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Strong lifting splits

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ABSTRACT

The concept of an enabling ideal is introduced so that an ideal I is strongly lifting if and only if it is lifting and enabling. These ideals are studied and their properties are described. It is shown that a left duo ring is an exchange ring if and only if every ideal is enabling, that Zhou's δ-ideal is always enabling, and that the right singular ideal is enabling if and only if it is contained in the Jacobson radical. The notion of a weakly enabling left ideal is defined, and it is shown that a ring is an exchange ring if and only if every left ideal is weakly enabling. Two related conditions, interesting in themselves, are investigated: the first gives a new characterization of δ-small left ideals, and the second characterizes weakly enabling left ideals. As an application (which motivated the paper), let M be an I-semiregular left module where I is an enabling ideal. It is shown that $m \in M$ is I-semiregular if and only if $m - q \in IM$ for some regular element q of M and, as a consequence, that if M is countably generated and IM is δ-small in M, then $M \cong \oplus_{i=1}^{\infty} Re_i$ where $e_i^2 = e_i \in R$ for each i.

Throughout this paper all rings are associative with unity, and all modules are unitary. We denote the Jacobson radical of a ring R by $J = J(R)$, the right and left socles by S_r and S_l, the right and left singular ideals by Z_r and Z_l, and the left δ-ideal [8] by $\delta = \delta(R)$. We write $I \lhd R$ to indicate that I is an ideal (right and left) of R. The ring of integers is denoted by $\mathbb{Z}$, and we write $\mathbb{Z}_n$ for the ring of integers modulo n. If X is a subset of R we write $\mathtt{r}(X)$ and $\mathtt{l}(X)$ for the right and left annihilators of X in R. Modules are left modules unless otherwise indicated, and homomorphisms are written on the right. The notation $S \subseteq^{\oplus} M$ indicates that S is a direct summand of M.

1. Enabling ideals

Let I be an ideal of a ring R. Then I is called *lifting* if, for each $a \in R$, the condition $a^2 - a \in I$ implies that $a - e \in I$ for some $e^2 = e \in R$. In [6], the ideal I is called *strongly lifting* if for each a the idempotent e can be chosen so that $e \in aR$, (equivalently $e \in Ra$, equivalently $e \in aRa$; see [6, Lemma 1]). The next result allows us to split the notion of strong lifting, and will be used extensively below.

Lemma 1. *Let $I \lhd R$. The following are equivalent.*

(1) *If $a - e \in I$, $a \in R$, $e^2 = e \in R$, then $a - f \in I$ for some $f^2 = f \in Ra$.*
(2) *If $a - e \in I$, $a \in R$, $e^2 = e \in R$, then $a - f \in I$ for some $f^2 = f \in aRa$.*
(3) *If $a - e \in I$, $a \in R$, $e^2 = e \in R$, then $a - f \in I$ for some $f^2 = f \in aR$.*

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Proof. We need only prove (1)⇒(2). If $x, y \in R$ write $x \equiv y$ when $x - y \in I$. Given (1), let $a \equiv e$ where $e^2 = e \in R$. We have $a^2 \equiv e$ so, replacing a by a^2 in (1), we obtain $g^2 = g \in Ra^2$ such that $g \equiv a^2 \equiv a$. Write $g = ba^2$, $b \in R$, where (since $g = g^2$) we may assume that $gb = b$. Define $f = aba \in aRa$. Then $f^2 = aba^2ba = agba = aba = f$, and $f = aba \equiv aba^2 = ag \equiv a^2 \equiv a$. This proves (2). □

We remark that Lemma 1 is true for any left ideal I provided the idempotent e is central.

An ideal $I \lhd R$ is called an **enabling ideal** of R if it satisfies the conditions in Lemma 1. Thus $I = 0$ and $I = R$ are both enabling ideals in any ring R. If $I \lhd R$ is strongly lifting then it is both lifting and enabling, and we claim that the converse is true. For if $a^2 - a \in I$ then, because I is lifting, there exists $e^2 = e \in R$ such that $a - e \in I$. But then there exists $f^2 = f \in Ra$ such that $a - f \in I$ because I is enabling. Hence I is strong lifting, and we have proved that strongly lifting "splits" in the following sense.

Theorem 2. *If $I \lhd R$, then I is strongly lifting if and only if I is both lifting and enabling.*

Since every strongly lifting ideal is enabling, Theorems 10 and 4 in [6], respectively, give the following examples.

Example 3. The right socle S_r and the left socle S_l are both enabling ideals.

Example 4. Every ideal in an exchange ring is enabling.

If R is commutative, the converse to Example 4 is true. In fact, we will show that a *left duo ring* R (left ideals are ideals) is exchange if and only if every ideal is enabling. We return to this in Corollary 23.

While the Jacobson radical may not be lifting, it is always enabling. In fact, we have the following proposition.

Proposition 5. *If $I \lhd R$ and $I \subseteq J$ then I is enabling. In particular, J is enabling.*

Proof. Write $r \equiv s$ to mean $r - s \in I$. If $a \equiv e = e^2$ we must find $f^2 = f \in Ra$ such that $a \equiv f$. We have $e - a \in I \subseteq J$, so $u = 1 - (e - a)$ is a unit in R and $u \equiv 1$. Moreover $ue = ae$ so $e = e^2 = e(u^{-1}ae)$. Hence $f = eu^{-1}a$ is an idempotent in Ra, and $f \equiv ea \equiv a^2 \equiv a$, as required. □

An enabling ideal need not be contained in the Jacobson radical. For example, if F is a field and $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ then $S_r = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ is enabling by Example 3, but $S_r \nsubseteq J$ because $J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$. However there is one important class of enabling ideals that must lie in the Jacobson radical. An ideal $I \lhd R$ is said to be *idempotent-free* if 0 is the only idempotent in I.

Proposition 6. *If $I \lhd R$ is idempotent-free, then $I \subseteq J(R)$ if and only if I is enabling.*

Proof. The forward implication is Proposition 5. For the converse, assume that I is enabling but $I \nsubseteq J$. Then $I \nsubseteq M$ for some maximal left ideal M of R, so $R = I + M$, say $1 = x + m$ where $x \in I$ and $m \in M$. Thus $m - 1 \in I$ so, since I is enabling, there exists $f^2 = f \in Rm$ with $m - f \in I$. But then $1 - f = (1 - m) + (m - f) \in I$. Since I is idempotent-free, it follows that $1 = f \in Rm \subseteq M$, a contradiction. □

We will improve on Proposition 6 later (Proposition 34).

It is worth noting that the proof of the reverse implication in Proposition 6 requires only that $a - 1 \in I$ implies $a - f \in I$ for some $f^2 = f \in Ra$. We will return to ideals with this property in the next section where they will be called *weakly enabling.*

Corollary 7. *If 0 and 1 are the only idempotents in a ring R, and if $I \lhd R$ is an enabling ideal, then $I \subseteq J$ or $I = R$.*

Proof. If $I \neq R$ is an enabling ideal, then I is idempotent-free by hypothesis, so $I \subseteq J$ by Proposition 6. □

Thus, for example, the only enabling ideals of $\mathbb{Z}$ are 0 and $\mathbb{Z}$.

The *left singular ideal* $Z_l = Z_l(R)$ of a ring R is defined by $Z_l = \{z \in R \mid \mathrm{l}(z) \text{ is essential in } {}_RR\}$, with a similar definition for the *right singular ideal* Z_r. If Z denotes Z_l or Z_r, it is often a mystery whether $Z \subseteq J$. Since Z is idempotent-free in both cases, we have the following corollary.

Corollary 8. *If Z denotes Z_l or Z_r, then Z is enabling if and only if $Z \subseteq J$.*

Recall that the *second left singular ideal* $Z_l^2 = Z_l^2(R)$ of a ring R is defined by $Z_l(R/Z_l) = Z_l^2/Z_l$. It follows in the same way that $Z_l^2 \subseteq J$ if and only if Z_l^2 is enabling.

The next example gives two important situations where the left singular ideal Z_l is enabling. Call a ring R *left principally injective* if every R-linear map $Ra \to R$, $a \in R$, extends to R; and call R *right Kasch* if every simple right module embeds in R, equivalently if $\mathrm{l}(T) \neq 0$ for all maximal right ideals T of R.

Example 9. Let R be a ring.

(1) If $\mathrm{l}(a) = 0$, $a \in R$, implies $aR = R$, then Z_l is enabling.
(2) In particular, if R is right Kasch or left P-injective, then Z_l is enabling.

Proof. (1) If $a \in Z_l$ then the fact that $\mathtt{l}(1-a) \cap \mathtt{l}(a) = 0$ implies that $\mathtt{l}(1-a) = 0$. Hence $(1-a)R = R$ by hypothesis. This proves that $Z_l \subseteq J$ so Corollary 8 applies.

(2) If R is left P-injective then $aR = \mathtt{rl}(a)$ for each a. Hence $\mathtt{l}(a) = 0$ implies $aR = \mathtt{r}(0) = R$, and (1) applies. Now assume that R is right Kasch, and suppose that $\mathtt{l}(a) = 0$ but $aR \neq R$. Let T be a maximal right ideal such that $aR \subseteq T$. Then $bT = 0$ for some $0 \neq b \in R$ by the Kasch condition. Hence $b \in \mathtt{l}(a) = 0$, a contradiction, and again we are done by (1). □

Example 10. Neither "lifting" nor "enabling" implies the other.

Proof. Let $p \neq q$ be primes in $\mathbb{Z}$, and let $R = \{\frac{k}{d} \in \mathbb{Q} \mid p \nmid d \text{ and } q \nmid d\}$. Write $P = pR$ and $Q = qR$. Then one verifies that $R/P \cong \mathbb{Z}_p$, $R/Q \cong \mathbb{Z}_q$, and $P \cap Q = pqR = J(R) = J$. Since $P + Q = R$, it follows from the Chinese remainder theorem that $R/J \cong \mathbb{Z}_p \times \mathbb{Z}_q$. As R is a domain, it follows that J is not lifting so, by Proposition 5, J is an enabling ideal that is not lifting.

On the other hand, the ideal P is lifting because $R/P \cong \mathbb{Z}_p$ is a field, but P is not enabling by Proposition 6. □

Note that if $p \in \mathbb{Z}$ is a prime, the ideal $p^n\mathbb{Z}$ of $\mathbb{Z}$ is another commutative example of a lifting ideal that is not enabling.

Proposition 11. *Let $A \subseteq I$ be ideals of R. Write $\bar{r} = r + A$ in $\bar{R} = R/A$.*

(1) *If A is lifting and I is enabling in R, then $\bar{I}$ is enabling in $\bar{R}$.*
(2) *If A is strongly lifting and $\bar{I}$ is enabling in $\bar{R}$, then I is enabling in R.*

Proof. (1) Let $\bar{a} - \bar{e} \in \bar{I}$, $\bar{e}^2 = \bar{e}$. As A is lifting we may assume that $e^2 = e$. Since $a - e \in I$ and I is enabling, there exists $f^2 = f \in Ra$ such that $a - f \in I$. Hence $\bar{f}^2 = \bar{f} \in \bar{R}\bar{a}$ and $\bar{a} - \bar{f} \in \bar{I}$.

(2) Assume that $\bar{I}$ is enabling in $\bar{R}$. If $a - e \in I$, $e^2 = e$, we want $g^2 = g \in Ra$ such that $a - g \in I$. We have $\bar{a} - \bar{e} \in \bar{I}$, $\bar{e}^2 = \bar{e}$ so, by hypothesis, let $\bar{f}^2 = \bar{f} \in \bar{R}\bar{a}$ satisfy $\bar{a} - \bar{f} \in \bar{I}$. As A is lifting we may assume that $f^2 = f$. Write $\bar{f} = \bar{r}\bar{a}$, $r \in R$, and observe that $(ra)^2 - ra \in A$. Since A is strongly lifting, there exists $g^2 = g \in Ra$ such that $g - ra \in A \subseteq I$. Hence, if $\equiv$ denotes congruence modulo I, we have $g \equiv ra \equiv f \equiv a$; that is $a - g \in I$. □

Corollary 12. *If J is lifting and $J \subseteq I \lhd R$, then I is enabling in R if and only if I/J is enabling in R/J.*

Proof. It follows by Proposition 11 because J is strongly lifting whenever it is lifting. □

Corollary 13. *Let $A \subseteq I$ be ideals of R where A is strongly lifting and $I/A \subseteq J(R/A)$. Then:*

(1) *The ideal I is enabling;*
(2) *I is strongly lifting if and only if it is lifting.*

Proof. Since (1) implies (2), it is enough to prove (1). But I/A is enabling by hypothesis, so I is enabling by (2) of Proposition 11. □

Following Zhou [8, Lemma 1.2] a submodule $K \subseteq M$ is called δ-**small** in M if the following equivalent conditions are satisfied.

(1) *If $M = K + X$ where M/X is singular then $X = M$.*
(2) *If $M = K + X$ then $M = Y \oplus X$ where $Y \subseteq K$ and Y is semisimple and projective.*

Note that (2) implies that a δ-small left ideal which is a direct summand is semisimple.

Zhou defines $\delta(M)$ to be the sum of all δ-small submodules of M and shows that δ is a preradical in the category of left R-modules. In particular $\delta({}_RR)$ is an ideal of R which we will call the **left** δ-**ideal** of R, and denote by $\delta = \delta(R)$. Zhou goes on to show [8, Theorem 1.6 and Corollary 1.7] that $\delta(R)$ is the intersection of the essential, maximal left ideals of R, and that $J(R/S_l) = \delta/S_l$. In particular, $\delta \supseteq J$. Since S_l is strongly lifting [6, Theorem 10], the following result follows from Corollary 13.

Corollary 14. *If R is any ring then $\delta(R)$ is enabling.*

Note that the ideal $\delta(R)$ need not be lifting. To see this, let $p \neq q$ be primes in $\mathbb{Z}$, and let $R = \{\frac{k}{d} \in \mathbb{Q} \mid p \nmid d \text{ and } q \nmid d\}$ be the ring in Example 10. Then pR and qR are the only maximal ideals in R, and they are essential in R, so $\delta(R) = pR \cap qR = J(R)$. Hence $\delta(R)$ is not lifting by Example 10.

We conclude this section with some results about related rings. The first result is clear from the definitions.

Proposition 15. *Let $I_k \lhd R_k$ where each R_k is a ring. Then $\Pi_k I_k$ is enabling in the direct product $\Pi_k R_k$ if and only if I_k is enabling in R_k for each k.*

Proposition 16. *Let $g^2 = g \in R$. If $I \lhd R$ is enabling, then $gIg \lhd gRg$ is enabling.*

Proof. Let $a \in gRg$ and assume that $a - e \in gIg$ where $e^2 = e \in gRg$. Then $a \in R$ and $a - e \in I$ so, by hypothesis, there exists $f^2 = f \in aRa$ with $f - a \in I$. As $a \in gRg$, it follows that $f \in gRg$, and hence that $f - a \in gRg$. Finally $f \in aRa = (ag)R(ga) = a(gRg)a$, completing the proof. □

If R is a general ring (possibly no unity), we define enabling ideals of R in the same way. Then an argument similar to the proof of Proposition 16 shows that if $I \lhd R$ is enabling then $I \cap S \lhd S$ is enabling for any left or right ideal S of R.

Proposition 17. *Let* $T = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ *be a split-null extension. If* $I \lhd R$ *and* $K \lhd S$*, and if* $\begin{bmatrix} I & V \\ 0 & K \end{bmatrix}$ *is enabling in* T*, then* I *and* K *are enabling in* R *and* S *respectively.*

Proof. Let $a - e \in I$ where $e^2 = e \in R$, and let $b - f \in K$ where $f^2 = f \in S$. Then we have $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} - \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \in \begin{bmatrix} I & V \\ 0 & K \end{bmatrix}$. By hypothesis, there exists $\begin{bmatrix} p & v \\ 0 & q \end{bmatrix}^2 = \begin{bmatrix} p & v \\ 0 & q \end{bmatrix} \in T \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ such that $\begin{bmatrix} p & v \\ 0 & q \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \begin{bmatrix} I & V \\ 0 & K \end{bmatrix}$. It follows that $p^2 = p \in Ra$ with $p - a \in I$, and $q^2 = q \in Sb$ with $q - b \in K$. □

In particular, if $T_n(R)$ denotes the ring of upper triangular matrices over R, and if $I \lhd R$ and $T_n(I)$ is enabling in $T_n(R)$, then I is enabling in R. We do not know if the converse is true.

Question. *If* $I \lhd R$ *is enabling, is* $M_n(I) \lhd M_n(R)$ *enabling where* $M_n(R)$ *is the matrix ring.*

Question. *If* $I \lhd R$ *is enabling, is* $I[x] \lhd R[x]$ *enabling?*

Question. *If* $I \lhd R$ *is enabling, is* $I[[x]] \lhd R[[x]]$ *enabling?*

2. Weakly enabling left ideals

In the discussion following Proposition 6, we mentioned a class of ideals with a weakened enabling requirement. Other characterizations are given in the following lemma.

Lemma 18. *The following are equivalent for a left ideal* L *of a ring* R.

(1) *If* $a - 1 \in L$, $a \in R$, *then* $a - f \in L$ *for some* $f^2 = f \in Ra$.
(2) *If* $a - 1 \in L$, $a \in R$, *then* $a - f \in L$ *for some* $f^2 = f \in aRa$.
(3) *If* $a - 1 \in L$, $a \in R$, *then* $a - f \in L$ *for some* $f^2 = f \in aR$.

Proof. We prove (1)⇒(2) and (3)⇒ (2). Write $x \equiv y$ to mean that $x - y \in L$.

(1)⇒(2). If $a \equiv 1$ one verifies that $a^2 \equiv 1$, so (1) produces $g^2 = g \in Ra^2$ such that $g \equiv a^2$. Write $g = ba^2$, $b \in R$, where (since $g = g^2$) we may assume that $gb = b$. If we define $f = aba \in aRa$, then $f^2 = aba^2ba = agba = aba = f$, so it remains to show that $f \equiv a$. Since $1 \equiv a^2$ we obtain $b \equiv ba^2 = g \equiv a^2 \equiv a$. Since $a \equiv 1$, we have $f = aba \equiv (ab)1 = ab \equiv a^2 \equiv 1 \equiv a$. This proves (2).

(3)⇒(2). If $a \equiv 1$ then $a^2 \equiv 1$ so (3) produces $g^2 = g \in a^2R$ such that $g \equiv a^2$. Write $g = a^2c$, $c \in R$, where $cg = c$. If we define $h = aca$ then $h^2 = h \in aRa$ as before. Note that, since $a \equiv 1$, we have $ah = a^2ca = ga \equiv g \equiv 1$. Finally, define $f = h + ah - hah$. Then $f^2 = f \in aRa$ and $f \equiv h + 1 - h \cdot 1 = 1 \equiv a$, proving (2). □

A left ideal L is called **weakly enabling** if the conditions in Lemma 18 are satisfied for every $a \in R$. A left ideal L is called *strongly lifting* [6] if $a^2 - a \in L$ implies $e - a \in L$ for some $e^2 = e \in Ra$, and, in this case, L is weakly enabling because $a - 1 \in L$ implies $a^2 - a \in L$. Obviously enabling ideals are weakly enabling both as a left and a right ideal, but we do not have an example showing that the converse is not true.

It turns out that weakly enabling left ideals admit a characterization in terms of the following important notion. If N is a submodule of a module M, we call N a **partial summand**[1] of M if the following condition holds:

If $M = N + X$, X *a submodule then* $M = S + X$ *for some* $S \subseteq^{\oplus} M$ *with* $S \subseteq N$.

As the terminology suggests, every direct summand N of M is a partial summand (take $S = N$), and every small submodule is a partial summand. In fact, if N is a partial summand of M, and K is small in M, then $N + K$ is a partial summand. Indeed, if $M = (N + K) + X$ then $M = N + X$ because K is small, and so there exists $S \subseteq^{\oplus} M$ such that $S \subseteq N \subseteq N + K$. We note in passing that by [5, Proposition 2.9] a projective module has the finite exchange property if and only if every submodule is a partial summand.

Before proceeding, we recall a lemma [4, Lemma 1.16] which will be used several times below.

Lemma 19. *Let* M *be a projective module and assume that* $M = P + K$ *where* P *and* K *are submodules and* P *is a direct summand of* M. *Then there exists* $Q \subseteq K$ *such that* $M = P \oplus Q$.

A left ideal L is called a **left partial summand** of R if it is a partial summand of ${}_RR$; that is

(1) *If* $R = L + X$, X *a left ideal, then* $R = Re + X$ *for some* $e^2 = e \in L$.
By Lemma 19 this is equivalent to
(2) *If* $R = L + X$, X *a left ideal, then* $R = Re \oplus Rf$ *for some* $e^2 = e \in L$ *and* $f^2 = f \in X$.
These left partial summands are precisely the weakly enabling left ideals.

1 In [1] a partial summand of M is said to be DM in M.

Theorem 20. *The following conditions are equivalent for a left ideal L of a ring R.*

(1) *L is weakly enabling.*
(2) *L is a left partial summand of R.*

Proof. (1)⇒(2). Let $R = L + {}_RX$, say $1 = a + x$, $a \in L$, $x \in X$. Then $x - 1 \in L$ so, by (1), there exists $f^2 = f \in Rx$ such that $x - f \in L$. Hence $f \in X$ and $1 - f = (1 - x) + (x - f) \in L$, so $R = R(1 - f) + X$. This shows that L is a left partial summand of ${}_RR$.

(2)⇒(1). Let $a - 1 \in L$; we must find $g^2 = g \in Ra$ such that $a - g \in L$. We have $R = L + Ra$ so (2) gives $R = Re + Ra$ where $e^2 = e \in L$. Since R is projective, Lemma 19, gives $R = Re \oplus Rf$ where $f^2 = f \in Ra$. Let $1 = re + sf$ where $r, s \in R$ and consider $g = f + sf - fsf$. Then $g^2 = g \in Ra$ and $1 - g = (1 - f)(1 - sf) = (1 - f)re \in L$. Hence $a - g = (a - 1) + (1 - g) \in L$, as required. □

For an ideal, Theorem 20 shows that "partial summand" is a left–right symmetric concept.

Theorem 21. *If $I \lhd R$ the following are equivalent.*

(1) *I is weakly enabling as a left ideal.*
(2) *I is weakly enabling as a right ideal.*
(3) *I is a left partial summand of R.*
(4) *I is a right partial summand of R.*

Proof. (1)⇔(3) and (2)⇔(4) follow from Theorem 20, and (1)⇔(2) follows from Lemma 18. □

With this we can give a new characterization of exchange rings.

Theorem 22. *The following are equivalent for a ring R.*

(1) *R is an exchange ring.*
(2) *Every left ideal of R is a weakly enabling left ideal.*

Proof. (1)⇒(2). Every left ideal of an exchange ring is strongly lifting [6, Theorem 4], hence weakly enabling.

(2)⇒(1). Given (2), every left ideal of R is a left partial summand of R by Theorem 20. But then [5, Proposition 2.9] shows that ${}_RR$ has the finite exchange property, that is, R is an exchange ring. □

Corollary 23. *A left duo ring R is an exchange ring if and only if every ideal is enabling.*

Proof. If R is exchange, use Example 4. Conversely, if $L \subseteq R$ is a left ideal then, by hypothesis, $L \lhd R$ so L is enabling. But then L is weakly enabling as a left ideal and Theorem 22 applies. □

If R is a ring in which every ideal is a left and right partial summand, we do not know whether every ideal is enabling in R. This clearly holds for commutative rings; indeed, it holds much more generally. The following notion is needed.

Following [3], a module M is said to have the **summand-sum property** (**SSP**) if $K + L$ is a direct summand of M whenever both K and L are direct summands. We say that a ring has **left SSP** if the module ${}_RR$ has the SSP. Clearly any von Neumann regular ring is left and right SSP, as is every ring in which 0 and 1 are the only idempotents. More generally, any ring with all idempotents central is left and right SSP (if $e^2 = e$ and $f^2 = f$, and we let $g = e + f - ef$, then $g^2 = g$ and $Re + Rf = Rg$). However, if $R = \begin{bmatrix} S & V \\ 0 & T \end{bmatrix}$ is any split-null extension and both S and T have no idempotents except 0 and 1, then R has left SSP but idempotents are not central. Indeed, except for 0 and 1, the only idempotents in R are of the form $e = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix}$, and one verifies that $Re + Rf = R$ in every case.

We can now state one situation where left partial summands are enabling. The proof requires the following sufficient condition that an ideal $I \lhd R$ is enabling.

Lemma 24. *Let $I \lhd R$ and let $a - e \in I$ where $a \in R$ and $e^2 = e \in R$. Assume that*

$$Ra + R(1 - e) + I = Rg + R(1 - e) + I \quad \text{for some } g^2 = g \in Ra.$$

Then there exists $f^2 = f \in Ra$ such that $a - f \in I$.

Proof. Write $R/I = \bar{R}$ and $\bar{r} = r + I$ for all $r \in R$. Since $\bar{a}^2 = \bar{a}$ and $\bar{g} \in \bar{R}\bar{a}$, we obtain $\bar{g}\bar{a} = \bar{g}$. We claim that also $\bar{a}\bar{g} = \bar{a}$. To see this, write $a = rg + s(1 - e) + x$ where $r, s \in R$ and $x \in I$. Since $\bar{a} = \bar{e}$ we obtain $\bar{a} = \bar{a}^2 = \bar{r}\bar{g}\bar{a} + \bar{s}(\bar{1} - \bar{e})\bar{a} + \bar{x}\bar{a} = \bar{r}\bar{g}\bar{a} = \bar{r}\bar{g}$, and it follows that $\bar{a}\bar{g} = \bar{a}$.

Now define $f = g + ag - gag$. Then $f^2 = f \in Ra$ and $\bar{f} = \bar{g} + \bar{a}\bar{g} - \bar{g}\bar{a}\bar{g} = \bar{g} + \bar{a} - \bar{g}^2 = \bar{a}$. This proves the Lemma. □

Theorem 25. *Let R be a left SSP ring. If $I \lhd R$ then I is weakly enabling if and only if I is enabling.*

Proof. By Theorem 20 it is enough to show that, if $I \lhd R$ is a left partial summand then I is enabling. So let $a - e \in I$ where $e^2 = e$; we want $f^2 = f \in Ra$ such that $a - f \in I$. We have $1 = (e - a) + a + (1 - e)$, so $R = I + Ra + R(1 - e)$. Since I is a left partial summand, there exists $S \subseteq^{\oplus} R$ such that $S \subseteq I$ and $R = S + [Ra + R(1 - e)]$. Now rewrite this as $R = Ra + [S + R(1 - e)]$, and observe that $S + R(1 - e)$ is a summand of R because R is left SSP. Hence Lemma 19 applies to give a summand $Rg \subseteq Ra$, $g^2 = g$, such that $R = Rg \oplus [S + R(1 - e)]$. Since $S \subseteq I$ this gives $R = Rg + R(1 - e) + I$. But we also have $R = Ra + R(1 - e) + I$. Since $g^2 = g \in Ra$, we are done by Lemma 24. □

The proof of Proposition 17 goes through to give the following example.

Example 26. If $I \subseteq R$ and $K \subseteq S$ are left ideals and $X = \begin{bmatrix} I & V \\ 0 & K \end{bmatrix} \subseteq \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ is a weakly enabling left ideal, then the same is true for $I \subseteq R$ and $K \subseteq S$.

3. δ-small ideals

We are going to characterize the ideals $I \lhd R$ such that I is a δ-small left ideal in terms of the following notion. If M is a module, a submodule N is said to be **summand-small** in M if it satisfies the following condition.

If $M = N + X$, X a submodule then $X \subseteq^{\oplus} M$.

Hence small submodules are summand-small, as are δ-small submodules, and submodules of summand-small submodules are again summand-small. A module M is summand-small in itself if and only if M is semisimple, and every submodule of a semisimple module M is summand-small in M. However, the Prüfer group $\mathbb{Z}_{p^\infty}$ has the property that every proper subgroup is summand-small (in fact small) but $\mathbb{Z}_{p^\infty}$ is not semisimple.

If M is projective, Lemma 19 yields the following stronger condition that a submodule $N \subseteq M$ is summand-small in M.

Lemma 27. *If M is projective the following are equivalent for $N \subseteq M$.*

(1) *N is summand-small in M.*
(2) *If $M = N + X$, X a submodule, then $M = S \oplus X$ for some $S \subseteq N$.* [2]

In particular, summand-small submodules of M are partial summands.

The following examples show that partial summands of a projective module need not be summand-small, and that in general, summand-small submodules need not be partial summands.

Example 28. Partial summands of a projective module need not be summand-small.

Proof. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is a field, take $M = {}_RR$, and take $N = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$. Then N is a partial summand of ${}_RM$ (it is a direct summand), but N is not summand-small in M. In fact, if $X = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ then $M = N + X$ but X is not a direct summand of M. □

Example 29. Summand-small submodules need not be partial summands.

Proof. Let ${}_{\mathbb{Z}}M = \mathbb{Z}_2 \oplus \mathbb{Z}_8$. Then the direct summands of M are 0, M, $\mathbb{Z}(1, 1)$, $\mathbb{Z}(1, 4)$, $\mathbb{Z}_2 \oplus 0$ and $0 \oplus \mathbb{Z}_8$. Let $N = \mathbb{Z}(1, 2)$. If $M = N + X$ then X must be M, $0 \oplus \mathbb{Z}_8$ or $\mathbb{Z}(1, 1)$, and so $X \subseteq^{\oplus} M$. This shows that N is a summand-small in M. But N is not a partial summand because there does not exist a direct summand $S \subseteq N$ such that $M = S + X$ whenever $M = N + X$. □

We have the following implications among submodule properties $N \subseteq M$ that we have defined:

(1) "δ-small" ⇒ "summand-small"
(2) "δ-small" ⇒ "partial summand"
(3) If M is projective: "summand-small" ⇒ "partial summand" (by Lemma 27).

We can now extend the fact that the sum of a partial summand and a small submodule is again a partial summand.

Proposition 30. *Let M be projective, and let K be summand-small in M. If N is a partial summand of M, then $N + K$ is a partial summand of M.*

[2] In [1] such a submodule N is said to be SDM in M.

Proof. Let $M = (N + K) + X$; we want $S \subseteq^{\oplus} M$ such that $M = S + X$ and $S \subseteq N + K$. We have $M = (N + X) + K$ so, as K is summand-small in M, we have $N + X \subseteq^{\oplus} M$. But then, since M is projective, Lemma 19 provides $Q \subseteq K$ such that $M = (N + X) \oplus Q$.

On the other hand, N is a partial summand of $N + X$ by [1, Lemma 3.2] because N is a partial summand of M. This means that $N + X = S_1 + X$ where $S_1 \subseteq^{\oplus} (N + X)$ and $S_1 \subseteq N$. Since $M = (N + X) \oplus Q$, it follows that $S_1 + Q \subseteq^{\oplus} M$. Now write $S = S_1 + Q$. We have shown that $S \subseteq^{\oplus} M$, and clearly $S = S_1 + Q \subseteq N + K$. Finally,

$$S + X = (S_1 + Q) + X = (S_1 + X) + Q = (N + X) + Q = M.$$

Hence we are done with $S = S_1 + Q$. □

Theorem 31. *The following are equivalent for a left ideal L of R.*

(1) *L is δ-small in R.*
(2) *L is summand-small in R.*
(3) *L is weakly enabling, and Re is summand-small in R whenever $e^2 = e \in L$.*

Proof. (1)⇒(2). This is clear as δ -small submodules are all summand-small.

(2)⇒(3). If L is as in (2), then L is a partial summand by Lemma 27, so L is weakly enabling by Theorem 20. If $e^2 = e \in L$ then Re is summand-small in R by (2).

(3)⇒(1). Let $R = L + X$, where R/X is singular; we must show that $X = R$. Since L is a partial summand of R by (3) and Theorem 20, we have $R = Re + X$ where $e^2 = e \in L$. But then $X \subseteq^{\oplus} R$ as Re is summand-small, so R/X is both projective and singular. Hence $X = R$. □

Let us say that an ideal $I \lhd R$ is **left summand-small** in R if it is summand-small in ${}_RR$, and that I is **left δ-small** if it is δ-small in ${}_RR$.

Corollary 32. *An ideal is left δ-small if and only if it is left summand-small.*

The next result shows that any left δ-small ideal is enabling.

Theorem 33. *The following are equivalent for $I \lhd R$:*

(1) *I is left δ-small in R.*
(2) *I is enabling and Re is summand-small in R whenever $e^2 = e \in I$.*

Proof. (2)⇒(1) is clear by Theorem 31.
(1)⇒(2). If $e^2 = e \in I$ then Re is summand-small by (1).

To show that I is enabling, let $a - e \in I$ where $e^2 = e$; we must find $f^2 = f \in Ra$ such that $a - f \in I$. We have $1 = (e - a) + a + (1 - e)$, so $R = I + K$ where $K = Ra + R(1 - e)$. Since I is a left summand-small, Lemma 27 shows that $R = S \oplus K$ where $S \subseteq I$. In particular K is projective so, since $R(1 - e)$ is a summand of K, applying Lemma 19 gives $K = Q \oplus R(1 - e)$ for some $Q \subseteq Ra$. But then Q is a summand of ${}_RR$ (as K is), so $K = Rg \oplus R(1 - e)$ where $g^2 = g \in Ra$. Hence $Ra + R(1 - e) = K = Rg \oplus R(1 - e)$, and we are done by Lemma 24. □

Since $\delta(R)$ is itself δ-small in ${}_RR$ by [8, Theorem 1.6], Theorems 20, 31 and 33 show that $\delta(R)$ is enabling, giving a new proof of Corollary 14. Also, $\delta(R)$ is an example of a δ-small left ideal that may not be strongly lifting (it may not be lifting; see the discussion following Corollary 14).

Theorem 33 also implies that every left summand-small ideal is enabling, but the converse is false. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is a field. Then $I = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ is enabling (in fact strongly lifting as R is an exchange ring). But if $X = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ then $I + X = R$ but X is not a summand of ${}_RR$. Indeed, apart from 0 and 1, the only idempotents π in X are $\pi = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}$, $t \in F$, and $R\pi = F\pi \neq X$ for each t.

In Theorems 20 and 31, respectively, we have established the following implications for $I \lhd R$.

$$I \text{ is left summand-small } \Rightarrow I \text{ is enabling } \Rightarrow I \text{ is a left partial summand.}$$

The converse to the first implication is false (example above), but we have no counter-example to the converse of the second implication. Note that if I is a left partial summand then it is both left and right weakly enabling by Theorem 21 but, as mentioned above, we do not know if the converse is true. However, we do have the following result (extending Proposition 6) when the ideal is idempotent-free.

Proposition 34. *The following are equivalent for an idempotent-free ideal I of R :*

(1) *$I \subseteq J(R)$.*
(2) *I is enabling.*
(3) *I is a left (right) partial summand of R.*
(4) *I is left (right) summand-small in R.*

Proof. (1)⇒(2). This is by Proposition 6.

(2)⇒(3). This is by Theorem 20.

(3)⇒(4). Let $R = I + {}_RX$; we must show that $X \subseteq^{\oplus} R$. By (3) there exists $e^2 = e \in I$ such that $Re + X = R$. Since $e = 0$ by hypothesis, we have $X = R$ and (4) follows.

(4)⇒(1). If $I \nsubseteq J$ then $I \nsubseteq M$ for some maximal left ideal M. Thus $R = I + M$ so Lemma 27 implies that $R = Re \oplus M$ for some $e^2 = e \in I$. Since I is idempotent-free this implies that $M = R$, a contradiction. □

Note that the implications (1)⇒(2)⇒(3) in Proposition 34 all hold without the idempotent-free hypothesis.

Regarding (4) of Proposition 34, we have the following example.

Example 35. Being summand-small is not left–right symmetric for ideals in general.

Proof. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is a field. and write S_r and S_l for the right and left socles of R. Thus $S_r = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$, so $S_r = \delta_r$ because $\frac{\delta_r}{S_r} = J(\frac{R}{S_r}) = 0$. Similarly $\delta_l = S_l = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$. If we let $I = S_l \lhd R$ then I is left summand-small in R by Theorem 31. But I is not right summand-small because I is not δ-small as a right ideal since $\delta_l \nsubseteq \delta_r$. □

Example 36. For an ideal I, neither strongly lifting nor left summand-small implies the other.

Proof. The ideal $\delta(R)$ is left summand-small because it is δ-small, but $\delta(R)$ may not be lifting (see the discussion following Corollary 14).

Conversely, let $I \lhd R$ be such that $R = I \oplus K$ for some left ideal K. Then I is strongly lifting by [6, Example 3]. But if I is not semisimple then it is not δ-small by [8, Lemma 1.2] . For example, if $R = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ and $I = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$, then $I \lhd R$, $R = I \oplus K$ for some left ideal K, but $I \nsubseteq S_l = \begin{bmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$. For a commutative example, let $R = F^{\Lambda}$ where F is a field and Λ is infinite, and choose $e^2 = e \in R - F^{(\Lambda)}$. If $I = Re$ then I is a direct summand ideal so, if I were δ-small, it would be semisimple. But $I \nsubseteq F^{(\Lambda)} = soc(R)$. □

The next result continues Proposition 11 by presenting more results about factor rings.

Proposition 37. *Let $I \lhd R$. If $K \supseteq I$ is a left ideal of R such that $K/I \subseteq J(R/I)$, then*

(1) *If I is left summand-small in R then K is summand-small in ${}_RR$;*
(2) *If I is a partial summand of ${}_RR$ then K is a partial summand of ${}_RR$;*
(3) *If every left ideal in I is summand-small in ${}_RR$, then every left ideal in K is a partial summand of ${}_RR$.*

Proof. Let I, K and R be as above, and write $\bar{R} = R/I$ and $\bar{r} = r + I$ for $r \in R$.

(1) Let $R = K + {}_RX$; we must show that $X \subseteq^{\oplus} {}_RR$. Write $1 = k + x$, $k \in K$, $x \in X$. Since $\bar{1} - \bar{x} = \bar{k} \in J(\bar{R})$, $\bar{x}$ is a unit in $\bar{R}$. It follows that $R = I + Rx$, whence $R = I + X$. By hypothesis, $X \subseteq^{\oplus} {}_RR$, as required.
(2) Let $R = K +_R X$; we want $R = Re + X$ for some $e^2 = e \in K$. If $1 = k + x$, $k \in K$, $x \in X$, then (as above) $\bar{x}$ is a unit in $\bar{R}$, so $R = Rx + I$. Write $1 = rx + a$, $r \in R$, $a \in I$, so that $rx - 1 \in I$. As I is weakly enabling there exists $f^2 = f \in Rrx$ such that $rx - f \in I$. Since $1 - f = (1 - rx) + (rx - f) \in I$, we have $R = I + Rf$. But $I \subseteq K$ so we have $R = K + Rf$. Finally, since R is projective and Rf is a summand, Lemma 19 shows that there exists $e^2 = e \in K$ such that $R = Re \oplus Rf$. Since $f \in X$ we have $R = Re + X$, as required.
(3) Let $R = M + {}_RX$ where $M \subseteq K$ is a left ideal; we must show that $R = Rh + X$ for some $h^2 = h \in M$. Write $1 = m+x$, $m \in M$, $x \in X$. Then $1-x \in M \subseteq K$ so, as in (1), $\bar{x}$ is a unit in $\bar{R}$. Hence $R = Rx+I$, say $1 = rx+a$, $r \in R$, $a \in I$. Then $rx - 1 \in I$. So, as I is weakly enabling (it is left summand-small by hypothesis), there exists $e^2 = e \in Rrx \subseteq X$ such that $rx - e \in I$. It follows that $e - 1 \in I$, whence $Re + I = R$. Since I is left summand-small, let $Re + Rf = R$ where $f^2 = f \in I$. As $e \in X$, the modular law asserts that $X = Re + (X \cap Rf)$ where $X \cap Rf \subseteq I$.

   Hence $R = M + Re + (X \cap Rf)$. But $X \cap Rf$ is summand-small in ${}_RR$ by hypothesis, so $R = (M + Re) \oplus Rg$ where $g^2 = g \in X \cap Rf$. In particular, $M + Re$ is projective so, as Re is a direct summand of $M + Re$, Lemma 19 shows that $M + Re = L \oplus Re$ where $L \subseteq M$. Hence $R = (L \oplus Re) \oplus Rg = L + X$ because $e \in X$ and $g \in X$. If $L = Rh$, $h^2 = h$, we are done because $L \subseteq M$. □

We summarize the implications for an ideal among the properties we have been discussing:

$$\begin{array}{ccccc} & & \text{Strong Lifting} & & \\ & & \Downarrow & & \\ \delta\text{-small} \Leftrightarrow \text{Summand-small} & \Rightarrow & \text{Enabling} & \Rightarrow & \text{Weakly Enabling} \Leftrightarrow \text{Partial Summand} \end{array}$$

4. I-semiregular modules

If $I \lhd R$ the I-semiregular modules are an important generalization of the semiregular modules. Several theorems about I-semiregular modules in general are strengthened if the ideal I is assumed to be enabling. Two such theorems (which motivated the study of enabling ideals) are discussed in this section.

Following Zelmanowitz [7], an element $q \in {}_RM$ is called *regular* in the left module ${}_RM$ if $q = (q\lambda)q$ for some $\lambda \in M^* = hom(M, R)$. If $M = {}_RR$ this recovers the definition of a regular element: $q = qbq$ for some $b \in R$. Zelmanowitz proves the following lemma.

Lemma 38. *Let $q \in {}_RM$ be a regular element, say $q = (q\lambda)q$ for some $\lambda \in M^*$. If $e = q\lambda$ then $e^2 = e$, $Rq \cong Re$, and $M = Rq \oplus K$ where $K = \{k \in M \mid (k\lambda)q = 0\}$.*

A version of the next lemma appears in [2, Proposition 2.2].

Lemma 39. *Let R be a ring, let $I \lhd R$, and let ${}_RM$ be a left R-module. The following are equivalent for $m \in M$.*

(1) *There exists a decomposition $M = P \oplus Q$ where P is projective, $P \subseteq Rm$ and $Rm \cap Q \subseteq IM$.*
(2) *There exists $\lambda \in M^*$ such that $m\lambda = e = e^2$ and $(1 - e)m \in IM$.*
(3) *There exists $\gamma^2 = \gamma \in end({}_RM)$ such that $M\gamma \subseteq Rm$, $M\gamma$ is projective, and $m - m\gamma \in IM$.*

An element m of a module ${}_RM$ is called *I-semiregular* if it satisfies the conditions in Lemma 39, and M is called an *I-semiregular module* if every element is I -semiregular.

If $I \lhd R$ and m is an I-semiregular element of a left module M, it is known that $m - q \in IM$ for some regular element $q \in M$. If I is enabling, the converse is true. More precisely, we have the following theorem.

Theorem 40. *Let $I \lhd R$ be an enabling ideal, and let ${}_RM$ be a module. The following are equivalent for $m \in M$.*

(1) *m is I-semiregular.*
(2) *There exists a regular element $q \in Rm$ such that $m - q \in IM$.*
(3) *There exists a regular element $q \in M$ such that $m - q \in IM$.*

Proof. (1)⇒(2). Choose $e = m\lambda$ as in part (2) of Lemma 39, and write $q = em$. Then $q\lambda = e(m\lambda) = e^2 = e$, so $(q\lambda)q = eq = q$. Hence q is regular, and $m - q = (1 - e)m \in IM$ by (2) of Lemma 39.

(2)⇒(3). This is obvious.

(3)⇒(1). Assume that I is enabling and that $m-q \in IM$ where q is regular, say $(q\lambda)q = q$ where $\lambda : M \to R$. Write $e = q\lambda$ so that, by Lemma 38, $e^2 = e \in R$, $Rq \cong Re$, and $M = Rq \oplus K$ where $K = \{k \in M \mid (k\lambda)q = 0\}$. Then $m\lambda - e = (m - q)\lambda \in I$ because $m - q \in IM$. Since I is enabling, there exists $f^2 = f \in R(m\lambda)$ such that $f - m\lambda \in I$. Write $f = a(m\lambda)$, $a \in R$, where we may assume that $fa = a$. Define $p = am$. Then $p\lambda = a(m\lambda) = f$, so $(p\lambda)p = fp = p$; that is p is regular.

For convenience, if $x, y \in M$ we write $x \equiv y$ to mean that $x - y \in IM$. Thus $m \equiv q$ by hypothesis, whence $m\lambda - q\lambda \in I$. Moreover, $am\lambda - m\lambda = f - m\lambda \in I$, so

$$aq = a(q\lambda)q \equiv a(m\lambda)q \equiv (m\lambda)q \equiv (q\lambda)q = q.$$

It follows that $p = am \equiv aq \equiv q$ so, since $q \equiv m$, we have

$$p \equiv m \quad \text{and} \quad Rm + IM = Rp + IM.$$

Since p is regular, let $M = Rp \oplus W$ where $W = \{w \mid (w\lambda)p = 0\}$. Now let $\gamma^2 = \gamma \in end(M)$ be such that $M\gamma = Rp$ and $M(1 - \gamma) = W$. Since $Rp \subseteq Rm$, the modular law gives

$$Rm = Rp \oplus (Rm \cap W).$$

Since Rp is projective, it remains to show that $Rm \cap W \subseteq IM$. But:

$$Rm \cap W \subseteq (Rm + IM) \cap W = (Rp + IM) \cap W$$

so it suffices to show that $(Rp + IM) \cap W \subseteq IM$. But if $x = rp + n \in W$, $r \in R$, $n \in IM$, then $0 = x\gamma = rp + n\gamma$, whence $rp = -n\gamma \in IM$. This means that $x = rp + n \in IM$, proving (1). □

Corollary 41. *Let $I \lhd R$ be enabling, and let $m, m_1 \in {}_RM$. If m is I-semiregular and $m - m_1 \in IM$ then m_1 is also I -semiregular.*

With Theorem 40 we can give a structure theorem for countably generated I-semiregular modules M where IM is δ-small in M. The result is analogous to [4, Theorem 1.12].

Theorem 42. *Let ${}_RM$ be a countably generated, I -semiregular module, and let $I \lhd R$ be an enabling ideal. If IM is δ-small in M, then $M \cong \oplus_{i=1}^{\infty} Re_i$ where $e_i^2 = e_i \in R$. In particular M is projective.*

Proof. Let $x_1, x_2, \ldots$ be a generating set for M. It suffices to construct regular elements q_i of M, submodules $W_i \subseteq M$, and submodules $Z_i \subseteq IM$ such that, for each $n \geq 1$,

$$(1_n) \quad M = Rq_1 \oplus \cdots \oplus Rq_n \oplus W_n$$

$$(2_n) \quad Rx_1 + \cdots + Rx_n \subseteq Rq_1 \oplus \cdots \oplus Rq_n + Z_n.$$

Indeed, in that case $\Sigma_i Rx_i$ is a direct sum, and $M \subseteq \Sigma_i Rx_i \subseteq \oplus_i Rq_i + \Sigma_i Z_i \subseteq \oplus_i Rq_i + IM$. Hence $M = \oplus_i Rq_i + IM$. As IM is δ-small, [8, Lemma 1.2] shows that $M = \oplus_i Rq_i \oplus S$ where $S \subseteq IM$ is a projective, semisimple module. With this we are done by Lemma 38.

Since I is enabling, Theorem 40 gives $x_1 = q_1 + z_1$ where q_1 is regular in M and $z_1 \in IM$. Hence, by Lemma 38, $M = Rq_1 \oplus W_1$ where $W_1 = \{w \in M \mid (w\lambda)q_1 = 0\}$. Hence (1_1) and (2_1) hold with $Z_1 = Rz_1$.

Now assume inductively that q_i, W_i and Z_i have been constructed for each $k = 1, 2, \ldots, n$ so that (1_n) and (2_n) hold. Write $P_n = Rq_1 \oplus \cdots \oplus Rq_n$ so that $M = P_n \oplus W_n$. Let $\pi : M \to M$ be the projection with $M\pi = P_n$ and $ker(\pi) = W_n$. It is routine to verify that

$$P_n + Rx_{n+1} = P_n \oplus Rx_{n+1}(1 - \pi).$$

Define $t_{n+1} = x_{n+1}(1 - \pi)$. Then $t_{n+1} \in W_n$ and W_n is I-semiregular by [1, Theorem 2.6], so $t_{n+1} = q_{n+1} + z_{n+1}$ where $q_{n+1} \in Rt_{n+1}$, q_{n+1} is regular in W_n (and so in M), and $z_{n+1} \in IW_n \subseteq IM$. By Lemma 38 we have $M = Rq_{n+1} \oplus K$ for some K, so $W_n = Rq_{n+1} \oplus (Rq_{n+1} \cap K)$, and (1_{n+1}) follows if we take $W_{n+1} = Rq_{n+1} \cap K$. Next $x_{n+1} = x_{n+1}\pi + t_{n+1} = x_{n+1}\pi + q_{n+1} + z_{n+1}$, so $Rx_{n+1} \subseteq P_n + Rq_{n+1} + Rz_{n+1}$. Using (2_n) it follows that

$$Rx_1 + \cdots + Rx_{n+1} \subseteq Rq_1 \oplus \cdots \oplus Rq_{n+1} + Z_n + Rz_{n+1},$$

and we obtain (2_{n+1}) with $Z_{n+1} = Z_n + Rz_{n+1}$. This completes the proof of the theorem. □

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