J. Differential Equations 254 (2013) 1193–1229

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Smooth attractors of finite dimension for von Karman evolutions with nonlinear frictional damping localized in a boundary layer

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A R T I C L E I N F O

Article history:
Received 18 January 2012
Revised 18 July 2012
Available online 14 November 2012

Keywords:
Dynamical systems
Long-time behavior
Global attractors
Nonlinear plates
Nonlinear damping
Localized damping

A B S T R A C T

In this paper dynamic von Karman equations with localized interior damping supported in a boundary collar are considered. Hadamard well-posedness for von Karman plates with various types of nonlinear damping are well known, and the long-time behavior of nonlinear plates has been a topic of recent interest. Since the von Karman plate system is of "hyperbolic type" with *critical nonlinearity* (noncompact with respect to the phase space), this latter topic is particularly challenging in the case of *geometrically constrained*, *nonlinear* damping. In this paper we first show the existence of a compact global attractor for finite energy solutions, and we then prove that the attractor is both *smooth* and finite dimensional. Thus, the hyperbolic-like flow is stabilized asymptotically to a smooth and finite dimensional set.

1. Introduction

We consider the evolution of a nonlinear von Karman plate subject to nonlinear frictional damping with essential support in a boundary collar. Our aim is to consider the long-time behavior of the corresponding evolution. This includes studying (a) existence of a global attractor which captures long-time behavior of the dynamics, and (b) properties of this attractor, such as smoothness and finite dimensionality.

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.10.016

In short, our goal is to show that the original infinite dimensional and non-smooth dynamics of hyperbolic type can be reduced (asymptotically) to a finite dimensional and regular set, with respect to the topology of "finite energy". The latter is associated with weak (or generalized) solutions of the underlying semigroup for the dynamics. This type of result then allows the implementation of tools from finite dimensional control theory in order to achieve a preassigned outcome for the dynamics.

1.1. Model and energies

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega = \Gamma$ taken to be sufficiently smooth. We consider a plate model where the real-valued function $u(x, y; t)$ models the out-of-plane displacement of a plate with negligible thickness. Then the von Karman model [18,45] requires that u satisfies

$$u_{tt} + \Delta^2 u + d(\mathbf{x})g(u_t) = f_V(u) + p \quad \text{in } \Omega \times (0, \infty) \equiv Q,$$
$$u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1. \tag{1.1}$$

The von Karman nonlinearity

$$f_V(u) = \big[v(u) + F_0, u\big] \tag{1.2}$$

is given in terms of (a) the Airy Stress function $v(u)$, satisfying

$$\Delta^2 v(u) = -[u, u] \quad \text{in } \Omega,$$
$$\partial_\nu v(u) = v(u) = 0 \quad \text{on } \Gamma, \tag{1.3}$$

and (b) the von Karman bracket given by

$$[u, w] = u_{xx}w_{yy} + u_{yy}w_{xx} - 2u_{xy}w_{xy}. \tag{1.4}$$

The internal force $F_0 \in H^\theta(\Omega) \cap H_0^1(\Omega)$, $\theta > 3$, and external force $p \in L_2(\Omega)$ play an essential role in shaping the nontrivial stationary solutions. (In this paper $H^s(D)$ denotes the Sobolev space of order $s \in \mathbb{R}$ on domain D.) In the absence of these forces, the stationary solution of the corresponding nonlinear boundary value problem becomes trivial and simply reduces to zero.

In this treatment we focus on the stabilizing properties of the damping term $d(\mathbf{x})g(u_t)$. In particular, we take $g(\cdot) \in C(\mathbb{R})$ to be a monotone increasing function, with $g(0) = 0$ and further boundedness and smoothness assumptions to be imposed later; additionally, $d(\mathbf{x}) \equiv d_\omega(\mathbf{x})$ is a nonnegative $L_\infty(\Omega)$ localizing function which restricts the damping term $g(u_t)$ to a particular subset $\omega \subset \Omega$. This is to say $\omega \subset \operatorname{supp} d$ or $d(x) \geqslant c_0 > 0$ for $x \in \omega$. Initially we will take ω to be a general set $\omega \Subset \Omega$, but more specifically, we are interested in taking ω to be an open collar of the boundary Γ. This type of damping represents localized, viscous damping active *near* the boundary Γ.

The boundary conditions we consider for the plate are:

1. Clamped, denoted (C)

$$u = \partial_\nu u = 0 \quad \text{in } \Gamma \times (0, \infty) \equiv \Sigma. \tag{1.5}$$

2. Hinged (simply-supported), which we denote by (H)

$$u = \Delta u = 0 \quad \text{in } \Sigma. \tag{1.6}$$

3. Free-type, denoted by (F)

$$\mathcal{B}_1 u \equiv \Delta u + (1-\mu) B_1 u = 0 \quad \text{on } \Gamma_1,$$
$$\mathcal{B}_2 u \equiv \partial_\nu \Delta u + (1-\mu) B_2 u - \mu_1 u - \beta u^3 = 0 \quad \text{on } \Gamma_1,$$
$$u = \partial_\nu u = 0 \quad (\text{clamped}) \text{ on } \Gamma_0, \tag{1.7}$$

where we have partitioned the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ (with Γ_0 possibly empty). For simplicity we assume that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Otherwise the regularity theory for elliptic problems with *mixed* boundary conditions must be invoked. The boundary operators B_1 and B_2 are given by [45]:

$$B_1 u = 2\nu_1 \nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx},$$
$$B_2 u = \partial_\tau \big[\big(\nu_1^2 - \nu_2^2\big) u_{xy} + \nu_1 \nu_2 (u_{yy} - u_{xx}) \big],$$

where $\nu = (\nu_1, \nu_2)$ is the outer normal to Γ, $\tau = (-\nu_2, \nu_1)$ is the unit tangent vector along Γ. The parameters μ_1 and β are nonnegative, the constant $0 < \mu < 1$ has the meaning of the Poisson modulus.

Notation. Note, when referencing the plate equation above in (1.1) we will write (1.1)(C), (1.1)(H), or (1.1)(F) to indicate which boundary conditions we are taking. We write the norm in $H^s(D)$ as $\|\cdot\|_s$ and $\|\cdot\|_0 \equiv \|\cdot\|_{L_2(D)}$; for simplicity (when the meaning is clear from context) norms and inner products written without subscript ($(\cdot,\cdot)$, $\|\cdot\|$), are taken to be $L_2(D)$ of the appropriate domain D. Additionally, we employ the notation that $H_0^s(D)$ gives the closure of $C_0^\infty(D)$ in the $\|\cdot\|_s$ norm.

The von Karman plate equation is well known in nonlinear elasticity, and constitutes a basic model to describe the nonlinear oscillations of a thin plate with large displacements [45] (and references therein). In particular, we take the thickness of the plate to be negligible (as is usual in the modeling of thin structures [18]).

Remark 1.1. It is worth noting that the von Karman plate model can accomodate plates with non-negligible thickness – the equation in (1.1) then gives the vertical displacement of the central plane of the plate. This is tantamount to adding the term $-\gamma \Delta u_{tt}$, $\gamma > 0$ to the LHS of (1.1). This term corresponds to rotational inertia in the filaments of the plate, and (a) is *regularizing* from the energetic point of view and (b) forces the dynamics of the plate to be hyperbolic. In this treatment we take $\gamma = 0$, since it constitutes the most challenging problem mathematically, however, a future manuscript will address the case $\gamma > 0$ and the limiting problem (convergence of solutions and attractors) as $\gamma \searrow 0$.

The energies associated to the above equation are given by (in the case of clamped (C) or hinged (H) boundary conditions)

$$E(t) = \frac{1}{2}\big(\|\Delta u\|^2 + \|u_t\|^2\big),$$
$$\widehat{E}(t) = E(t) + \frac{1}{4}\big\|\Delta v(u)\big\|^2,$$
$$\mathscr{E}(t) = E(t) + \Pi(u),$$

where

$$\Pi(u) = \frac{1}{4}\int_{\Omega}\left(\left|\Delta v(u)\right|^2 - 2[F_0, u]u - 4pu\right). \tag{1.8}$$

The above (linear) energy $E(t)$ dictates our state space $\mathcal{H}$, which depends on boundary conditions. In the case of clamped boundary conditions (C) we have $\mathcal{H}_1 \equiv H_0^2(\Omega) \times L_2(\Omega)$. For hinged boundary conditions (H) we have $\mathcal{H}_2 \equiv (H^2 \cap H_0^1)(\Omega) \times L_2(\Omega)$.

Lastly, for free boundary conditions (F) we have $\mathcal{H}_3 \equiv (H^2 \cap H^2_{0,\Gamma_0})(\Omega) \times L_2(\Omega)$ (where $H^2_{0,\Gamma_0}(\Omega)$ is the Sobolev space $H^2(\Omega)$ with clamped conditions on Γ_0); the potential energy in this case is given by the bilinear form

$$a(u, v) = \int_{\Omega} \widetilde{a}(u, v) + \mu_1 \int_{\Gamma_1} uv, \tag{1.9}$$

where

$$\widetilde{a}(u, v) \equiv u_{xx}v_{xx} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1-\mu)u_{xy}v_{xy}. \tag{1.10}$$

Then the energy becomes

$$E(t) = \frac{1}{2}\left\{\|u_t\|^2 + a\big(u(t), u(t)\big)\right\},$$
$$\widehat{E}(t) \equiv E(t) + \frac{1}{4}\left\|\Delta v(u)\right\|^2 + \frac{\beta}{2}\int_{\Gamma_1} u^4\,d\Gamma.$$

The total energy becomes

$$\mathscr{E}(t) = E(t) + \Pi\big(u(t)\big) + \frac{1}{4}\beta\int_{\Gamma_1} u^4(t).$$

Remark 1.2. We note that this last form of the energy described by the bilinear form $a(u, v)$ can also be applied to clamped or hinged boundary conditions. Indeed, in this latter case the bilinear form $a(u, u)$ collapses just to $\|\Delta u\|^2$.

It will be convenient to introduce an elliptic operator $A : \mathscr{D}(A) \subset L_2(\Omega) \to L_2(\Omega)$ given by $Au = \Delta^2 u$, where $\mathscr{D}(A)$ incorporates the corresponding boundary conditions (clamped, hinged, or free). It is useful to note that by elliptic regularity

$$\mathscr{D}\big(A^{1/2}\big) = \begin{cases} H_0^2(\Omega) & \text{clamped B.C.,} \\ (H^2 \cap H_0^1)(\Omega) & \text{hinged B.C.,} \\ (H^2 \cap H^2_{0,\Gamma_0})(\Omega) & \text{free B.C.} \end{cases}$$

It is important to note the total potential energy may not be positive, or even not bounded from below. This is due to the presence of internal force F_0 which may drive the energy to $-\infty$. However, the presence of the von Karman bracket in the model, along with appropriate regularity properties imposed on F_0, assures that the energy is bounded from below. This can be seen from the following lemma [14,15]:

Lemma 1.1. *Let $u \in \mathscr{D}(A^{1/2})$, $p \in L_2(\Omega)$, and $F_0 \in H_0^1(\Omega) \cap H^\theta(\Omega)$, $\theta > 3$. Then, $\forall \epsilon > 0$ there exists $M(\epsilon, \|p\|, \|F_0\|_\theta) = M_{\epsilon,p,F_0} < \infty$ such that in the clamped and hinged case*

$$\|u\|^2 \leqslant \epsilon\left(\left\|A^{1/2}u\right\|^2 + \left\|\Delta v(u)\right\|^2\right) + M_{\epsilon,p,F_0}$$

and in the free case with $\beta > 0$,

$$\|u\|^2 \leqslant \epsilon\left(\left\|A^{1/2}u\right\|^2 + \left\|\Delta v(u)\right\|^2 + \frac{\beta}{2}\|u\|^4_{L_4(\Gamma)}\right) + M_{\epsilon,p,F_0,\beta}.$$

As a consequence we have the following bounds from below for the energy:
There exist positive constants m, c, M, C such that

$$-m + c\widehat{E}(t) \leqslant \mathscr{E}(t) \leqslant M + C\widehat{E}(t), \tag{1.11}$$

$$-m + cE(t) \leqslant \mathscr{E}(t) \leqslant h\big(E(t)\big) \tag{1.12}$$

where $h(s)$ denotes a continuous and increasing function.

1.2. Motivation and literature

Well-posedness for von Karman's plate equation with interior and/or boundary dissipation has been known for some time for smooth solutions in the case of homogeneous [10] or inhomogeneous nonlinear boundary conditions [14,22] and references therein. The issue of well-posedness for 'weak' (finite energy) solutions is more recent [14,22]. In this paper, we are interested in homogeneous type boundary conditions and we will be considering *generalized* nonlinear semigroup solutions [5,54] which also can be shown to be *weak* variational solutions. For a detailed and complete discussion regarding the well-posedness and regularity of von Karman solutions the reader is referred to [14,37]. In the context of this paper we will need the following well-posedness result, which is contingent upon the recently shown sharp regularity of the Airy Stress function in (1.3) [22,14]:

Theorem 1.2. *With reference to problem 1.1(C) with initial data $(u_0, u_1) \in \mathcal{H}_1$, or 1.1(H) with initial data $(u_0, u_1) \in \mathcal{H}_2$, or 1.1(F) with initial data $(u_0, u_1) \in \mathcal{H}_3$, there exists a* unique *global solution of finite energy (i.e. $(u, u_t) \in C([0, T]; \mathcal{H}_i)$ for $i = 1, 2, 3$ respectively, for any $T > 0$). Additionally, (u, u_t) depends continuously on $(u_0, u_1) \in \mathcal{H}_i$.*

Thus, for any initial data in the finite energy space $(u_0, u_1) \in \mathcal{H}$, there exists a well-defined semiflow (nonlinear semigroup) $S_t(u_0, u_1) \equiv (u(t), u_t(t)) \in \mathcal{H}$ which varies continuously with respect to the initial data in $\mathcal{H}$. The domain of the corresponding generator $\mathcal{A}(u, v) \equiv (v, -Au - d(\mathbf{x})\, g(v) + f_V(u) + p)$ is given by $\mathscr{D}(\mathcal{A}) = \{(u, v) \in \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/2});\ Au + d(\mathbf{x})\, g(v) \in L_2(\Omega)\}$. For initial data taken in $\mathscr{D}(A)$, the corresponding solutions are regular and remain invariant in $\mathscr{D}(\mathcal{A})$ [5,49,54]. With an additional assumption that $g(s)$ is bounded polynomially at infinity, one has $\mathscr{D}(\mathcal{A}) \subset H^4(\Omega) \times H^2(\Omega)$. Equipped with the regularity of the domain $\mathscr{D}(\mathcal{A})$, one derives the energy identity for all regular solutions. Due to the density of the embedding $\mathscr{D}(\mathcal{A}) \subset \mathcal{H}$, monotonicity of the damping, and sharp regularity of the Airy stress function (see Lemma 3.5) the same energy equality remains valid for all generalized solutions corresponding to any boundary conditions under consideration. Thus we have the energy identity for boundary conditions (C), (H), or (F) satisfied for all generalized (semigroup) solutions (complete details of this argument are given in [14]).

This equality reads: for all $0 < s < t$, strong and generalized solutions u to (1.1) satisfy

$$\mathscr{E}(t) + \int_s^t \int_\Omega d(\mathbf{x})\, g(u_t) u_t = \mathscr{E}(s). \tag{1.13}$$

With the well-posedness of the semiflow established in Theorem 1.1, it is natural to investigate long time behavior of the dynamical system generated by (1.1). It is clear from (1.13) that the essential mechanism for dissipating the energy is the damping term $d(x)\,g(u_t)$. In the simplest possible scenario when $p = F_0 = 0$ the energy function $\mathscr{E}(t)$ is equivalent topologically to the norm of the phase space $\mathcal{H}$. Since $\mathscr{E}(t)$ is nonincreasing on the trajectories, it becomes a Lyapunov function for the corresponding nonlinear dynamical system, whose only equilibrium is the zero point. If one assumes that $d(x) > 0$, *a.e.* in Ω, it is well known that $\mathscr{E}(t)$ becomes a *strict* Lyapunov function and zero equilibrium is strongly stable. However, the above condition imposed on $d(x)$ is *not sufficient* to guarantee uniform convergence to the equilibrium (this is also the case for *linear* dynamics without the von Karman term). In order to secure uniform convergence or, more generally, convergence to a compact attractor, a stronger form of the damping is necessary. For example, $d(x) \geqslant c_0 > 0$, $x \in \Omega$ and $g(s) = as$, $a > 0$, provides a classical model for which uniform convergence to zero in the absence of external/internal forcing (or more generally to an attractor) can be shown [9,10,14,45]. The goal in this paper is to consider nonlinear damping of a reduced essential support whereby the inequality $d(x) \geqslant c_0 > 0$ will be enforced only in a small set $\omega \Subset \Omega$, while the dynamics will be forced by nontrivial sources p, F_0. Existence of a compact and possibly smooth finite dimensional attracting set for the dynamics generated by (1.1) with boundary conditions (C), (H), or (F) and geometrically constrained dissipation is of great physical interest. Such a result is tantamount to asserting that the infinite dimensional, non-smooth dynamics are asymptotically reduced to a *smooth and finite dimensional set.* While such a reduction is expected for dynamical systems that exhibit some smoothing effects (e.g. parabolic-like) [55,47,19,51,3,48,33], it is a much less evident phenomena in the case of hyperbolic-like dynamics, where the 'taking-off' of the dynamics produces no smoothing effect. The role of the frictional damping in such a system is instrumental; in fact, it is the induced friction that creates a stabilizing and asymptotically regularizing effect on the evolution, ultimately reducing it to a compact set. On the other hand it is well known that the hyperbolic-like dynamics cannot be stabilized by a compact feedback operator [39] (and references therein). This is due to the fact that instabilities in the system are inherently infinite dimensional and the essential part of the spectrum cannot be dislodged by a compact perturbation. Thus, any effective damping cannot be compact (with respect to the phase space). The above feature combined with (a) nonlinearity of the damping and (b) lack of compactness of the nonlinear von Karman source makes the analysis of long-time behavior for this class of systems challenging. In fact, critical exponent nonlinearities and nonlinear dissipation are known to constitute endemic difficulties in the study of hyperbolic-like systems [23].

To orient the reader and to provide some perspective for the problem studied, we shall briefly describe some of the principal contributions to this area of research. A detailed account is given in [14].

In the discussion of global attractors for von Karman evolution equations, we must distinguish between two types of dynamics for the problem: (a) the rotational case (as addressed above) when the term $-\gamma\Delta u_{tt}$, $\gamma > 0$ is added to the LHS of (1.1) and (b) nonrotational ($\gamma = 0$). In case (a), we note that the von Karman nonlinearity (in the finite energy topology) is *compact*, which considerably simplifies the analysis of long-time dynamics. In the latter case (b) (which we consider here), a very different type of analysis is needed. Here, we shall focus on part (b) only. In fact, the very first contribution to this problem is a pioneering paper [10] where the existence of *weak* attractors with a linear, fully supported damping was demonstrated. Later on, owing to new results on the regularity of Airy's stress function [22,14], *weak* attractors were improved to *strong* attractors, and the restriction of linear damping was removed in order to allow nonlinear, monotone damping [13]. In order to incorporate fully nonlinear interior damping, [13] assumes that the dissipation parameter is sufficiently *large*. This restriction was later removed in [34], whose paper introduces a very clever way of bypassing a lack of compactness and replacing it with an "iterated convergence" trick. Further studies of the attractor (including properties such as dimensionality and smoothness) in the fully nonlinear setup, without "size" restrictions imposed on the parameters, are presented in [16] and in monograph form in [12,14].

It should be noted that the results described above pertain to the interior and *fully supported* dissipation. The situation is much more delicate when the dissipation is *geometrically constrained*, where the essential support of the damping is localized to a *subset* of the spatial domain Ω. In that case, the issue of propagating the damping from one area to another becomes the critical one. While this sort of problems has been previously studied in the context of stabilization to equilibria [45,29,30], the estimates needed for attractors are much more demanding. Previous methods developed in the context of stabilization no longer apply. Some long-time behavior results with boundary damping are presented in [15,16], wherein nonlinear dissipation on the boundary acting via *free* boundary conditions is considered. These works, however, impose the rather stringent geometric restrictions of the entire boundary being star-shaped. Such restrictions are removed in [14], where dissipation via hinged boundary conditions is considered; however this is done at the expense of limiting the class of dissipation to those of *linearly bounded* type. This restriction is needed since the elimination of the geometric condition is achieved via microlocal estimates [41], which in turn force velocity dependent nonlinear terms to be linearly bounded.

Localized interior damping arises naturally in the control and long-time behavior of PDEs (in particular, wave and plate equations [35,17]). Use of such damping, for general localization, constitutes a physically motivated attempt to obtain controllability and stability results for "small" subsets of the domain. These results can be more demanding than the use of full interior damping, i.e. $d(x) \geqslant c_0 > 0$ for all x in Ω, since energy methods require the use of commutators to reconstruct the full energy in observability type estimates. More specifically, the use of geometrically constrained damping in the form of damping active in a collar of the boundary has arisen in the study of coupled dynamics [56,43,6].

This brings us to the main contribution of the present manuscript. Our goal is to show that the fully *nonlinear* damping with essential support in an arbitrarily small layer near the boundary provides not only the existence of compact attractors but also desirable properties such as C^∞ smoothness and finite dimensionality. Thus the original hyperbolic-like non-smooth flow is asymptotically reduced to smooth and finite dimensional dynamics. The result is valid for *all types of boundary conditions* with *geometrically constrained dissipation*, which can be nonlinear of *any polynomial growth* at infinity and with *no restriction on the size* of the damping parameter.

We obtain this result by proving that the dynamics are *quasistable* – a concept introduced in [12] and [14]. The ability to show quasistability is dependent upon a new method of localization of multipliers that allows smooth propagation of the damping from the boundary collar into the interior (even in the presence of boundary conditions – free – that do not comply with the Lopatinski conditions [53]) the latter in the context of geometrically constrained dissipation for wave dynamics.

Lastly, we would like to note that while some of the methods developed for *boundary* dissipation [14,16,38] can also be used in the case of partially localized dissipation and Dirichlet – clamped boundary conditions, this is not the case with Neumann type (free) boundary conditions which violate strong Lopatinski [53] condition. In this latter case, propagation of the damping from the boundary layer via boundary damping estimates is obstructed by the well-known lack of sufficient regularity (the absence of so called "hidden" regularity [40]) of boundary traces corresponding to the linear model [41,42]. Our aim in this paper is to develop a method which is effective for all kind of boundary conditions and does not depend on hidden regularity, where the latter restricts the analysis to Lopatinski type of models. The key element for this are suitably localized multipliers estimates.

1.3. Statement of results

Equipped with well-posedness of finite energy and regular solutions corresponding to (1.1) under one of the boundary conditions (C), (H), or (F), we are now ready to state our main results pertaining to long time behavior of solutions. To accomplish this we shall introduce Lyapunov function

$$V(u_0, u_1) \equiv \mathcal{E}(u_0, u_1) = \frac{1}{2}\left(\left\|A^{1/2}u_0\right\|^2 + \|u_1\|^2\right) + \Pi(u_0).$$

By Lemma 1.1, $V(u_0,u_1)$ is bounded from below and above on bounded sets in $\mathcal{H}$. It is also continuous (Theorem 1.2) and radially unbounded, i.e, $V(u_0,u_1)\to\infty$ when $\|(u_0,u_1)\|_{\mathcal{H}}\to\infty$ (Lemma 1.1). We introduce next the set

$$W_R\equiv\bigl\{u=(u_0,u_1)\in\mathcal{H}\colon\ V(u_0,u_1)\leqslant R\bigr\}.$$

The following properties are immediate from Lemma 1.1 and energy inequality (1.13):

(1) There exists $R_0>0$ such that W_R is non-empty for all $R>R_0$.
(2) For every bounded set B in $\mathcal{H}$ there exists $R>0$ so that $B\subset W_R$.
(3) W_R is bounded for every $R>0$.
(4) W_R is invariant with respect to the flow $S_t(u_0,u_1)$, i.e $S_t(W_R)=W_R$.

The above properties allows us to consider for $R>R_0$ the dynamical system (W_R,S_t), which is a restriction of $(\mathcal{H},S_t)$.

In order to formulate our results we shall assume validity of an *asymptotic* growth condition *from below* imposed on $g(s)$. Such condition is typical [45] and necessary in order to obtain uniform decay rates of solutions in hyperbolic-like dynamics. It allows control of the kinetic energy for large frequencies.

Assumption 1. There exist positive constants $0<m\leqslant M<\infty$ and a constant $p\geqslant 1$ such that

$$m\leqslant g'(s)\leqslant M|s|^p,\quad |s|\geqslant 1.$$

We now state the primary result in this treatment.

Theorem 1.3. *Take Assumption 1 to be in force. Let* $\operatorname{supp} d\supset\omega$ *and* $d(\mathbf{x})\geqslant\alpha_0>0$ *in* ω*, where* $\omega\Subset\Omega$ *is any full collar near the boundary* Γ*. Then for all generalized solutions corresponding to solutions with initial data* $\|(u_0,u_1)\|_{\mathcal{H}}\leqslant R$*, there exist compact attractor* $\mathbf{A}_R\in\mathcal{H}$*. This is to say that for any* $R>R_0$ *the dynamical system* (W_R,S_t) *admits a global compact attractor* $\mathbf{A}_R$*.*

Properties of the attractor $\mathbf{A}_R$ such as smoothness and finite dimensionality are addressed in the two theorems below.

Theorem 1.4. *In addition to Assumption 1 assume that there exists* $m,M>0$*, and* $\gamma<1$ *such that* $0<m\leqslant g'(s)\leqslant M[1+sg(s)]^\gamma$ *for all* $s\in\mathbb{R}$*. Then,*

(a) *the attractor* $\mathbf{A}_R$ *is regular, which is to say* $\mathbf{A}_R\subset H^4(\Omega)\times H^2(\Omega)$ *is a bounded set in that topology.*
(b) *The fractal dimension of* $\mathbf{A}_R$ *is finite.*

Remark 1.3. If we consider $g(s)=|s|^ps$, then one can show that $\gamma=\frac{p}{p+2}$ satisfies the above condition.

Theorem 1.5. *In addition to the assumptions of Theorem 1.4 we assume that* F_0, g *are* C^∞*. Then, the attractor is also* C^∞*. More precisely* $\mathbf{A}_R$ *is a bounded set in* $H^{k+2}(\Omega)\times H^k(\Omega)$ *for all* $k=1,2,\ldots.$

The second part of our results addresses the question of existence of global attractor $\mathbf{A}$ – or more precisely independence of $\mathbf{A}_R$ on R for R sufficiently large. For this, we shall introduce the following unique continuation condition, denoted UC.

Definition 1. We say that the system satisfies the UC property iff the following implication is valid for any weak solution (u, u_t) to (1.1): There exists $T > 0$ such that

$$u_t = 0 \quad \text{a.e. in } \operatorname{supp} d \times (0, T) \quad \Rightarrow \quad u_t = 0 \quad \text{a.e. in } \Omega \times (0, T).$$

It is clear that the UC property holds if $d(\mathbf{x}) > 0$ a.e. in Ω. However, this assumption *does not lead* to uniform stability, even in the case of the linear model. For the latter it is needed that $d(\mathbf{x}) \geqslant d_0 > 0$ for all $\mathbf{x} \in \Omega$.

Remark 1.4 *(Comments on the UC property).* First, such property has been used extensively in the context of wave equation with semilinear local nonlinear terms. The validity of UC property for these models stems from Carleman's estimates [52,31,20,21] developed for the wave equation with potential term. Carleman's estimates have been also derived for plate equations with biharmonic principal part [36,1,32] and lower order terms up to second order. These estimates were obtained by reiterating Carleman's estimates obtained first for the Schrodinger equation [32]. The resulting weighted inequalities allow one to prove the UC property for nonlinear plates with *local semilinear* terms [36] or for some non-local problems such as Berger's plates where nonlinear term is of the form $f_B(u) = \|\nabla u(t)\|^2 \Delta u(t)$ [12]. Space-independent nonlinear terms enable the propagation of the zero solution across the entire region [52,1].

The main obstacle in obtaining the UC property for the von Karman plate with localized damping is the completely non-local character of von Karman bracket that prevents the applicability of Carleman's estimates for the purpose of obtaining the UC property. Exception to these are some special models with well-tailored lower order terms [45], so that classical Pohozaev's inequality applies. However, for this to hold, one needs to consider lower order terms that are sufficiently structured. For instance, adding *static dissipation* to a boundary collar alleviates the problem. For the von Karman plate with $F_0 = 0$ calculations on p. 110 [45] allow one to deduce the UC property after adding to the equation a term of the form $d_0(x)g_0(u)$, where $\omega \subset \operatorname{supp} d_0$ and g_0 is any smooth and monotone increasing function.

However, in the general case, as considered in this paper, the unique continuation property for the von Karman plate is poorly understood. A now classical set of tools developed for plate equations and based on Carleman estimates [20,1,31,21,36] do not apply. The non-locality of the von Karman bracket prevents propagation across the entire domain of *weak* damping localized to a "small" set. Therefore, we have the question: *if the damping in the equation (represented by $d(\mathbf{x})u_t$) is zero in an open set of positive measure inside of Ω, does this imply that the solution u must also be 0 in Ω?*; it remains open. In relation to our analysis here, if the general unique continuation property holds for the von Karman plate, then it immediately strengthens our result by allowing $d(\mathbf{x})$ to vanish away from an open collar of the boundary. However, at present, the best we can state is a sufficient condition, namely that $d(\mathbf{x}) > 0$ a.e. in order to satisfy the UC property (in the absence of additional static dissipation or a small constant in front of von Karman bracket).

The validity of UC property allows to show that the dynamical system under consideration has a gradient structure. In such case, one shows that there exist global attractor $\mathbf{A}$ and local results become global, leading to the equality $\mathbf{A}_R = \mathbf{A}$ for some $R > 0$. This result is stated below.

Theorem 1.6. *Assume that the UC property holds. Then the attractor is global, i.e. $\mathbf{A}_R = \mathbf{A}$ for some $R > 0$. Moreover, all the results of Theorem 1.3, Theorem 1.4 and Theorem 1.5 apply to the global attractor $\mathbf{A}$.*

Under the assumption that the UC property holds, the system under consideration is a gradient system. As a consequence, the trajectories from the attractor stabilize asymptotically to the unstable manifold.

Since $\mathbf{A} = \mathscr{M}^u(\mathscr{N})$ where $\mathscr{N} = \{(\varphi, 0)\colon\ \varphi \in \mathscr{N}^*\}$ is the set of equilibria points and $\mathscr{N}^* \subset \mathcal{D}(A^{1/2})$ is the set of weak solutions to the stationary problem corresponding to (1.1)–(1.4), we have that

$$\lim_{t\to\infty} d_{\mathcal{H}}(S_t W|\mathcal{N}) = 0 \quad \text{for any } W \in \mathcal{H} \tag{1.14}$$

which implies the closedness to the equilibria points.

Of course, an interesting question is whether the individual trajectories stabilize to specific equilibria (rather than to the set of equilibria). In fact, such property is known, provided that the set of equilibria is *finite* (see Corollary 2.32 [12]). Thus, under the assumption that *UC* property holds and the set of equilibria $\mathcal{N}$ is *finite*, one has that any $x \in \mathbf{A}$ belongs to some full trajectory $\gamma = \{(u(t), u_t(t)), t \in \mathbb{R}\}$ and for any $\gamma \in \mathbf{A}$ there exists a pair $\{e, e*\} \in \mathcal{N}$ such that

$$\big(u(t), u_t(t)\big) \to (e, 0), \quad \text{in } H \text{ as } t \to \infty; \qquad \big(u(t), u_t(t)\big) \to (e*, 0), \quad \text{in } H \text{ as } t \to -\infty. \tag{1.15}$$

While the property of finiteness of equilibria points is generic with respect to the loads p (Sard's Theorem), it is interesting to know under which condition this property is valid for each individual trajectory. And in fact, there is a new tool addressing this issue that has been developed in series of papers [26–28,7,8,57,2] and references therein which is based on the validity of the so-called Lojasiewicz inequality. The Lojasiewicz gradient inequality refers to an analytic function defined on a real Hilbert space V, $F : V \to R$ and states that for any point $a \in V$ there is a neighborhood $U(a) \in V$ and two constants $\theta \in (0, 1/2]$, $C > 0$ such that

$$\big|F(u) - F(a)\big|^{1-\theta} \leqslant C\big\|DF(u)\big\|_{V'}, \quad \forall u \in U(a) \subset V. \tag{1.16}$$

In the case of dynamical systems, the functional F is related to potential energy of the system. The advantage of having Lojasiewicz inequality is that it provides a tool for proving stabilization of trajectories to specific equilibria which are stationary points of the dynamics [26,27,7,8,57]. It turns out that Lojasiewicz inequality is satisfied in the case of the Karman problem. Indeed, considering the functional $\Pi^*(u) \equiv \Pi(u) + 1/2a(u, u)$, where $\Pi(u)$ is given by (1.8) and $a(u, u)$ is given by (1.9), it has been shown in [11] that there exist $\delta > 0$, $C > 0$, $\theta \in (0, 1/2]$ such that

$$\big|\Pi^*(u) - \Pi^*(e)\big|^{1-\theta} \leqslant C\big\|A^{-1/2}D\Pi^*(u)\big\|, \quad \forall u \in B_{\mathscr{D}(A^{1/2})}(e, \delta). \tag{1.17}$$

Here $e \in \mathscr{D}(A)$ is a stationary point satisfying the nonlinear elliptic problem $\Delta^2 e = f_V(e) + p$ with appropriate boundary conditions. The above result follows from Corollary 6.5 in [27], after the conditions imposed in that Corollary have been verified. This was accomplished in [11]. We note that analyticity of Π^* on $\mathscr{D}(A^{1/2})$ follows from sharp regularity of Airy's stress function. It was also shown in [11] that by assuming hyperbolicity of stationary solutions, the Lojasiewicz exponent θ is optimal and equal to $1/2$. By using Lojasiewicz inequality, [11] proves that the trajectories of von Karman evolution with nonlinear fully interior damping that is mildly degenerate at the origin stabilizes asymptotically to equilibria. In the non-degenerate case, the rate of convergence to equilibria are also established in [11].

However, the arguments related to convergence to equilibria depend strongly on the fact that (i) the damping has full geometric support and (ii) only mild degeneracy of the damping at the origin is allowed. At the present time, it is not known whether similar result should be expected for geometrically constrained damping.

Regarding damping which is degenerate at the origin ($g'(0) = 0$), it is known that under the additional assumption of *finite number of equilibria (generic property) and hyperbolicity of equilibria*, the trajectories converge to equilibria at a specified rate depending on degeneracy of the damping at the origin.

Theorem 1.7. *(See [14].) In addition to the assumptions of Theorem 1.2 assume that* (i) $d(x) > c_0 > 0$, *a.e. in* Ω, (ii) *The set of equilibria is finite and hyperbolic. Then, for any* $U_0 = (u_0, u_1) \in \mathcal{H}$ *there exists* $e \in \mathcal{N}$ *such that the following decay rate holds for the trajectory* $S_t U_0$,

$$\|S_t U_0 - e\|_{\mathcal{H}} \leqslant C\sigma(t), \quad t > T_0$$

where σ satisfies the ODE equation $\sigma_t + Q(\sigma) = 0$, $\sigma(0) = \sigma_0 = C(e, U_0)$ and $Q(s) \sim h^{-1}(s)$ where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, concave and monotone increasing function such that $s^2 \leqslant h(g(s)s)$ for $|s| \leqslant 1$.

This result can be proved by repeating the arguments of similar result Theorem 10.4.10 in [14] with the observability estimates replaced by the estimates of the present work.

The result in Theorem 1.7 gives decay rates to equilibria. The rates depend on the dissipation at the origin which is not required to be qualified a priori. Clearly when $g(s)$ is linear at the origin, the corresponding decay rates are exponential.

Theorem 1.7 follows from Theorem 9.5.3 in [14]. We note that the result also holds when strict positivity of $d(x)$ a.e. Ω is replaced by the UC property. This follows from the treatment of localized damping for the nonlinear plate presented in this paper, along with the approach taken in [14]; the forthcoming manuscript [24] addresses convergence to equilibria (under comparable assumptions as Theorem 1.7) for both the Berger and von Karman plates.

1.3.1. Comments

There are three main difficulties/novelties pertaining to the proof of the results stated above:

(a) The nonlinear source is of critical exponent (lack of compactness).
(b) The essential damping is geometrically constrained to a small subset ω.
(c) The damping is genuinely nonlinear (any polynomial growth at the infinity is allowed).

These three difficulties are well-recognized in the context of studying long time behavior of hyperbolic-like systems where there is no inherent smoothing mechanism present in the model. In order to provide some perspective, it helps to add that geometrically constrained damping forces to use higher order multipliers which become *supercritical* when dealing with energy terms and nonlinear critical terms. Thus, any successful approach must rely on suitable cancellations, which must be uncovered for the specific dynamics in question.

Similar issues appear when dealing with nonlinear damping. The damping term must be critical (in hyperbolic dynamics) in order to be effective (we recall that the essential spectrum of an operator cannot be altered by a compact perturbation). The property of monotonicity of the problem does help when dealing with a single solution at the energy level. However, when dealing with long-time behavior, the protagonist is not a single solution but the difference of two solutions. In the study of the corresponding dynamics at the non-energetic levels (resulting from multipliers), monotonicity is destroyed. There is a "spillover" of the noncompact (in fact, supercritical) damping that must be absorbed. For this issue, different mechanisms need to be discovered (e.g. backward smoothness of trajectories, compensated compactness, etc.).

While recent developments in the field provide tools enabling us to handle a combination of *any two* of the difficulties listed above, the *inclusion of the third* prevents us from utilizing existing mathematical technology. The principal contribution of this treatment is to develop method which is capable of dealing with all three aforementioned difficulties simultaneously. The main ingredients of this new approach are (i) a localization method which allows us to show propagation of the damping without any requiring that the Lopatinski condition be satisfied, and (ii) a compactness/density argument applied on the attractor which yields the necessary quasistability estimate.

We conclude this section by listing few problems that are of interest to pursue and still open.

(1) *Damping restricted to a portion of an open collar.* Dissipation localized to part of the collar could be considered by assuming certain geometric conditions imposed on the uncontrolled part of the collar. Certain ideas presented in [17,6] should prove useful.
(2) *The UC property for a larger class of dampings*.
(3) *Convergence* (*and rate of convergence*) *to equilibrium points under minimal assumptions.* The analysis in [24] may be performed without assuming finiteness or hyperbolicity of equilibria points. The method will have to exploit the Lojasiewicz inequality, as mentioned above.

2. Long-time behavior of dynamical systems

In this manuscript we will make ample use dynamical systems terminology (see [3,48,51,9,14,44,25]); let $(\mathcal{H}, S_t)$ be a dynamical system on a complete metric space $\mathcal{H}$ with $\mathscr{N} \equiv \{x \in \mathcal{H}\colon\ S_t x = x$ for all $t \geqslant 0\}$ the set of its stationary points. $(\mathcal{H}, S_t)$ is said to be dissipative iff it possesses a bounded absorbing ball.

We say that a dynamical system is *asymptotically compact* if there exists a compact set K which is uniformly attracting: for any bounded set $D \subset \mathcal{H}$ we have that

$$\lim_{t\to+\infty} d_{\mathcal{H}}\{S_t D \mid K\} = 0 \tag{2.1}$$

in the sense of the Hausdorff semidistance.

$(\mathcal{H}, S_t)$ is said to be *asymptotically smooth* if for any bounded, forward invariant $(t > 0)$ set D there exists a compact set $K \subset \overline{D}$ such that (2.1) holds.

Global attractor $\mathbf{A}$ is a closed, bounded set in $\mathcal{H}$ which is invariant (i.e. $S_t\mathbf{A} = \mathbf{A}$ for all $t > 0$) and uniformly attracting (as defined above).

The following *if and only if* characterization of global attractors is standard and well known [4]

Theorem 2.1. *Let $(\mathcal{H}, S_t)$ be a dissipative dynamical system in a complete metric space $\mathcal{H}$. Then $(\mathcal{H}, S_t)$ possesses a compact global attractor $\mathbf{A}$ if and only if $(\mathcal{H}, S_t)$ is asymptotically smooth.*

An *asymptotically smooth* dynamical system for which there is a Lyapunov function $\Phi(x)$ that is bounded from above on any bounded set can be thought of as one which possesses *local attractors*. More precisely (see [12], p. 33),

Theorem 2.2. *Assume that $(\mathcal{H}, S_t)$ be an asymptotically smooth dynamical system in a Banach space $\mathcal{H}$. Let $\Phi(x)$ be an associated Lyapunov function that is bounded from above on any bounded set. Assume that the set $\Phi_R \equiv \{x \in \mathcal{H}, \Phi(x) \leqslant R\}$ is bounded for every $R > 0$. Then, the dynamical system (Φ_R, S_t) possesses a compact global attractor $\mathbf{A}_R$ for every $R > 0$.*

Theorem 2.2 provides an existence of *local attractors*, i.e. for each bounded set of initial data. However, these sets need not be uniformly bounded with respect to R. The latter is guaranteed by the existence of an absorbing set. However, establishing existence of an absorbing set may be technically demanding. Fortunately, there is a way of circumventing this difficulty by taking advantage of the good structure of a Lyapunov function.

A *strict Lyapunov function* for $(\mathcal{H}, S_t)$ is a functional Φ on $\mathcal{H}$ such that (a) the map $t \to \Phi(S_t x)$ is nonincreasing for all $x \in \mathcal{H}$, and (b) $\Phi(S_t x) = \Phi(x)$ for all $t > 0$ and $x \in \mathcal{H}$ implies that x is a stationary point of $(\mathcal{H}, S_t)$. If the dynamical system has a strict Lyapunov function defined on the entire phase space, then we say that $(\mathcal{H}, S_t)$ is *gradient*.

In the context of this paper we will use a few keys theorems (which we now formally state) to prove the existence of the attractor and determine its properties. (For proofs and references, see [14] and references therein.) First, we address attractors for gradient systems and characterize the attracting set. The following result follows from Theorem 2.28 and Corollary 2.29 in [12].

Theorem 2.3. *Suppose that $(\mathcal{H}, S_t)$ is a gradient, asymptotically smooth dynamical system. Suppose its Lyapunov function $\Phi(x)$ is bounded from above on any bounded subset of $\mathcal{H}$ and the set $\Phi_R \equiv \{x \in \mathcal{H}\colon\ \Phi(x) \leqslant R\}$ is bounded for every R. If the set of stationary points for $(\mathcal{H}, S_t)$ is bounded, then $(\mathcal{H}, S_t)$ possesses a compact global attractor $\mathbf{A}$ which coincides with the unstable manifold, i.e.*

$$\mathbf{A} = \mathscr{M}^u(\mathscr{N}) \equiv \Big\{x \in \mathcal{H}\colon\ \exists U(t) \in \mathcal{H},\ \forall t \in \mathbb{R} \text{ such that } U(0) = x \text{ and } \lim_{t\to-\infty} d_{\mathcal{H}}\big(U(t)\big|\mathscr{N}\big) = 0\Big\}.$$

Secondly, we state a useful criterion (inspired by [34]) which reduces asymptotic smoothness to finding a suitable functional on the state space with a compensated compactness condition:

Theorem 2.4. *(See [12], Proposition 2.10.) Let $(\mathcal{H}, S(t))$ be a dynamical system, $\mathcal{H}$ Banach with norm $\|\cdot\|$. Assume that for any bounded positively invariant set $B \subset \mathcal{H}$ and for all $\epsilon > 0$ there exists a $T \equiv T_{\epsilon,B}$ such that*

$$\|S_T x_1 - S_T x_2\|_{\mathcal{H}} \leqslant \epsilon + \Psi_{\epsilon,B,T}(x_1, x_2), \quad x_i \in B$$

with Ψ a functional defined on $B \times B$ depending on ϵ, T, and B such that

$$\liminf_m \liminf_n \Psi_{\epsilon,T,B}(x_m, x_n) = 0$$

for every sequence $\{x_n\} \subset B$. Then $(\mathcal{H}, S_t)$ is an asymptotically smooth dynamical system.

In order to establish both smoothness of the attractor and finite dimensionality, a stronger estimate on the difference of two flows is needed. We now cite [14, pp. 381–387]:

Theorem 2.5. *Suppose X_1 and X_2 are Banach spaces with X_1 compactly embedded into X_2. Take $\mathcal{H} \equiv X_1 \times X_2$ with norm $\|y\|^2_{\mathcal{H}} = \|u_0\|^2_{X_1} + \|u_1\|^2_{X_2}$. Assume that $(\mathcal{H}, S_t)$ is a dynamical system with the evolution defined by $S_t y = (u(t), u_t(t))$ for $y = (u_0, u_1) \in \mathcal{H}$, where $u \in C(\mathbb{R}, X_1) \cap C^1(\mathbb{R}, X_2)$.*

Now assume that $B \subset \mathcal{H}$, and that there exists a compact seminorm $\mu_{X_1}(\cdot)$ on X_1 and nonnegative scalar functions $a(t)$, $b(t)$, and $c(t)$ on $\mathbb{R}_+$ such that (i) *$a(t)$ and $c(t)$ are locally bounded,* (ii) *$b(t) \in L_1(\mathbb{R}_+)$ with $\lim_{t\to\infty} b(t) = 0$, and* (iii) *for all $y_1, y_2 \in B$ and $t > 0$ the following relations hold*

$$\|S_t y_1 - S_t y_2\|^2_{\mathcal{H}} \leqslant a(t)\|y_1 - y_2\|^2_{\mathcal{H}}, \tag{2.2}$$

$$\|S_t y_1 - S_t y_2\|^2_{\mathcal{H}} \leqslant b(t)\|y_1 - y_2\|^2_{\mathcal{H}} + c(t) \sup_{[0,t]} \big[\mu_{X_1}\big(u^1(s) - u^2(s)\big)\big]^2 \tag{2.3}$$

where we have denoted $S_t y_i = (u^i(t), u^i_t(t))$. (In this case we say that the dynamical system is "quasistable".)

Then, assuming the dynamical system $(\mathcal{H}, S_t)$ possesses a compact global attractor $\mathbf{A}$ and is quasistable on $\mathbf{A}$, the following hold

- *The attractor $\mathbf{A}$ has finite fractal dimension.*
- *Assuming that the function $c(t) \in L_\infty(\mathbb{R}_+)$, then any full trajectory $\{(u(t), u_t(t)):\ t \in \mathbb{R}\}$ that belongs to the attractor possess the following regularity:*

$$u_t \in L_\infty(\mathbb{R}; X_1) \cap C(\mathbb{R}; X_2), \quad \text{and} \quad u_{tt} \in L_\infty(\mathbb{R}; X_2).$$

We will utilize the following specialization of the theorem above.

Theorem 2.6. *Let $x_1, x_2 \in B \subset \mathcal{H}$ where B is a forward invariant set for the flow $S_t x_i$. Assume that the following inequality holds for all $t > 0$ with positive constants $C_1(B)$, $C_2(B)$, ω_B,*

$$\|S_t x_1 - S_t x_2\|^2_{\mathcal{H}} \leqslant C_1(B)e^{-\omega_B t}\|x_1 - x_2\|^2_{\mathcal{H}} + C_2(B) \max_{\tau\in[0,t]} \|S_\tau x_1 - S_\tau x_2\|^2_{\mathcal{H}_1} \tag{2.4}$$

where $\mathcal{H} \subset \mathcal{H}_1$ is compactly embedded. Then the attractor $\mathbf{A}$ associated with the flow S_t possesses the following properties:

(a) *The fractal dimension of $\mathbf{A}$ is finite.*
(b) *For any $x \in \mathbf{A}$ one has $\frac{d}{dt}(S_t x) \in L_\infty(\mathbb{R}, \mathcal{H})$.*

Remark 2.1. The estimate in (2.4) is often referred to (in practice) as the "quasistability" estimate. It reflects the fact that the flow can be stabilized exponentially to a compact set. Alternatively, we might say that the flow is exponentially stable, modulo a compact perturbation (lower order terms). We note that the lower order terms being quadratic is important for the validity of Theorem 2.6.

The proof of Theorem 2.6 employs the idea of "piecewise" trajectories introduced in [46,50]. This allows to generalize previous criteria for finite dimensionality [3,55,51,19] by reducing the problem to validity of quasistable estimate.

2.1. The approach and outline of the paper

To show our main result on the existence of the global attractor for (1.1) with boundary conditions (C), (H), or (F) we make use of the theorems above. First, we note that in the case of any boundary conditions, the von Karman system in (1.1) has Lyapunov function $V(u_1,u_2)=\mathscr{E}(u_1,u_2)$ which is bounded on the set $W_R=\{(u_1,u_2)\in\mathcal{H};\ \mathcal{E}(u_1,u_2)\leqslant R\}$ for all $R>0$ and which contains any bounded set for sufficiently large R. Thus, the existence of local attractors $\mathbf{A}_R$ is equivalent proving asymptotic smoothness. For this later task we shall appeal to Theorem 2.4. We will analyze z, taken to be the difference of strong solutions, and bound the linear energy $E_z(t)=\|\Delta z\|^2+\|z_t\|^2$ (then, via a standard limiting procedure obtain our estimate for generalized solutions as well); this estimation will produce our functional Ψ in Theorem 2.4. Our main tool in estimating $E_z(t)$ will be the use of two multipliers: $f_1 z$ and $h\cdot\nabla((f_2)z)$, where h will be a suitably chosen C^2 vector field and f_i are appropriate localization functions.

First, we perform multiplier analysis as generally as possible, without imposing boundary conditions. Later on, we shall use boundary conditions (either clamped, or hinged or free) in order to obtain the smoothness inequality in Theorem 2.4.

After establishing the existence of the attractor $\mathbf{A}_R$, we proceed to show that it has additional regularity than that of the state space, and also that it has finite fractal dimension. The ultimate goal is to prove a "quasistability" estimate for the difference of general trajectories and apply abstract Theorem 2.6, however doing so directly in this case is not straightforward. Proving this will depend upon a trajectory being 'close' to smooth elements on already established attractor. Thus existence and compactness of the attractor are perquisites for carrying the estimates. We can then establish the sought after quasistability estimate in Theorem 2.6, which will produce the regularity and finite fractal dimension of the attractor.

C^∞ regularity of the attractor requires appropriate bounds on higher derivatives of solutions *on* the attractor, which in turn depends on careful tracing of critical terms in the inequalities. The special structural decomposition of the von Karman bracket plays a critical role here.

Existence of global attractor requires that the UC property is satisfied (e.g. in the case that $d(\mathbf{x})>0$ a.e. in Ω). We refer to [14] for the details. Moreover, the set of stationary points for the dynamical system generated by (1.1)(C), (1.1)(H), or (1.1)(F) is bounded. This latter fact follows from (1.11) (see [14]). Hence we are in a position to use Theorem 2.3 after referring to asymptotic smoothness proved earlier.

3. Asymptotic smoothness

In this section we prove that the dynamical system generated by (1.1) is asymptotically smooth. We will refrain from imposing boundary conditions until absolutely necessary in the hope of unifying the treatment of (C), (H), and (F).

Lemma 3.1. *The dynamical system* $(\mathcal{H},S_t)$ *generated by* (1.1)–(1.3)*, under any boundary conditions listed in* (C), (H), (F)*, is asymptotically smooth.*

Proof relies on application of Theorem 2.4. For this we need rather extensive background and several auxiliary estimates.

Note that the new variable $z = u - w$, where $(u(t), u_t(t)) = S_t(u_0, u_1)$, $(w(t), w_t(t)) = S_t(w_0, w_1)$ are solutions to (1.1) with initial data taken in bounded set in $B \subset \mathcal{H}$. On the strength of Lemma 1.1 and (1.11) we may assume that there exists $R > 0$ such that

$$\big\|u(t), u_t(t)\big\|_{\mathcal{H}} \leqslant R, \qquad \big\|w(t), w_t(t)\big\|_{\mathcal{H}} \leqslant R, \quad t > 0. \tag{3.1}$$

Note, for the remainder of this treatment, we will denote constants depending on R by $C(R)$. The difference of two trajectories $z = u - w$ solves the following PDE:

$$\begin{aligned} &z_{tt} + \Delta^2 z + \mathcal{G}(z) + \mathcal{F}(z) = 0 \quad \text{in } Q, \\ &z(0) = u_0 - w_0; \qquad z_t(0) = u_1 - w_1 \end{aligned} \tag{3.2}$$

where

$$\mathcal{F}(z) \equiv -\big(f_V(u) - f_V(w)\big), \quad \text{and} \quad \mathcal{G}(z) \equiv d(\mathbf{x})\big(g(u_t) - g(w_t)\big).$$

The above evolution is equipped with appropriate boundary conditions (C), (H), or (F) which will be specified later.

3.1. Multipliers

Ultimately, we will need a pointwise bound (in time) on the functional $E_z(t)$ as defined above. To achieve this bound, we will employ multiplier methods based on specially chosen cut-off functions λ and μ. These functions are taken to be $C^\infty(\Omega)$. Later, we will choose the supports of the derivatives of λ and μ to be contained in the damping region ω, where the damping $g(u_t)$ is effectively localized; the cut-off functions will be chosen in this way so as to reconstruct the full energy $E_z(t)$ via the multipliers, bounded in terms of the damping. However, for now, we can consider $\operatorname{supp}\lambda \subset \Omega$ to be arbitrary.

We define the variables $\phi = \lambda z$ and $\psi = \mu z$. The use of the cut-off functions will produce commutators active in the regions of ω where the cut-off functions are non-constant. Lastly, we will make use of the following notational conventions. First, to describe (a) lower order terms:

$$l.o.t.^f \equiv \sup_{[0,T]} \big\|f(t)\big\|^2_{2-\eta}, \qquad l.o.t._1^f \equiv \sup_{[0,T]} \big\|f(t)\big\|_{2-\eta},$$

where $0 < \eta < 1/2$, and (b) boundary terms: $B.T.^f = \{\Delta f \partial_\nu f - \partial_\nu(\Delta f) f\}$.

Remark 3.1. We note that the use of different notations for lower order terms is necessary in the handling of dissipation estimates. Specifically, we must treat the dissipation terms differently when dealing with asymptotic smoothness type estimates, and the estimates which will ultimately yield the quasistability estimate.

3.1.1. ϕ Multiplier

Let P and Q be two differential operators. We will make use of the commutator symbol given by

$$[P, Q]f = P(Qf) - Q(Pf).$$

We shall work with smooth solutions guaranteed by Theorem 1.2. Multiplying the PDE in (3.2) by λ we arrive at

$$\phi_{tt} + \Delta^2\phi + \lambda\mathcal{G}(z) + \lambda\mathcal{F}(z) = \big[\Delta^2, \lambda\big]z.$$

Now, we employ the multiplier ϕ. This is an equipartition multiplier which allows us to reconstruct the difference between the potential and kinetic energies. The following Green's identities are available [45] for sufficiently smooth functions z and ϕ:

$$\begin{cases} \int\limits_{\Omega} \Delta^2 z\phi = \int\limits_{\Omega} \Delta z\Delta\phi + \int\limits_{\Gamma} (\partial_\nu \Delta z\phi - \Delta z\partial_\nu\phi), & \text{clamped and hinged B.C.,} \\ \int\limits_{\Omega} \Delta^2 z\phi = a(z,\phi) + \beta\int\limits_{\Gamma_1} z^3\phi + \int\limits_{\Gamma_1} (\mathcal{B}_2 z\phi - \mathcal{B}_1 z\partial_\nu\phi), & \text{free B.C.} \end{cases}$$

Using the first formula for clamped or hinged boundary conditions yields:

$$\int\limits_{Q} \{|\Delta\phi|^2 - |\phi_t|^2\} = \int\limits_{Q} [\Delta^2,\lambda]z\phi - \int\limits_{Q} \lambda\{\mathcal{G}(z) + \mathcal{F}(z)\}\phi + \int\limits_{\Sigma} \{\Delta\phi\partial_\nu\phi - \partial_\nu(\Delta\phi)\phi\} - (\phi_t,\phi)\big|_0^T. \tag{3.3}$$

Making use of standard splitting and Sobolev embeddings, we arrive at

$$\int\limits_0^T \{\|\Delta\phi\|^2 - \|\phi_t\|^2\} \leqslant \int\limits_{\Sigma} B.T.^{\phi} + \int\limits_{Q} \big([\Delta^2,\lambda]z\big)\phi + \int\limits_{Q} \lambda\{\mathcal{G}(z) + \mathcal{F}(z)\}\phi + C\big(E(T) + E(0)\big). \tag{3.4}$$

In the case of free boundary conditions, the equipartition of energy takes the form

$$\int\limits_0^T \{a(\phi,\phi) + \beta|\phi|^4_{L_4(\Gamma)} - \|\phi_t\|^2\} \leqslant \int\limits_{\Sigma_1} (\mathcal{B}_1\phi\phi - \mathcal{B}_2\phi\partial_\nu\phi) + \int\limits_{Q} \big([\Delta^2,\lambda]z\big)\phi + \int\limits_{Q} \lambda\{\mathcal{G}(z) + \mathcal{F}(z)\}\phi + C\big(E(T) + E(0)\big). \tag{3.5}$$

We note for all boundary conditions (C), (H), the boundary terms $B.T.^{\phi} \equiv 0$. In the free case (F) we have $\mathcal{B}_1\phi = 0$, $\mathcal{B}_2\phi = 2\beta\phi uw$ where the latter term contributes a lower order term to the estimate.

To continue with our observability estimation, we must explicitly bound the commutator $\int_Q [\Delta^2,\lambda]z\phi$. Purely algebraic calculations give

$$\begin{aligned} [\Delta^2,\lambda]f &= \Delta^2(\lambda f) - \lambda\Delta^2 f \\ &= \big(\Delta^2\lambda\big)f + 2\Delta\lambda\Delta f + 2\big(\nabla\lambda,\nabla(\Delta f)\big) + 2\big(\nabla(\Delta\lambda),\nabla f\big) + 2\Delta(\nabla\lambda\nabla f). \end{aligned} \tag{3.6}$$

The calculation above implies that the commutator $[\Delta^2,\lambda]$ is a differential operator of order three. In order to exploit this in the calculations with the energy, we need to reduce the order of differential operator acting on a solution via integration by parts. This is done below.

This computation makes sole use of Green's Theorem. For the sake of exposition, we do not impose any boundary conditions:

$$\int_\Omega \nabla \Delta u(\phi \nabla \lambda) = -\int_\Omega (\Delta u)\operatorname{div}(\phi \nabla \lambda) + \int_\Gamma (\phi \Delta u)\nabla \lambda \cdot \nu, \tag{3.7}$$

$$\int_\Omega \Delta(\nabla \lambda \nabla u)\phi = -\int_\Omega \nabla(\nabla \lambda \nabla u)\nabla \phi + \int_\Gamma \partial_\nu(\nabla u \nabla \lambda)\phi. \tag{3.8}$$

Note that here we assume that the support of $\nabla\lambda$ is away from the boundary (i.e. λ is constant near the boundary), and thus all of the boundary terms in the above expressions (3.7) and (3.8) will vanish. Moreover,

$$\left|\int_\Omega \nabla \lambda \nabla \Delta u \phi\right| + \left|\int_\Omega \Delta(\nabla \lambda \nabla u)\phi\right| \leqslant C_\lambda \|u\|_2 \|\phi\|_1. \tag{3.9}$$

Hence to conclude our ϕ multiplier estimate, we have the following technical lemma:

Lemma 3.2 *(Preliminary ϕ estimate). Let $\phi \equiv \lambda z$, as defined above, where z solves* (3.2) *with boundary conditions* (C) *or* (H). *Then, there exists* $0 < C < \infty$ *such that*

$$\int_0^T \left\{\|\Delta \phi\|^2 - \|\phi_t\|^2\right\} \leqslant C(T, \lambda) l.o.t.^z + \int_Q \lambda\bigl\{\mathcal{G}(z) + \mathcal{F}(z)\bigr\}\phi + C\bigl(E_z(T) + E_z(0)\bigr). \tag{3.10}$$

In the free case (F)

$$\int_0^T \left\{a(\phi, \phi) + \beta \int_\Gamma \phi^4 - \|\phi_t\|^2\right\}$$
$$\leqslant C(T, \lambda, R) l.o.t.^z + \int_Q \lambda\bigl\{\mathcal{G}(z) + \mathcal{F}(z)\bigr\}\phi + C\bigl(E_z(T) + E_z(0)\bigr). \tag{3.11}$$

Proof. Taking into account (3.9) in (3.4), we have

$$\int_0^T \left\{\|\Delta \phi\|^2 - \|\phi_t\|^2\right\} \leqslant C(T, \lambda) l.o.t.^z + \int_Q \lambda\bigl\{\mathcal{G}(z) + \mathcal{F}(z)\bigr\}\phi + \int_\Sigma B.T.^\phi$$
$$+ \int_\Sigma \left\{\partial_\nu\bigl(\Delta(\lambda z)\bigr)(\lambda z) - \Delta(\lambda z)\partial_\nu(\lambda z) - \partial_\nu(\Delta z)\lambda^2 z\right.$$
$$\left. + 2\lambda z(\Delta z)\partial_\nu z + \lambda^2(\Delta z)\partial_\nu z\right\} + C\bigl(E(T) + E(0)\bigr).$$

Taking into consideration boundary conditions (C) or (H) in (3.4), noting that $B.T.^\phi = 0$ and accounting for the fact that the boundary terms resulting from the commutators vanish leads to the first statement in the lemma. Calculations in the free case are analogous, and result from (3.5) and $\mathcal{B}_1\phi = 0$, $\mathcal{B}_2\phi = 2\beta\phi u w$, where the latter term contributes a lower order term to the estimate:

$$\left|\int_{\Gamma_1} \mathcal{B}_2\phi\phi\right| \leqslant 2\beta \int_{\Gamma_1} |\phi|^2 |u||w| \leqslant 2\beta R^2 \|\phi\|_1^2 \leqslant C(R) l.o.t.^z. \qquad \square$$

3.1.2. Multiplier 2: $h \cdot \nabla\psi$

For the first part of this section, we specify only that $\operatorname{supp}\mu \cap \Gamma = \emptyset$; otherwise, we keep μ as general as possible, specifying it at the last possible moment. Additionally, define a set $M \equiv \operatorname{supp}\nabla\mu = \overline{\{x \in \Omega \mid \mu \not\equiv \text{constant}\}}$. Now, if we multiply (3.2) by μ, and recall that $\psi \equiv \mu z$, we obtain

$$\psi_{tt} + \Delta^2\psi + \mu\mathcal{G}(z) + \mu\mathcal{F}(z) = \left[\Delta^2, \mu\right]z$$

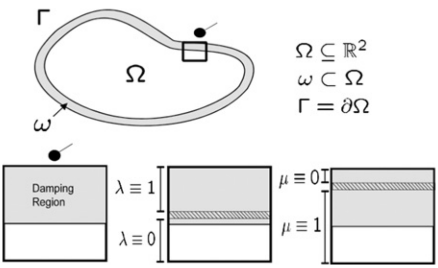
where $\mathcal{G}(z) = d(\mathbf{x})(g(u_t) - g(w_t))$ and $\mathcal{F}(z) = -(f_V(u) - f_V(w))$, as before. We now make use of the multiplier $h \cdot \nabla\psi$, which we write as $h\nabla\psi$ henceforth; there are various choices for the vector field h, situationally dependent, however here we need only take $h = \mathbf{x} - \mathbf{x}_0 \in \mathbb{R}^2$ in order to obtain control on the potential energy of the plate. Now, as in the previous section, we multiply the last equality by our multiplier and use Green's Theorem to obtain

$$\int_Q \left(|\psi_t|^2 + |\Delta\psi|^2\right) \leqslant C\left(E_z(T) + E_z(0)\right) + \int_Q \mu\mathcal{G}(z)(h\nabla\psi) + \int_Q \mu\mathcal{F}(z)(h\nabla\psi) + \int_Q \left[\Delta^2, \mu\right]z(h\nabla\psi).$$

By explicitly writing out the commutator, and taking into account the support of $\nabla\mu$, upon splitting we obtain:

$$\int_Q \left[\Delta^2, \mu\right]z(h\nabla\psi) = \int_0^T \int_M \left[\Delta^2, \mu\right]z(h\nabla\psi) \leqslant C(\mu)\int_0^T \int_M |\Delta z|^2 + C(T, \mu)l.o.t.^z. \tag{3.12}$$

Now, at this point we specify the specific structure of the supports for λ and μ (which up to now have been general). The following picture illustrates our choice for these supports and their relationship to the damping region ω:



We emphasize that (a) the set $M \subset \operatorname{supp}\lambda$ and (b) $\operatorname{supp}\lambda$ and $\operatorname{supp}\mu$ overlap inside the damping region ω and that $\operatorname{supp}\lambda \cup \operatorname{supp}(\mu) = \Omega$. Since $M \subset \{x \in \Omega\colon\ \lambda(x) \equiv 1\}$, we have the following inequality:

$$\begin{aligned}
\int_Q \left[\Delta^2, \mu\right]z(h\nabla\psi) &\leqslant C(\mu)\int_0^T \int_M |\Delta z|^2 + C(\mu, T)l.o.t.^z \\
&\leqslant C(\mu)\int_0^T \int_{\lambda\equiv 1} |\Delta z|^2 + C(\mu, T)l.o.t.^z \\
&\leqslant C(\mu)\int_Q |\Delta\phi|^2 + C(\mu, T)l.o.t.^z.
\end{aligned} \tag{3.13}$$

3.2. Energy recovery estimate

We may now appeal to our calculation with the ϕ multiplier previously, to obtain our preliminary ψ estimate:

Lemma 3.3 *(Preliminary ψ estimate). Let $\psi \equiv \mu z$, as defined above, where z solves* (3.2) *with any boundary conditions under considerations. Moreover, assume* $\operatorname{supp}(\mu)$ *is bounded away from Γ. Then, in the case of clamped* (C) *or hinged* (H) *boundary conditions we have*

$$\int_Q \left(|\psi_t|^2 + |\Delta\psi|^2\right) \leqslant C(\mu,\lambda)\left\{\left(E_z(T)+E_z(0)\right) + \int_Q \mu\left\{\mathcal{G}(z)+\mathcal{F}(z)\right\}(h\nabla\psi) + \int_Q \lambda\left\{\mathcal{G}(z)+\mathcal{F}(z)\right\}\phi + C(T)l.o.t.^z\right\}.$$

In the case of free boundary conditions (F)

$$\int_0^T \left(\|\psi_t\|^2 + a(\psi,\psi)\right) \leqslant C(\mu,\lambda)\left\{\left(E_z(T)+E_z(0)\right) + \int_Q \mu\left\{\mathcal{G}(z)+\mathcal{F}(z)\right\}(h\nabla\psi) + \int_Q \lambda\left\{\mathcal{G}(z)+\mathcal{F}(z)\right\}\phi + C(T)l.o.t.^z\right\}. \tag{3.14}$$

We note that in the nonlinear boundary term associated with the operator $\mathcal{B}_2$ vanishes due to the fact that the support of μ is away from the boundary.

We may now combine the estimates from Lemma 3.2 and Lemma 3.3 to obtain an estimate on the total energy (with either form of boundary conditions (C) or (H) or (F)):

$$\begin{aligned}\int_0^T &\left\{\|\psi_t\|^2 + \|\phi_t\|^2 + a(\phi,\phi) + a(\psi,\psi) + \beta\int_{\Gamma_1}|\phi|^4\right\} \\ &\leqslant C(\mu,\lambda)\left\{\left(E_z(T)+E_z(0)\right) + \int_Q \mu\left\{\mathcal{G}(z)+\mathcal{F}(z)\right\}(h\nabla\psi) \right. \\ &\quad \left. + \int_Q \lambda\left\{\mathcal{G}(z)+\mathcal{F}(z)\right\}\phi + \int_0^T \|\phi_t\|^2 + C(T,R)l.o.t.^z\right\}.\end{aligned}$$

By our choice of supports for μ and λ we note that the LHS of the above equation *overestimates* the total energy $E_z(t)$. On the RHS of the estimate we have the term $\int_Q |\phi_t|^2$, which we replace by $\int_0^T\int_\omega |z_t|^2$ since $\operatorname{supp}\lambda \subset \omega$ and on $\operatorname{supp}\lambda$, $\lambda \leqslant 1$, so we have that

$$\int_Q |\phi_t|^2 \leqslant \int_0^T\int_\omega |z_t|^2.$$

Making the appropriate changes above in Lemma 3.2 and Lemma 3.3, we have the analogous result for the free boundary conditions (F). Hence we can conclude

Lemma 3.4 *(Preliminary energy estimate). For any boundary condition* (C), (H), *or* (F) *we have*

$$\int_0^T E_z(t) \leqslant C(\mu,\lambda)\left\{\big(E_z(T)+E_z(0)\big)+\int_Q \mu\big\{\mathcal{G}(z)+\mathcal{F}(z)\big\}(h\nabla\psi)+\int_Q \lambda\big\{\mathcal{G}(z)+\mathcal{F}(z)\big\}\phi\right\}$$
$$+\int_0^T\int_\omega |z_t|^2 + C(T)l.o.t.^z. \tag{3.15}$$

Remark 3.2. At this point, we impose clamped (C) or hinged (H) boundary conditions, in order to simplify and streamline the analysis. At the end of this section, we discuss the boundary conditions (F).

If we take into account the supports of λ and μ (dropping dependence of the constants on μ, λ, and Ω) then (3.15) with clamped boundary conditions becomes

$$\int_0^T E_z(t) \leqslant C\left\{E_z(T)+E_z(0)+\int_Q \big\{\mathcal{G}(z)+\mathcal{F}(z)\big\}(h\nabla\psi)+\int_Q \big\{\mathcal{G}(z)+\mathcal{F}(z)\big\}z\right\}$$
$$+C\int_0^T\int_\omega |z_t|^2 + C(T)l.o.t.^z. \tag{3.16}$$

Remark 3.3. At this point we pause to point out that the estimate we have shown above in (3.16) will be used in the sections to follow, specifically in the quasistability estimate. In particular, we must handle the damping terms (involving u_t, w_t) differently in the estimation for asymptotic smoothness, versus the estimation for quasistability.

By the assumptions on g in Assumption 1, for every δ there exists $C_\delta > 0$ such that

$$|u_t - w_t|^2 \leqslant \delta + C_\delta\big(g(u_t)-g(w_t)\big)(u_t-w_t).$$

This gives that

$$\int_0^T\int_\omega |z_t|^2 \leqslant T\delta|\Omega| + C(\delta)\int_0^T\int_\omega \big(g(u_t)-g(w_t)\big)z_t$$

or, simplifying, and taking into account $\omega \subset \operatorname{supp} d$ and that $d(\mathbf{x}) \geqslant \alpha_0 > 0$, we have

$$\int_0^T\int_\omega |z_t|^2 \leqslant \delta + C(\delta,T,\Omega)\int_Q \mathcal{G}(z)z_t.$$

So taking into account the last inequality in (3.16), we obtain

$$\int_0^T E_z(t) \leqslant \delta + C\left\{E_z(T) + E_z(0) + \int_0^T \int_\omega \big(\mathcal{G}(z) + \mathcal{F}(z)\big)z \right.$$
$$\left. + \int_Q \big(\mathcal{G}(z) + \mathcal{F}(z)\big)h\nabla\psi + C(\delta, T)\int_Q \mathcal{G}(z)z_t\right\}$$
$$+ C(T)l.o.t.^z$$

where the constant C does not depend on T. Recall, u and w are solutions to (1.1) corresponding to some initial conditions y_1 and y_2, satisfying $S_t y_1 = (u(t), u_t(t))$ and $S_t y_2 = (w(t), w_t(t))$ for the evolution S_t associated to the plate dynamics. We can assume that $y_1, y_2 \in \mathcal{W}_R$ for some $R > R_*$, where the invariant set $\mathcal{W}_R = \{(u, v) \in \mathcal{H}, \mathscr{E}(u, v) \leqslant R\}$. Assuming the solutions u and w are strong, by the invariance of $\mathcal{W}_R$ we have

$$\|u(t)\|_2 + \|u_t(t)\| + \|w(t)\|_2 + \|w_t(t)\| \leqslant C(R), \quad t \geqslant 0, \tag{3.17}$$
$$\|u(t)\|_{C(\Omega)} + \|w(t)\|_{C(\Omega)} \leqslant C(R), \quad t \geqslant 0. \tag{3.18}$$

Recent developments in the area of Hardy–Lizorkin spaces and compensated compactness methods allow one to show the following 'sharp' regularity of the Airy stress function v:

Theorem 3.5 *(Sharp regularity of the Airy stress function). (See [14].)*

$$\|v(u)\|_{W^{2,\infty}} \leqslant C\|u\|_2^2, \qquad \|v(u, w)\|_{W^{2,\infty}} \leqslant C\|u\|_2\|w\|_2$$

where we have denoted $v(u, w) \equiv -\Delta^{-2}[u, w]$ and $\mathscr{D}(\Delta^2) = H^4(\Omega) \cap H_0^2(\Omega)$.

Making use of the above inequalities, we have the estimate

$$\|[v(u), z]\| \leqslant C\|u(t)\|_2^2\|z\|_2 \leqslant C(R)\|z\|.$$

Additionally, we have

$$\|v(u) - v(w)\|_{W^{2,\infty}} = \|v(z, u + w)\|_{W^{2,\infty}} \leqslant C\|z\|_2\big(\|u\|_2 + \|w\|_2\big).$$

Therefore,

$$\begin{aligned} \|\mathcal{F}(z)\| &= \|[u, v(u)] - [w, v(w)] + [z, F_0]\| \\ &= \|[v(u) - v(w), z] + [v(w), z] + [z, F_0]\| \\ &\leqslant C(R)\|z\|_2, \quad t \geqslant 0. \end{aligned}$$

So we obtain

$$\int_0^T \int_\omega \mathcal{F}(z)z \leqslant \int_Q \mathcal{F}(z)z \leqslant \epsilon \int_0^T \|z(t)\|_2^2\,dt + C(T, \epsilon)l.o.t.^z \tag{3.19}$$

and similarly

$$\int_Q \mathcal{F}(z)h\nabla\psi \leqslant C(R)\int_0^T \big\|z(t)\big\|_2\big\|\psi(t)\big\|_1 \leqslant \epsilon\int_0^T \big\|z(t)\big\|_2^2 + C(T,\epsilon)l.o.t.^z \tag{3.20}$$

(where again, dependence of constants on Ω, ω, and h are suppressed). To proceed, we need estimates on the dissipation. By the energy equality

$$E_z(T) + \int_s^T\int_\Omega \mathcal{G}(z)z_t = E_z(s) + \int_s^T\int_\Omega \mathcal{F}(z)z_t, \tag{3.21}$$

we have

$$\int_Q \mathcal{G}(z)z_t \leqslant C(R) + \left|\int_Q \mathcal{F}(z)z_t\right|. \tag{3.22}$$

Taking into account the embedding $H^{2-\eta}(\Omega) \subset C(\Omega)$ for $0<\eta<1$, we see

$$\begin{aligned}\int_Q \mathcal{G}(z)z &\leqslant \int_Q d(\mathbf{x})\left(\big|g(u_t)\big| + \big|g(w_t)\big|\right)|z| \\ &\leqslant C\|z\|_{C(0,T;C(\Omega))}\int_Q d(\mathbf{x})\left(\big|g(u_t)\big| + \big|g(w_t)\big|\right) \\ &\leqslant C\|z\|_{C(0,T;H^{2-\eta}(\Omega))}\int_Q d(\mathbf{x})\left(\big|g(u_t)\big| + \big|g(w_t)\big|\right).\end{aligned}$$

Splitting the region of integration according to $|u_t| \leqslant 1$ and $|u_t| > 1$, and similarly according to $|w_t| \leqslant 1$ and $|w_t| > 1$, we obtain

$$\int_Q d(\mathbf{x})\left(\big|g(u_t)\big| + \big|g(w_t)\big|\right) \leqslant g(1)\|d\|_{L_\infty(\Omega)}\,\mathrm{meas}(Q) + \int_Q d(\mathbf{x})\left(g(u_t)u_t + g(w_t)w_t\right) \leqslant C(R,T).$$

Hence

$$\int_Q \mathcal{G}(z)z \leqslant C(R,T)l.o.t._1^z. \tag{3.23}$$

Now applying Holder's inequality with the exponent $r>1$ we see

$$\int_Q \mathcal{G}(z)h\nabla\psi \leqslant C\sup_{[0,T]}\big\|\nabla\psi(t)\big\|_{r'}\int_Q d(\mathbf{x})^r\left(\big|g(u_t)\big|^r + \big|g(w_t)\big|^r\right)$$

where $\frac{1}{r}+\frac{1}{r'}=1$. Taking $r = 1+\frac{1}{p+1}$, and again splitting the region of integration according to $|u_t| \leqslant 1$ and $|u_t| > 1$, and using the polynomial growth condition imposed on g in Assumption 1, we obtain

$$\int_Q d(\mathbf{x})^r \left|g(u_t)\right|^r \leqslant C(d)\left\{g(1)\,\mathrm{meas}(Q) + \int_Q d(\mathbf{x})\,g(u_t)u_t\right\} \leqslant C(R)(T+1).$$

Since the same computations hold for terms in w, and we have the continuous embedding $H^{1-\delta}(\Omega) \hookrightarrow L_{r'}(\Omega)$ for sufficiently small δ, we have

$$\int_Q \mathcal{G}(z)h\nabla\psi \leqslant C(R,T)l.o.t._1^z. \tag{3.24}$$

Hence by the above estimates, we have

$$\int_0^T E_z(t) \leqslant C\left\{E_z(T) + E_z(0) + \delta + C(R,\delta) + C(\delta)\int_Q \mathcal{F}(z)z_t + C(R,T)\big(l.o.t.^z + l.o.t._1^z\big)\right\}$$

and eventually by (3.21) we have

$$\int_0^T E_z(t) \leqslant C_*\left\{E(T) + \delta + C(R,\delta) + C(\delta)\left|\int_0^T\int_\Omega \mathcal{F}(z)z_t\right| + C(R,T)\big(l.o.t.^z + l.o.t._1^z\big)\right\} \tag{3.25}$$

where we write C_* to emphasize that this constant *does not* depend on T. If we integrate (3.21) over $(0,T)$ with respect to the variable s, and take into account (3.25), we may choose T sufficiently large $(T > 2C_*)$ and ϵ sufficiently small (with respect to T) such that

Lemma 3.6 *(Asymptotic smoothness estimate).*

$$E_z(T) \leqslant \epsilon + \frac{C(R,\epsilon)}{T}\left(1 + \left|\int_Q \mathcal{F}(z)z_t\right| + \left|\int_0^T\int_s^T\int_\Omega \mathcal{F}(z)z_t\right|\right) + C(\epsilon,R,T)\big(l.o.t.^z + l.o.t._1^z\big). \tag{3.26}$$

3.2.1. Completion of the proof of Lemma 3.1 – asymptotic smoothness

We are now in a position to prove Lemma 3.1 – necessary for the proof of Theorem 1.3 on the existence of a compact attracting set $\mathbf{A}_R$. For this we shall invoke the abstract Theorem 2.4.

To apply Theorem 2.4 we need to construct a functional $\Phi_{\epsilon,R,T}$ such that

$$\liminf_{m\to\infty}\liminf_{n\to\infty}\Phi_{\epsilon,R,T}(y_n,y_m) = 0$$

for every sequence $\{y_n\}$ from B (following from Theorem 2.4). The functional will contain "noncompact and not-small" terms in the inequality (3.26). More specifically, for any initial data $U_0 = (u_0,u_1)$, $W_0 = (w_0,w_1) \in B$ we define

$$\widetilde{\Phi}_{\epsilon,R,T}(U_0,W_0) = \left|\int_0^T \big(\mathcal{F}(z),z_t\big)\right| + \left|\int_0^T\int_t^T \big(\mathcal{F}(z),z_t\big)\right|$$

where the trajectory $z = u - w$ has initial data $U_0 - W_0$. The key to compensated compactness is the following representation for the bracket:

$$\begin{aligned}\big(\mathcal{F}(z), z_t\big) = \frac{1}{4}\frac{d}{dt}\big\{-\big\|\Delta v(u)\big\|^2 - \big\|\Delta v(w)\big\|^2 + 2\big([z,z], F_0\big)\big\} - \big(\big[v(w), w\big], u_t\big)\\ - \big(\big[v(u), u\big], w_t\big).\end{aligned} \tag{3.27}$$

Integrating the above expression in time and evaluating on the difference of two solutions $z^{n,m} = w^n - w^m$, where $w^i \rightharpoonup w$ yields:

$$\begin{aligned}\lim_{n\to\infty}\lim_{m\to\infty}\int_t^T \big(\mathcal{F}\big(z^{n,m}\big), z_t^{n,m}\big) = \frac{1}{2}\big\{\big\|\Delta v(w)(t)\big\|^2 - \big\|\Delta v(w)(T)\big\|^2\big\} \\ - \lim_{n\to\infty}\lim_{m\to\infty}\int_0^T \big\{\big(\big[v\big(w^n\big), w^n\big], w_t^m\big) + \big(\big[v\big(w^m\big), w^m\big], w_t^n\big)\big\},\end{aligned} \tag{3.28}$$

where we have used (a) the weak convergence in $H^2(\Omega)$ of $z^{n,m}$ to 0, and (b) compactness of $\Delta v(w)$ from $H^2(\Omega) \to L_2(\Omega)$. The iterated limit in (3.28) is handled via iterated weak convergence, as follows:

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T \big\{\big(\big[v\big(w^n\big), w^n\big], w_t^m\big) + \big(\big[v\big(w^m\big), w^m\big], w_t^n\big)\big\}\\ &= 2\int_t^T \big(\big[v(w), w\big], w_t\big)\\ &= \frac{1}{2}\big\|\Delta v(w)(t)\big\|^2 - \frac{1}{2}\big\|\Delta v(w)(T)\big\|^2.\end{aligned}$$

This yields the desired conclusion, that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_t^T \big(\mathcal{F}\big(z^{n,m}\big), z_t^{n,m}\big) = 0.$$

The second integral term in $\widetilde{\Phi}$ is handled similarly. As a consequence we obtain

$$\lim_{m\to\infty}\inf\lim_{n\to\infty}\inf \widetilde{\Phi}_{\epsilon,R,T}(y_n, y_m) = 0.$$

Now, we define

$$\Phi_{\epsilon,R,T} = \widetilde{\Phi} + \big(l.o.t.^z + l.o.t._1^z\big),$$

and noting that the terms $(l.o.t.^z + l.o.t._1^z)$ in (3.26) are compact with respect to $H^2(\Omega)$ via the Sobolev embeddings, the final conclusion follows by taking T sufficiently large and ϵ sufficiently small. This concludes the proof of smoothness estimate required by Theorem 2.4. Thus, the dynamical system is asymptotically smooth.

3.2.2. Completion of the proof of Theorem 1.3 – local attractors

For this we refer to Theorem 2.2. The Lyapunov function $V(u_1, u_2) = \mathcal{E}(u_1, u_2)$ satisfies all the properties required by this theorem. On the strength of Lemma 1.1, $\mathcal{E}(u_1, u_2)$ is bounded on bounded sets. The set $\{(u_1, u_2) \in \mathcal{H},\ \mathscr{E}(u_1, u_2) \leqslant R\}$ is positively invariant by (1.13) and bounded (Lemma 1.1). Thus, Theorem 2.2 applies and gives the existence of local attractors $\mathbf{A_R}$ – the statement in Theorem 1.3.

4. Regularity and finite dimensionality of the attractor

Let $\mathbf{A}$ (resp. $\mathbf{A}_R$) be the global (resp. local) attractor corresponding to the flow S_t, as established in Section 3. To prove finiteness of the fractal dimension of the set $\mathbf{A}$ (resp. $\mathbf{A}_R$) we shall use Theorem 2.6 which is based on the *quasistability* estimate formulated below.

4.1. Quasistability estimate

We shall follow a general program developed in [16] and supported by PDE estimates derived in previous sections and specific to localized dissipation.

With the previous notation, we state the following lemma which gives a preliminary estimate for quasistability inequality:

Lemma 4.1. *Let $z \equiv u - w$ where $(u(t), u_t(t)), (w(t), w_t(t)) \in \mathbf{A}_R$. Then, there exists $T_0 > 0$ such that for all $-\infty < s < \infty$ the following inequality holds*:

$$E_z(s + T_0) + \int_s^{s+T_0} E_z(\tau) \leqslant C(\mathbf{A}_R, T_0) D_s^{s+T_0} + C(\mathbf{A}_R, T_0) \sup_{\tau \in [s, s+T_0]} \|z(\tau)\|_{2-\eta}^2 \tag{4.1}$$

for $\eta > 0$, where

$$D_{t_1}^{t_2} \equiv \int_{t_1}^{t_2} \int_\Omega d(\mathbf{x}) \big(g(u_t) - g(w_t)\big) z_t.$$

This lemma will be the key step in showing quasistability of the attractor. It states that the total energy can be recovered from the damping and lower order terms.

Thus the crux of the proof of regularity and dimensionality of the attractor reduces to the demonstration of Lemma 4.1. We also note that in comparison with the asymptotic smoothness inequality, the inequality in Lemma 4.1 is more demanding. This is due to necessity of keeping *at least quadratic forms* in the lower order terms. This very demand forces the damping to have at least linear growth at the origin $g'(0) > 0$. (Such restriction is typical – if not necessary – whenever regularity or finite dimensionality of attractors becomes a concern).

4.2. Preparation for the proof of Lemma 4.1 – quasistability estimate

In the proof of Lemma 4.1, we will make use of the recovery estimate in (3.16) and the energy relation (3.21) as our main tools. Beginning with (3.16), and taking into account estimates involving $\mathcal{F}(z)$ in (3.19), (3.20) and the linear growth condition $g'(0) > 0$ in (3.16), we arrive at

$$\int_0^T \int_\Omega |z_t|^2 \leqslant C D_0^T(z).$$

Applying the above inequality in (3.16) gives:

$$\int_0^T E_z(\tau) \leqslant C\left\{D_0^T(z) + E_z(T) + E_z(0) + \left|\int_Q \mathcal{G}(z)z\right| + \left|\int_Q \mathcal{G}(z)h\nabla z\right|\right\} + C(R,T)l.o.t.^z. \tag{4.2}$$

Now, in tackling quadratic dependence of the dissipation terms, we give the following proposition

Proposition 4.2. *Let assumptions of Theorem* 1.4 *be satisfied, and take z be a solution to* (3.2)*. Then there exists $\delta > 0$ such that*

$$\left|\int_Q \mathcal{G}(z)z\right| \leqslant \delta D_0^T(z) + C(\delta, R, T)\sup_{[0,T]}\|z\|^2_{2-\eta}, \quad 0 < \eta < 2-\gamma, \tag{4.3}$$

$$\left|\int_Q \mathcal{G}(z)h\nabla z\right| \leqslant \delta D_0^T(z) + C(\delta, R, T)\sup_{[0,T]}\|z\|^2_{2-\eta}, \quad 0 < \eta < 1-\gamma \tag{4.4}$$

where $\mathcal{G}(z) = d(\mathbf{x})(g(u_t) - g(w_t))$ and $D_0^T(z) = \int_Q \mathcal{G}(z)z_t$.

Proof. We note that the assumptions on the damping function g (namely, montonicity and the polynomial growth condition in Assumption 1) imply that

$$\frac{g(s_2) - g(s_1)}{s_2 - s_1} \leqslant C\left[1 + g(s_1)s_1 + g(s_2)s_2\right]^\gamma. \tag{4.5}$$

Using the Jensen inequality we estimate

$$|z| \leqslant \delta|z_t| + C(\delta)\frac{|z|^2}{|z_t|}.$$

The above, along with (4.5), gives

$$\left|\int_Q \mathcal{G}(z)z\right| \leqslant \delta D_0^T(z) + C(\delta, M)\int_Q d(x)\left(1 + \left(g_0(u_t)u_t\right)^\gamma + \left(g_0(w_t)w_t\right)^\gamma\right)|z|^2.$$

Now, applying the Holder inequality with exponent $p = \gamma^{-1}$ and Sobolev's embedding $H^{2-\eta}(\Omega) \subset L_{\frac{2}{1-\gamma}}(\Omega)$, and taking into account energy equality (1.13) we arrive at

$$\left|\int_Q \mathcal{G}(z)z\right| \leqslant \delta D_0^T(z) + C(\delta, R)l.o.t.^z.$$

The inequality in (4.4) can be shown analogously. $\square$

So, taking into account (4.3) and (4.4) in (4.2) we obtain

$$\int_0^T E_z(\tau) \leqslant C\big\{D_0^T(z) + E_z(T) + E_z(0)\big\} + C(R,T) l.o.t.^z.$$

We note that C does not depend on T, and $l.o.t.^z$ is of quadratic order. By using semigroup property and reiterating the same argument on the intervals $[s, s+T]$ one obtains

$$\int_s^{T+s} E_z(\tau) \leqslant C\big\{D_s^{T+s}(z) + E_z(T+s) + E_z(s)\big\} + C(R,T) l.o.t.^z(s, T+s) \tag{4.6}$$

where $l.o.t.^z(s, T+s)$ denote lower order terms collected over the interval $[s, T+s]$.

In order to prove (4.1), we have to handle the noncompact term $(\mathcal{F}(z), z_t)$. A technical calculation based on the symmetry properties of von Karman bracket gives us the following proposition whose proof is given in [14].

Proposition 4.3. *If $u, w \in C([0,t]; H^2(\Omega)) \cap C^1([0,t]; L_2(\Omega))$ and $z = u - w$ then*

$$\big(\mathcal{F}(z), z_t\big) = \frac{1}{4}\frac{d}{dt} Q(z) + \frac{1}{2} P(z) \tag{4.7}$$

where

$$\begin{aligned} Q(z) &= \big(v(u) + v(w), [z,z]\big) - \big\|\Delta v(u+w, z)\big\|^2, \\ P(z) &= -\big(u_t, \big[u, v(z,z)\big]\big) - \big(w_t, \big[w, v(z,z)\big]\big) - \big(u_t + w_t, \big[z, v(u+w, z)\big]\big). \end{aligned} \tag{4.8}$$

Now, we can state the following lemma:

Lemma 4.4. *Let $u(\tau)$ and $w(\tau)$ be two functions from the class*

$$C\big([s,t]; H_0^2(\Omega)\big) \cap C^1\big([s,t]; L_2(\Omega)\big)$$

for $s, t \in \mathbb{R}$, $s < t$, such that

$$\big\|u(\tau)\big\|_2^2 + \big\|u_t(\tau)\big\|^2 \leqslant R^2, \qquad \big\|w(\tau)\big\|_2^2 + \big\|w_t(\tau)\big\|^2 \leqslant R^2, \quad \tau \in [s,t].$$

Let $z(\tau) = u(\tau) - w(\tau)$. Then for $\eta > 0$,

$$\left|\int_s^t \big(\mathcal{F}(z), z_t\big)\right| \leqslant C(R) \sup_{\tau\in[s,t]} \big\|z(\tau)\big\|_{2-\eta}^2 + C(R) \int_s^t P(z). \tag{4.9}$$

Proof. The inequality follows from the basic properties of von Karman bracket [14] and the decomposition in Proposition 4.3. □

4.3. Completion of the proof of Lemma 4.1 – quasistability estimate

Let $\gamma_u = \{(u(t), u_t(t)) \colon t \in \mathbb{R}\}$ and $\gamma_w = \{(w(t), w_t(t)) \colon t \in \mathbb{R}\}$ be trajectories from the attractor $\mathbf{A}_R$. It is clear that for the pair $w(t)$ and $u(t)$ satisfy the hypotheses of Lemma 4.4 for every interval $[s,t]$. We shall estimate the energy $E_z(t)$ of $z(t) \equiv u(t) - w(t)$. Here we critically use the estimates for the noncompact term involving $\mathcal{F}(z)$. By (4.9), we have

$$\left| \int_s^t \big(\mathcal{F}(z), z_t\big) \right| \leqslant C(R) \sup_{\tau \in [s,t]} \big\| z(\tau) \big\|_{2-\eta}^2 + C(R) \int_s^t P(z) \tag{4.10}$$

for all $-\infty < s \leqslant t < +\infty$. Our main goal is to handle the second term on the RHS of (4.10) which is of *critical regularity*. To accomplish this we shall use the already established *compactness* of the attractor. We recall the attractor is bounded in $H^2(\Omega) \times L_2(\Omega)$. Our ultimate goal is to obtain the boundedness of the attractor in $H^4(\Omega) \times H^2(\Omega)$. The starting point is formula (4.10).

Since for every $\tau \in \mathbb{R}$, the element $u_t(\tau)$ belongs to a compact set in $L_2(\Omega)$, by density of $H_0^2(\Omega)$ in $L_2(\Omega)$ we can assume, without a loss of generality, that for every $\epsilon > 0$ there exists a finite set $\{\phi_j\} \subset H_0^2(\Omega)$, $j = 1, 2, \dots, n(\epsilon)$, such that for all $\tau \in \mathbb{R}$ we can find indices $j_1(\tau)$ and $j_2(\tau)$ so that

$$\big\| u_t(\tau) - \phi_{j_1(\tau)} \big\| + \big\| w_t(\tau) - \phi_{j_2(\tau)} \big\| \leqslant \epsilon, \quad \text{for all } \tau \in \mathbb{R}. \tag{4.11}$$

Let $P(z)$ be given by (4.8) with the pair $w(t)$ and $u(t)$ and

$$P_{j_1,j_2}(z) \equiv -\big(\phi_{j_1}, \big[u, v(z,z)\big]\big) - \big(\phi_{j_2}, \big[w, v(z,z)\big]\big) - \big(\phi_{j_1} + \phi_{j_2}, \big[z, v(u+w, z)\big]\big)$$

where $z(t) = w(t) - u(t)$. It can be easily shown that for all $j_1, j_2 \leqslant n(\epsilon)$,

$$\big\| P\big(z(\tau)\big) - P_{j_1(\tau), j_2(\tau)}\big(z(\tau)\big) \big\| \leqslant \epsilon C(R) \big\| z(\tau) \big\|_2^2 \tag{4.12}$$

uniformly in $\tau \in \mathbb{R}$.

We shall need negative norm estimates for von Karman brackets. Starting with the estimate (6.6) on p. 141 [12] or (1.4.17), p. 41 [14],

$$\big\| [u, w] \big\|_{-2} \leqslant C \|u\|_{2-\beta} \|w\|_{1+\beta}, \quad \forall \beta \in [0, 1) \tag{4.13}$$

and exploiting elliptic regularity one obtains

$$\begin{aligned} \big\| \big[u, v(z, w)\big] \big\|_{-2} &\leqslant C \|u\|_{2-\beta} \big\| \Delta^{-2}[z, w] \big\|_{\beta+1} \leqslant C \|u\|_{2-\beta} \big\| \Delta^{-2}[z, w] \big\|_2 \\ &\leqslant C \|u\|_{2-\beta} \big\| [z, w] \big\|_{-2} \leqslant C \|u\|_{2-\beta} \|z\|_{2-\beta_1} \|w\|_{1+\beta_1} \end{aligned} \tag{4.14}$$

where above inequality holds for any $\beta, \beta_1 \in [0, 1)$.

Recalling the additional smoothness of $\phi_j \in H_0^2(\Omega)$, along with the estimate in (4.14) applied with $\beta = \beta_1 = \eta$, and accounting the structure of P_j terms one obtains:

$$\big\| P_{j_1,j_2}(z) \big\| \leqslant C(R) \big(\|\phi_{j_1}\|_2 + \|\phi_{j_2}\|_2 \big) \big\| z(\tau) \big\|_{2-\eta}^2 \tag{4.15}$$

for $\eta > 0$. So we have for any $\eta > 0$,

$$\sup_{j_1, j_2} \big\| P_{j_1,j_2}(z) \big\| \leqslant C(\epsilon) \big\| z(\tau) \big\|_{2-\eta}^2 \tag{4.16}$$

where $C(\epsilon) \to \infty$ when $\epsilon \to 0$. Taking into account (4.12) and (4.16) in (4.10) we obtain

$$\left| \int_s^t (\mathcal{F}(z), z_t) \right| \leqslant C(\epsilon, T, R) \sup_{\tau \in [s,t]} \|z(\tau)\|_{2-\eta}^2 + \epsilon \int_s^t \|z(\tau)\|_2^2 \tag{4.17}$$

for all $s \in \mathbb{R}$ with $\eta > 0$ and $t > s$.

Using the energy relation (3.21), we find from (4.17) that

$$E_z(s) \leqslant E_z(t) + \int_s^t \int_\Omega \mathcal{G}(z) z_t + C(\epsilon, R) \sup_{\tau \in [s,t]} \|z(\tau)\|_{2-\eta}^2 + \epsilon \int_s^t \|z(\tau)\|_2^2 \, d\tau \tag{4.18}$$

and similarly

$$E_z(t) \leqslant E_z(s) + C(R) \sup_{\tau \in [s,t]} \|z(\tau)\|_{2-\eta}^2 + \epsilon \int_s^t \|z(\tau)\|_2^2 \, d\tau \tag{4.19}$$

for all $s \leqslant t$. Now, if we apply (4.6) on each subinterval $[s, s + T_0]$, we have

$$\int_s^{s+T_0} E_z(\tau) \leqslant C\left\{D_s^{s+T_0}(z) + \left(E_z(s+T_0) + E_z(s)\right)\right\} + C(R, T_0) \sup_{\tau \in [s,s+T_0]} \|z(\tau)\|_{2-\eta}^2.$$

Taking into account (4.18) in the last inequality and choosing ϵ sufficiently small we arrive at

$$\int_s^{s+T_0} E_z(\tau) \leqslant C\left\{D_s^{s+T_0}(z) + E_z(s+T_0) + C(R, T_0) \sup_{\tau \in [s,s+T_0]} \|z(\tau)\|_{2-\eta}^2\right\}.$$

Now, integrating (4.19) we have

$$T_0 E_z(s+T_0) \leqslant \int_s^{s+T_0} E_z(\tau)\, d\tau + C(\epsilon, R, T_0) \sup_{\tau \in [s,s+T_0]} \|z(\tau)\|_{2-\eta}^2 + \epsilon \int_s^{s+T_0} \|z(\tau)\|_2^2 \, d\tau;$$

considering the two previous inequalities, taking T_0 sufficiently large, we have

$$E_z(s+T_0) + \int_s^{s+T_0} E_z(\tau) \leqslant C(\mathbf{A}_R, T_0) D_s^{s+T_0}(z) + C(\mathbf{A}_R, T_0) \sup_{\tau \in [s,s+T_0]} \|z(\tau)\|_{2-\eta}^2$$

which gives (4.1) and thus proves Lemma 4.1.

4.4. Proof of regularity and finite dimensionality of the attractor – Theorem 1.4

Having established Lemma 4.1 we now proceed with the proof of the quasistability estimate. This is done as follows.

By (4.1), (4.17) and energy relation (3.21) written on the interval $[s, s+T_0]$ there exists a constant $0<\mu<1$ (μ depending on T_0 and R) such that $z(t)$ satisfies the following estimate

$$E_z(s+T_0) \leqslant \mu E_z(s) + C(T_0) \sup_{\tau\in[0,T_0]} \big\| z(s+\tau)\big\|^2_{2-\eta}. \tag{4.20}$$

Indeed, we have

$$D_s^{s+T_0} = E_z(s) - E_z(s+T_0) + \int_s^{s+T_0} \big(\mathcal{F}(z), z_t\big)\, d\tau.$$

Now, plugging in the above equality into (4.1), and utilizing the bound in (4.17), we have

$$E_z(s+T_0) + \int_s^{s+T_0} E_z(\tau)\, d\tau \leqslant C(\mathbf{A}_R, T_0)\big[E_z(s) - E_z(s+T_0)\big] + C(\mathbf{A}_R, T_0, \epsilon) \sup_{\tau\in[0,T_0]} \big\| z(s+\tau)\big\|^2_{2-\eta}$$
$$+ \epsilon \int_s^{s+T_0} E_z(\tau)\, d\tau.$$

Absorbing the ϵ term for ϵ sufficiently small, and rearranging terms, we have

$$E_z(s+T_0) \leqslant \frac{C(\mathbf{A}_R, T_0)}{1+C(\mathbf{A}_R, T_0)} E_z(s) + C(\mathbf{A}_R, T_0) \sup_{\tau\in[0,T_0]} \big\| z(s+\tau)\big\|^2_{2-\eta}. \tag{4.21}$$

We then note that (4.20) yields

$$E_z\big((m+1)T_0\big) \leqslant \gamma^m E_z(mT_0) + C(\mathbf{A}_R, T_0) b_m, \quad m = 0, 1, 2, \ldots$$

with $0<\gamma=\gamma(\mathbf{A}_R, T_0)<1$, where

$$b_m \equiv \sup_{\tau\in[mT,(m+1)T_0]} \big\| z(\tau)\big\|^2.$$

This yields

$$E_z(mT_0) \leqslant \gamma^m E_z(0) + c\sum_{l=1}^{m} \gamma^{m-l} b_{l-1}.$$

Since $\gamma<1$, by a standard argument [14, pp. 745–747], there exists constants C_1, C_2 and σ possibly depending on R such that for all $t \geqslant 0$ we have

$$E_z(t) \leqslant C_1 E_z(0) e^{-\sigma t} + C_2 \sup_{\tau\in[0,t]} \big\| z(\tau)\big\|^2_{2-\eta}$$

which yields (2.2). Finally, on the strength of Theorem 2.6, applied with $B = \mathbf{A}_R$, $\mathcal{H} = \mathscr{D}(\mathcal{A}^{1/2}) \times L_2(\Omega)$, $\mathcal{H}_1 = H^{2-\eta}(\Omega) \times \{0\}$ we conclude that $\mathbf{A}_R$ has a finite fractal dimension. Additionally, Theorem 2.6 guarantees that

$$\|u_{tt}(t)\|^2 + \|u_t(t)\|_2^2 \leqslant C \quad \text{for all } t \in \mathbb{R}.$$

Hence, for $u_t \in H^2(\Omega) \subset C(\Omega)$, we have $g(u_t) \in C(\Omega) \subset L_2(\Omega)$ by the continuity of g. Hence, elliptic regularity theory for $\Delta^2 u = -u_{tt} - d(x)\, g(u_t) - f(u)$ with the boundary conditions (C), (H), or (F) give that

$$\|u(t)\|_4^2 \leqslant C \quad \text{for all } t \in \mathbb{R}.$$

We have now completed the proof of Theorem 1.4.

5. C^∞ regularity of attractors – proof of Theorem 1.5

We have shown in Theorem 1.4 that the attractor is a bounded set in $H^4(\Omega) \times H^2(\Omega)$ with appropriate boundary conditions. The goal of Theorem 1.5 is to establish C^∞ regularity for C^∞ data. This is done by induction. In fact, the method of proof follows from a general inductive argument presented in Theorem 8.7.4, p. 427 [12]. Since this general framework requires that (i) the dissipation be linear and (ii) the dissipation directly controls the kinetic energy, which is not the case with localized damping, we need some adaptation of this argument.

We begin by considering the evolution corresponding to smooth solutions evolving on the attractor and obtained in Theorem 1.4. Let us denote $z^{(m)}(t) \equiv \frac{d^m}{dt^m} u(t)$, $m = 1, 2, \ldots$. We must consider $m = 1$ and $m = 2$ in our base case. When $m = 1$ then $z(t) = z^{(1)}(t)$ satisfies

$$\begin{gathered} z_{tt} + Az + d(x)\, g'(u_t) z_t = f'(u) z, \\ f'(u) z = 2\big[\Delta^{-2}[u, z], u\big] + \big[v(u) + F_0, z\big]. \end{gathered} \tag{5.1}$$

This equation is obtained rigorously by considering first $z^h(t) \equiv h^{-1}[u(t+h) - u(t)]$, and then passing the limit, when $h \to 0$, as on p. 428 [12]. We recall that we work under the assumption that $g'(s) \geqslant c_0 > 0$, such that the (linear) evolution corresponding to

$$w_{tt} + Aw + d(x)\, g'(u_t) w_t = 0$$

is exponentially stable. This is to say, for all $s \leqslant t$,

$$W(t) \equiv \big(w(t), w_t(t)\big) = U(t, s) W(s)$$

where the evolution operator $U(t, s)$ satisfies

$$\|U(t, s)\|_{\mathscr{L}(\mathcal{H})} \leqslant C e^{-\omega(t-s)}, \quad \omega > 0. \tag{5.2}$$

This estimate follows from the same multiplier method used in the proof of asymptotic smoothness, as presented in Section 3. Note that due to the regularity $u_t \in H^2(\Omega)$, one has $d(x)\, g'(u_t) \in C(\Omega \times \mathbb{R})$, which is needed in proving (5.2).

Now, consider $m=2$, and denote $z=z^{(2)}=u_{tt}$.

$$z_{tt}+Az+d(x)\,g'(u_t)z_t=-d(x)\,g''(u_t)u_{tt}^2+f'(u)z+R(u,u_t) \tag{5.3}$$

$$\equiv f'(u)z+G(u_t,u_{tt})+R(u,u_t) \tag{5.4}$$

where we have

$$f'(u)z=\big[v(u)+F_0,z\big]+2\big[\Delta^{-2}[u,z],u\big],$$
$$R(u,u_t)=2\big[\Delta^{-2}[u_t,u_t],u\big]+4\big[\Delta^{-2}[u,u_t],u_t\big],$$
$$G(u_t,u_{tt})=-d(x)\,g''(u_t)u_{tt}^2=-d(x)\,g''(u_t)u_{tt}z.$$

We now write $Z(t)=(z(t),z_t(t))$ and decompose $Z(t)=Z_1(t)+Z_2(t)$ where

$$Z_1(t)=U(t,s)Z(s)+\int_s^t U(t,\tau)\big\{0,D(\tau)\big\}\,d\tau,$$

$$Z_2(t)=\int_s^t U(t,\tau)\big\{0,B(\tau)\big\}\,d\tau$$

with

$$D(t)=2\big[\Delta^{-2}\big[u(t),z_1(t)\big],u(t)\big]+R\big(u(t),u_t(t)\big)+G\big(u_t(t),z_1(t)\big),$$
$$B(t)=\big[v(u)(t)+F_0,z_1(t)+z_2(t)\big]+2\big[\Delta^{-2}\big[u(t),z_2(t)\big],u(t)\big]+G\big(u_t(t),z_2(t)\big).$$

Since the trajectories lie on the attractor and the evolution is exponentially stable, letting $s\to-\infty$ (here we perform the argument with approximation by finite difference – see p. 429 [14]) one obtains for $t\in\mathbb{R}$,

$$\big\|Z_1(t)\big\|_{\mathcal{H}}\leqslant C\int_{-\infty}^t e^{-\omega(t-s)}\big\|D(\tau)\big\|\,ds. \tag{5.5}$$

As shown in [12], the already obtained regularity of the elements on the attractor renders $R(u,u_t)$ subcritical. The remaining components of $D(t)$ are "almost subcritical", meaning they consist of subcritical terms and small quantities of critical norms. More specifically, the following estimates hold:

$$\big\|\big[v(u)+F_0,z\big]\big\|\leqslant C\|z\|_2\big[\|u\|_2^2+\|F_0\|_4\big]\leqslant C_R\|z\|_2,$$
$$\big\|\big[\Delta^{-2}[u,z],u\big]\big\|\leqslant C\|u\|_{W^{2,\infty}}\big\|[u,z]\big\|_{-2}\leqslant C\|u\|_4\|u\|_{\beta+1}\|z\|_{2-\beta}\leqslant C_R\|z\|_{2-\beta}\leqslant C_{R,\epsilon}+\epsilon\|z\|_2,$$
$$\big\|R(u,u_t)\big\|\leqslant C\|u_t\|_2^2\|u\|_2\leqslant C_R. \tag{5.6}$$

Contribution of the nonlinear dissipation G is estimated as follows:

Since $u_t(t) \in H^2(\Omega) \subset C(\Omega)$ and $u_{tt}(t) \in L_2(\Omega)$, we have

$$\begin{aligned} \left\|d(x)\, g''(u_t)u_{tt}^2\right\|_{L_2(\Omega)} &\leqslant C\left\|u_{tt}^2\right\|_{L_2(\Omega)} \leqslant C\|u_{tt}z\|_{L_2(\Omega)} \leqslant C\|z\|_{C(\Omega)}\|u_{tt}\|_{L_2(\Omega)} \\ &\leqslant C_R\|z\|_{1+\epsilon} \leqslant C_{R,\epsilon} + \epsilon\|z\|_2, \end{aligned} \tag{5.7}$$

which is almost subcritical. This gives almost subcritical estimates for the D term:

$$\left\|D(s)\right\| \leqslant C_{R,\epsilon} + \epsilon\left\|z_1(s)\right\|_2 \tag{5.8}$$

and from (5.5), absorbing the ϵ term

$$\left\|Z_1(t)\right\|_{\mathcal{H}} \leqslant C_R, \quad t \in \mathbb{R}. \tag{5.9}$$

Regarding the variable Z_2, the term $B(t)$ remains critical. This follows from the first estimate in (5.6). However, the following equality is useful:

$$\begin{aligned} \int_s^t \left(\left[v(u)+F_0, z\right], z_t\right) &= \frac{1}{2}\int_s^t \left(v(u)+F_0, \frac{d}{dt}[z,z]\right) \\ &= \frac{1}{2}\left(v(u)+F_0,[z,z]\right)\Big|_s^t - \frac{1}{2}\int_s^t \left(\frac{d}{dt}v(u),[z,z]\right) \\ &= \frac{1}{2}\left(v(u)+F_0,[z,z]\right)\Big|_s^t - \frac{1}{2}\int_s^t \left(\left[\frac{d}{dt}v(u), z\right], z\right). \end{aligned} \tag{5.10}$$

But from sharp Airy's regularity (3.5) and noting $\frac{d}{dt}v(u) = 2\Delta^{-2}[u,u_t]$,

$$\left(\left[\frac{d}{dt}v(u), z\right], z\right) \leqslant \|z\|\|z\|_2\|u\|_2\|u_t\|_2 \leqslant C_R\|z\|_2.$$

This leads to

$$\int_s^t \left(\left[v(u)+F_0, z\right], z_t\right) \leqslant C_R \int_s^t \|z\|_2 + C_R\|z\|_2\big|_s^t. \tag{5.11}$$

Note that the last term is just linear (not quadratic).

Returning to the Z_2 variable, we have

$$z_{2,tt} + Az_2 + d(x)\, g'(u_t)z_{2,t} = B(t), \qquad Z_2(s) = 0.$$

Expanding $B(t)$ we obtain

$$B(t) = \left[v(u)+F_0, z_1\right] + \left[v(u)+F_0, z_2\right] + 2\left[v(u,z_2), u\right] + G(u_t, z_2)$$

and using (5.9), we have

$$\left\|\left[v(u)+F_0, z_1\right](t)\right\| \leqslant C_R\left\|z_1(t)\right\|_2 \leqslant C_R. \tag{5.12}$$

From the second estimate in (5.6), (5.7), (5.12) and (5.11)

$$\begin{aligned}\int_s^t \left(B(\tau), z_{2,t}\right) &\leqslant \int_s^t \|z_{2,t}\|\left(C_R+\epsilon\|z_2\|_2\right)+\int_s^t \left(\left[v(u)+F_0, z_2\right], z_{2,t}\right) \\ &\leqslant C_{R,\epsilon}(1+t-s)+\epsilon \sup_{s<\tau\leqslant t}\left[\left\|z_2(\tau)\right\|_2^2+\left\|z_{2,t}(\tau)\right\|^2\right].\end{aligned} \tag{5.13}$$

Tracing the arguments which yield Lemma 4.1, it follows that there exists $T>0$ such that for all $s \in \mathbb{R}$,

$$\begin{aligned}&E_2(T+s)+\int_s^{T+s} E_2(\tau) \\ &\quad\leqslant C_T\left[E_2(s)-E_2(T+s)\right]+C_T\int_s^{s+T}\left\{\left(B(\tau), z_{2,t}\right)+\left\|B(\tau)\right\|\cdot\left\|z_2(\tau)\right\|_1\right\}+C_{R,T}\end{aligned}$$

where $E_2(s)$ is the energy corresponding to z_2, $E_2(s)=\|\Delta z_2(s)\|^2+\|z_{2,t}(s)\|^2$. Due to subcriticality of the forcing term $B(\tau)z_{2,t}(\tau)$ and the subcriticality of $\|z_2\|_1 \leqslant \epsilon\|z\|_2+C_{R,\epsilon}$ (5.9) one obtains the quasistability estimate

$$E_2(T+s)+\int_s^{T+s} E_2(\tau) \leqslant C_T\left(E_2(s)-E_2(T+s)\right)+C_{R,T}$$

which then yields

$$E_2(t) \leqslant C_R, \quad t \in \mathbb{R}.$$

Combining this estimate with (5.9) gives

$$\left\|Z(t)\right\|_{\mathcal{H}} \leqslant C_R$$

and thus

$$\left\|u_{tt}(t)\right\|_2 \leqslant R, \qquad \|u_{ttt}\| \leqslant R, \quad t \in \mathbb{R}.$$

Returning to the equation, we have

$$z_{tt}^{(1)}+Az^{(1)}+d(x)\,g'(u_t)z_t^{(1)}=f'(u)z^{(1)}.$$

This implies that $\|Az^{(1)}(t)\| \leqslant R$, and hence

$$\left\|z^{(1)}\right\|_4=\|u_t\|_4 \leqslant R, \qquad \left\|z_t^{(1)}\right\|_2 \leqslant R.$$

Reiteration of this argument yields the final conclusion. Indeed,

$$z_{tt}^{(n)} + Az^{(n)} + d(x)\,g'(u_t)z_t^{(n)} = \left[v(u) + F_0, z^{(n)}\right] + R\left(u, u_t, \dots, \frac{d^n}{dt^n}u\right)$$

where the term $R(\cdot \dots \cdot)$ contains *only* subcritical terms, and the term $[v(u) + F_0, z^{(n)}]$ is handled identically as in the previous step.

6. Proof of Theorem 1.6 – global attractor

The UC property allows us to conclude that the Lyapunov function $\Phi(u_1, u_2) \equiv \mathcal{E}(u_1, u_2)$ is strict; if we assume $\Phi(t) = \Phi(0)$ for all $t \geqslant 0$, the energy identity in (1.13) forces us to conclude that

$$\int_0^t d(x)\,g(u_t)u_t = 0,$$

and by the UC property, the solution u must be stationary. Hence the dynamical system $(\mathcal{H}, S_t)$ is a gradient system. Having shown the property of asymptotic smoothness for the dynamical system $(\mathcal{H}, S_t)$, we can then conclude the existence of a global attractor, which coincides with the unstable manifold $\mathbf{A} = \mathcal{M}^u(\mathcal{N})$ once we establish the boundedness of the set of stationary points $\mathcal{N}$ (follows from Theorem 2.3 or Corollary 2.29 in [12]). Boundedness of stationery points is obtained from Lemma 1.1 and the following argument:

Estimating stationary solutions of

$$\Delta^2 u = f_V(u) + p$$

gives

$$\left\|A^{1/2}u\right\|^2 = \left(\left[v(u) + F_0, u\right], u\right) + (p, u).$$

Making use of the properties of the von Karman bracket

$$\begin{aligned}\left\|A^{1/2}u\right\|^2 + \frac{1}{2}\left\|\Delta v(u)\right\|^2 &\leqslant \|F_0\|_2\|u\|_2\|u\| + 4\|p\|\|u\| \\ &\leqslant \epsilon\|u\|_2^2 + C_\epsilon\left(\|F\|^2\|u\|^2 + \|p\|^2\right).\end{aligned}$$

By Lemma (1.1)

$$\left\|A^{1/2}u\right\|^2 + \frac{1}{2}\left\|\Delta v(u)\right\|^2 \leqslant \epsilon_1\left(\|u\|_2^2 + \left\|\Delta v(u)\right\|^2\right) + C_{\epsilon_1}\left[\|p\|^2 + M_{p,F_0}\right].$$

Taking ϵ_1 small enough gives

$$\left\|A^{1/2}u\right\|^2 \leqslant C\left(\|p\|, \|F_0\|_\theta\right), \quad \text{with } \theta > 3.$$

The fact that the statements of Theorems 1.4 and 1.5 apply to global attractor $\mathbf{A}$ follows from the inclusion $\mathcal{N} \subset \mathbf{A}_{R_0}$ for some $R_0 > 0$. Since the Lyapunov function is strict, $\mathbf{A}_R = \mathcal{M}^u(\mathcal{N})$ for $R > R_0$. Thus $\mathbf{A}_R$ does not depend on R and coincides with the global attractor $\mathbf{A}$. In view of this, all the properties obtained for $\mathbf{A}_R$ are inherited by $\mathbf{A}$. The proof is now completed.

Acknowledgments

The authors wish to express their gratitude to Alain Haraux and Ralph Chill for a very useful discussion (and for providing references) regarding the applicability of Lojasiewicz inequality in the context of studying convergence to equilibria states.

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