Topology and its Applications 44 (1992) 57-62
North-Holland

Semi-sequentially normal bitopological spaces

Lawrence M. Brown
Department of Mathematics, Hacettepe University, Beytepe, Ankara, Turkey

Received 1 October 1989
Revised 23 September 1990 and 5 November 1990

Abstract

Brown, L.M., Semi-sequentially normal bitopological spaces, Topology and its Applications 44 (1992) 57-62.

In this paper we introduce a weak form of full normality for bitopological spaces, and consider its relationship to pairwise paracompactness in the sense of S. Romaguera and J. Marin, and to the notion of σ-bicushioned refinement for dual covers.

Keywords: Dual cover, sequentially normal, p-q-metrizable, semi-sequentially normal, semi-open dual cover, bicushioned refinement, σ-bicushioned refinement.

AMS (MOS) Subj. Class.: 54E55.

1. Introduction

The notion of sequential normality for bitopological spaces is considered in [2-4]. It is shown to be a generalization of topological full normality which, unlike full binormality, is satisfied by all p-q-metric bitopological spaces. In [4] it is shown that a bitopological space is p-q-metrizable if and only if it is bidevelopable and sequentially normal, thereby providing a generalization of Bing's metrization theorem. In Section 2 we consider an apparently weaker form of sequential normality, namely semi-sequential normality, and show that the above p-q-metrization theorem remains valid in this case. We also relate this form of normality to the notion of pairwise paracompactness introduced recently by Romaguera and Marin [9]. In Section 3 we consider the notion of σ-bicushioned refinement of a dual cover, and relate this to semi-sequential normality.

2. Semi-sequential normality

We recall [1-5] that a *dual family* d on a set X is a binary relation on the power set of X, and that $\mathrm{jc}(d)=\bigcup\{U\cap V \mid UdV\}$. A dual family with $\mathrm{jc}(X)=X$ is called a *dual cover* of X. There will be no loss of generality in assuming that $UdV \Rightarrow U\cap V\neq \emptyset$. We call d *open* for a bitopological space (X, u, v) if $d\subseteq u\times v$. Refinement is defined in the obvious way, while in this paper star refinement is based on the stars

$$\mathrm{St}(d, A)=\bigcup\{U \mid \exists\, V,\ UdV \text{ and } V\cap A\neq\emptyset\}$$

and

$$\mathrm{St}(A, d)=\bigcup\{V \mid \exists\, U,\ UdV \text{ and } U\cap A\neq\emptyset\}$$

of $A\subseteq X$. This is the form of star refinement involved in the covering characterization of quasi-uniformities given by Gantner and Steinlage [6], and used by the author in defining various generalizations of quasi-uniformity, see for example [1, 2]. We write $e<(*)d$ if e is a star refinement of d, and say the dual cover d is *normal* for (X, u, v) if there is a sequence $\langle d_n\rangle$ of open dual covers with $d_0<d$ and $d_{n+1}<(*)d_n$ for all n.

Let us recall that a bitopological space (X, u, v) is called *sequential normal* [4] if given an open dual cover d of X there exist (open) normal dual covers e_n and open dual families d_n satisfying

(i) $\bigcup \mathrm{jc}(d_n)=X$ and

(ii) $e_n * f_n=\{(\mathrm{St}(e_n, U), \mathrm{St}(V, e_n)) \mid Ud_nV\}<d,\ n\in N$.

We now wish to generalize the notion of sequential normality. Let us call a dual cover d of (X, u, v) *semi-open* if for each $x\in X$, $\mathrm{St}(d, x)$ is a u-neighborhood and $\mathrm{St}(x, d)$ is a v-neighborhood of x. Since a (normal) open dual cover is semi-open the following notion is apparently weaker than sequential normality:

Definition 2.1. (X, u, v) is *semi-sequentially normal* if given an open dual cover d there exist sequences (d_n) of open dual families and (e_n) of semi-open dual covers satisfying (i) and (ii) above.

Theorem 2.2. *(X, u, v) is p-q-metrizable if and only if it is semi-sequentially normal and bidevelopable.*

Proof. Necessity is clear, so we outline the proof of the sufficiency. First let us note that a bidevelopable space is pairwise R_0. Hence by [5, Theorem 2.1] it will be sufficient to show the existence of a countable bineighborhood basis $\{(R(n, x), S(n, x)) \mid n\in N\}$ at each $x\in X$ satisfying the following two conditions:

(D) Given $x\in X$ and n there exist $x\in U_1\in u$, $x\in V_1\in v$ so that

$$y\in U_1 \;\Rightarrow\; x\in S(n, y) \quad\text{and}\quad y\in V_1 \;\Rightarrow\; x\in R(n, y),$$

(E) given $x\in X$, $x\in U\in u$, $x\in V\in v$ there exist $x\in U_2\in u$, $x\in V_2\in v$ and r so that

$$y\in U_2 \;\Rightarrow\; R(r, y)\subseteq U \quad\text{and}\quad y\in V_2 \;\Rightarrow\; S(r, y)\subseteq V.$$

Let a bidevelopment for (X, u, v) be (b_m). Then for each m we have semi-open dual covers $f_{m,n}$ and open dual families $d_{m,n}$ so that

(i) $\bigcup\{\mathrm{jc}(d_{m,n}) \mid n \in N\} = X$, and

(ii) $f_{m,n} * d_{m,n} < b_m$ for all $n \in N$.

If we define $R(m, n, x) = \mathrm{St}(f_{m,n}, x)$ and $S(m, n, x) = \mathrm{St}(x, f_{m,n})$, then it is easy to see that $\{(R(m, n, x), S(m, n, x)) \mid m, n \in N\}$ is a countable bineighborhood base at x. Clearly (D) may be satisfied by taking $U_1 \subseteq R(m, n, x)$ and $V_1 \subseteq S(m, n, x)$. Now take $x \in U \in u$ and $x \in V \in v$. We have m with $\mathrm{St}(b_m, x) \subseteq U$ and $\mathrm{St}(x, b_m) \subseteq V$, and for this m we have n with $x \in \mathrm{jc}(d_{m,n})$. (E) is now satisfied by taking $U_2 d_{m,n} V_2$ with $x \in U_2 \cap V_2$. □

We now compare sequential and semi-sequential normality with the notion of pairwise paracompactness introduced by Romaguera and Marin [9], which is based on a characterization of regular paracompact (i.e., fully normal) topological spaces due to Junnila [7]. The restriction to T_1 spaces and quasi-metrization in [9] seems unnecessarily restrictive as it excludes such a fundamental space as (R, s, t)—the reals with the lower and upper topologies. Also, when regarded as a generalization of full normality, the assumption of pairwise regularity may be omitted. Hence for convenience we shall refer to a not necessarily T_1 nor pairwise regular bitopological space—otherwise satisfying the conditions of [9, Definition 4]—as an R-M-normal bitopological space. We now have:

Theorem 2.3. *Sequentially normal* $\Rightarrow$ *R-M-normal* $\Rightarrow$ *semi-sequentially normal.*

Proof. First let d be an open dual cover in the sequentially normal space (X, u, v). Then by [2, Theorem 1.4.2] there exists a p-q-metric p with the property that $H_n(x) = \{y \mid p(x, y) < 2^{-n}\} \in u$, $K_n(x) = \{y \mid p(y, x) < 2^{-n}\} \in v$ for each n, and such that, given $x \in X$, there exist n and UdV with

$$H_n(x) \subseteq U \quad \text{and} \quad K_n(x) \subseteq V. \tag{1}$$

Hence $\{(H_n(x), K_n(x)) \mid n \in N\}$ is a countable family of bineighborhoods of x, and

(i) $y \in H_n(x) \Leftrightarrow x \in K_n(y)$.

(ii) For $x \in X$ we have $n \in N$ and UdV with $H_n(x) \subseteq U$, $K_n(x) \subseteq V$ by (1), and then $H^2_{n+1}(x) = \bigcup\{H_{n+1}(y) \mid y \in H_{n+1}(x)\} \subseteq H_n(x) \subseteq U$, and likewise $K^2_{n+1}(x) \subseteq V$. This verifies [9, Definition 4], so showing that (X, u, v) is R-M-normal.

Now let d be an open dual cover in an R-M-normal space, and let $\{(U_n(x), V_n(x)) \mid n \in N\}$ be a family of bineighborhoods satisfying [9, Definition 4]. Define

$$f_n = \{(\{x\}, V_n(y)) \mid x \in V_n(y)\}.$$

Then the condition $x \in U_n(y) \Leftrightarrow y \in V_n(x)$ implies f_n is a semi-open dual cover with $\mathrm{St}(f_n, x) = U_n(x)$ and $\mathrm{St}(x, f_n) = V_n(x)$. Finally let

$$d_n = \{(U_n(x), V_n(x)) \mid \exists\, UdV,\ x \in U \cap V,\ U_n(x) \subseteq U,\ V_n(x) \subseteq V\}.$$

Then Definition 2.1 (i) and (ii) are easily verified for this choice of (f_n), (d_n). □

In view of this result [9, Theorem 1] is also a consequence of Theorem 2.2. However I do not know if either of the implications in Theorem 2.3 is reversible.

3. σ-bicushioned refinements of dual covers

In this section we consider the notions of bicushioned and σ-bicushioned refinement for dual covers.

Definition 3.1. The (faithfully indexed) dual cover $e = \{(R_a, S_a) \mid a \in A\}$ is said to be *bicushioned* in the dual cover d, or to be a *bicushioned refinement* of d, if for each $a \in A$ there exists $U_a d V_a$ so that

$$\mathrm{cl}_v(\bigcup \{R_a \mid a \in A'\}) \subseteq \bigcup \{U_a \mid a \in A'\},$$

and

$$\mathrm{cl}_u(\bigcup \{S_a \mid a \in A'\}) \subseteq \bigcup \{V_a \mid a \in A'\}$$

for all $A' \subseteq A$.

It is clear that the assumption that e be faithfully indexed may be removed. The notion of bicushioned refinement is closely related to that of semi-open dual cover, as the next proposition shows.

Proposition 3.2. *The open dual cover d has a bicushioned refinement if and only if there exists a semi-open dual cover f with $f < (\triangle) d$.*

Proof. If f exists with the stated properties, then clearly $e = \{(\{x\}, \{x\}) \mid x \in X\}$ is a bicushioned refinement of d. Conversely let $e = \{(R_a, S_a) \mid a \in A\}$ be a bicushioned refinement of d, and let $U_a d V_a$ be as in Definition 3.1. For $x \in X$ choose $a(x) \in A$ with $x \in R_{a(x)} \cap S_{a(x)}$. Then $f = \{(\{x\}, V_{a(x)} \cap \{y \mid x \in U_{a(y)}\}) \mid x \in X\}$ is easily seen to satisfy the stated properties. □

Contrary to the single topology case an open dual cover of a p-q-metric bitopological space need not have a bicushioned refinement. To see this we consider the following example.

Example 3.3 [2, 3]. Let X be the closed first quadrant of the plane. Let u consist of $\emptyset$ and all sets G satisfying

(i) $(x, y) \in G,\ 0 < x' \leq x \Rightarrow (x', y) \in G$,

(ii) $(x, y) \in G,\ 0 < y \leq y' \Rightarrow (x, y') \in G$, and

(iii) $\exists y > 0,\ (0, y) \in G$.

Clearly u is a topology on X, and so is $v = \{G^{-1} \mid G \in u\}$. It is shown in [2, 3] that the bitopological space (X, u, v) is p-q-metrizable. Consider the finite open dual cover

$$d = \{(G_1, X), (G_2, X)\},$$

where $G_1 = \{(x, y) \mid y > 0\}$ and $G_2 = \{(x, 0) \mid x \geq 0\} \cup \{(0, y) \mid y \geq 0\}$. Suppose that $e = \{(R_a, S_a) \mid a \in A\}$ is bicushioned in d, and for $x \geq 0$ choose $a(x) \in A$ so that $(x, 0) \in R_{a(x)} \cap S_{a(x)}$, and put $A' = \{a(x) \mid x \geq 0\}$. Clearly every nonempty v-open set meets $\bigcup \{R_a \mid a \in A'\}$, and so $\mathrm{cl}_v(\bigcup \{R_a \mid a \in A'\}) = X$. On the other hand $(x, 0) \in R_{a(x)} \cap S_{a(x)} \subseteq U_{a(x)} \cap V_{a(x)} \Rightarrow U_{a(x)} = G_2$ for all $x \geq 0$, and this gives an immediate contradiction.

This example shows that the notion of bicushioned refinement is too powerful to consider in the context of p-q-metric spaces, and so we make the following:

Definition 3.4. The dual family e is said to be *σ-bicushioned* in d if we may write $e = \bigcup \{e_n \mid n \in N\}$ with each e_n bicushioned in d. A σ-bicushioned refinement of a dual cover d is a dual cover e which is σ-bicushioned in d.

We may now state:

Theorem 3.5. *In a semi-sequentially normal space every open dual cover has an open σ-bicushioned refinement.*

Proof. Let d be an open dual cover, and d_n, e_n as in Definition 2.1. It is trivial to verify that $\bigcup \{d_n \mid n \in N\}$ is the required open σ-bicushioned refinement of d. □

It may be verified that if every open dual cover of (X, u, v) has an open σ-bicushioned refinement, then with respect to the joint topology $u \vee v$ on X every open cover has a σ-cushioned open refinement. Hence by a standard theorem of general topology [8, Theorem V.4] we may state:

Corollary. *A weakly pairwise T_1 semi-sequentially normal bitopological space is jointly paracompact.*

It is natural to wonder about the converse of Theorem 3.5. If every open dual cover has an open σ-bicushioned refinement, must the space be semi-sequentially normal? The answer is not known, but we do have the following result:

Theorem 3.6. *Suppose that in (X, u, v) every (finite) open dual cover has an open σ-bicushioned refinement. Then (X, u, v) is pairwise normal.*

Proof. Take a u-closed set P and a v-closed set Q with $P \cap Q = \emptyset$, and consider the open dual cover $d = \{(X - P, X), (X, X - Q)\}$. Let e_n be open bicushioned refinements of d whose union is a dual cover of X. Let

$$U_n = \bigcup \{R \mid \exists R e_n S,\ R \cap S \cap Q \neq \emptyset\},$$

$$V_n = \bigcup \{S \mid \exists R e_n S,\ R \cap S \cap P \neq \emptyset\}.$$

Then since e_n is bicushioned in d we see that $\mathrm{cl}_v(U_n) \cap P = \emptyset$ and $\mathrm{cl}_u(V_n) \cap Q = \emptyset$. Let

$$U_n^* = U_n - \mathrm{cl}_u(\bigcup \{V_k \mid 0 \leq k \leq n\}),$$

$$V_n^* = V_n - \mathrm{cl}_v(\bigcup \{U_k \mid 0 \leq k \leq n\}),$$

$U = \bigcup \{U_n^* \mid n \in N\}$ and $V = \bigcup \{V_n^* \mid n \in N\}$. Then clearly $Q \subseteq U \in u$, $P \subseteq V \in v$ and $U \cap V = \emptyset$. Hence (X, u, v) is pairwise normal. □

Corollary. *A semi-sequentially normal, and hence an R-M-normal, bitopological space is pairwise normal.*

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