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Some issues on HPM and HAM methods: A convergence scheme

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ABSTRACT

The homotopy method for the solution of nonlinear equations is revisited in the present study. An analytic method is proposed for determining the valid region of convergence of control parameter of the homotopy series, as an alternative to the classical way of adjusting the region through graphical analysis. Illustrative examples are presented to exhibit a vivid comparison between the homotopy perturbation method (HPM) and the homotopy analysis method (HAM). For special choices of the initial guesses it is shown that the convergence-control parameter does not cover the HPM. In such cases, blindly using the HPM yields a non convergence series to the sought solution. In addition to this, HPM is shown not always to generate a continuous family of solutions in terms of the homotopy parameter. By the convergence-control parameter this can however be prevented to occur in the HAM.

1. Introduction

The non-stopping continuing search for a better and easy to use tool for the solution of nonlinear equations illuminating the nonlinear phenomena of our life has been the focus of recent studies in the literature.

Various methods, therefore were devised, such as perturbation techniques [1], variational approaches [2], decomposition [3], Exp-function [4] and harmonic balance based methods [5]. One of the most recent popular technique is the homotopy method, which is a combination of the classical perturbation technique and homotopy concept as used in topology. Liao in [6] proposed for the first time this technique, named the homotopy analysis method (HAM). Afterwards, He in [7] gave the homotopy perturbation method (HPM), which is well-understood today as the special interpretation of the HAM.

Unlike the aforementioned traditional perturbation methods, the homotopy technique does not require a small perturbation parameter in the equation. In this method, a homotopy with an imbedding parameter $p \in [0, 1]$ is constructed, and the imbedding parameter is considered as a small parameter. Thus the original nonlinear problem is converted into an infinite number of linear problems without using the perturbation techniques; see the book by Liao [8]. Different from other methods, the HAM provides a simple way to control and adjust the convergence region of solution series by means of an auxiliary parameter [9,10]. Unfortunately, Sajid and Hayat [11] pointed out that the so-called homotopy perturbation method has nothing new except its new name, because the HPM is only a special case of the homotopy analysis method (HAM) so that all results given by the HPM can also be obtained by the HAM as a special case. Moreover, as a special case of the HAM, the HPM cannot give convergent series solution for some strongly nonlinear problems. Abbasbandy in [12] gave a simple example to show that, like perturbation approximations, the results given by the HPM are divergent when the physical parameters in the governing nonlinear equations become large. Among many other authors, VanGorder and Vajravelu in [13] also pointed out the fact of disadvantages of the HPM. More recently, Liang and Jeffrey in [14] pointed out

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for a linear partial differential equation that in some cases the series solution given by the HPM and VIM (another method proposed also by He in [2]) is divergent at all points except $t = 0$ that however defines the initial condition.

In the present paper, we revisit the homotopy method. The main aim is to analyze the HAM and HPM from a mathematical point of view by means of some basic nonlinear problems of algebraic or differential kind, and hence show a presentable comparison via obtaining the exact solution of the homotopy equations. In the case of HPM, it is demonstrated that the constructed homotopy may not yield a continuous family of solutions in terms of the imbedding parameter, that consequently results in divergence of the HPM. On the other hand, by means of the convergence-control parameter this can always be avoided in the HAM. Moreover, a scheme is proposed to explicitly obtain the interval of convergence-control parameter, as an alternative to the frequently used constant h-curves approach suggested by Liao. Examples prove that the proposed scheme and the constant h-curves approaches generate the equivalent regions. Determination of the interval of convergence-control parameter via this scheme explains why or why not HPM can be utilized in some problems.

In the rest of the paper, Section 2 lays the basis of the homotopy method for nonlinear algebraic and differential equations. Having introduced a convergence scheme for the homotopy series in Section 3, examples are shown to compare the HPM and HAM methods in Section 4. Conclusions eventually follow in Section 5.

2. The homotopy method

The homotopy method as applied to the nonlinear equations was first proposed by the Chinese mathematician Liao [6]. The essential idea of this method is to introduce a homotopy parameter, say p, which varies continuously from 0 to 1. At $p = 0$, the system of equations usually is reduced to a simplified form which normally admits a rather simple solution. As p gradually increases continuously toward 1, the system goes through a sequence of deformations, and the solution at each stage is close to that at the previous stage of the deformation. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of the deformation gives the desired solution. To outline the method initially on nonlinear algebraic equations, let us consider the root-finding problem

$$f(x) = 0, \tag{2.1}$$

where $f(x)$ is an infinitely many continuously differential function over a subset of real numbers. According to the homotopy technique, a homotopy $H(x, p)$ with $p \in [0, 1]$ can be constructed via

$$H(x, p) = (1 - p)[F(x)] + phf(x), \tag{2.2}$$

where F is a suitable auxiliary equation whose roots (possibly near those of (2.1)) are easy to solve and h is a parameter to control the convergence of the subsequent homotopy series, named as the convergence-control parameter. It is obvious from Eq. (2.2) that

$$H(x, 0) = F(x), \qquad H(x, 1) = f(x), \tag{2.3}$$

hence, as p moves from 0 to 1, $x(p)$ moves from x_0 of $F(x) = 0$ to x of $f(x) = 0$. Upon expansion of the solution $x(p)$ into power series at $p = 0$, the following series is obtained

$$x(p) = x_0 + \sum_{k=1}^{\infty} x_k p^k. \tag{2.4}$$

At $p = 1$, a general assumption is to accept in prior the convergence of the series (2.4), if that is the case, then an analytic series solution is found in the form

$$x = \sum_{k=0}^{\infty} x_k, \tag{2.5}$$

where $x_k = \left.\frac{\partial^k x}{\partial p^k}\right|_{p=0}$. As a result of which, an approximate solution to the solution of (2.1) can always be written at the Mth-order of approximation

$$x = \sum_{k=0}^{M} x_k. \tag{2.6}$$

Next, to illuminate the application of the homotopy method on the nonlinear differential equations (ordinary or partial), consider the general problem

$$N(u) = 0, \qquad B\left(u, \frac{\mathrm{d}u}{\mathrm{d}n}\right) = 0, \tag{2.7}$$

where u is the function to be solved under the boundary constraints given in B. The homotopy on $u(r, p) : R \times [0, 1] \to R$ can be constructed via

$$H(u, p) = (1 - p)[L(u) - L(u_0)] + phN(u), \tag{2.8}$$

where L is a suitable auxiliary linear operator to approximate the solution and u_0 is an initial approximation of Eq. (2.7) satisfying exactly the boundary conditions. It is again clear from Eq. (2.8) that

$$H(u, 0) = L(u) - L(u_0), \qquad H(u, 1) = N(u). \tag{2.9}$$

Thus, as p moves from 0 to 1, $u(r, p)$ deforms from $u_0(r)$ to $u(r)$. Besides, the solution $u(r, p)$ can be expressed as a power series in p at $p = 0$ in the form

$$u(r, p) = u_0(r) + \sum_{k=1}^{\infty} u_k(r)p^k. \tag{2.10}$$

At $p = 1$, on the condition that the convergence of series (2.10) is guaranteed, an analytic series solution is determined in the form

$$u(r) = \sum_{k=0}^{\infty} u_k(r), \tag{2.11}$$

where $u_k(r) = \left.\frac{\partial^k u(r,p)}{\partial p^k}\right|_{p=0}$. Hence, the approximate solution to (2.7) can always be written at the Mth-order of approximation by truncating the series (2.11)

$$u(r) = \sum_{k=0}^{M} u_k(r). \tag{2.12}$$

It should be remarked that when h is set to unity in (2.2) and (2.8), solutions will be independent of h and the resulting homotopy is termed the homotopy perturbation method (HPM), whereas as a general case of h, it is named the homotopy analysis method (HAM).

3. A convergence scheme

It has been already proven that if the resultant series in (2.5) and (2.11) are convergent, then they represent converged solutions to the exact solutions of (2.1) and (2.7), respectively; see for instance [8]. On the other hand, the convergence of the homotopy series is not always guaranteed. Liao [8] suggested to examine the behavior of a certain quantity of the exact solution as a function of the parameter h, once the homotopy series approximating the exact solution is computed. This approach is termed later as the constant h-curves approach. The constant h-curve is quite rational since the homotopies given in (2.2) and (2.8) indicate firmly that whenever convergence takes place at $p = 1$, the quantities of exact solution should be independent of the parameter h. Therefore, this graphical way of adjusting the convergence interval of h has been often employed amongst the researchers in the literature.

In place of the above mentioned methodology of identifying the convergence-control parameter, we here propose that the traditional ratio test valid for power series might well be adopted to determine the region of convergence-control parameter h. To this end, for algebraic equations of the form (2.1), the convergence of the truncated homotopy series (2.6) at the Mth-order can be enforced to hold true, that is,

$$\left| \frac{x_M(h)}{x_{M-1}(h)} \right| < 1, \tag{3.13}$$

which, naturally yields the exact interval for the convergence-control parameter h in the limiting case $M \to \infty$.

When nonlinear differential equations of type (2.7) are taken into account, the question of convergency of the homotopy series (2.11) gets harder. Because in this case the coefficients in the homotopy series (2.11) are not only function of h but also of the space and time coordinates r and t. The convergence then should be looked for every point r and t in the domains of definition. If it is possible, one can show the uniform convergence of the series (2.11) applying the well-known Abel's uniform convergence test or Weierstrass M-test. On the other hand, in the community working on the homotopy problems the graphical way of adjusting the convergence region of h is frequently made use on a prescribed value of exact solution, such as $u'(x = a)$ (in the ordinary differential case, unless it is constant for all h) or $u_{xt}(x = \alpha, t = \beta)$ (in the partial differential case, unless it is constant for all h) or similar. Therefore, instead of graphical way of determining the convergence region h for the specified quantity, the same ratio test as in (3.13) can be applied here on that quantity. This will not only generate the interval of convergency, but also it will let one to beforehand know whether the HPM can be used in the stage of homotopy.

Table 1
Illustrating the right end γ of the interval of convergence parameter h for Example 1 with the initial guesses $x_0 = 1$ and $x_0 = 2$, respectively.

	$M = 2$	$M = 10$	$M = 50$	$M = 100$	$M = 250$	$M = 500$
$x_0 = 1$	1.3333333	0.7247327	0.6769496	0.6717364	0.6686777	0.6676694
$x_0 = 2$	2.1333333	2.0485649	2.0099924	2.0049991	2.0019999	2.0009999

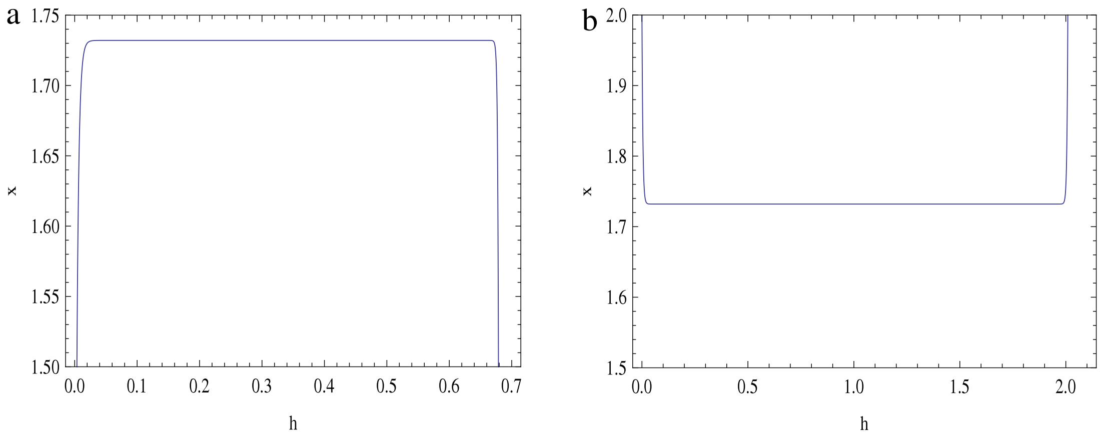

Fig. 1. The constant h-curves of $u'(0)$ for Example 1, with the corresponding initial approximations (a) $x_0 = 1$ and $x_0 = 2$.

4. Illustrative examples

To illustrate the use of (3.13) in the homotopy method, we take into consideration the following examples.

Example 1. Let us consider the root-finding problem (2.1), for which we assign $f(x) = x^2 - 3$. Taking the auxiliary function $F(x) = f(x) - f(x_0)$ in (2.2), we select two distinct initial approximations to the positive root, respectively $x_0 = 1$ and $x_0 = 2$. Table 1 tabulates the interval of convergence $0 < h < \gamma$, where γ is evaluated by means of (3.13) at the order M of homotopy series (2.6). The presented data in Table 1 clearly indicates that the interval for the convergence-control parameter h tends to $\left(0, \frac{2}{3}\right)$ for $x_0 = 1$ and $(0, 2)$ for $x_0 = 2$, provided that sufficiently large number of homotopy terms are evaluated in the series (2.6). The constant h-curves taken at $M = 250$ are also plotted in Fig. 1((a)–(b)). It should be noticed that with the initial guess $x_0 = 1$, the HPM method corresponding to $h = 1$ in the homotopy (2.2) would fail to converge, unlike the case $x_0 = 2$. Therefore, this example points to the fact that if the HPM method is desired to utilize for the solution, then the initial approximation cannot be freely chosen, whereas it is not the case for the HAM method.

Example 2. Let us now consider the ordinary differential equation

$$u' + u^2 = 1, \qquad u(0) = 0, \tag{4.14}$$

that governs the steady free convection flow over a vertical semi-infinite flat plate which is imbedded in a fluid saturated porous medium of ambient temperature [15] and also the steady-state boundary layer flows over a permeable stretching sheet [16]. Eq. (4.14) admits an exact solution expressed by

$$u(x) = -\frac{1 - e^{2x}}{1 + e^{2x}}. \tag{4.15}$$

In subsequent, we choose different initial conditions and linear operators to approximate this solution via the homotopy in Eq. (2.8).

Case 1. $u_0(x) = 0$ and $L = \frac{d}{dx}$. In this case the homotopy (2.8) becomes

$$\frac{\partial u(x,p)}{\partial x} + p\,u(x,p)^2 - p = 0, \qquad u(0,p) = 0. \tag{4.16}$$

Eq. (4.16) results in the exact homotopy solution

$$u(x,p) = -\frac{1 - e^{2px}}{1 + e^{2px}}. \tag{4.17}$$

As mentioned in Section 2, this gives the initial solution for $p = 0$ and the exact solution (4.15) for $p = 1$. Having formed the homotopy series in (2.11) with the help of (4.16), it is found from the ratio test (3.13)

$$\lim_{M\to\infty}\left|\frac{u'_{M+1}(0)}{u'_M(0)}\right| < 1$$

that the region of convergence of h for the considered problem is obtained exactly as $0 < h < 2$.

Case 2. $u_0(x) = 1 - \mathrm{e}^{-2x}$ and $L = \frac{\mathrm{d}}{\mathrm{d}x} + 2$. In this case the homotopy (2.8) becomes

$$\frac{\partial u(x,p)}{\partial x} + p\,u(x,p)^2 + 2(1-p)u(x,p) + p - 2 = 0, \qquad u(0,p) = 0. \tag{4.18}$$

Solving (4.18) exactly generates the homotopy solution

$$u(x,p) = \frac{\left(-1+\mathrm{e}^{2x}\right)(-2+p)}{\mathrm{e}^{2x}(-2+p) - p}. \tag{4.19}$$

As expected, this gives the initial solution for $p = 0$ and the exact solution (4.15) for $p = 1$. In this case, the interval of convergence of h in the homotopy series (2.11) is also evaluated from (3.13) as $0 < h < 2$.

Example 3. Let us now consider the following heat transfer problem governed by the nonlinear ordinary differential equation [12]

$$(1+\epsilon u)u' + u = 0, \qquad u(0) = 1, \tag{4.20}$$

where $\epsilon \geq 0$ is a physical parameter [14].

When the auxiliary parameters are taken as $u_0(x) = 1$ as well as $L = \frac{\mathrm{d}}{\mathrm{d}x}$, the homotopy (2.8) becomes

$$(1 - p + h\,p(1+\epsilon u(x,p)))\frac{\partial u(x,p)}{\partial x} + h\,p\,u(x,p) = 0, \qquad u(0,p) = 1. \tag{4.21}$$

Solving (4.20) exactly generates the homotopy solution

$$u(x,p) = \frac{1-h+hp}{hp\epsilon}\mathrm{ProductLog}\left(hp\epsilon\,\mathrm{e}^{\frac{hp(\epsilon-x)}{1-p+hp}}\right), \tag{4.22}$$

where the function $\mathrm{ProductLog}(z)$, which can be viewed as a generalization of a logarithm, gives the solution for w in $z = w\mathrm{e}^{w}$. As expected, this gives the initial solution for the limiting case of $p = 0$ and the exact solution of (4.20) for $p = 1$. In this case the interval of convergence of h for $u'(0)$ in the homotopy series (2.11) is evaluated from (3.13) as $|1 - h(1+\epsilon)| < 1$, which produces for the physical values of $\epsilon \geq 0$,

$$0 < h < \frac{2}{1+\epsilon}.$$

For the HPM case ($h = 1$), this points to the convergence only for $\epsilon < 1$, which supports the argument of [12]. Consequently, the HPM case is restricted to a small interval, as compared to the HAM method.

Example 4. Consider the second-order nonlinear ordinary differential equation

$$2u'' + u - u^2 = 0, \qquad u(0) = 0, \qquad u(\infty) = 1 \tag{4.23}$$

that governs the steady mixed convection flow past a plane of arbitrary shape under the boundary layer and Darcy–Boussinesq approximations [17]. Eq. (4.23) admits an exact solution given by

$$u(x) = \frac{1}{2}\left(-1 + 3\tanh\left[\frac{1}{4}\left(\sqrt{2}x + 4\,\mathrm{arctan\,h}\left[\frac{1}{\sqrt{3}}\right]\right)\right]^2\right). \tag{4.24}$$

In order to approximate this exact solution, we choose the auxiliary parameters as follows

$$u_0(x) = 1 - \mathrm{e}^{-x}, \qquad L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 1.$$

Then, the homotopy (2.8) turns out to be

$$(1+p)\frac{\partial^2 u(x,p)}{\partial x^2} - pu(x,p)^2 - (1-2p)u(x,p) - p + 1 = 0, \qquad u(0,p) = 0, \qquad u(\infty,p) = 1. \tag{4.25}$$

Table 2
Illustrating the right end γ of the interval of convergence parameter h for Example 4.

$M = 2$	$M = 6$	$M = 10$	$M = 15$	$M = 20$	$M = 40$
1.2631579	1.0501313	1.0272286	1.0174646	1.0101213	1.0023145

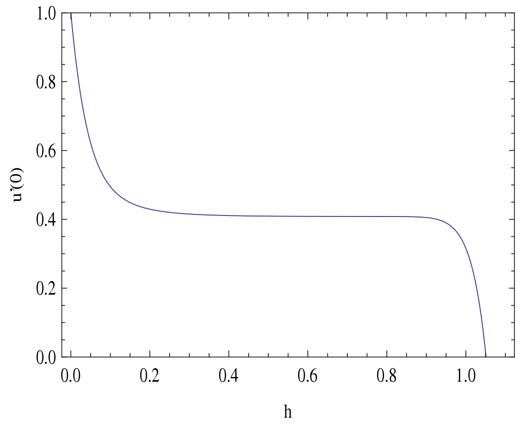

Fig. 2. The constant h-curve for $u'(0)$ for Example 4.

Once solved, Eq. (4.25) produces the exact homotopy solution

$$u(x,p) = \frac{-3 + 2p + 3\tanh\left[\frac{1}{2}\left(\frac{x}{\sqrt{1+p}} + 2\,\mathrm{arctan\,h}\left[\frac{\sqrt{3-2p}}{\sqrt{3}}\right]\right)\right]^2}{2p}. \tag{4.26}$$

As aforementioned, this gives the initial solution in the limiting case $p = 0$ and the exact solution (4.24) for $p = 1$. Using (3.13) together with the homotopy series (2.11), the convergence-control parameter h is found to be restricted in the interval $0 < h < \gamma$ for $u'(0)$, the values of γ are tabulated in Table 2. The h-curve is also depicted in Fig. 2 taken at the $M = 15$th-order homotopy approximation. The prediction of convergence interval from (3.13) is again pretty fine for the physical problem.

Example 5. Consider now the nonlinear partial differential Burger's equation

$$u_t + uu_x = u_{xx}, \qquad u(x,0) = 2x, \quad (x,t) \in R \times [0, 1/2), \tag{4.27}$$

that has been found to describe various kinds of phenomena, such as a mathematical model of turbulence and the approximate theory of the flow through a shock wave traveling in a viscous fluid [18]. Eq. (4.27) admits an exact solution given by

$$u(x,t) = \frac{2x}{1+2t}. \tag{4.28}$$

To approximate the exact solution (4.28), we choose the auxiliary parameters $u_0(x,t) = 2x$ and $L = \frac{\partial}{\partial t}$. Then, the homotopy (2.8) turns out to be

$$(1 - p + hp)u_t(x,t,p) + hp(u(x,t,p)u_x(x,p,t) - u_{xx}(x,t,p)) = 0, \qquad u(x,0,p) = 2x. \tag{4.29}$$

Eq. (4.29) produces the exact homotopy solution

$$u(x,t,p) = \frac{2x}{1 + 2\frac{hp}{1-p+hp}t}. \tag{4.30}$$

This gives rise to the initial solution in the case $p = 0$ and the exact solution (4.28) for $p = 1$. Using (3.13) together with the homotopy series (2.11), it appears that

$$\lim_{M\to\infty} \frac{(u_{xt})_{M+1}(0,0)}{(u_{xt})_M(0,0)} = 1 - h,$$

thus resulting in interval of the convergence-control parameter h as $0 < h < 2$.

Example 6. Consider now the well-known KdV-Burger's equation involving both dispersion and dissipation terms

$$u_t + 2(u^3)_x - u_{xxx} + u_{xx} = 0, \qquad u(x, 0) = \frac{1}{6}\left(1 + \tanh\left[\frac{1}{6}\left(x - \frac{2}{9}t\right)\right]\right), \tag{4.31}$$

whose exact traveling-wave solution is given by

$$u(x, t) = \frac{1}{6}\left(1 + \tanh\left[\frac{1}{6}\left(x - \frac{2}{9}\frac{ph}{1 - p + ph}t\right)\right]\right). \tag{4.32}$$

To approximate the exact solution (4.32), we choose the auxiliary parameters $u_0(x, t) = \frac{1}{6}\left(1 + \tanh\left[\frac{x}{6}\right]\right)$ and $L = \frac{\partial}{\partial t}$. Then, the homotopy (2.8) turns out to be

$$(1 - p + hp)u_t(x, t, p) + hp(2(u(x, t, p)^3)_x - u_{xxx}(x, p, t) + u_{xx}(x, t, p)) = 0, \tag{4.33}$$
$$u(x, 0, p) = \frac{1}{6}\left(1 + \tanh\left[\frac{x}{6}\right]\right).$$

After solving Eq. (4.33), the following exact homotopy solution is obtained

$$u(x, t, p) = \frac{1}{6}\left(1 + \tanh\left[\frac{1}{6}\left(x - \frac{2}{9}\frac{ph}{1 - p + ph}t\right)\right]\right). \tag{4.34}$$

This gives rise to the initial solution in the case $p = 0$ and the exact solution (4.32) for $p = 1$. Using (3.13) together with the homotopy series (2.11), it is found that the convergence-control parameter h satisfies $0 < h < 2$.

Example 7. As a final example, the restriction on the HPM method can be further demonstrated. Consider, the linear partial differential equation

$$u_t + u_x - 2u_{xxt} = 0, \qquad u(x, 0) = e^{-x}, \tag{4.35}$$

whose exact separable solution is given by

$$u(x, t) = e^{-x-t}. \tag{4.36}$$

To approximate the exact solution (4.35), if we choose the auxiliary parameters $u_0(x, t) = e^{-x}$ and $L = \frac{\partial}{\partial t}$, then the homotopy (2.8) turns out to be

$$(1 - p + hp)u_t(x, t, p) + hp(u_x(x, p, t) - u_{xxt}(x, t, p)) = 0, \qquad u(x, 0, p) = e^{-x}. \tag{4.37}$$

The exact homotopy solution of (4.37) can be found as

$$u(x, t, p) = e^{-x + \frac{ph}{1-p-ph}t}. \tag{4.38}$$

This leads to the initial solution in the case $p = 0$ and the exact solution (4.32) for $p = 1$. However, when $h = 1$, the homotopy family (4.38) becomes

$$u(x, t, p) = e^{-x + \frac{p}{1-2p}t},$$

which breaks down at $p = 1/2$. This, consequently contradicts with the necessity that the homotopy solutions should be continuous over $p \in [0, 1]$. Although when p is set to unity in (4.37), there seems no problem in the homotopy, but this does not prevent the homotopy solution having a singularity. Therefore, the initial solution (obtained for $p = 0$) cannot be extended up to the exact solution (obtained for $p = 1$) due to the discontinuity inherent in the homotopy solution. As a result, if the solution is tried in the HPM case, the obtained series will obviously diverge except may be at time $t = 0$, which only corresponds to the initial solution; see [14].

To remedy the problem occurred in the HPM case, addition of the convergence-control parameter h is obligatory in (4.37). It can be seen from the form (4.38) that a particular h can be selected allowing the continuous solutions over $p \in [0, 1]$. Using (3.13) together with the homotopy series (2.11), the interval of convergence-control parameter h for $u(0, 1)$ is computed as presented in Table 3, for $\gamma < h < 0$. The h-curve is also depicted in Fig. 3 taken at the $M = 20$th-order homotopy approximation. Table 3 and Fig. 3 are open evidences to point to the failure of HPM for this specific example.

The exact form of the homotopy families in Examples 1 through 7 clearly show further why the homotopy method would work neatly and converge to the exact solution of the relevant equations, provided that a particular value of h is assigned in the HAM case.

Table 3
Illustrating the left end γ of the interval of convergence parameter h for Example 7.

$M = 2$	$M = 6$	$M = 10$	$M = 15$	$M = 20$	$M = 40$
-1.3333333	-1.5660116	-1.6457586	-1.7000346	-1.7341822	-1.7501305

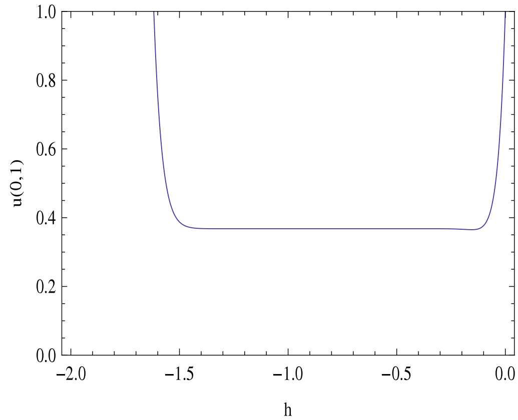

Fig. 3. The constant h-curve of $u(0, 1)$ for Example 7.

5. Concluding remarks

In this paper, the homotopy method is revisited with a prime aim to demonstrate why it should work well to obtain approximate analytical solutions for nonlinear equations governing nonlinear phenomena. To serve to this purpose, a convergence criterion for the homotopy series has been proposed, with verifications of it via certain selected algebraic and differential equations. It is shown that the constant h-curves approach of Liao is equivalent to the approach adopted. The analytic solutions presented reveal clearly the distinction between the HPM and HAM methods. The limitations and failure in some cases of the HPM have been illuminated and to obtain converged approximate solutions, HAM has been shown to be a better tool.

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