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Series solution of nonlinear two-point singularly perturbed boundary layer problems

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ABSTRACT

The present paper is concerned with the approximate analytic series solution of the nonlinear two-point second-order singularly perturbed boundary value problems. In place of the traditional numerical, perturbation or asymptotic methods, a homotopy technique is employed. It is shown that proper choices of an auxiliary linear operator and also an initial approximation during the implementation of the homotopy analysis method (HAM) can yield uniformly valid and accurate solutions. The fast convergence of the method is ensured by the optimal convergence control parameter obtained through the absolute residual error concept. To demonstrate the favor of the HAM over the traditional finite-difference techniques several nonlinear problems have been solved and compared.

1. Introduction

Singularly perturbed second-order two-point boundary value problems, which received a significant amount of attention in past and recent years arise very frequently in fluid mechanics and other branches of science and engineering. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts. One of the well-known examples is the Navier–Stokes equation of computational fluid dynamics, which is singularly perturbed at high Reynolds number. Equations of this type typically exhibit solutions with layers; that is, the domain of the differential equation contains narrow regions where the solution varies very fast, whereas away from this region the solution behaves smoothly and varies slowly. To handle this type of problem the basic idea is, to divide the domain of integration into two sub-domains and then to apply different schemes on each sub-domain [1] and [2]. References [3–5] and [6] contain a good analytical discussion of the subject. For some further numerical methods one may refer to books [7–10].

Liao in [11] proposed a new technique which is based on the homotopy concept in topology, named the homotopy analysis method (HAM). Unlike the traditional perturbation methods, this technique does not require a small perturbation parameter in the equation. In this method, according to the homotopy technique, a homotopy with an embedding parameter is constructed, and the embedding parameter is considered as a small parameter. Thus the original nonlinear problem is converted into an infinite number of linear problems without using the perturbation techniques; see the book by Liao [12]. Different from other methods, the HAM provides a simple way to control and adjust the convergence region of solution series by means of an auxiliary parameter [13,14].

There are many physical situations in which the sharp changes occur inside the domain of interest, and the narrow regions across which these changes take place are usually referred to as shock layers in fluid and solid mechanics, transition points in quantum mechanics, and strokes lines and surfaces in mathematics. These rapid changes cannot be handled by slow scales, but they can be handled by fast or magnified or stretched scales. Thus, in general, finding numerical solution

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of a boundary value problem is more difficult than finding numerical solution of corresponding initial value problem. In the present paper we use the homotopy analysis technique for the solution of the nonlinear two-point singular boundary layers, without the introduction of extra scales or the need of converting the equations into asymptotic first-order initial value problem. Better auxiliary linear operators and initial approximations are the essential target to be used here within the homotopy concept in order to obtain highly accurate solutions. The optimal convergence is achieved by picking up the best approximating convergence control parameter via the absolute residual error. The proposed linear operators together with the homotopy analysis method provide analytic series solutions which are valid for all the perturbation parameters and are also more accurate than those already available in the literature. Several nonlinear problems are accounted to demonstrate the applicability of the method.

The following strategy is adopted in the rest of the paper. In Section 2 the idea of homotopy analysis method is laid out. Application of the method is implemented in Section 3, in which analytic expressions are derived yielding better results than those in [3,6]. Finally conclusions follow in Section 4.

2. The homotopy analysis method

Liao in [11] proposed a new kind of analytic technique for nonlinear problems, namely the homotopy analysis method. This method is based on the homotopy and has several advantages. To underline, firstly its validity does not depend upon whether or not nonlinear equations under consideration contain small or large parameters, hence it can solve more of strongly nonlinear equations than the perturbation techniques. Secondly, it provides us with a great freedom to select proper auxiliary linear operators and initial guesses so that uniformly valid approximations can be obtained. Thirdly, it gives a family of approximations which are convergent in a larger region. Liao successfully applied the homotopy analysis method to solve some nonlinear problems in mechanics. Fascinating examples are provided within Ref. [12]. To briefly revisit and describe the method let us consider the following nonlinear differential equation

$$N(u) - f(x) = 0, \quad x \in \Omega \tag{2.1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad x \in \Gamma \tag{2.2}$$

where N is an operator having linear and nonlinear parts, B is a boundary operator, u is an unknown analytic function and Γ is the boundary of the domain Ω. By this technique, we construct a homotopy $v(x, p)$ from the cartesian set $\Omega \times [0, 1]$ to R which satisfies

$$H(v, p) = (1 - p)L(v - u_0) + ph[N(u) - f(x)] = 0, \tag{2.3}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (2.1) that satisfies the boundary condition (2.2) and h is a constant that can be adjusted to speed the convergence. It is clear from Eq. (2.3) that for $p = 0$ and $p = 1$ respectively the following hold

$$\begin{aligned} H(v, 0) &= L(v - u_0) = 0, \\ H(v, 1) &= N(u) - f(x) = 0. \end{aligned} \tag{2.4}$$

Hence, it can be deduced from (2.4) that the deformation process of p from zero to unity is just that of the solution from $u_0(x)$ to $u(x)$. This kind of continuous variation is called deformation in topology so that we call Eq. (2.3) the zeroth-order deformation equation. Next, differentiating (2.3) successively and eventually imposing at $p = 0$, the kth-order deformation equations follow as

$$L(u_k - \gamma_k u_{k-1}) = -hR_k, \tag{2.5}$$

with the proper boundary conditions. The constant function γ in (2.5) is defined by $\gamma = 0$ if $k \leq 1$ and $\gamma = 1$ otherwise. Additionally, the function R_k can be found by differentiating the nonlinear operator N.

Considering p as a parameter, the solution to system (2.1) and (2.2) can be naturally expressed taking into account a Taylor expansion of the solution $v(x, p)$ at $p = 0$ and later imposing the expansion at $p = 1$, that is

$$u(x) = u_0(x) + \sum_{k=1}^{\infty} u_k(x), \tag{2.6}$$

where u_k are defined by $u_k = \frac{1}{k!}\frac{\partial^k u}{\partial p^k}|_{p=0}$.

It is well known, as also shown in [15] that properly chosen auxiliary parameters can ensure the convergence of the homotopy analysis method. Thus, it is the auxiliary parameter that provides us, a simple way to ensure the convergence of series solution. Actually, the region of validity of the convergence control parameter can be worked out via displaying the constant-h curves for some certain fixed quantities of physical interest, such as $u''(0)$, $u'''(0)$, etc., as long as they are not zero. However, this technique does not tell one how to pick up a specific value. To avoid this, and to obtain accurate

Table 1
The absolute residual errors at different orders of homotopy approximation M, corresponding to two values of convergence control parameters for the singular problem (3.8).

	$M = 1$	$M = 5$	$M = 10$	$M = 13$
$h = -1$	2.974×10^{-1}	8.128×10^{-3}	1.527×10^{-4}	7.260×10^{-6}
$h = -1.33$	2.385×10^{-1}	1.102×10^{-3}	2.331×10^{-6}	7.601×10^{-8}

series solutions in the relatively lower-order approximations, in the present paper the convergence control parameter h was chosen optimally from the absolute residual error

$$Res(h) = \int_{t_a}^{t_b} |N(u(t)) - f(t)| \mathrm{d}t, \tag{2.7}$$

which is evaluated at the Mth-order of homotopy approximation. Here, the interval $[t_a, t_b]$ is the physical domain of interest. The minimum of (2.7) gives rise to the optimal value of h.

3. Application to boundary layer problems

To demonstrate the applicability of the above presented method we have applied it to three linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in the literature and also approximate solutions are available for a concrete comparison. The approximate solution obtained through the HAM method is compared here with the numerical (so-called exact here) solution computed using the contemporary software package MATHEMATICA.

It was proven in [12] that the convergent solution of the HAM series converges to the true solution of the relevant differential equation. Therefore, there is no doubt that the converged solutions as shown in the figures below fully represent the solutions of the corresponding differential equations.

Example 1. Consider the following singular perturbation problem from [3] (p. 463; Eq. (9.7.1))

$$\epsilon u'' + 2u' + \mathrm{e}^u = 0, \quad u(0) = 0,\ u(1) = 0. \tag{3.8}$$

The structure of differential equation suggests an auxiliary linear operator $L = \epsilon \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}}{\mathrm{d}x}$ with an assumption of the initial solution of the form $u_0 = 0$, that enables us to easily handle the nonlinear term e^u. The term R_k on the right-hand side of deformation equation (2.5) is constructed by

$$R_k = \epsilon u''_{k-1} + 2u'_{k-1} + D\mathrm{exp}_{k-1},$$

where $D\mathrm{exp}_k$ is given by the recurrence formula

$$D\mathrm{exp}_0 = \mathrm{e}^{u_0}, \qquad D\mathrm{exp}_k = \sum_{m=0}^{k-1} \left(1 - \frac{m}{k}\right) u_{k-m} D\mathrm{exp}_m.$$

Taking into account all these and the homotopy introduced in (2.5)–(2.6), the optimal value of h computed from the absolute residual error at the order of $M = 10$ homotopy approximation is $h = -1.33$. To investigate the influence of the convergence control parameter on the solutions, we present the absolute residual errors at different orders of approximation in Table 1. An efficient and fast convergence of the method with the optimum value of h is observed from the table.

With the optimum parameter $h = -1.33$, we obtain a first-order approximation to the solution in subsequent form

$$u = -\frac{133}{200}\left(\frac{\mathrm{e}^{\frac{2}{\epsilon} - \frac{2x}{\epsilon}}}{\left(-1 + \mathrm{e}^{2/\epsilon}\right)} - \frac{\mathrm{e}^{2/\epsilon}}{\left(-1 + \mathrm{e}^{2/\epsilon}\right)} + x\right), \tag{3.9}$$

Fig. 1 demonstrates how the HAM method sufficiently resolves the boundary layer region with a convergent solution. For this example, we have a boundary layer of thickness $\mathrm{O}(\epsilon)$ at $x = 0$. Therefore, Fig. 1 only concentrates on the boundary layer region for the selected value of $\epsilon = 10^{-4}$. In this figure, first-order (dashed curve), fourth-order (dotted curve), 13th-order (thick-dashed curve) HAM results calculated with the optimum convergence control parameter and the exact result (solid curve) are shown. It is seen from the figure that the homotopy solution converges rapidly to the numerical solution, even the fourth-order solution almost graphically collides onto the exact one. The 13th-order HAM result, the numerical result from the MATHEMATICA (exact) and the literature results are also presented in Table 2 for $\epsilon = 10^{-4}$.

Example 2. Now we consider the following singular perturbation problem from [5] (p. 9; Eq. (1.10) Case 2)

$$\epsilon u'' - uu' = 0, \quad u(-1) = 0,\ u(1) = -1. \tag{3.10}$$

Keeping in mind that outside the boundary layer region solution does not change abruptly with the behavior $u(1) = -1$, it is natural to select the linear operator as $L = \epsilon \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}$ and assume the initial approximation to the solution satisfying

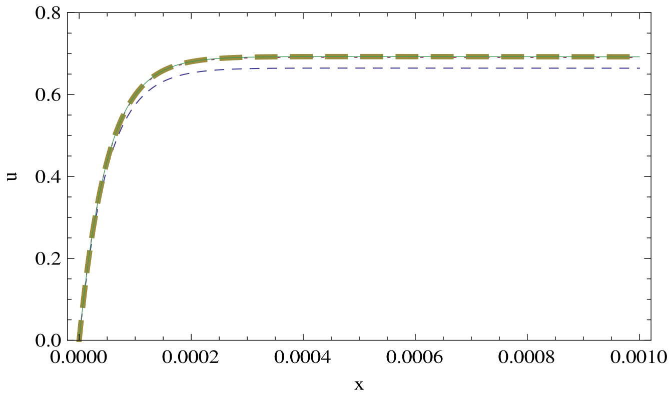

Fig. 1. Solution of boundary layer equation (3.8). For the legends refer to the text.

Table 2
Illustrating the comparisons of (3.8) for $\epsilon = 10^{-4}$. HAM was obtained at 13th-order approximation with the optimum convergence control parameter $h = -1.33$.

x	[6]	[3]	HAM	Exact
0.00	0.0000000	0.0000000	0.0000000	0.0000000
0.01	0.6866081	0.6831968	0.6832306	0.6832306
0.02	0.6766733	0.6733446	0.6733775	0.6733775
0.04	0.6570964	0.6539265	0.6539579	0.6539578
0.06	0.6378974	0.6348783	0.6349082	0.6349081
0.08	0.6190615	0.6161861	0.6162146	0.6162146
0.10	0.6005756	0.5978370	0.5978641	0.5978641
0.20	0.5129692	0.5108256	0.5108469	0.5108469
0.30	0.4324506	0.4307829	0.4307995	0.4307995
0.40	0.3579564	0.3566750	0.3566877	0.3566877
0.50	0.2886464	0.2876821	0.2876917	0.2876917
0.60	0.2238446	0.2231436	0.2231505	0.2231505
0.70	0.1629993	0.1625189	0.1625237	0.1625237
0.80	0.1056545	0.1053605	0.1053634	0.1053634
0.90	0.0514289	0.0512933	0.0512946	0.0512946
1.00	0.0000000	0.0000000	0.0000000	0.0000000

exactly the boundary condition as

$$u_0 = \frac{e^{\frac{1}{\epsilon} - \frac{x}{\epsilon}}}{-1 + e^{2/\epsilon}} - \frac{e^{2/\epsilon}}{-1 + e^{2/\epsilon}},$$

together with the term R_k on the right-hand side of deformation equation (2.5) as

$$R_k = \epsilon u''_{k-1} - \sum_{j=0}^{k-1} u_j u'_{k-1-j}.$$

In this example, the optimal value of h computed from the absolute residual error at the order of $M = 10$ homotopy approximation is $h = -1.021$, which yields 10^{-14} as the absolute residual error.

For this example, we have a boundary layer of width $O(\epsilon)$ at $x = -1$(see [5], p. 9–10, Eq. (1.10), (1.13) and (1.14), Case 2). Therefore, Fig. 2 only concentrates on the boundary layer region for the fixed value of $\epsilon = 10^{-4}$. In figure, leading-order (dashed curve), first-order (dash-dotted curve), third-order (dotted curve), 11-order (thick-dashed curve) HAM results calculated with the optimum convergence control parameter and the exact result (solid curve) are shown. It is seen from the figure that the homotopy solution converges rapidly to the numerical solution, even the third-order solution almost graphically collides onto the exact one. Fig. 2 also demonstrates how the HAM method sufficiently resolves the boundary layer region with a convergent solution. The 11th-order HAM result, the numerical result from the MATHEMATICA (exact) and the literature results are also presented in Table 3 for $\epsilon = 10^{-4}$.

Example 3. Finally, we consider the following singular perturbation problem from [4] (p. 56; Eq. (2.5.1))

$$\epsilon u'' + uu' - u = 0, \quad u(0) = -1, \; u(1) = 3.9995. \tag{3.11}$$

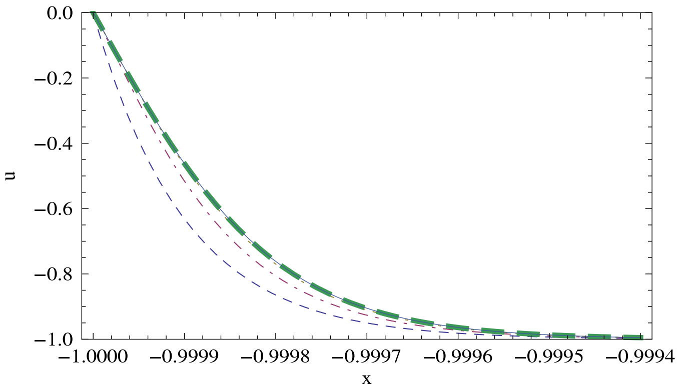

Fig. 2. Solution of boundary layer equation (3.10). For the legends refer to the text.

Table 3
Illustrating the comparisons of (3.10) for $\epsilon = 10^{-4}$. HAM was obtained at 11th-order approximation with the optimum convergence control parameter $h = -1.021$.

x	[6]	[5]	HAM	Exact
−1.000	0.0000000	0.0000000	0.0000000	0.0000000
−0.999	–	–	−0.9999092	−0.9999092
−0.800	−1.0000000	−1.0000000	−1.0000000	−1.0000000
−0.500	−1.0000000	−1.0000000	−1.0000000	−1.0000000
−0.200	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.0000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.2000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.4000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.6000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.8000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
1.0000	−1.0000000	−1.0000000	−1.0000000	−1.0000000

If we now choose the linear operator as $L = \epsilon \frac{d^2}{dx^2} + c\frac{d}{dx}$ and assume the initial approximation to the solution as

$$u_0 = -\frac{c e^{\frac{c}{\epsilon} - \frac{cx}{\epsilon}}}{-1 + e^{c/\epsilon}} + \frac{1 - e^{c/\epsilon} + c e^{c/\epsilon}}{-1 + e^{c/\epsilon}} + x$$

with $c = 3.9995$, the term R_k on the right-hand side of deformation equation (2.5) is computed as

$$R_k = \epsilon u''_{k-1} - u_{k-1} + \sum_{j=0}^{k-1} u_j u'_{k-1-j}.$$

The optimal value of the convergence control parameter is found to be $h = -0.988$ computed from the absolute residual error (2.7) at the order of $M = 10$ homotopy approximation. The absolute residual error at this order is evaluated as 10^{-8}.

For this example we also have a boundary layer of width $O(\epsilon)$ at $x = -1$ ([4] pp. 56–66). Therefore, Fig. 3 only concentrates on the boundary layer region for the fixed value of $\epsilon = 10^{-4}$. In figure, leading-order (dashed curve), first-order (dash-dotted curve), third-order (dotted curve), 14-order (thick-dashed curve) HAM results calculated with the optimum convergence control parameter and the exact result (solid curve) are shown. It is seen from the figure that the homotopy solution converges rapidly to the numerical solution, even the third-order solution almost graphically collides onto the exact one. The 20th-order HAM result, the numerical result from the MATHEMATICA (exact) and the literature results are further presented in Table 4 for $\epsilon = 10^{-4}$.

It can be readily deduced from the considered examples that singular two-point boundary value problems exhibiting stronger singularities or possessing larger physical intervals can also be tackled by the powerful HAM method as implemented here. It should be emphasized that only a few order approximate solutions that we obtained reveal excellent agreement with the exact numerical solutions. Addition of higher approximations from the homotopy technique would naturally yield more remarkable agreement. It is furthermore worthwhile to state that the homotopy solutions obtained here are valid for all the values of the parameter ϵ. To conclude, the advantage of the HAM solutions obtained here is that they represent solutions all over the domain both inside the boundary layer and outside it. Therefore, values corresponding to any ϵ are calculated from a single formula. However, the conventional numerical techniques such as the finite differences require finer meshes particularly distributed inside the boundary layer region to resolve the field as the values of ϵ gets decreased drastically.

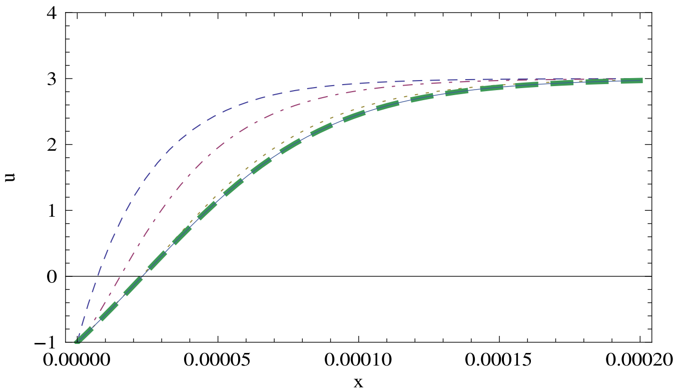

Fig. 3. Solution of boundary layer equation (3.11). For the legends refer to the text.

Table 4
Illustrating the comparisons of (3.11) for $\epsilon = 10^{-4}$. HAM was obtained at 20th-order approximation with the optimum convergence control parameter $h = -0.988$.

x	[6]	[4]	HAM	Exact
0.00	−1.000000	−1.000000	−1.000000	−1.000000
0.01	3.0161450	3.0095000	3.0095000	3.0095000
0.02	3.0194790	3.0195000	3.0195000	3.0195000
0.04	3.0395010	3.0395000	3.0395000	3.0395000
0.06	3.0595010	3.0595000	3.0595000	3.0595000
0.08	3.0795010	3.0795000	3.0795000	3.0795000
0.10	3.0995010	3.0995000	3.0995000	3.0995000
0.20	3.1995010	3.1995000	3.1995000	3.1995000
0.30	3.2995010	3.2995000	3.2995000	3.2995000
0.40	3.3995010	3.3995000	3.3995000	3.3995000
0.50	3.4995010	3.4995000	3.4995000	3.4995000
0.60	3.5995000	3.5995000	3.5995000	3.5995000
0.70	3.6995000	3.6995000	3.6995000	3.6995000
0.80	3.7995000	3.7995000	3.7995000	3.7995000
0.90	3.8995000	3.8995000	3.8995000	3.8995000
1.00	3.9995000	3.9995000	3.9995000	3.9995000

4. Concluding remarks

In this paper the homotopy analysis method (HAM) is employed, for the first time, to the nonlinear boundary value problems with singularity, i.e. the highest-order derivative is multiplied by a very small parameter.

Taking the advantage of free selection of the linear operator and the initial approximation to the solution, three nonlinear examples with boundary layers have been treated. The homotopy method with these choices are shown to generate analytic approximations that are more accurate than the numerical results available in the literature. The optimum convergence control parameters calculated using the absolute residual error concept make the method more feasible in terms of rapid convergence. The graphical displays are clear evident that the homotopy analysis method adhered is able to adequately resolve the boundary layer region.

References

[1] S.M. Roberts, A boundary value technique for singular perturbation problems, J. Math. Anal. Appl. 89 (1982) 489–508.
[2] M.K. Kadalbajo, Y.N. Reddy, Numerical treatment of singularly perturbed two point boundary value problems, Appl. Math. Comput. 21 (1987) 93–110.
[3] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientist and Engineers, McGraw-Hill, Inc., 1978.
[4] J. Kevorkian, J.D. Cole, Perturbation Methods in Applied Mathematics, Springer-Verlag, 1981.
[5] R.E. O'Maley, Singular Perturbation Methods for Ordinary Differential Equations, Springer, 1991.
[6] Y.N. Reddy, P.P. Chakravarthy, An exponentially fitted finite difference method for singular perturbation problems, Appl. Math. Comput. 21 (2003) 235–284.
[7] J.J.H. Miller, E. ORiordan, G.I. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems, World Scientific, 1996.
[8] P.W. Hemkarr, J.J.H. Miller, Numerical Analysis of Singular Perturbation Problems, Academic Press, 1979.
[9] E.P. Doolan, J.J.H. Miller, W.H.A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, 1980.
[10] H.G. Roos, M. Stynes, L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Springer-Verlag, 1991.
[11] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.
[12] S.J. Liao, Beyond Perturbation: Introduction to Homotopy Analysis Method, Chapman & Hall/CRC, 2003.
[13] S.J. Liao, I. Pop, Explicit analytic solution for similarity boundary layer equations, Int. J. Heat Mass Transfer 47 (2004) 75–85.
[14] Y. Cheng, S.J. Liao, On the explicit, purely analytic solution of von karman swirling viscous flow, Commun. Nonlinear Sci. Numer. Simulat. 47 (2006) 75–85.
[15] S.J. Liao, Notes on the homotopy analysis method: Some definitions and theorems, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 983–997.