinant, inverse matrix, $LU$ factorization, Gaussian $q$-binomial coefficient, Zeilberger's algorithm

## 1. Introduction

As mentioned in [8], there are many combinatorial matrices defined by a given sequence $\{a_n\}$. One of them is known as the Hankel matrix and is defined as follows:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \vdots \\ \vdots & \vdots & \cdots & \ddots \end{bmatrix}$$

for more details see [6]. Considering some special number sequences instead of $\{a_n\}$, there are many special matrices with nice algebraic properties. Moreover, some authors, such as [10], studied the Hankel matrix considering the reciprocal sequence of $\{a_n\}$

$$\begin{bmatrix} \frac{1}{a_0} & \frac{1}{a_1} & \frac{1}{a_2} & \cdots \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} & \cdots \\ \frac{1}{a_2} & \frac{1}{a_2} & \frac{1}{a_4} & \vdots \\ \vdots & \vdots & \cdots & \ddots \end{bmatrix}.$$

For the sequence $\{a_{i,j}\}$, a matrix can be defined by taking $(i,j)$th entries $a_{i,j}$. Well-known types of these sequences typically include binomial coefficients. As examples, we give the family of Pascal matrices whose entries are defined via the usual binomial coefficients [2, 3]. The Pascal matrices are mainly two kinds: the first is the left adjusted Pascal matrix $P_n = (p_{ij})$ and the second is the right adjusted Pascal matrix $Q_n = (m_{ij})$,

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where

$$p_{ij} = \binom{i}{j} \text{ and } m_{ij} = \binom{i}{n-1-j}, \quad 0 \le i, j < n.$$

The Gaussian $q$-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where $(x;q)_n$ is the $q$-Pochhammer symbol defined by

$$(x;q)_n = (1-x)(1-xq)\dots\left(1-xq^{n-1}\right).$$

Note that

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

where $\binom{n}{k}$ is the usual binomial coefficient.

We recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^{n} \left(1+xq^k\right),$$

and *Rothe's formula* [1] is

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n = \prod_{k=0}^{n-1} \left(1-xq^k\right).$$

Recently, Prodinger [8] defined a matrix whose entries consist of super Catalan numbers. He also defined its reciprocal analogue as well as its $q$-versions whose $(i,j)$th entries are defined for $0 \le i, j < n$

$$\binom{2i}{i}^{-1}\binom{2j}{j}^{-1}\binom{i+j}{i},$$

$$\binom{2i}{i}\binom{2j}{j}\binom{i+j}{i}^{-1},$$

$$\begin{bmatrix} 2i \\ i \end{bmatrix}_q^{-1}\begin{bmatrix} 2j \\ j \end{bmatrix}_q^{-1}\begin{bmatrix} i+j \\ i \end{bmatrix}_q$$

and

$$\begin{bmatrix} 2i \\ i \end{bmatrix}_q\begin{bmatrix} 2j \\ j \end{bmatrix}_q\begin{bmatrix} i+j \\ i \end{bmatrix}_q^{-1},$$

respectively. Then he gave some algebraic properties of these matrices.

Recently, Kılıç et al. [4] defined and studied a variant of the reciprocal super Catalan matrix with two additional parameters whose entries are defined as

$$\binom{2i+r}{i}^{-1}\binom{2j+s}{j}^{-1}\binom{i+j}{i}^{-1}.$$

Explicit formulae for its LU-decomposition, LU decomposition of its inverse, and the Cholesky decomposition are obtained. For all results, $q$-analogues are also presented.

In this paper, for nonnegative integers $r$ and $s$, we define two $n\times n$ matrices $M=[M_{kj}]$ and $T=[T_{kj}]$ with entries

$$M_{kj}=\binom{k+j}{k}\binom{2k+r}{k}^{-1}\binom{2j+s}{j}^{-1}$$

and

$$T_{kj}=\binom{2k+r}{k}\binom{2j+s}{j}\binom{k+j}{k}^{-1}$$

for $0\le k,j<n$, respectively.

First, we give the matrices $\mathcal{M}$ and $\mathcal{T}$ which are the $q$-analogues of the matrices $M$ and $T$, respectively. For both matrices, we derive explicit expressions for the $LU$-decomposition, which leads to a formula for the determinant via $\prod_{0\le i<n} U_{i,i}$. Further, we have expressions for $L^{-1}$ and $U^{-1}$, for $LU$-decomposition of the inverse matrix and their inverses, and for the Cholesky decomposition when the matrix is symmetric, that is, the case $r=s$. Afterwards, when $q\to 1$, we get the results for the matrices $M$ and $T$. Our results generalize the results of [8] for the case $r=s=0$.

Firstly, we list the result related to the matrix $\mathcal{M}$ in the next section and secondly prove them in Section 3. Then we list results related to the matrix $\mathcal{T}$ and then give related proofs in the next section. Finally, we give the results related to the matrices $M$ and $T$ as special cases of the results related to the matrices $\mathcal{M}$ and $\mathcal{T}$. To prove the claimed results, our main tool is to guess relevant quantities and then we will use the $q$-version of Zeilberger's celebrated algorithm (for more details see [7, 9]) and Rothe's formula to justify relevant equalities. All identities we will obtain hold for general $q$ and generalized Fibonomial analogue of our results could be obtained by using the application of $q$-identities for Fibonacci numbers. We refer to [5] to give an idea.

## 2. The matrix $\mathcal{M}$

We denote matrices $L$ and $U$ by $A$ and $B$ in $LU$-decomposition of any inverse matrix, respectively, that is, $\mathcal{M}^{-1}=AB$. For the Cholesky decomposition of a matrix $G$, we will use the letter $C$ such that $G=CC^T$.

The matrix $\mathcal{M}$ is defined with entries for $0\le k,j<n$,

$$\mathcal{M}_{kj}=\begin{bmatrix}k+j\\k\end{bmatrix}_q\begin{bmatrix}2k+r\\k\end{bmatrix}_q^{-1}\begin{bmatrix}2j+s\\j\end{bmatrix}_q^{-1}.$$

Firstly, we list here the formulae related to matrix $\mathcal{M}$ that were found for $0\le k,j<n$ :

$$L_{kj}=\begin{bmatrix}2k+r\\k\end{bmatrix}_q^{-1}\begin{bmatrix}2j+r\\j\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q,$$

$$L^{-1}_{kj}=(-1)^{k+j}\,q^{\binom{k-j}{2}}\begin{bmatrix}2k+r\\k\end{bmatrix}_q^{-1}\begin{bmatrix}2j+r\\j\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q,$$

$$U_{kj}=q^{k^2}\begin{bmatrix}2k+r\\k\end{bmatrix}_q^{-1}\begin{bmatrix}2j+s\\j\end{bmatrix}_q^{-1}\begin{bmatrix}j\\k\end{bmatrix}_q,$$

$$U_{kj}^{-1} = (-1)^{k+j}\, q^{k(k+1)/2-j(j+1)/2-kj} \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q,$$

$$A_{kj} = (-1)^{k+j}\, q^{k(k+3)/2-j(j+3)/2-n(k-j)} \frac{1-q^{2j+1}}{1-q^{k+j+1}} \begin{bmatrix} n-j-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q$$
$$\times \begin{bmatrix} k+j \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q,$$

$$A_{kj}^{-1} = q^{(k-j)(k-n+1)} \begin{bmatrix} k+j \\ k \end{bmatrix}_q \begin{bmatrix} n-j-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q \begin{bmatrix} k+s \\ s \end{bmatrix}_q^{-1},$$

$$B_{kj} = (-1)^{k+j}\, q^{(j+1)(j+2)/2-n(k+j+1)+3k(k+1)/2} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} n+k \\ k+j+1 \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q$$
$$\times \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q \begin{bmatrix} k+s \\ s \end{bmatrix}_q^{-1},$$

$$B_{kj}^{-1} = q^{(k+j+1)(n-j-1)} \frac{1-q^{2j+1}}{1-q^{n-k}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} n+j \\ k+j \end{bmatrix}_q^{-1} \begin{bmatrix} j \\ k \end{bmatrix}_q$$
$$\times \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q,$$

for $r = s$,

$$C_{kj} = q^{j^2/2} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} k \\ j \end{bmatrix}_q$$

and

$$\det \mathcal{M} = q^{n(n-1)(2n-1)/6} \prod_{k=0}^{n-1} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q^{-1}.$$

## 3. Proofs related to the matrix $\mathcal{M}$

For $L$ and $L^{-1}$,

$$\sum_{j\le d\le k} L_{kd} L_{dj}^{-1} = \sum_{j\le d\le k} (-1)^{d+j}\, q^{\binom{d-j}{2}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2d+r \\ d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q$$
$$\times \begin{bmatrix} 2d+r \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} d \\ j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q$$
$$= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \sum_{0\le d\le k-j} \begin{bmatrix} k-j \\ d \end{bmatrix}_q (-1)^d\, q^{\binom{d}{2}}.$$

By Rothe's formula, if $k \neq j$ then we have $(1;q)_{k-j} = 0$, and, if $k = j$, then the last sum on the RHS of the above equation is equal to $1$. Thus we conclude

$$\sum_{j \le d \le k} L_{kd} L_{dj}^{-1} = \delta_{k,j},$$

where $\delta_{k,j}$ is Kronecker delta, as claimed.

For $U$ and $U^{-1}$,

$$\sum_{k \le d \le j} U_{kd} U_{dj}^{-1} = q^{k^2 - \binom{j+1}{2}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q$$
$$\times q^{k(2j+k)/2} (-1)^{k+j} \sum_{0 \le d \le j-k} \begin{bmatrix} j-k \\ d \end{bmatrix}_q (-1)^d q^{\binom{d+1}{2} + d(k-j)}.$$

By the Cauchy binomial theorem, if $k \neq j$, then the last sum on the RHS of the above equation equals $\prod_{d=1}^{j-k} \left(1 - q^{(k-j)+d}\right) = 0$. Thus we have

$$\sum_{k \le d \le j} U_{kd} U_{dj}^{-1} = \delta_{k,j},$$

as desired.

For $LU$-decomposition, we have to prove that

$$\sum_{0 \le d \le \min\{k,j\}} L_{kd} U_{dj} = \mathcal{M}_{kj}.$$

Consider

$$\sum_{0 \le d \le \min\{k,j\}} L_{kd} U_{dj} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} (q;q)_k (q;q)_j$$
$$\times \sum_{0 \le d \le k} q^{d^2} \frac{1}{(q;q)_d^2 (q;q)_{k-d} (q;q)_{j-d}}.$$

Denote the last sum in the equation just above by $\texttt{SUM}_k$. The Mathematica version of the $q$-Zeilberger algorithm [7] produces the recursion

$$\texttt{SUM}_k = \frac{1 - q^{j+k}}{(1-q^k)^2} \texttt{SUM}_{k-1}.$$

Since $\texttt{SUM}_0 = (q;q)_k^{-1} (q;q)_j^{-1}$, we obtain

$$\texttt{SUM}_k = (q;q)_k^{-1} (q;q)_j^{-1} \begin{bmatrix} k+j \\ k \end{bmatrix}_q.$$

Therefore, we get

$$\sum_{0 \le d \le \min\{k,j\}} L_{kd} U_{dj} = \mathcal{M}_{kj},$$

which completes the proof.

For $A$ and $A^{-1}$, consider

$$\sum_{j\le d\le k} A_{kd}A_{dj}^{-1} = (-1)^k\, q^{k(k+3)/2-j+n(j-k)} \frac{(q;q)_{n-j-1}}{(q;q)_{n-k-1}} \times \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} k \\ j \end{bmatrix}_q \times \sum_{j\le d\le k} \begin{bmatrix} k-j \\ d-j \end{bmatrix}_q (-1)^d\, q^{d(d-1)/2-jd} \frac{(q;q)_{d+j}}{(q;q)_{d-j}} \frac{1-q^{2d+1}}{1-q^{k+d+1}}.$$

By the $q$-Zeilberger algorithm for the second sum in the last equation, we obtain that it is equal to 0 provided that $k\neq j$. If $k=j$, it is obvious that $A_{kk}A_{kk}^{-1}=1$. Thus

$$\sum_{j\le d\le k} A_{kd}A_{dj}^{-1} = \delta_{k,j},$$

as claimed.

Similarly, we have

$$\sum_{k\le d\le j} B_{kd}B_{dj}^{-1} = \delta_{k,j}.$$

For the Cholesky decomposition, we examine the equation

$$\sum_{0\le d\le \min\{k,j\}} C_{kd}C_{jd} = \mathcal{M}_{kj}.$$

Here

$$\sum_{0\le d\le \min\{k,j\}} C_{kd}C_{jd} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \sum_{0\le d\le \min\{k,j\}} q^{d^2} \begin{bmatrix} k \\ d \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q.$$

Note that the sum on the RHS of the equation just above is the same as the sum in the $LU$-decomposition, which was proven before.

For the $LU$-decomposition of $\mathcal{M}^{-1}$, we should show that $\mathcal{M}^{-1}=AB$, which is same as $\mathcal{M}=B^{-1}A^{-1}$. Hence, it is sufficient to show that

$$\sum_{\max\{k,j\}\le d\le n-1} B_{kd}^{-1}A_{dj}^{-1} = \mathcal{M}_{kj}.$$

After some arrangements, we have

$$\sum_{\max\{k,j\}\le d\le n-1} B_{kd}^{-1}A_{dj}^{-1} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \sum_{j\le d\le n-1} q^{(j+k+1)(n-1-d)} \times \frac{1-q^{2d+1}}{1-q^{n-k}} \begin{bmatrix} d \\ k \end{bmatrix}_q \begin{bmatrix} n+d \\ k+d \end{bmatrix}_q^{-1} \begin{bmatrix} d+j \\ d \end{bmatrix}_q \begin{bmatrix} n-j-1 \\ d-j \end{bmatrix}_q,$$

which, by replacing $(n-1)$ with $n$ and apart from the constants factors, equals

$$\sum_{j\le d\le n} q^{(j+k+1)(n-d)} \frac{1-q^{2d+1}}{1-q^{n+1-k}} \begin{bmatrix} d \\ k \end{bmatrix}_q \begin{bmatrix} n+1+d \\ k+d \end{bmatrix}_q^{-1} \begin{bmatrix} d+j \\ d \end{bmatrix}_q \begin{bmatrix} n-j \\ d-j \end{bmatrix}_q.$$

Denote this sum by $\mathtt{SUM}_n$. The $q$-Zeilberger algorithm gives the following recursion provided that $k \neq n$ and $j \neq n$

$$\mathtt{SUM}_n = \mathtt{SUM}_{n-1}.$$

Therefore, $\mathtt{SUM}_n = \mathtt{SUM}_j = \begin{bmatrix} k+j \\ k \end{bmatrix}_q$ which completes the proof except for the case $(k,j) = (n-1, n-1)$, which could be easily checked. Thus the proof is complete.

## 4. The matrix $\mathcal{T}$

The matrix $\mathcal{T}$ is defined with entries for $0 \le k, j < n$,

$$\mathcal{T}_{kj} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q \begin{bmatrix} k+j \\ k \end{bmatrix}_q^{-1}.$$

For $0 \le k, j < n$, we have

$$L_{kj} = \begin{bmatrix} 2k+r \\ k+j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} k+r \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q,$$

$$\begin{aligned} L_{kj}^{-1} &= (-1)^{k+j} q^{\binom{k-j}{2}} \frac{1-q^{2k}}{1-q^{k+j}} \begin{bmatrix} k+j \\ k-j \end{bmatrix}_q \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \\ &\times \begin{bmatrix} j+r \\ r \end{bmatrix}_q \text{ for } j \ge 1, \\ L_{k0}^{-1} &= (-1)^k \left(1+q^k\right) q^{\binom{k}{2}} \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \text{ and } L_{00}^{-1} = 1, \end{aligned}$$

$$\begin{aligned} U_{kj} &= (-1)^k q^{k(3k-1)/2} \left(1+q^k\right) \begin{bmatrix} 2j+s \\ k+j \end{bmatrix}_q \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} j-k+s \\ s \end{bmatrix}_q \\ &\times \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \text{ for } k \ge 1,\ U_{0j} = \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q, \end{aligned}$$

$$\begin{aligned} U_{kj}^{-1} &= (-1)^k q^{k(k+1)/2-j(k+j)} \frac{1-q^j}{1-q^{k+j}} \begin{bmatrix} k+j \\ j-k \end{bmatrix}_q \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q \\ &\times \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q, \end{aligned}$$

$$A_{kj} = (-1)^{k+j}\, q^{(k+1)(k+2)/2-(j+1)(j+2)/2+n(j-k)} \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q$$
$$\times \begin{bmatrix} n+j-1 \\ 2j \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q,$$

$$A_{kj}^{-1} = q^{(k-j)(k-n+1)} \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q \begin{bmatrix} n+j-1 \\ 2j \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q^{-1}$$
$$\times \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q,$$

$$B_{kj} = q^{(j+1)(j+2)/2-n(n-1)/2-jn+k^2-1} \begin{bmatrix} n+j-1 \\ 2j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q^{-1}$$
$$\times \begin{bmatrix} j+r \\ r \end{bmatrix}_q \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1},$$

$$B_{kj}^{-1} = (-1)^{n+j+1}\, q^{k-kj-j(j+1)/2+kn+n(n-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q$$
$$\times \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1},$$

for $r = s$ and $j \geq 1$,

$$C_{kj} = \mathbf{i}^j\, (1+q)^{j/2}\, q^{j(3j-1)/4} \begin{bmatrix} 2k+r \\ k+j \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} k-j+r \\ r \end{bmatrix}_q,$$

where $\mathbf{i} = \sqrt{-1}$ and for $j = 0$,

$$C_{k0} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q$$

and

$$\det \mathcal{T} = (-1)^{\binom{n}{2}} \prod_{k=1}^{n-1} q^{k(3k-1)/2} \begin{bmatrix} 2k+s \\ 2k \end{bmatrix}_q \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q^{-1}.$$

## 5. Proofs related to the matrix $\mathcal{T}$

For $L$ and $L^{-1}$, it should be shown

$$\sum_{j \leq d \leq k} L_{kd} L_{dj}^{-1} = \delta_{k,j}.$$

By the definitions of the matrices $L$ and $L^{-1}$, for the case $j = 0$, we have

$$\sum_{0 \leq d \leq k} L_{k,d} L_{d,0}^{-1} = L_{k0} L_{0,0}^{-1} + \sum_{1 \leq d \leq k} L_{k,d} L_{d,0}^{-1}.$$

If $k=0$, we get $1$ as $(0,0)$th entry of matrix $LL^{-1}$. If $k>0$, after some rearrangements we have

$$\begin{aligned}\sum_{1\le d\le k} L_{kd}L^{-1}_{d0} &= \sum_{0\le d\le k-1} L_{k,d+1}L^{-1}_{d+1,0} = \sum_{0\le d\le n} L_{n+1,d+1}L^{-1}_{d+1,0}\\ &= \sum_{0\le d\le n} (-1)^{d+1}\left(1+q^{d+1}\right) q^{\left(d^2+d\right)/2} \begin{bmatrix} 2n+2+r \\ n+d+2 \end{bmatrix}_q \\ &\times \begin{bmatrix} n+1 \\ d+1 \end{bmatrix}_q \begin{bmatrix} n+1+r \\ d+1 \end{bmatrix}_q^{-1},\end{aligned}$$

which, by using the $q$-Zeilberger algorithm, equals $-\left[\begin{smallmatrix} 2n+2+r \\ n+1 \end{smallmatrix}\right]_q$. By changing $n+1$ with $k$ again, we get $-\left[\begin{smallmatrix} 2k+r \\ k \end{smallmatrix}\right]_q$. Finally if $k>0$,

$$\begin{aligned}\sum_{0\le d\le k} L_{kd}L^{-1}_{d0} &= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q + \sum_{1\le d\le k} L_{kd}L^{-1}_{d0}\\ &= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q - \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q = 0,\end{aligned}$$

as desired. For the case $j>0$, we have

$$\begin{aligned}\sum_{j\le d\le k} L_{kd}L^{-1}_{dj} &= \sum_{j\le d\le k} (-1)^{d+j} q^{\binom{d-j}{2}} \frac{1-q^{2d}}{1-q^{d+j}} \begin{bmatrix} 2k+r \\ k+d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q \\ &\times \begin{bmatrix} k+r \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} d+j \\ d-j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q.\end{aligned}$$

By the $q$-Zeilberger algorithm, we obtain that it is equal to $0$ provided that $k\neq j$. The case $k=j$ could be easily checked. Thus

$$\sum_{j\le d\le k} L_{kd}L^{-1}_{dj} = \delta_{k,j},$$

which completes the proof.

Verification of the inverse of $U$ could be similarly done. Inverses of the matrices $A$ and $B$ could be shown as in Section 3.

For $LU$-decomposition, we have to prove that

$$\sum_{0\le d\le \min\{k,j\}} L_{kd}U_{dj} = \mathcal{T}_{kj}.$$

The cases $k=0,\ 0\le j<n$, and, $j=0,\ 0\le k<n$ could be easily shown. For other cases, consider

$$\sum_{0\le d\le \min\{k,j\}} L_{kd}U_{dj} = L_{k0}U_{0j} + \sum_{1\le d\le \min\{k,j\}} L_{kd}U_{dj} = \begin{bmatrix}2k+r\\k\end{bmatrix}_q \begin{bmatrix}2j+s\\j\end{bmatrix}_q$$

$$+ \sum_{1\le d\le \min\{k,j\}} (-1)^d \left(1+q^d\right) q^{(3d-1)d/2} \begin{bmatrix}2k+r\\k+d\end{bmatrix}_q \begin{bmatrix}k\\d\end{bmatrix}_q$$

$$\times \begin{bmatrix}k+r\\d\end{bmatrix}_q^{-1} \begin{bmatrix}2j+s\\j+d\end{bmatrix}_q \begin{bmatrix}j-d+s\\s\end{bmatrix}_q \begin{bmatrix}j+s\\s\end{bmatrix}_q^{-1}$$

$$= \begin{bmatrix}2k+r\\k\end{bmatrix}_q \begin{bmatrix}2j+s\\j\end{bmatrix}_q + \begin{bmatrix}2k+r\\k\end{bmatrix}_q \begin{bmatrix}2j+s\\j\end{bmatrix}_q$$

$$\times \begin{bmatrix}2k\\k\end{bmatrix}_q^{-1} \begin{bmatrix}2j\\j\end{bmatrix}_q^{-1} \sum_{1\le d\le \min\{k,j\}} (-1)^d \left(1+q^d\right) q^{(3d-1)d/2}$$

$$\times \begin{bmatrix}2k\\k+d\end{bmatrix}_q \begin{bmatrix}2j\\j+d\end{bmatrix}_q .$$

Without loss of generality, we may consider $k\le j$. Hence, consider the sum

$$\mathtt{SUM}_k = \sum_{-k\le d\le k} (-1)^d \left(1+q^d\right) q^{(3d-1)d/2} \begin{bmatrix}2k\\k+d\end{bmatrix}_q \begin{bmatrix}2j\\j+d\end{bmatrix}_q .$$

The $q$-Zeilberger algorithm gives the recurrence relation

$$\mathtt{SUM}_k = \frac{\left(1+q^k\right)\left(1-q^{2k-1}\right)}{\left(1-q^{j+k}\right)} \mathtt{SUM}_{k-1}.$$

Since $\mathtt{SUM}_0 = 2\begin{bmatrix}2k\\k\end{bmatrix}_q$, we obtain

$$\mathtt{SUM}_k = 2\begin{bmatrix}2k\\k\end{bmatrix}_q \begin{bmatrix}2j\\j\end{bmatrix}_q \begin{bmatrix}k+j\\k\end{bmatrix}_q^{-1}.$$

Since the summand of the $\mathtt{SUM}_k$ is symmetric with respect to $k$ and $-k$, we have

$$\sum_{1\le d\le k} (-1)^d \left(1+q^d\right) q^{(3d-1)d/2} \begin{bmatrix}2k\\k+d\end{bmatrix}_q \begin{bmatrix}2j\\j+d\end{bmatrix}_q = \frac{1}{2}\mathtt{SUM}_k - \begin{bmatrix}2k\\k\end{bmatrix}_q \begin{bmatrix}2j\\j\end{bmatrix}_q .$$

Finally consider

$$\sum_{0\le d\le k} L_{kd}U_{dj} = \begin{bmatrix}2k+r\\k\end{bmatrix}_q \begin{bmatrix}2j+s\\j\end{bmatrix}_q + \begin{bmatrix}2k+r\\k\end{bmatrix}_q \begin{bmatrix}2j+s\\j\end{bmatrix}_q$$

$$\times \begin{bmatrix}2k\\k\end{bmatrix}_q^{-1} \begin{bmatrix}2j\\j\end{bmatrix}_q^{-1} \left(\frac{1}{2}\mathtt{SUM}_k - \begin{bmatrix}2k\\k\end{bmatrix}_q \begin{bmatrix}2j\\j\end{bmatrix}_q\right)$$

$$= \begin{bmatrix}2k+r\\k\end{bmatrix}_q \begin{bmatrix}2j+s\\j\end{bmatrix}_q \begin{bmatrix}k+j\\k\end{bmatrix}_q^{-1} = \mathcal{T}_{kj},$$

as desired.

For $LU$-decomposition of the inverse of the matrix $\mathcal{T}$, the argument in Section 3 could be similarly used. We omit it here.

**6. The matrix $M$**

Recall that the $n\times n$ matrix $M=[M_{kj}]$ is defined for $0\le k,j<n$ and nonnegative integers $r$ and $s$,

$$M_{kj}=\binom{k+j}{k}\binom{2k+r}{k}^{-1}\binom{2j+s}{j}^{-1}.$$

In Section 2, by taking $q\to 1$, we get the following results for $0\le k,j<n:$

$$L_{kj}=\binom{2k+r}{k}^{-1}\binom{2j+r}{j}\binom{k}{j},$$

$$L_{kj}^{-1}=(-1)^{k+j}\binom{2k+r}{k}^{-1}\binom{2j+r}{j}\binom{k}{j},$$

$$U_{kj}=\binom{2k+r}{k}^{-1}\binom{2j+s}{j}^{-1}\binom{j}{k},$$

$$U_{kj}^{-1}=(-1)^{k+j}\binom{2k+s}{k}\binom{2j+r}{j}\binom{j}{k},$$

$$A_{kj}=(-1)^{k+j}\frac{1+2j}{k+j+1}\binom{n-j-1}{k-j}\binom{2k+s}{k}\binom{k+j}{k}^{-1}$$
$$\times\binom{2j+s}{s}^{-1}\binom{j+s}{s},$$

$$A_{kj}^{-1}=\binom{k+j}{k}\binom{n-j-1}{k-j}\binom{2j+s}{j}^{-1}\binom{2k+s}{s}\binom{k+s}{s}^{-1},$$

$$B_{kj}=(-1)^{k+j}\binom{2j+r}{j}\binom{n+k}{k+j+1}\binom{j}{k}\binom{2k+s}{s}\binom{k+s}{s}^{-1},$$

$$B_{kj}^{-1}=\frac{2j+1}{n-k}\binom{2k+r}{k}^{-1}\binom{n+j}{k+j}^{-1}\binom{j}{k}\binom{2j+s}{s}^{-1}\binom{j+s}{s},$$

for $r=s$,

$$C_{kj}=\binom{2k+r}{k}^{-1}\binom{k}{j}$$

and

$$\det\mathcal{M}=\prod_{k=0}^{n-1}\binom{2k+r}{k}^{-1}\binom{2k+s}{k}^{-1}.$$

**7. The matrix $T$**

Recall that the $n\times n$ matrix $T=[T_{kj}]$ is defined for $0\le k,j<n$, and nonnegative integers $r$ and $s$,

$$T_{kj}=\binom{2k+r}{k}\binom{2j+s}{j}\binom{k+j}{k}^{-1}.$$

In the Section 4, by taking $q\to 1$, we obtain the following results. For $0\le k,j<n$,

$$L_{kj}=\binom{2k+r}{k+j}\binom{k}{j}\binom{k+r}{j}^{-1}\binom{2j+r}{r}^{-1}\binom{j+r}{r},$$

for $j\ge 1$,

$$L_{kj}^{-1}=(-1)^{k+j}\,\frac{2k}{k+j}\binom{k+j}{k-j}\binom{2k+r}{r}\binom{k+r}{r}^{-1}\binom{2j+r}{r}^{-1}\binom{j+r}{r},$$

$$L_{k0}^{-1}=2\,(-1)^{k}\binom{2k+r}{r}\binom{k+r}{r}^{-1}\text{and } L_{00}^{-1}=1,$$

for $k\ge 1$,

$$U_{kj}=(-1)^{k}\,2\binom{2j+s}{k+j}\binom{2k+r}{r}\binom{j-k+s}{s}\binom{k+r}{r}^{-1}\binom{j+s}{s}^{-1}$$

$$\text{and } U_{0j}=\binom{2j+s}{j},$$

$$U_{kj}^{-1}=(-1)^{k}\,\frac{j}{k+j}\binom{k+j}{j-k}\binom{2j+r}{r}^{-1}\binom{j+r}{r}\binom{2k+s}{s}^{-1}\binom{k+s}{s},$$

$$A_{kj}=(-1)^{k+j}\binom{k}{j}\binom{n+k-1}{2k}\binom{n+j-1}{2j}^{-1}\binom{2k+s}{s}^{-1}\binom{k+s}{s}$$
$$\times\binom{2j+s}{s}\binom{j+s}{s}^{-1},$$

$$A_{kj}^{-1}=\binom{k}{j}\binom{n+k-1}{2k}\binom{n+j-1}{2j}^{-1}\binom{2k+s}{s}^{-1}\binom{k+s}{s}$$
$$\times\binom{2j+s}{s}\binom{j+s}{s}^{-1},$$

$$B_{kj}=\binom{n+j-1}{2j}\binom{j}{k}\binom{2k+s}{k}^{-1}\binom{j+r}{r}\binom{2j+r}{r}^{-1},$$

$$B_{kj}^{-1}=(-1)^{n+j+1}\binom{j}{k}\binom{n+k-1}{2k}^{-1}\binom{2j+s}{s}\binom{2k+r}{r}\binom{k+r}{r}^{-1},$$

for $r=s$ and $j\ge 1$,

$$C_{kj}=(-2)^{j/2}\binom{2k+r}{k+j}\binom{k+r}{r}^{-1}\binom{k-j+r}{r},$$

for $j = 0$,

$$C_{k0} = \binom{2k+r}{k}.$$

Thus

$$\det \mathcal{T} = (-1)^{\binom{n}{2}} \prod_{k=1}^{n-1} \binom{2k+s}{2k} \binom{2k+r}{r} \binom{k+r}{r}^{-1} \binom{k+s}{s}^{-1}.$$

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